Lecture notes: Linear control theory and state space design

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Outline

This lecture note is based on

- Chapter 6 of John Lygeros and Federico A. Ramponi, (2015). Lecture Notes on Linear System Theory.
- Karl Johan Aström Richard M. Murray, *Feedback Systems, An introduction to Scientists and Engineers*. Chapter 3. Cruise control example, Chap5, The Matrix Exponential.

$$\dot{x} = Ax$$

If there are control input

$$\dot{x} = Ax + Bu$$

- u: input vector (can be multi-input).

Reachability

let us include the input.

$$\dot{x} = Ax + Bu$$

Reachable set: the set of all points x_f such that given $x(0)$ there exists $u(t), t \in [0, T]$ that steers the system from $x(0)$ to $x(T) = x_f$.

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Reachability A linear system is reachable if for **any** $x_f \in R^n$ there exists a $T > 0$ and $u : [0, T] \to R^n$ such that the corresponding solution satisfies

$$x(0) = x_0, x(T) = x_f$$

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equivalent to: for $x(0) = 0$, the reachable set is $\mathbb{R}^n$.

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equivalent to: for $x(0) = 0$, the reachable set is $\mathbb{R}^n$.

- How to decide if a LTI system is **reachable**?
- Given x_0 and x_f, how to design the input u to **stablize to** x_f under disturbances?

Reachability matrix

Q: How to decide if a LTI system is **reachable**?

Under zero initial condition, the solution of $\dot{x} = Ax + \mathbf{Bu}$ is

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau.$$

$\uparrow \delta(\tau)$

$$X(s) =$$

The solution under an impulse $u(t) = \delta(t)$:

$$\mathcal{L}^{-1}$$

$$x_\delta(t) = \int_0^t e^{A(t-\tau)} B \delta(\tau)\, d\tau = e^{At} B$$

$$u = \dot{\delta}(t) \quad x_{\dot{\delta}}(t) = A e^{At} B$$

$$u = \ddot{\delta}(t) \quad x_{\ddot{\delta}}(t) = A^2 e^{At} B$$

$$\vdots$$

$$u = \delta^{(n-1)}(t) \quad X_{\delta^{(n-1)}}(t) = A^{(n-1)} e^{At} B$$

particular input: $u(t) = a_0 \delta(t) + a_1 \dot{\delta}(t) + \cdots + a_{n-1} \delta^{(n-1)}(t)$

$$a_0 e^{At} B + a_1 A e^{At} B + a_2 A^2 e^{At} B + \cdots + a_{n-1} A^{(n-1)} e^{At} B$$

$t := T$: $\stackrel{\Delta}{=} x_f$ $\qquad e^{AT} = I$

$$\lim_{t \to 0^+} \left(\quad \right) = a_0 B + a_1 AB + a_2 A^2 B + \cdots a_{n-1} A^{(n-1)} B$$

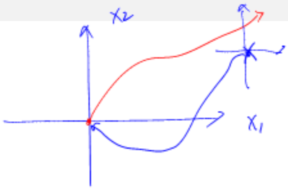
$$= x_f$$

$$\underbrace{\begin{bmatrix} B & AB & \cdots & A^{(n-1)} B \end{bmatrix}}_{\text{Full rank.}} \underbrace{\begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}}_{\in \mathbb{R}^n} = x_f$$

Reachability rank condition

Theorem A linear system is **reachable** if and only if the reachability matrix $[B, AB, \ldots, A^{n-1}B]$ is **invertible.**

Controllability



Controllability A linear system is **controllable** if for any $x_0 \in R^n$ there exists a $T > 0$ and $u : [0, T] \to R$ such that the corresponding solution satisfies

$$x(T) = x_e.$$

where x_e is the equilibrium (usually the origin.)

$$\dot{x} = Ax + Bu$$

$$Ax = 0$$

$$x_e = 0$$

$$\dot{x}_1 = x_1 + x_2 + u$$

$$u = -x_2 - 2x_1$$

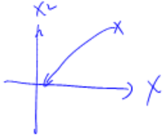
$$\dot{x}_1 = -x_1$$

- The reachability implies the controllability.
- The controllability does not imply the reachability.

ex.

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = \begin{matrix} x_1 + x_2 + u \\ -x_2 \end{matrix}$$

- Is the system reachable? $\operatorname{rank}\begin{bmatrix} B & AB \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 1 < 2$
- Is the system controllable ~~at the equilibirium of unforced system~~ $\cancel{\dot{x} = Ax}$?



$$\dot{x}_2 = -x_2 \qquad e^{-t} x_2(0)$$

$$\boxed{\dot{x}_1 = x_1 + u} \quad \text{when } x_2 = 0.$$

Consider a system

$$\dot{x} = Ax + Bu$$

Objective: design the control input u such that the system eventually converge to the origin?

