Lecture notes: Linear control theory and state space design

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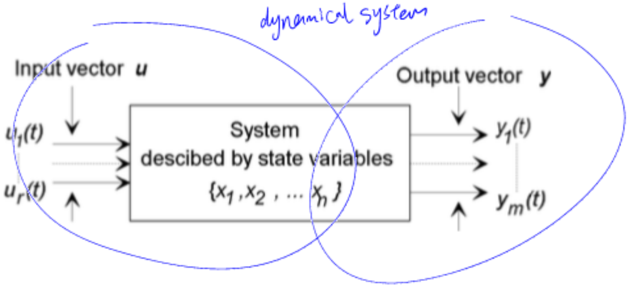
Outline

This lecture note is based on

- Chapter 6 of John Lygeros and Federico A. Ramponi, (2015). Lecture Notes on Linear System Theory.
- Karl Johan Aström Richard M. Murray, *Feedback Systems, An introduction to Scientists and Engineers*. Chapter 3. Cruise control example, Chap5, The Matrix Exponential.

State space form

State: A **minimum** set of variables, known as state variables, that fully describe the system and its response to **any given set of inputs.**



sensor
observation
of the system
dynamics

Figure 1: System inputs and outputs.

State space form

consider a general nonlinear, system:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$$

for time t and state vector $\mathbf{x}$ and input vector $\mathbf{u}$.

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}, \mathbf{u}, t) \\ \dot{x}_2 &= f_2(\mathbf{x}, \mathbf{u}, t) \\ \vdots &= \vdots \\ \dot{x}_n &= f_n(\mathbf{x}, \mathbf{u}, t) \end{aligned}$$

A system is **time invariant** if $f(\mathbf{x}, \mathbf{u}, t) = f(\mathbf{x}, \mathbf{u})$.
A system is **autonomous** if $f(\mathbf{x}, \mathbf{u}, t) = f(\mathbf{x}, t)$. unforced

Linear time invariant (LTI) system

The function $f_i(\mathbf{x}, \mathbf{u})$ is **linear** in the state **x** and **u**, and the parameter does not change with time.

$$
\begin{array}{ccccccccccccccc}
\dot{x}_1 & = & a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & + & b_{11}u_1 & + & \ldots & + & b_{1r}u_r \\
\dot{x}_2 & = & a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & + & b_{21}u_1 & + & \ldots & + & b_{2r}u_r \\
\vdots & & \vdots & & & & & & & & & & & & \\
\dot{x}_n & = & a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & + & b_{n1}u_1 & + & \ldots & + & b_{nr}u_r
\end{array}
$$

Linear time invariant (LTI) system

The function $f_i(\mathbf{x}, \mathbf{u})$ is **linear** in the state **x** and **u**, and the parameter does not change with time.
which can be compactly represented as:

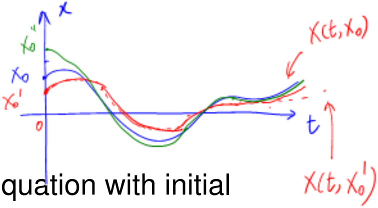
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1r} \\ b_{21} & & b_{2r} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

and $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$.

Stability: Basic definitions

consider a general nonlinear, time varying and autonomous system:

$$\dot{x} = f(x, t)$$



for time t and state vector x.
Stability:
Let $x(t, x_0)$ be a solution to the differential equation with initial condition $x(0) = x_0$.

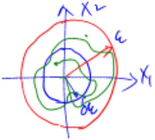
A solution is **stable** if **other solutions** starting near $x(t, x_0)$, stay close to $x(t, x_0)$.

Asymptotically stable: Stable and $\lim_{t\to\infty}(x(t, b) - x(t, a)) = 0$ for b is sufficiently close to a.

Let x_0 be the initial state and $x(t, x_0)$ be the solution of the ODE.

$x_0' \neq x_0$ and $x(t, x_0')$ be the solution of the ODE with x_0' as the initial state.

Question: Which formula is stability condition, and which is asymptotically stable?



Stable

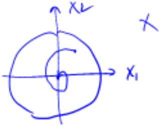
$$\forall \epsilon > 0, \exists \delta_\epsilon > 0, \|x_0' - x_0\| \leq \delta_\epsilon, \implies \|x(t, x_0') - x(t, x_0)\| \leq \epsilon, \forall t \geq t_0.$$

Local

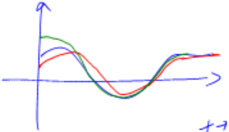
A.S: $\exists \delta > 0, \|x_0' - x_0\| \leq \delta, \implies \|x(t, x_0') - x(t, x_0)\| \to 0$ as $t \to \cancel{0}\,\infty$.

