Lecture notes: Robust Control

Jie Fu

Department of Electrical and Computer Engineering
Robotics Engineering Program
Worcester Polytechnic Institute

RBE502, 2018

Outline

This lecture note is based on

- Chapter 8 in M. Spong **Robot modeling and control.**

Recap: Computed torque control

The dynamic model of the robot manipulator is

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

The inverse dynamic control input

$\ddot{q} = a_q$

$$\tau = M(q)a_q + C(q,\dot{q})\dot{q} + N(q)$$

If the model is incorrect, we have

The inverse dynamic control input

$$\tau = \bar{M}(q)a_q + \bar{C}(q,\dot{q})\dot{q} + \bar{N}(q)$$

And subsititute into the dynamic model

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \bar{M}(q)a_q + \bar{C}(q,\dot{q})\dot{q} + \bar{N}(q)$$

yields

$$M(q)\ddot{q} = \bar{M}(q)a_q + (\bar{C} - C)\dot{q} + \bar{N} - N.$$

and let $\hat{X} = \bar{X} - X$

$$\ddot{q} = \underline{M^{-1}(\bar{M}(q)}a_q + \underline{\hat{C}}\dot{q} + \underline{\hat{N}}). \qquad \underline{\ddot{q} = a_q}$$

$$= a_q \underbrace{- a_q + M^{-1}(\bar{M}(q)a_q + \hat{C}\dot{q} + \hat{N})}_{\eta(q,\dot{q},a_q)}$$

Bound on the "disturbance"

$$\ddot{q} = a_q + \eta(a_q, q, \dot{q})$$

The disturbance term can be

$$\eta(a_q, q, \dot{q}) = (M^{-1}\bar{M} - I)a_q + M^{-1}(\hat{C}\dot{q} + \hat{N})$$

previously, with accurate modeling, we have

$e = q - q_d$

$$a_q = \ddot{q}_d - K_p e - K_D \dot{e}.$$

adding additional input. we have

$$a_q = \ddot{q}_d - K_p e - K_D \dot{e} + \underset{\Delta}{\mathbf{v}}.$$

Bound on the “disturbance”

Since we do not know $\eta(a_q, q, \dot{q})$, can we bound the size?
why? — Robust control of linear systems 101—Lyapunov second method.
Given $x = [e, \dot{e}]$.

$\dot{e} = \dot{e}$

$\ddot{e} = -K_P - K_D + (v + \eta)$

$$\dot{x} = Ax + B(v + \eta)$$

Since A is hurwitz, we select a candidate Lyapunov function for $\dot{x} = Ax$:

$$V(x) = x^T P x$$

$\ddot{q} = a_q$

$a_q = \ddot{q}_d - K_P(q - q_d) - K_D(\dot{q} - \dot{q}_d)$

$A = \begin{bmatrix} 0 & 1 \\ -K_P & -K_D \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

P : pos. def. $V(x)$

$\dot{V} = x^T P A x + x^T A^T P x$

$= x^T (PA + A^T P) x$ neg. def.

introduce $Q = -(PA + A^T P)$

$= -x^T Q x$

Q : pos. def.

$V = x^T P x$ P : pos. def symmetric

$\dot{V} = 2x^T P \dot{x}$ $\dot{x} = Ax + B(\underbrace{v}_{\text{added input}} + \eta)$ ← model error

$= 2x^T P (Ax + B(v+\eta))$

$= 2x^T P A x + 2x^T P B (v + \eta)$

$= \underbrace{-x^T Q x}_{\leq 0} + \underbrace{2x^T PB(v+\eta)}_{<0} < 0$

insight: as long as our input v can make $\dot{V}$ negative definite then :-).

Question: Is $\eta(a_q, q, \dot{q})$ bounded?

$\tau = \bar{M} a_q + \bar{C}\dot{q} + \bar{N}$

let's try to find $\rho(e, t)$ such that

$$\|\eta\| \leq \rho(x, t)$$

First, substitute $\eta(a_q, q, \dot{q})$ with our controller input a_q:

note $a_q = \ddot{q}_d - K_p e - K_D \dot{e} + \mathbf{v}$. and

$$\eta(a_q, q, \dot{q}) = (M^{-1}\bar{M} - I)a_q + M^{-1}(\hat{C}\dot{q} + \hat{N})$$

$v = -\frac{w}{\|w\|} \rho(x, t)$

$\|E\| \rho(e, t) + \gamma_1$ ..

