Robot Control

Assignment 0

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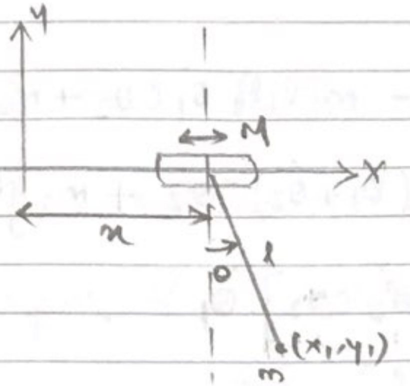
Q.1 a)



$$x_1 = x + l\sin\theta$$
$$y_1 = -l\cos\theta$$

$$\dot{x}_1 = \dot{x} + l\dot{\theta}C\theta$$
$$\dot{y}_1 = l\dot{\theta}S\theta$$

$$v = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} \dot{x} + l\dot{\theta}C\theta \\ l\dot{\theta}S\theta \end{bmatrix}$$

$$v^2 = \dot{x}_1^2 + \dot{y}_1^2 = \dot{x}^2 + 2l\dot{x}\dot{\theta}C\theta + l^2\dot{\theta}^2$$

$\rightarrow$ $L = K - P$: $K_1 = \frac{1}{2}M\dot{x}^2$, $P_1 = 0$

$$K_2 = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\theta}C\theta + \frac{1}{2}ml^2\dot{\theta}^2$$

$$P_2 = mgy_1 = -mglC\theta$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}C\theta + \frac{1}{2}ml^2\dot{\theta}^2 + mglC\theta$$

$\Rightarrow$ $$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

$$\tau_1 = \frac{d}{dt}\left(ml\dot{x}C\theta + ml^2\dot{\theta}\right) - \left(-ml\dot{x}\dot{\theta}S\theta - mglS\theta\right)$$

$$\tau_1 = ml\ddot{x}C\theta - \cancel{ml\dot{x}\dot{\theta}S\theta} + ml^2\ddot{\theta} + \cancel{ml\dot{x}\dot{\theta}S\theta} + mglS\theta$$

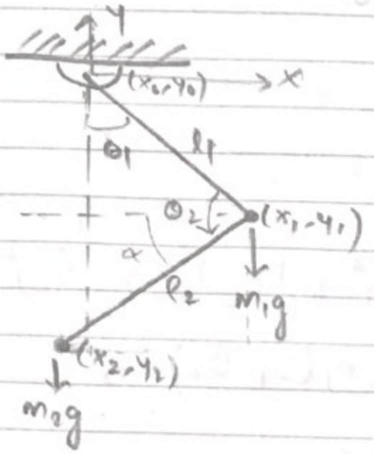
$$\tau_1 = ml\ddot{x}C\theta + ml^2\ddot{\theta} + mglS\theta$$

$$\Rightarrow \quad \tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

$$\tau_2 = \frac{d}{dt}\left((M+m)\dot{x} + ml\dot{\theta}C\theta \right) - 0$$

$$\tau_2 = (M+m)\ddot{x} + ml\ddot{\theta}C\theta - ml\dot{\theta}^2 S\theta$$

b)



$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1 \cos\theta_1$$

$$x_2 = x_1 - l_2 \cos((\theta_2+\theta_1) - 90)$$
$$= l_1 S\theta_1 - l_2 S(\theta_2+\theta_1)$$

$$y_2 = y_1 - l_2 \sin((\theta_2+\theta_1) - 90)$$
$$= -l_1 C\theta_1 + l_2 C(\theta_2+\theta_1)$$

$$v_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} l_1 \dot{\theta}_1 C\theta_1 \\ l_1 \dot{\theta}_1 S\theta_1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} l_1 \dot{\theta}_1 C\theta_1 - l_2 (\dot{\theta}_2+\dot{\theta}_1) C(\theta_2+\theta_1) \\ l_1 \dot{\theta}_1 S\theta_1 - l_2 (\dot{\theta}_2+\dot{\theta}_1) S(\theta_2+\theta_1) \end{bmatrix}$$

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\theta}_1^2$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 (\dot{\theta}_2+\dot{\theta}_1)^2 - 2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_2+\dot{\theta}_1) C\theta_2$$

$$\Rightarrow \cancel{(l_1 \dot{\theta}_1 C\theta_1)^2} - 2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_2+\dot{\theta}_1) C\theta_1 C(\theta_2+\theta_1)$$
$$+ \cancel{(l_2 (\dot{\theta}_2+\dot{\theta}_1) C(\theta_2+\theta_1))^2}$$
$$+ \cancel{(l_1 \dot{\theta}_1 S\theta_1)^2} - 2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_2+\dot{\theta}_1) S\theta_1 S(\theta_2+\theta_1)$$
$$+ (l_2 (\dot{\theta}_2+\dot{\theta}_1) S(\theta_2+\theta_1))^2$$
$$= l_1^2 \dot{\theta}_1^2 + l_2^2 (\dot{\theta}_2+\dot{\theta}_1)^2 - 2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_2+\dot{\theta}_1) C(\theta_2)$$

