Lecture notes: Introduction to Lyapunov Stability and position regulation for robot

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Outline

This lecture note is based on

- Chapter 8 in M. Spong **Robot modeling and control.**

Robot manipulator dynamics

Given the model of n-link robot manipulator:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

Equilibrium states of a robot

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

thus

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ M^{-1}(q)[\tau - C(q,\dot{q})\dot{q} - N(q)] \end{bmatrix} \qquad \boxed{u = \tau}$$

$$= f(x) + g(x_1)u$$

control-affine

$$= \begin{bmatrix} x_2 \\ M^{-1}(x_1)(-C(x_1,x_2)x_2 - N(x_1)) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(x_1) \end{bmatrix} u$$

- x_e **unforced**: $x_2^e = 0$

  $\tau = u = 0$

  $$-M^{-1}(x_1)\left[C(x_1, x_2)\underbrace{x_2}_{0} + \underbrace{N(x_1)}_{N(x_1^e)=0}\right] = 0 \;;$$

- x_e **forced**: $x_2^e = 0$

  $\tau \neq 0$

  $$-M^{-1}(x_1)\left[C(x_1, \cancel{x_2})\underbrace{x_2}_{0} + N(x_1)\right] + M^{-1}(x_1)\, u = 0;$$

  $$-M^{-1}(x_1)\left[N(x_1) - u\right] = 0$$

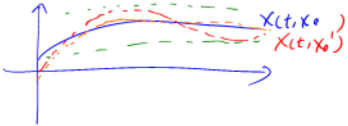
- all equilibrium states: $u^e = N(x_1^e)$.

  $x_2^e = 0 \Rightarrow \dot{q} = 0$ zero velocity

- joint torque: must balance the gravity at the equilibrium.

Recall the notion of stability

- $x_0 \neq x_0$ and $x(t, x_0)$ be the solution of the ODE with x_0 as the initial state.
- $x_0' \neq x_0$ and $x(t, x_0')$ be the solution of the ODE with x_0' as the initial state.



Stability of x_0:

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \|x_0' - x_0\| \leq \delta_\varepsilon, \implies \|x(t, x_0') - x(t, x_0)\| \leq \varepsilon, \forall t \geq t_0.$$

Asymptotic stability of x_0:

$$\exists \delta > 0, \|x_0' - x_0\| \leq \delta, \implies \|x(t, x_0') - x(t, x_0)\| \to 0 \text{ as } t \to \infty$$

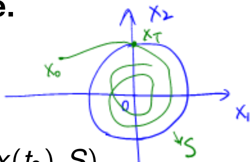
Global A.S.: $\forall \delta > 0$,

Stability

exponential stability:

$$\exists \delta, c, \lambda > 0 : \quad \|x_0' - x_0\| < \delta \rightarrow \|x(t, x_0') - x(t, x_0)\| \leq c \exp^{-\lambda t} \|x_0' - x_0\|.$$

for nonlinear system, this may hold up to a maximum finite δ — **called the region of attraction, which is hard to estimate.**



"practical " stability of a set S

$$\exists T(x(t_0), S) \in \mathbb{R} : x(t, x_0) \in S, \forall t \geq t_0 + \underbrace{T(x(t_0), S)}_{\text{bounded time}}$$

also known as **u.u.b. stability** ("ultimately uniformly bounded.")

The direct method of Lyapunov

Problem: How to determine the stability of a system?

$$\dot{x} = f(x)$$

Previously, we learned about the stability verification method of LTI system

$$\dot{x} = Ax$$

negative eigenvalues $\longrightarrow$ A stable

For unforced, time invariant system $\dot{x} = f(x)$, we can analyze the local stability around the equilibrium $x_e : f(x_e) = 0$ by local linearization and the stability of the linear system.

$$\dot{x} \approx f(x_e) + \underbrace{\frac{\partial f}{\partial x}\Big|_{x_e}}_{A} (x - x_e) \underbrace{\boxed{\cdots}}_{\text{omited.}}$$

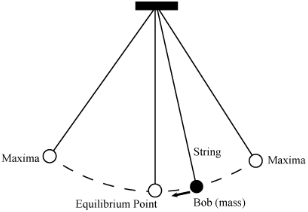
What about global stability?

