Problem 1

(20pt) Consider the system

$$\dot{x} = -y - x^3$$
$$\dot{y} = x - y^3$$

- 5pt : show that the origin is an equilibrium. $f(0)=0.$
- 15pt: Take $V(x,y) = \frac{x^2+y^2}{2}$, and show the system asymptotically converge to the origin.

$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$

$= x(-y-x^3) + y(x-y^3)$

$= -x^4 - y^4 < 0$

$\dot{V}=0$ only at 0.

Problem 2

(20pt)
Show that $(x(t), y(t)) = (0,0)$ is an asymptotically stable solution of

$$\dot{x} = -x^3 + 2y^3$$
$$\dot{y} = -2xy^2$$

using LaSalle's invarianc principle and function $V = \frac{1}{2}(x^2+y^2)$.

$\dot{V} = x(-x^3+2y^3) + y(-2xy^2)$

$= -x^4$ neg. semi def.

$\dot{V}=0$ at $x=0$.

Largest invariance set: $\begin{bmatrix} x \\ y \end{bmatrix}$ $\dot{V}\equiv 0$

$x=0$ & $\dot{x}=0$

$0 = 0 + 2y^3 \Rightarrow y=0$

$\left\{\begin{bmatrix} x \\ y \end{bmatrix} \middle| \dot{V}\equiv 0\right\} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

Problem 3

(30pt) Consider a scalar system

$$\dot{x} = ax^3$$

- 10pt: Show that linearization method fails to determine stability of the origin.

  $\frac{\partial f}{\partial x} = 3ax^2 \Big|_{x=0} = 0$

  $\dot{x}=0$

- 15pt: Use the Lyapunov function
  $$V(x) = x^4$$
  to show that the system is stable for $a < 0$ and unstable for $a > 0$.
- 5pt: What can you say about the system for $a = 0$? $\dot{x}=0$ stable but not Asy. stable.

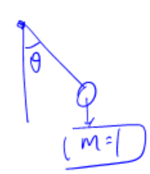
Problem 4

(30pt) Consider the pendulum equation $\rightarrow \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pi \\ 0 \end{pmatrix}\right\}$

$k\neq 0.$ $x_2=0 \Rightarrow$ largest invariant set

$\dot{x}_2=0$

$-\frac{g}{l}\sin x_1 = 0 \Rightarrow x_1 = 0, \pi$



$x_1=\theta$

$x_2=\dot{\theta}$

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{l}x_2. \tag{2}$$

Assume zero friction, i.e. let $k = 0$, and show that the origin is stable. (Hint. Show that the energy of the pendulum is constant along all system trajectories.)

Lyapunov candidate V = energy $= \frac{x_2^2}{2} + \frac{g}{l}(1-\cos x_1)$

$\dot{V} = x_2\dot{x}_2 + \frac{g}{l}(\sin x_1)\dot{x}_1$

$= x_2(-\frac{g}{l}\sin x_1 - \frac{k}{l}x_2) + \frac{g}{l}(\sin x_1)x_2 = -\frac{k}{l}x_2^2$

$k=0$: $\dot{V}=0, \forall t.$ Stable

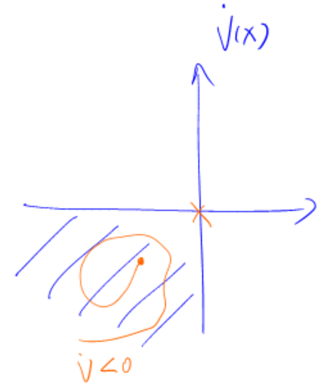
$k\neq 0$: $\dot{V}\leq 0$; $=0$ $x_2=0$

$\dot{x} = ax^3$

$V = x^4$

$\dot{V} = 4x^3\dot{x} = 4ax^6$ if $a > 0$.

$\exists\, w$, $\underbrace{V(w)}_{\dot{V} < 0} < V(0)$



Find another V: $V = x$ $V(0) = 0$ $V(w) < V(0)$

$\dot{V} = \dot{x} = ax^3$ $w = -1$ $\dot{V}(w) = aw^3 = -a$

$$\dot{x} = f(x) \xrightarrow{(0)} \dot{x} = \underbrace{\underset{\sim}{A}}_{\underbrace{\left.\frac{\partial f}{\partial x}\right|_{(0)}}_{\substack{\uparrow \\ \text{if it is unstable}}}} x$$

then it implies the $\dot{x} = f(x)$ is not stable at the origin.

instability: $\exists w.$ $\quad V(w) < V(\underset{\frown}{x_0})$ $\quad$ ↑origin.

$\dot{V} < 0.$