$\dot{V} < 0$

$\dot{V} = -x^T Q x + 2x^T PB(v + \eta)$

$x = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$

$2x^T PB(v + \eta) < 0$

$= 2x^T PB v + 2x^T PB \eta \to \rho(e,t)$

$\leq 2\|x^T PB\| v + 2x^T PB \|\eta\| \leq 2\|x^T PB\| v + 2x^T PB \rho(e,t) \leq 0$

$$\Rightarrow v = \begin{cases} -\dfrac{x^T PB \rho(x,t)}{\|x^T PB\|} & \text{if } \|x^T PB\| \neq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$v = -\frac{x^T PB\, \rho(x,t)}{\|x^T PB\|}$$

Denote $\underline{w = x^T PB}$

$$\dot{V} = -x^T Q x + 2x^T PB\,(v + \eta)$$

$$= -x^T Q x + \underline{2w\left(-\frac{w}{\|w\|}\rho(x,t) + \eta\right)}$$

$$2\left(-\frac{\|w\|^2}{\|w\|}\rho(x,t) + w\eta\right)$$

$a + b$

$\leq \|a\| + \|b\|$

$$2\left(-\|w\|\,\rho(x,t) + w\eta\right)$$

Cauchy-schwartz inequality

$$\leq 2\left(-\|w\|\,\rho(x,t) + \|w\|\,\|\eta\|\right)$$

$$= 2\|w\|\left(\underbrace{-\rho(x,t) + \|\eta\|}_{\leq 0}\right)$$

$\rho(e,t)$?

E

$\hat{C} = C - \bar{C}$

$$\eta(a_q, q, \dot{q}) = \underline{(M^{-1}\bar{M} - I)}\, a_q + M^{-1}(\hat{C}\dot{q} + \hat{N})$$

$$? \quad a_q = \ddot{q}^d - K_p e - K_D \dot{e} + \underline{\underline{v}}$$

$$\eta(a_q, q, \dot{q}) = \eta(q, \dot{q}, v)$$

$$= E(\ddot{q}^d - K_p e - K_D \dot{e} + v) + M^{-1}(\hat{C}\dot{q} + \hat{N})$$

$$= Ev + E\ddot{q}^d - EK_p e - EK_D \dot{e} + M^{-1}(\hat{C}\dot{q} + \hat{N})$$

$$\le \|E\| v + \gamma_1 \|x\| + \gamma_2 \|x\|^2 + \gamma_3 \quad (\text{Assumption})$$

select $\gamma_1, \gamma_2, \gamma_3$. to determine $\underline{\rho(e,t)}$

use $\rho(x,t)$ to determine v.

check always that $\|\eta\| \le \rho(x,t)$

$$\le \|E\| \|v\| + \gamma_1 \|x\| + \gamma_2 \|x\|^2 + \gamma_3$$

$$v = -\frac{x^T PB\, \rho(x,t)}{\|x^T PB\|} \qquad \rho(x,t) \geq \|\eta\|$$

$$\|v\| = \rho(x,t)$$

$$\|\eta\| \leq \|E\|\, \rho(x,t) + \gamma_1 \|x\| + \gamma_2 \|x\|^2 + \gamma_3 \leq \rho(x,t)$$

$$\|E\| < 1$$

$$E = M^{-1}\bar{M} - I$$

$$\|E\| = \|\underset{\substack{\uparrow \\ \text{true}}}{M^{-1}}\ \underset{\substack{\uparrow \\ \text{model}}}{\bar{M}} - I\| < 1$$

M and $\bar{M}$ must be close.

suppose: $M^L \precsim M^{-1} \precsim M^u$ $\qquad \begin{bmatrix} x & x \\ x & x \end{bmatrix} \precsim \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$

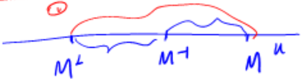
$$\boxed{\bar{M} \approx \frac{2}{M^L + M^u} \quad \text{elementwise.}}$$

$x \leq 0$

$$\|M^{-1}\bar{M} - I\| \leq \left\|M^{-1} \cdot \frac{2}{M^L + M^u} - I\right\| = \left\|\frac{2M^{-1} - M^L - M^u}{M^L + M^u}\right\| < 1$$

