# Lecture notes: Passity-based Control (Adaptive & Robust)

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# Outline

This lecture note is based on

- Chapter 8 in M. Spong **Robot modeling and control.**

# A general theorem

Let $q^d(t)$ be a twice differentiable function, define

$$e(t) = q(t) - q^d(t).$$

Consider the differential equation

$$M(q)\dot{r} + C(q,\dot{q})r + K_v r = \Phi$$

where $K_v = K_v^T > 0$ is a positive definite matrix.
Suppose

- $\int_{t=0}^{T} -r^{\intercal}(t)\Phi(t)dt \geq -\beta$ for all $T > 0$ and for some $\beta \geq 0$.
- and $r(t) = f(e(t))$ for some proper mapping $f(\cdot)$ such that $r(t) \to 0$ means $e(t) \to 0$ and $\dot{e}(t) \to 0$,

Then as $t \to \infty$, $r(t) \to 0$, $e(t) \to 0$ and $\dot{e}(t) \to 0$.

# Proof

$$M(q)\dot{r} + C(q,\dot{q})r + K_v r = \tilde{\Phi}$$

Consider a Lyapunov candidate $V$ defined by

$$V = \frac{1}{2}r^T M(q) r + \beta - \int_{t=0}^{T} r^\intercal(t)\Phi(t)dt$$

Clearly, $V \geq 0$. $\dot{V} < 0$

Differentiate $V$ along the sytem traj.

$$\dot{V} = r^T M(q)\dot{r} + \frac{1}{2} r^T \dot{M}(q) r - r^T \phi$$

$$= r^T M \dot{r} + \frac{1}{2} r^T \dot{M} r - r^T (M(q)\dot{r} + C(q,\dot{q})r + K_v r)$$

$$= \frac{1}{2} r^T (\dot{M} - 2C) r - r^T K_v r$$

$$= -r^T K_v r \leq 0 \text{ neg} \quad \text{and } \dot{V} = 0 \text{ iff } r = 0$$

as $t \to \infty$, $r(t) \to 0$ converge.

# Passivity based motion control

$r = f(e)$

$r \to 0$

$e \to 0$

$\dot{e} \to 0$

$$\textcircled{1} \quad M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

select $\tau$ and a definition of $r$ to make the system dynamics as

$$\textcircled{2} \quad M(q)\dot{r} + C(q,\dot{q})r + K_v r = 0$$

Note: A special case for $\Phi = 0$. Let $\tau = M(q)a + C(q,\dot{q})v + N(q) - K_v r$

$\textcircled{1}: \; M\ddot{q} + C\dot{q} + N - \tau = 0 = M\dot{r} + Cr + K_v r \quad \textcircled{2}$

Then

$\tau = M(\ddot{q} - \dot{r}) + C(\dot{q} - r) + N(q) - K_v r$

- $a = \ddot{q} - \dot{r}$
- $v = \dot{q} - r$
- $r = \dot{q} - v$

$\to \tau = Ma + Cv + N(q) - K_v r$

select a choice of $v$: $\boxed{v = \dot{q}^d - \Lambda e}$

$a = \ddot{q}^d - \Lambda \dot{e}$

$r = f(e, \dot{e})$ such that $r \to 0$, $e \to 0$, $\dot{e} \to 0$

$$r = \dot{q} - \dot{q}^d + \Lambda e$$

$$= \dot{e} + \Lambda e \qquad \Lambda \text{ be positive. def.}$$

$$\begin{aligned} r = 0 &\Rightarrow \\ \dot{r} = 0 &\Rightarrow \end{aligned} \quad \boxed{\begin{aligned} \dot{e} + \Lambda e &= 0 \\ \ddot{e} + \Lambda \dot{e} &= 0 \end{aligned}}$$

let $X = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \rightarrow \dot{X} = \begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} -\Lambda e \\ -\Lambda \dot{e} \end{bmatrix} = \begin{bmatrix} -\Lambda & 0 \\ 0 & -\Lambda \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$

$$\dot{X} = \underbrace{\begin{bmatrix} -\Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}}_{\text{neg. def.}} X$$

$X \to 0$ as $t \to \infty$.

$r(t) \to 0$.

$$r = \dot{e} + \Lambda e$$

$$v = \dot{q} - r = \dot{q} - (\dot{q} - \dot{q}^d) - \Lambda e = \dot{q}^d - \Lambda e$$

$$a = \ddot{q}^d - \Lambda \dot{e}$$

$$\tau = M(q)\underset{\uparrow \atop \ddot{q}^d - \Lambda\dot{e}}{a} + C(q,\dot{q})\underset{\uparrow \atop \dot{q}^d - \Lambda e}{v} + N(q) - K_v \underset{\uparrow \atop \dot{e} + \Lambda e}{r}$$

① General thm. $M(q)\dot{r} + C(\ )r + K_v r = \phi$

$\int_0^T -r^T \phi \, dt \geq -\beta$ then as $t \to \infty$. $r \to 0$

② $M\ddot{q} + C\dot{q} + N = \tau$

$M\dot{r} + Cr + K_v r = 0$

introduce $\tau$, and $r$.