The direct method of Lyapunov

Problem: How to determine the stability of a system without explicitly integrating the ODE?

$$\dot{x} = f(x)$$

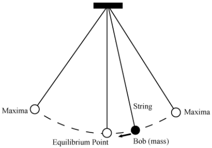
- Lyapunov formalized the idea: If the total energy is dissipated, then the system must be stable.
- Lyapunov function: “a measure of energy”.



The direct method of Lyapunov

Problem How to determine the stability of a system without explicitly integrating the ODE?

$$\dot{x} = f(t, x)$$



The energy is dissipated along the state trajectory of the system.
insight: To verify stability,

- Find a Lyapunov function ("energy").
- Show that as the system evolves, the energy dissipated.

Lyapunov and level sets

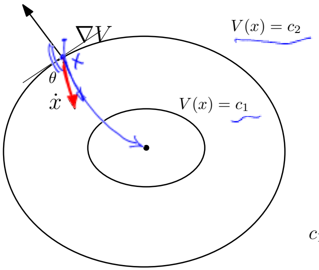
How to interpret stability using Lyapunov function?

scalar function of x

- $V(x) \geq 0$ — energy is always nonnegative.
- $V(x_e) = 0$ — lowest energy at the stable equilibrium.
- $\frac{dV}{dt} \leq 0$ — $\lim_{\Delta t \to 0} \frac{V(t+\Delta t) - V(t)}{\Delta t} \leq 0$ energy is nonincreasing.
- $\frac{dV}{dt} < 0$ — energy is strictly decreasing.

$\dot{x} = f(x)$

$v \cdot w$
$= \|v\| \|w\| \cos\theta$



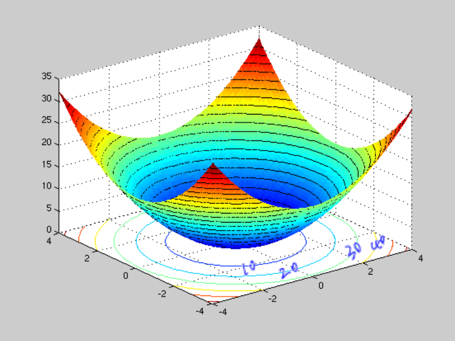
$c_1 < c_2$

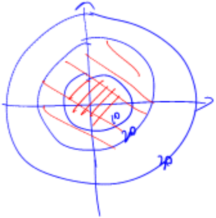
$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$
$$= \nabla V \cdot \dot{x}$$
$$= \|\nabla V\| \|\dot{x}\| \cos\theta$$

if

Level sets, contour plot of Lyapunov function

$V(x) = x_1^2 + x_2^2$.





level set; S is a level set of V for a given c_0;

$$S = S(c) = \{x \in \mathbb{R}^n : V(x) \leq c_0\}$$

Change of coordinates

Often we change the coordinate to make $x_e = 0$.
That is, we introduce a new variable $\tilde{x} = x - x_e$.

stability of $x = x_e$ $\Leftrightarrow$ stability of $\tilde{x} = 0$

$\dot{\tilde{x}}$

Preliminaries: Positive definite functions

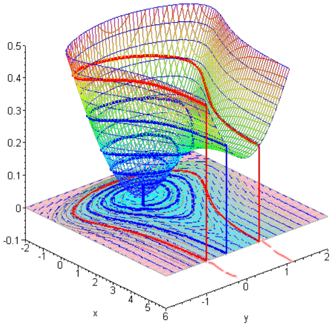
A function $V : \mathbb{R}^n \to \mathbb{R}$ is **positive definite** (PD) if

- $V(x) \geq 0 \quad \forall x$
- $V(x) = 0 \quad$ iff $x = 0$
- all levelsets of $V(x)$ have to be bounded.

$V(x) = x^T P x$ with $P = P^T$, is PD if and only if $P > 0$.

A function V is negative definite if and only if $-V$ is PD.