①

use : $\| M^{-1} - M^{u} + M^{-1} - M^{L} \| < \| M^{u} - M^{L} \|$



$$\frac{\| M^{u} - M^{L} \|}{\| M^{L} + M^{u} \|} < 1$$

let $\|E\| = \alpha$

$$\alpha \rho(x,t) + \gamma_1 \|x\| + \gamma_2 \|x\|^2 + \gamma_3 \leq \rho(x,t)$$

$$\rho(x,t) \leq \left(\frac{1}{1-\alpha}\right) \left[\gamma_1 \|x\| + \gamma_2 \|x\|^2 + \gamma_3 \right]$$

$$1 - \alpha > 0$$

$$= \gamma_1' \|x\| + \gamma_2' \|x\|^2 + \gamma_3'$$

Avoiding **Chattering**

$$x = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$$

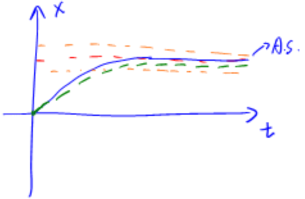
Approximate discontinous control by a continuous control

$$v = \begin{cases} -\rho(x,t)\frac{B^T Px}{\|B^T Px\|} & \text{if } \|B^T Px\| > \varepsilon \\ -\rho(x,t)\frac{B^T Px}{\varepsilon} & \text{if } \|B^T Px\| \le \underset{\Delta}{\varepsilon} \end{cases},$$

$\rightarrow$ upon switching: $-\rho(x,t)\frac{B^T Px}{\varepsilon}$

The system is **uniformly ultimately bounded** (U.U.D.) under the continuous control using Lyapunov theory.

Graphical interpration of U.U.D.



Denote $B^T P X = w$

Proof: when $\|B^T P x\| \le \varepsilon$

$w = B^T P x$

$$V = x^T P x$$

$$\dot{V} = \underline{x^T P A x + x^T A^T P x} + 2 x^T P B (v + \eta)$$

$$= -x^T Q x + 2 \underset{\smile}{x^T P B} (v + \eta)$$

with $Q = -(PA + A^T P)$

$$v = -\rho(x,t) \frac{w}{\varepsilon}$$

$x^T P B = w^T$ with symmetric P

$$\dot{V} = -x^T Q x + 2 w^T \left(-\rho(x,t) \frac{w}{\varepsilon} + \eta \right)$$

$$\|\eta\| \le \rho(x,t)$$

$$\le -x^T Q x + 2 \left(-\rho(x,t) \frac{w^T w}{\varepsilon} + \underset{\smile}{w^T \rho(x,t)} \right)$$

$$w^T w = \|w\|^2 \quad ; \quad w^T \le \|w\|$$

$$\le -\underbrace{x^TQx}_{\checkmark} + \underbrace{2\rho(x,t)}_{>0}\Big[\underbrace{-\frac{\|w\|^2}{\varepsilon}+\|w\|}_{g(\|w\|)}\Big)$$

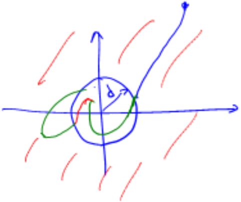
$$\le -x^TQx + 2\rho(x,t)\underbrace{\max_{\|w\|}\Big[-\frac{\|w\|^2}{\varepsilon}+\|w\|\Big]}_{\|w\|^*=\frac{\varepsilon}{2}}$$

$$= -x^TQx + \rho(x,t)\frac{\varepsilon}{2} < 0$$

$$x^TQx > \rho(x,t)\frac{\varepsilon}{2}$$

$$\lambda_{max}\|x\|^2 \ge x^TQx \ge \lambda_{min}\|x\|^2 > \rho(x,t)\frac{\varepsilon}{2}$$

$$\|x\|^2 > \underbrace{\frac{\rho(x,t)}{\lambda_{min}}\cdot\frac{\varepsilon}{2}}_{\delta} \Rightarrow \boxed{\dot V<0}$$



Obtain the bound $\rho(x, t)$

$$\rho(x,t) = \gamma_1' \|e\| + \gamma_2' \|e\|^2 + \gamma_3'$$

recall $x = [e; \dot{e}]$, where $\gamma_1', \gamma_2', \gamma_3'$ are positive reals.

- First, generate a candidate for the bound $\rho(x,t)$ by selecting some values for $\gamma_i', i = 1, 2, 3$;
- Checked as a posteriori the bound during runtime, update value for γ_i if the bound is violated.

Conclusion

A robust controller for computed torque control:

- maintain good performance in terms of stability, tracking error, or other specifications, despite **parametric uncertainty** and **model mismatch.**
- is a fixed controller, dynamic function of time, satisfy the performance specification over a **given range of uncertainties**: $\|\eta\| \leq \rho(q, \dot{q}, t)$.