Global asymptotical stability:

$\forall \delta > 0$, ( )



Stability



exponential stability of $x = a$: x_0 $X(t, x_0)$

$t \to \infty$

$\to 0$

$$\exists \delta, c, \lambda > 0: \quad \|x_0' - x_0\| < \delta \to \|x(t, x_0') - x(t, x_0)\| \le c \exp^{-\lambda t} \|x_0' - x_0\|.$$

$e^{-\lambda t} \to e^{-t}$ const. $< \delta$

Equilibrium: A special case of solution $f(x_e, t) = 0$ for all $t \geq 0$.

- An equilibrium is **stable** if the solution starting from the equilbrium is **stable.**
- An equilibrium is **Asymptotically stable** if the solution starting from the equilbrium is **Asymptotically stable**.

Example

Q: Can you find a solution for the system with initial state $(1, 0)$?

Q: What is the equilbrium? $AX_e = 0$ $\therefore X_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Q: Is the equilibrium stable? Is the equilbrium asymptotically stable?

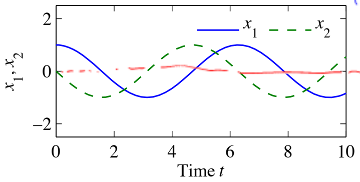
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

$x_1 = \cos(\omega t + \phi)$ $\omega = 1$

$x_2 = \sin(\omega t + \phi)$

$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\dot{X} = AX$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

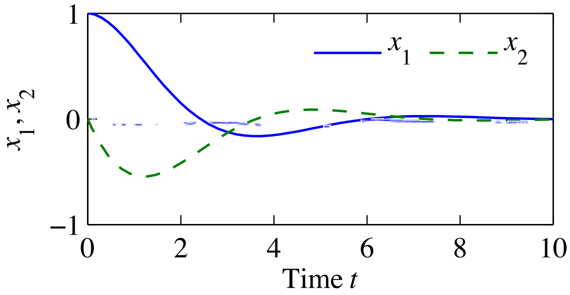


$\rightarrow X(t, 0)$

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$

$X_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



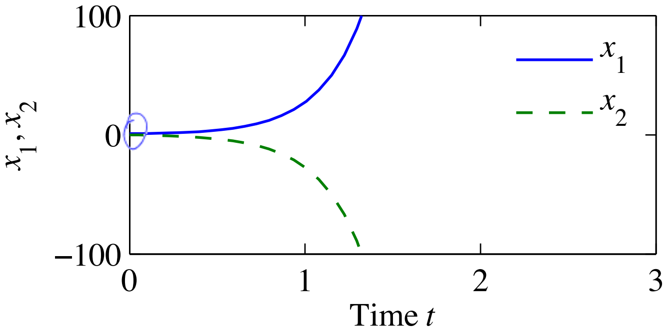
Example

unstable

$$\dot{x}_1 = 2x_1 - x_2$$
$$\dot{x}_2 = -x_1 + 2x_2$$

$x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_0 = \begin{bmatrix} 0.001 \\ 0.002 \end{bmatrix}$.



Stability of a linear system

A linear system has the form

$$\dot{x} = Ax, \quad x(0) = x_0$$

Equilibrium: $x_e = 0$

Question: Given a linear system, how to know if the system is stable or not?

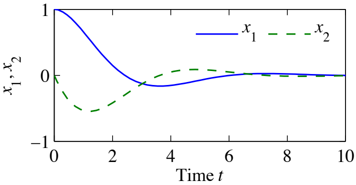
Direct method: Solve the ODE.

- consider the simple scalar system $\dot{x} = ax$, $x(0) = x_0$. The solution is $x(t) = e^{at}x_0$.
- For a linear system $\dot{x} = Ax$ for A is a matrix, x is a vector. The solution is $x(t) = e^{At}x_0$.

Example

Consider the example: Given the initial state $(1, 0)$, what is the solution $x(t)$?

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$



$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$x(t) = e^{At} x_0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The matrix exponential

matrix exponential

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k,$$

and differentiate w.r.t t

Question: How to determine the stability **without** solving the ODE?

Relating e^{At} with the eigenvalue of A

Given a matrix A, v is an **eigenvector** of A, and λ is the corresponding **eigenvalue**.

$$Av = \lambda v$$

$(A-\lambda I)v$

$\cancel{Ax} = 0$

- v is in the nullspace of $A - \lambda I$. ← identity.
- For $v \neq 0$ to exist, $A - \lambda I$ must not be full rank
  - $A - \lambda I$ not invertible.
  - and $\det(A - \lambda I) = 0$.

Relating e^{At} with the eigenvalue of A

Given a matrix A, v is an **eigenvector** of A, and λ is the corresponding **eigenvalue.**

$$Av = \lambda v$$

- v is in the nullspace of $A - \lambda I$.
- For $v \neq 0$ to exist, $A - \lambda I$ must not be full rank
  - $A - \lambda I$ not invertible.
  - and $\det(A - \lambda I) = 0$.

