Lecture notes: Observability of linear systems

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Outline

This lecture note is based on

- Karl Johan Aström Richard M. Murray, *Feedback Systems, An introduction to Scientists and Engineers*. Chapter 6-7. http://www.cds.caltech.edu/~murray/amwiki/index.php/Second_Edition

A brief review

so far, we learned state feedback control for linear system.

- Determine if the system is reachable by the rank condition.
- controller is a feedback law using the state of the system and a reference input.
- Pole placement design for achieving stability.

in some cases, we do not directly measure the state of the system — observability and output feedback.

Observability: Basic definitions

consider a LTI system:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

for time t, state vector x, output y.

Observability: A linear system is **observable** if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$.
(same definition for nonlinear system.)

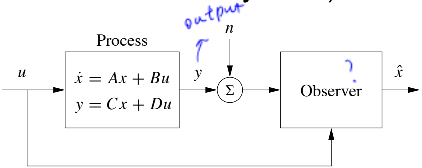

Figure 7.1: Block diagram for an observer. The observer uses the process measurement y (possibly corrupted by noise n) and the input u to estimate the current state of the process, denoted $\hat{x}$.

Testing for observability

Under which condition we can determine the state ~~estimate $\hat{x}$~~ x from the output y?

$$\dot{x} = Ax$$
$$\underset{\sim}{y} = Cx$$

$$\Rightarrow \quad y = Cx$$
$$\dot{y} = C\dot{x} = CAx$$
$$\ddot{y} = C\ddot{x} = CA^2x$$
$$\vdots$$
$$y^{(n-1)} = Cx^{(n-1)} = CA^{(n-1)}x$$

$$\underbrace{\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{(n-1)} \end{bmatrix}}_{W_o \text{ Full rank}} x \quad \Rightarrow \quad v = W_o x, \quad x = W_o^{-1} v$$

Observability of LTI systems

Theorem 7.1 (Observability rank condition). A linear system is observable if and only if the observability matrix $W_O = [C; CA; CA^2; \ldots; CA^{n-1}]$ is **full rank.**

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

To prove the necessary condition, we briefly review Carley-Hamilton theorem.

Cayley-Hamilton theorem:

Recall the characteristic polynomial of a square matrix A is

$$p(\lambda) = \det[\cancel{A\lambda I}]\ [A - \lambda I]$$

The Cayley-Hamilton theorem states: $P(A) = \mathbf{0}$ where $\mathbf{0}$ is the zero matrix [1]

Observability of LTI systems

$\dot{x} = Ax$

$x(t) = e^{At} x_0$

$y(t) = C e^{At} x_0$

$= C \sum_{k=0}^{\infty} \frac{(At)^k}{k!} x_0$

$= \sum_{k=0}^{n-1} \alpha_k(t)\, C A^k x_0.$

$$= [\alpha_0(t)\ \alpha_1(t) \cdots\ \alpha_{n-1}(t)] \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0}_{W_0 X_0} = 0$$

$W_0 X_0 \neq 0$

if W_0 is not full rank.

Then $\exists\, x_0 \neq 0$

$\Rightarrow y(t) = 0 \quad \forall t.$

$\underline{P(A) = 0}$

$A^n = \alpha_0 + \alpha_1 A^1 + \alpha_2 A^2 + \cdots + \alpha_{n-1} A^{n-1}$

$= \sum_{i=0}^{n-1} \alpha_i A^i$

The null space W_0 is not singleton $\{0\}$

Observer design

Consider a simple system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

insight: If we know the input, then we can determine the state by simulate the dynamics:

- the estimated state is $\hat{x}$: $\dot{\hat{x}} = A\hat{x} + Bu$.
- the estimated output should be $C\hat{x}$.

what if y does not equal $C\hat{x}$ — there is estimation error $x - \hat{x}$.

$$\tilde{x} = x - \hat{x}$$

$$\frac{d\tilde{x}}{dt} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu$$

$$= A(x - \hat{x}) = A\tilde{x}$$

if A is stable then $\tilde{x}(t) \to 0$ as $t \to \infty$

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Design the observer such that the **estimation error goes to zero.**

Observer design

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Design the observer such that the **estimation error goes to zero.**

Equivalent to: design the observer such that the system with state $x - \hat{x}$ stabilizes to the origin.

Observer design == Stabilizing the dynamical system with $x - \hat{x}$ as the state.

$\tilde{x} = x - \hat{x}$

- What is the dynamics of the "new" system?
- What is the input of the "new" system?
- Is it stablizable? How to design "stabilizing control"?

$\dot{\hat{x}} = A\hat{x} + Bu$

output $\underline{y - C\hat{x}}$

$\dot{\hat{x}} = \underline{A\hat{x} + Bu} + L(y - C\hat{x})$

$\frac{d\tilde{x}}{dt} = Ax + Bu - A\hat{x} - Bu - L(y - C\hat{x})$ $\quad y = Cx$

$= A(x - \hat{x}) - L(Cx - C\hat{x}) = (A - LC)(x - \hat{x}) = \underline{(A - LC)}\tilde{x}$

$A - LC$ is a stable matrix

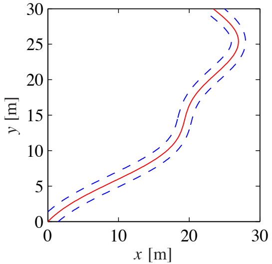
Example: Vehicle steering

a normalized linear model for vehicle steering is

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \gamma \\ 1 \end{bmatrix}}_{B} u$$

u: steering angle.

where x_1 — the lateral path deviation, x_2 — the turning rate.
goal: determine the turning rate x_2 from the measured path deviation x_1.
output: $y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$. $= x_1$



observer:

① Decide observability.