$$\Rightarrow \quad L = (K_1 + K_2) - (P_1 + P_2)$$

$$K_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$

$$K_2 = \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) c\theta_2$$

$$P_1 = m_1 g y_1 = -m_1 g l_1 c\theta_1$$

$$P_2 = m_2 g y_2 = -m_2 g l_1 c\theta_1 + m_2 g l_2 c(\theta_1 + \theta_2)$$

$$\Rightarrow \quad L = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) c\theta_2 + (m_1 + m_2) g l_1 c\theta_1 - m_2 g l_2 c(\theta_1 + \theta_2)$$

$$\Rightarrow \quad \vec{\tau} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{q}}}\right) - \frac{\partial L}{\partial \vec{q}} \Rightarrow \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} \end{bmatrix}$$

$$\tau_1 = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) - m_2 l_1 l_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) c\theta_2 + m_2 l_1 l_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) s\theta_2 - \left[-(m_1 + m_2) g l_1 s\theta_1 + m_2 g l_2 s(\theta_1 + \theta_2) \right]$$

$$= \left[(m_1 + m_2) l_1^2 + m_2 l_2^2 - 2 m_2 l_1 l_2 c\theta_2 \right] \ddot{\theta}_1 + \left[m_2 l_2^2 - m_2 l_1 l_2 c\theta_2 \right] \ddot{\theta}_2 + m_2 l_1 l_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) s\theta_2 + (m_1 + m_2) g l_1 s\theta_1 - m_2 g l_2 s(\theta_1 + \theta_2)$$

$$\tau_2 = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) - m_2 l_1 l_2 \ddot{\theta}_1 C\theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 S\theta_2$$

$$- \left[+ m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) S\theta_2 + m_2 g l_2 S(\theta_1 + \theta_2) \right]$$

$$= \left[m_2 l_2^2 - m_2 l_1 l_2 C\theta_2 \right] \ddot{\theta}_1$$

$$+ \left[m_2 l_2^2 \right] \ddot{\theta}_2$$

$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 S\theta_2$$

$$- m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) S\theta_2 - m_2 g l_2 S(\theta_1 + \theta_2)$$

$$\Rightarrow \vec{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} (m_1 + m_2) l_1^2 + m_2 l_2^2 - 2 m_2 l_1 l_2 C\theta_2 & m_2 l_2^2 - m_2 l_1 l_2 C\theta_2 \\ m_2 l_2^2 - m_2 l_1 l_2 C\theta_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 2 m_2 l_1 l_2 \dot{\theta}_2 S\theta_2 & m_2 l_1 l_2 \dot{\theta}_2 S\theta_2 \\ - m_2 l_1 l_2 \dot{\theta}_1 S\theta_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} (m_1 + m_2) g l_1 S\theta_1 - m_2 g l_2 S(\theta_1 + \theta_2) \\ - m_2 g l_2 S(\theta_1 + \theta_2) \end{bmatrix}$$

Q.2 a) Given: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

$\dot{x} = Ax$

$\dot{x} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

To compute eigenvalue we have two conditions

$A - \lambda I$ not invertible

$\det(A - \lambda I) = 0$

$\therefore \det\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$

$\det\left(\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} \right) = 0$

$(2-\lambda)(3-\lambda) - (2\times 1) = 0$

$6 - 2\lambda - 3\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 5\lambda + 4 = 0$

$\lambda = 4$ or 1

$\therefore$ Eigenvalue are $\lambda_2 = 4$ and $\lambda_1 = 1$

For eigenvector $(A - \lambda I)v = 0$

$\lambda_1 = 1$

$\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) V_1 = 0$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix} = 0 \quad \therefore \begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = 4$

$$\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right) v_2 = 0$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix} = 0$$

$$-2v_{2x} + 2v_{2y} = 0 \quad \text{or} \quad v_{2x} = v_{2y}$$
$$v_{2x} - v_{2y} = 0 \quad \text{or} \quad v_{2x} = v_{2y}$$

$$\therefore \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b) $T = [v_1, v_2]$

$$T = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \ \det(T) = 3$$

$$T^{-1} = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

$$T^{-1}AT = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

c) Given : $z = T^{-1}x$, $\dot{x} = Ax$

Taking derivative on both sides

$$\dot{z} = T^{-1}\dot{x}$$

$$\frac{dz}{dt} = T^{-1}Ax$$

replace $x = Tz$

$$\frac{dz}{dt} = T^{-1}ATz$$

But we know $T^{-1}AT = \lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$$\frac{dz}{dt} = \lambda z$$

$$\frac{dz_1}{dt} = \lambda_1 z_1$$

$$\frac{dz_2}{dt} = \lambda_2 z_2$$

d) $\frac{dz_1}{dt} = z_1$ & $\frac{dz_2}{dt} = 4z_2$

Integration on both sides

$z_1 = c_1 e^{t}$ & $z_2 = c_2 e^{4t}$ where c_1 & c_2 are const

$$x = Tz = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{t} \\ c_2 e^{4t} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2c_1 e^{t} + c_2 e^{4t} \\ c_1 e^{t} + c_2 e^{4t} \end{bmatrix} , \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ c_2 e^{4t} \end{bmatrix}$$