An example of unbounded level sets



Lyapunov Stability

Lyapunov candidate: $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that

$$V(0) = 0, \quad V(x) > 0, \forall x \neq 0$$

a positive definite function.
sufficient condition of stability

$$\exists V \text{ candidate} : \dot{V}(x) \leq 0 \text{ along the trajectory of } \dot{x} = f(x)$$

negative semi-definite $\dot{V}$

sufficient condition of asymptotic stability

$$\exists V \text{ candidate} : \dot{V}(x) < 0 \text{ along the trajectory of } \dot{x} = f(x)$$

negative definite $\dot{V}$

A Lyapunov exponential stability theorem

suppose a function V that is

- V is lyapunov candidate.
- $\dot{V}(x) \leq -\alpha V(x)$ for all x. $\alpha > 0$

then, there exists an M such that every trajectory of $\dot{x} = f(x)$ satisfies $\|x(t)\| \leq Me^{-\alpha t/2}\|x(0)\|$. (globally exponential stable.)

interpretation $V(x)$ large $\rightarrow$ $-\alpha V(x) \ll 0$

$V(x)$ small $\rightarrow$ $-\alpha V(x) < 0$

A Lyapunov instability theorem

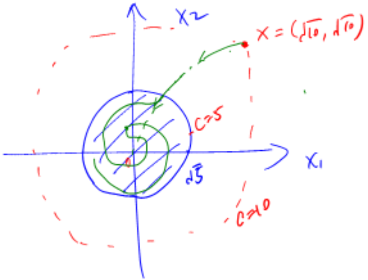
suppose a function V that is

- $\dot{V}(x) \leq 0 \quad \forall x$
- $\exists\, w \neq 0$, such that $V(w) < V(0)$

then,the trajectory of $\dot{x} = f(x)$ with $x(0) = w$ does not converge to zero.

$$V(\Delta t) \leq V(t_0) = V(w) < V(0)$$
$\Delta t > 0$

interpretation



$\exists V$ candidate such that

- S is a level set of V for a given c_0.
- $\dot{V} < 0$, along the trajectories of $\dot{x} = f(x)$, $x \notin S$.

level set: $S = \{x \mid V(x) \le c_0\}$

$V = x_1^2 + x_2^2$

$x_1^2 + x_2^2 \le 5$

Challenges

Lyapunov theory is only sufficient but not necessary.

- ☺If we find a Lyapunov function, the system is stable.
- ☹But if we cannot find a Lyapunov function, it does not mean the system is unstable.

Example: Lyapunov stability applies to LTI system

$$\dot{x} = Ax$$

A is stable

Define $V(x) = x^T P x$ P : positive definite

① $V(x) > 0 \quad \forall x \neq x_e$ ✓

② $V(x_e) = 0 \Rightarrow x_e = 0$; $V(0) = 0$ ✓

③ $\frac{dV}{dt} < 0$

$\frac{dV}{dt} = \nabla V \cdot \dot{x} = x^T P \dot{x} + x^T P^T \dot{x}$

$= x^T P A x + x^T P^T A x$

$= x^T \underbrace{(PA + P^T A)}_{\text{negative definite}} x$

$\underline{x^T P x}$

$\alpha = y^T P x$

$\frac{\partial \alpha}{\partial x} = y^T P$

$\frac{\partial \alpha}{\partial y} = x^T P^T$

P symmetric $\longrightarrow = 2x^T PA x$

A must be stable.

Example

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$\dot{x} = -x + y + xy$$
$$\dot{y} = x - y - x^2 - y^3$$

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

Question: is $(0, 0)^T$ a stable equilibrium?

Local stability: $f(x_e) = 0$.

$\dot{\vec{x}} = f(\vec{x}) \Rightarrow \frac{\partial f}{\partial \vec{x}}\Big|_{\vec{x}_e} = \begin{bmatrix} -1+y & 1+x \\ 1-2x & -1-3y^2 \end{bmatrix}\Bigg|_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\downarrow$ linearization.

$\dot{\vec{x}} = Ax \Leftarrow A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$\det[\lambda I - A] = \det\begin{bmatrix} \lambda+1 & -1 \\ -1 & \lambda+1 \end{bmatrix}$

$= (\lambda+1)^2 - 1 = 0$

$\lambda_1 = 0, \quad \lambda_2 = -2$

$$\dot{x} = -x + y + xy$$

$$\dot{y} = x - y - x^2 - y^3$$

Lyapunov: $V = x^2 + y^2$ candidate.