$\tau = M(\ddot{q} - \dot{r}) + C(\dot{q} - r) + N(q) - K_v r.$

③ Find the relation between $r(t)$ and $e(t)$, $\dot{e}(t)$

such that as $r(t) \to 0$ $e(t) \to 0$ and $\dot{e}(t) \to 0$

$$r = \dot{e} + \Lambda e$$

# What is the mapping $r = f(e, \dot{e})$?

Select $v = \dot{q}^d - \Lambda e$ where $\Lambda > 0$ is a positive definite matrix.

# Lyapunov stability analysis

Consider a lyapunov function

$$V = \frac{1}{2} r^T M(q) r + e^T \Lambda K_v e$$

## Passitivity-based adaptive control

No error:

$$M(q)\dot{r} + C(q,\dot{q})r + K_v r = 0$$

with error :

$$\tau = \bar{M}(q)a + \bar{C}(q,\dot{q})v + \bar{N}(q) - K_v r$$

subsititute.

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \bar{M}(q)a + \bar{C}(q,\dot{q})v + \bar{N}(q) - K_v r$$

$$a = \ddot{q} - \dot{r} \qquad v = \dot{q} - r$$

$$\ddot{q} = a + \dot{r} \qquad \dot{q} = v + r$$

$$M(q)(a+\dot{r}) + C(q,\dot{q})(v+r) + N(q) = \bar{M}(q)a + \bar{C}v + \bar{N} - K_v r$$

$$M(q)\dot{r} + C(q,\dot{q})r + K_v r = (\bar{M}(q) - M)a + (\bar{C} - C)v + \bar{N} - N$$

$$= Y(a, v, q, \dot{q})(\bar{\theta} - \theta)$$

↳ parameter mis match

linearity: $M(q)\underbrace{\ddot{q}}_{a} + C\underbrace{\dot{q}}_{v} + N(q) = Y(q, \dot{q}, \ddot{q})\theta$

$$Y(q, \dot{q}, a, v)\theta$$

Select an update law for $\tilde{\theta}$ to achieve passivity.

$$\int_{t=0}^{T} -r^{\mathsf{T}}(t)\Phi(t)dt \geq -\beta$$

$$M(q)\dot{r} + Cr + K_v r = \underbrace{Y(q,\dot{q},a,v)\tilde{\theta}}_{} = \phi(t)$$

$$\tilde{\theta} = \bar{\theta} - \theta$$

Recall: $M\dot{r} + Cr + K_v r = \phi$

$$r = \dot{e} + \Lambda e$$

$$\int_0^T -r^T \phi(t)\, dt = \int_0^T -r^T Y \tilde{\theta}\, dt$$

pick: $\boxed{\dot{\tilde{\theta}} = -\Gamma^{-1} Y^T r}$ $\Gamma$ pos. def.

$$r^T \phi = r^T Y \tilde{\theta} = -\dot{\tilde{\theta}}^T \Gamma Y^{-1} Y \tilde{\theta}$$

$$= -\dot{\tilde{\theta}}^T \Gamma \tilde{\theta}$$

$$\int_0^T -r^T \phi \, dt = + \int_0^T \dot{\tilde{\theta}}^T \Gamma \tilde{\theta} \, dt$$

$$= \frac{1}{2} \int_0^T \frac{d}{dt} \left( \tilde{\theta}^T \Gamma \tilde{\theta} \right) dt$$

$$= \frac{1}{2} \left[ \tilde{\theta}^T(T) \Gamma \tilde{\theta}(T) - \tilde{\theta}^T(0) \Gamma \tilde{\theta}(0) \right].$$

$\Gamma$. pos. symm.

$$\geq \underbrace{-\frac{1}{2} \tilde{\theta}^T(0) \Gamma \tilde{\theta}(0)}_{\beta}$$

$$\int_0^T -r^T \phi \, dt \geq -\beta$$

using the General thm: as $t \to \infty$ $r(t) \to 0$.

$$r = \dot{e} + \Lambda e$$

# Lyapunov analysis

Consider the Lyapunov candidate

$$V = \frac{1}{2} r^T M(q) r + e^T \Lambda K_v e + \frac{1}{2} \tilde{\theta}^T \Gamma \tilde{\theta}$$

Robust control. $M(q)\dot{r} + C(q,\dot{q})r + K_r = Y(a,v,q,\dot{q})(\bar{\theta}-\theta)$

$V =$