- Is the system controllable? rank( ) =
- Consider a feedback control: $u = Kx$, selecting the value for K so that the system is stabilized to the origin. Feedback

$$\dot{x} = \underbrace{(A + BK)}_{A'} x$$

A' eigenvalues of A' have neg. real parts.

Feedback control

exercise:

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

How to design the control u to stabilize the system to the origin?

Error dynamics

Consider a system

$$\dot{x} = Ax + Bu$$

Objective: design the control input u such that the system eventually converge to **the goal** x_f?

- Is the system reachable? rank.
- Will the same feedback control: $u = Kx$ do that job?

① change of state variable: $e = x - x_f$ error signal / state

② design control u such that $\lim_{t \to \infty} e(t) = 0$

$$\dot{e} = \dot{x} - \dot{x}_f = \dot{x} = Ax + Bu$$
$$= A(e + x_f) + Bu$$

$$\dot{e} = Ae + \boxed{Bu + Ax_f} \quad Bv$$

input for the error dynamics

$$\dot{e} = Ae + Bv$$

goal: $\lim_{t\to\infty} e(t) = 0$

$$v = Ke$$

$$= (A + BK)e$$

stable.

$$BKe = Bu + Ax_f$$

solve for u.

$$\boxed{u = Kx + K_r \cdot r}$$

gain

reference signal

$$BKe = BKx + BK_r \cdot r + Ax_f$$

$$\cancel{BKx} - BKx_f = \cancel{BKx} + BK_r \cdot r + Ax_f$$

let $r = x_f$

$$BK_r x_f + Ax_f + BKx_f = 0$$

$$\left[\underline{BK_r + A + BK} \right] x_f = 0$$

$$B\underline{K_r} + A + BK = 0 \rightarrow \text{zero matrix}$$

$$K_r = \underline{\cancel{-B^{+}(A+BK)}}$$

$$= -\frac{1}{(A+BK)^{+}B}$$

Error dynamics

Consider a system

$$\dot{x} = Ax + Bu$$

Objective: design the control input u such that the system eventually converge to **the goal** x_f?

- Is the system reachable?
- Will the same feedback control: $u = Kx$ do that job?

Introducing Error dynamics: Let $e = x - x_f$ called the **error state.**
Q: How does the error evolves over time?

Goal: Design u so that the error converges to the zero.

Error dynamics

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

design a controller to reach the goal $x_f = (10, 10)^T$.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} [k_1 \quad k_2]$

$k = [k_1 \quad k_2]$

$u = Kx + K_r x_f$

$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad A + BK = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix}$

$= \underbrace{\begin{bmatrix} 1+k_1 & 1+k_2 \\ k_1 & -1+k_2 \end{bmatrix}}_{\text{stable matrix}}.$

$K_r = -\dfrac{1}{(A+BK)^{-1} B}$

Trajectory tracking

Consider a system

$$\dot{x} = Ax + Bu$$

Objective: design the control input u such that the system eventually converge to **a desired trajectory** $x^d(t)$?

① Feasible. ② . . . optimal ③ obstacle free

Similar to previous case, define $e(t) = x(t) - x^d(t)$ and aims to drive the error to the origin.

$$\dot{x}^d = Ax^d + B\underbrace{u^d}_{\text{Feasible control input}}$$

Q: How does the error evolves over time?

$$\dot{e}(t) = \dot{x}(t) - \dot{x}^d(t)$$

$$= Ax + Bu - Ax^d - Bu^d$$

$$\dot{e} = Ae + \underbrace{B(u - u^d)}_{Bv}$$

$$v = Ke \qquad \dot{e} = (\underbrace{A + BK}_{\text{stable}})e$$

$$u - u^d = v = Ke$$

$$\boxed{u_{(t)} = \underbrace{Ke_{(t)}}_{\text{FB}} + \underbrace{u^d_{(t)}}_{\text{FF (Feed-forward) Reference input}}}$$

Trajectory tracking

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Design a controller so that the system track the trajectory of $x_1 = \sin t$ and $x_2 = \sin 2t$.

$x_1^d = \sin t$: $\quad \dot{x}_1^d = \cos t = x_1^d + x_2^d + u^d$

$\cos t = \sin t + \sin 2t + u^d$

$x_2^d = \sin 2t$: $\quad \dot{x}_2^d = 2\cos 2t = -\sin 2t + u^d$

Non-feasible traj.

Design feasible traj :

note: goal-reaching is a spectral case of trajectory tracking.

Example: Feedback control, pole placement

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Choose a feedback gain K such that the closed loop system is stable and has good control performance (pole placement.)

Summary

- stability criteria of linear time invariant systems.
- reachability, controllability.
- Stablization, goal-reaching, and trajectory tracking.