$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

② $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

$\dot{\tilde{x}} = (A - LC)\tilde{x}$ where $\tilde{x} = x - \hat{x}$

$$A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}$$

$$\det\left(\lambda I - (A - LC)\right) = \det\left(\begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} - \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}\right)$$

$$J = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$$

$$= \det\left(\begin{bmatrix} \lambda + l_1 & -1 \\ l_2 & \lambda \end{bmatrix}\right) = \underline{\lambda^2 + l_1 \lambda + l_2}$$

$te^{\lambda t}$

$$\boxed{\lambda^2 + 2 \xi_0 w_0 \lambda + w_0^2}$$

$$\frac{-l_1 \pm \sqrt{l_1^2 - 4 l_2}}{2}$$

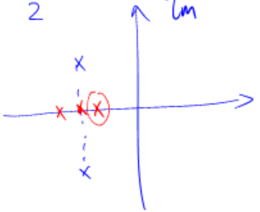
$$\lambda_{1,2} = \frac{-2\xi_0 w_0 \pm \sqrt{4 \xi_0^2 w_0^2 - 4 w_0^2}}{2}$$

$$= -\xi_0 w_0 \pm w_0 \sqrt{\underbrace{\xi_0^2 - 1}}$$

> 0

< 0

$= 0$



$\dot{x} = -5x$ and $\dot{x} = -x$

Control with the estimated state

In general, a system is

$$\dot{x} = Ax + Bu, \quad y = Cx.$$

suppose the system is both controllable and observable.
goal: How to determine the control u as a feedback from the **estimated state** such that

- the controlled system stabilizes the system and tracks a reference value $r(t)$. e.g., $r = 3(\sin(w_1 t) + sin(2w_1 t))$ can be used to approximate a curved road.

Assumption: A controller is of the form

$$u(t) = -K\hat{x}(t) + k_r r(t)$$

where $\hat{x}$ is the output of an observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

Control with the estimated state: Separation principle

The tracking error: $\tilde{x} = x - \hat{x}$.

$$\frac{dx}{dt} = Ax + Bu = Ax + B(-K\hat{x} + K_r r(t))$$

$$\frac{d\tilde{x}}{dt} = \frac{dx}{dt} - \frac{d\hat{x}}{dt} = (A - LC)\tilde{x}$$

$$\frac{dx}{dt} = Ax + B(-K(x - \tilde{x}) + K_r r(t))$$

$$= (A - BK)x + BK\tilde{x} + BK_r r(t)$$

$\tilde{x} \to 0$

$$\boxed{\frac{dx}{dt} = (A - BK)x + BK_r \underset{x_f}{r(t)}.}$$

$\to$ state feedback goal reaching control.

Augmented state: $z = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$

$$\frac{dz}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{d\tilde{x}}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} (A - BK) & BK \\ 0 & (A - LC) \end{bmatrix}}_{A'} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} BK_r \\ 0 \end{bmatrix} r(t)$$

A' is stable

$$\underbrace{\det[\lambda I - A']}_{} = \det(\lambda I - (A - BK)) \cdot \det[\lambda I - (A - LC)]$$

eig. value of A' neg real parts.

$A - BK$

$A - LC$

Separation Principle

Theorem 7.3 (Eigenvalue assignment by output feedback). *Consider the system*

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

The controller described by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) = (A - BK - LC)\hat{x} + Ly,$$
$$u = -K\hat{x} + k_r r$$

gives a closed loop system with the characteristic polynomial

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC).$$

This polynomial can be assigned arbitrary roots if the system is reachable and observable.

The overall observation-based control system

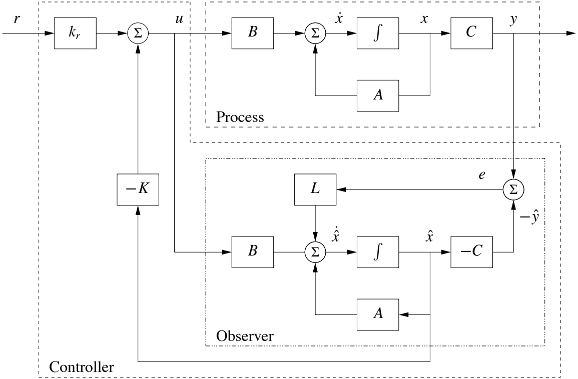


Figure 7.7: Block diagram of an observer-based control system. The observer uses the measured output y and the input u to construct an estimate of the state. This estimate is used by a state feedback controller to generate the corrective input. The controller consists of the observer and the state feedback; the observer is identical to that in Figure 7.5.

Vehicle steering example

Example 7.4 from Chapter 7.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} \gamma \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$\underline{A-LC}$ Stable: $\quad L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$

Control: $\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \gamma & 1 \\ 1 & 0 \end{bmatrix}$

$A - BK \qquad K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$

$u = -K\hat{x} + k_r r(t)$

Summary

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

$$\boxed{\ddot{q} = a} = M(q)^{-1}(\tau - C(q,\dot{q})\dot{q} - N(q))$$

- Learned how to estimate the state using an observer.
- Closed-loop design for both controller and observer.
- The separation principal.

- For dynamic trajectory tracking, we need to include feedforward component.
- Many important topics are not covered yet: Kalman decomposition, Kalman filter (observer with noise measurement), etc. (will do if time permitted. Read Chapter 6 and 7 to learn more.)