—- which means λ is are the roots of the **characteristic polynomial** $\det(sI - A)$.

$$\lambda(A) = \{s : \det(sI - A) = 0\}.$$

Example

let $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$. Exercise: What are the eigenvalues and eigenvectors of A.

$$|A - I\lambda| = \begin{vmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(-1-\lambda) + 1 = 0$$

$$\Rightarrow \lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

Eigen decomposition

$A \in \mathbb{R}^{n\times n}$

$\lambda_1 \cdots \lambda_n$

$$(A - \lambda I)v = 0.$$

Let $\lambda_1, \ldots, \lambda_k$ be k different eigenvalues and $v_1, \ldots, v_k$ be their corresponding eigenvector.

$$A \begin{bmatrix} v_1 & v_2 & \ldots & v_k \end{bmatrix} = [v_1 \cdots v_k] \Lambda$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

Eigen decomposition

$$(A - \lambda I)v = 0.$$

Let $\lambda_1, \ldots, \lambda_k$ be k different eigenvalues and $v_1, \ldots, v_k$ be their corresponding eigenvector.

$$A \begin{bmatrix} v_1 & v_2 & \ldots & v_k \end{bmatrix} =$$

Thus, let $T = \begin{bmatrix} v_1 & v_2 & \ldots & v_k \end{bmatrix}$, we have

$$AT = T\Lambda$$

with $\Lambda = \mathrm{diag}(\lambda_1, v_2, \ldots, v_k)$. So that $A = T\Lambda T^{-1}$.

Eigen decomposition

$$(A - \lambda I)v = 0.$$

Let $\lambda_1, \ldots, \lambda_k$ be k different eigenvalues and $v_1, \ldots, v_k$ be their corresponding eigenvector.

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with $\Lambda = \cancel{\mathrm{diag}(\lambda_1, v_2, \ldots, v_k)}$. So that $A = T\Lambda T^{-1}$.

diag(lambda_1, lambda_2, ..., lambda_k)

Relating e^{At} with the eigenvalue of A

What is the relation between the eigenvalue of A and the solution of $\dot{x} = Ax$?

$x_e = 0$

$\dot{x} = Ax$ transforms to $\dot{x} = T\Lambda T^{-1}x$.

$z \to 0$
$x = Tz \to 0$

Introducing the state-transformation $z = T^{-1}x$.

$$\dot{z} = \Lambda z.$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Simple case: Λ is a diagonal matrix.

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

$z(t) = e^{\Lambda t} z_0$, $z_0 = T^{-1}x_0$

$$e^{\lambda t} \begin{cases} \lambda > 0 & \uparrow \infty \\ \lambda = 0 & | \\ \lambda < 0 & \searrow 0 \end{cases}$$

Stable or Hurwitz matrix

- A square matrix of A is called stable if and only if every eigenvalue λ_i of A has strictly negative real part.
- $x(t) \to 0$ as $t \to \infty$.

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2$$

- What is A matrix?
- What are the eigenvalues of A? Is A stable.
- Find the state transformation $z = T^{-1}x$ such that $\dot{z} = \Lambda z$ for a diagonal matrix Λ.

$A \in \mathbb{R}^{n\times n}$

If Λ is Jordan form J: Each eigenvalue corresponds to multiple eigenvector.

$\lambda_1 \quad \cdots \quad \lambda_n$

$\downarrow \qquad\quad \downarrow$

$v_1 \qquad\quad v_n$

$\lambda_i \neq \lambda_j$

$\exists i, j: \lambda_i = \lambda_j$

$$J = \begin{bmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & J_{k-1} & 0 \\ 0 & 0 & \dots & 0 & J_k \end{bmatrix}, \quad \text{where } J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}$$

$$e^J = \begin{bmatrix} e^{J_1} & 0 & \dots & 0 \\ 0 & e^{J_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & e^{J_k}. \end{bmatrix}. \qquad e^{J_i t} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & & 1 & \ddots & \vdots \\ & & & \ddots & t \\ 0 & \dots & & 0 & 1 \end{bmatrix} e^{\lambda_i t}.$$

Stability of a linear time invariant (LTI) system

Given linear system

$$\dot{x} = Ax, \quad x(0) = x_0$$

The solution is $x_0 e^{At}$.

Theorem:

Local AS. = global A.S. = Exp. stable

The linear system $\dot{x} = Ax$ is

- **asymptotically stable** at the equilibrium if and only if all eigenvalues of A ___have neg. real parts.___.
- **unstable** if and only if **any** eigenvalues of A ___has positive real parts___.

Exercise: check the stability of previous examples.