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= 2x(-x + y + xy) + 2y(x - y - x^2 - y^3)$$

$$= -2x^2 + 2xy + \cancel{2x^2y} + 2xy - 2y^2 - \cancel{2x^2y} - 2y^4$$

$$= -2(x^2 - 2xy + y^2) - 2y^4$$

$$= -2(x-y)^2 - 2y^4 \le 0$$

when $\dot{V} = 0 \Rightarrow y = 0$, $x - y = 0$ iff $x = y = 0$

$\Rightarrow$ system is globally asymptotically stable.

Example

consider the system

$$\dot{x}_1 = -x_1 + g(x_2) \quad (1)$$
$$\dot{x}_2 = -x_2 + h(x_1). \quad (2)$$

where $|g(z)| \leq |z|/2$ and $|h(z)| \leq |z|/2$.

$|g(x_2)| \leq |x_2|/2$

$V = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V} = x_1 \cdot \dot{x}_1 + x_2 \dot{x}_2$

$= x_1(-x_1 + g(x_2)) + x_2(-x_2 + h(x_1))$

$= -x_1^2 - x_2^2 + \underbrace{x_1 g(x_2)}_{\leq |x_1||g(x_2)|} + x_2 h(x_1) \quad \cdots\cdots \leq 0$

$\leq -x_1^2 - x_2^2 + |x_1| \cdot |g(x_2)| + |x_2| \cdot |h(x_1)|$

$$\leq -x_1^2 - x_2^2 + \frac{|x_1|\cdot|x_2|}{2} + \frac{|x_2||x_1|}{2} \overset{?}{\leq} 0$$

$$\leq -(|x_1| - |x_2|)^2 - |x_1||x_2| \qquad \left(-(x_1^2 + x_2^2 - |x_1||x_2|)\right.$$

$$-x_1^2 - x_2^2 + 2|x_1||x_2| \qquad - |x_1||x_2|$$

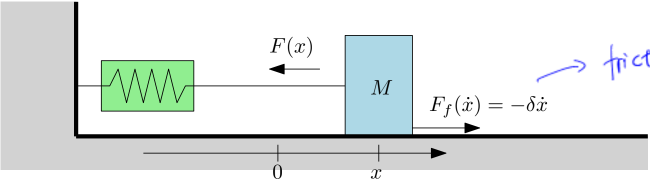
$$= -|x_1|^2 - |x_2|^2 - 2|x_1||x_2| + 2|x_1||x_2| + |x_1||x_2|$$

$$= -(|x_1| - |x_2|)^2 - |x_1||x_2| \leq 0$$

$$|x_1| = |x_2| \qquad |x_1||x_2| = 0$$

$$|x_1|^2 = 0 \Rightarrow x_1^2 = 0$$

G.A.S.



→ friction coeff. constant > 0

The linear harmonic oscillator:

$$\underline{M\ddot{x} = -F(x) - \delta\dot{x}}$$

$x_2 = \dot{x}$
$\dot{x}_1 = \ddot{x} = -F(x) - \delta\dot{x}$
$= -F(x_1) - \delta x_2$

let $M = 1$, $x_1 = x$ and $x_2 = \dot{x}$.
The state space equation is

$\dot{x}_1 = x_2$
$\dot{x}_2 = -\delta x_2 - F(x_1)$

consider the Lyapunov function candidate:

$$V(x) = \underline{\int_0^{x_1} F(s)ds} + \frac{1}{2}x_2^2$$

Show the system is stable:

$$\dot{V} = F(x_1)\dot{x}_1 + x_2 \cdot \dot{x}_2$$

$$= F(x_1)x_2 + x_2 \cdot (-\delta x_2 - F(x_1))$$

$$= -\delta x_2^2 \leq 0$$

$\therefore$ stable

$\dot{V} = 0$ if $x_2 = 0$.

Finding Lyapunov functions

- there are many different types of Lyapunov theorems
- the key in all cases is to find a Lyapunov function and verify that it has the required properties

one common approach:

- decide form of Lyapunov function (e.g., quadratic), parametrized by some parameters (called a Lyapunov function candidate)
- try to find values of parameters so that the required hypotheses hold.