For the integral action,

$$\dot{x} = \bar{A}x + \bar{B}u.$$

$\bar{A}, \bar{B}$ are matrices in the original system.

Augmenting the state space $\xi = \begin{bmatrix} x \\ z \end{bmatrix}$. $\dot{z} = y - r = x_1 - r$

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \bar{A}x + \bar{B}u \\ x_1 - r \end{bmatrix} = \begin{bmatrix} \bar{A}x + \bar{B}u \\ [1\ 0]x - r \end{bmatrix}$$

$$= \left[\begin{array}{c:c} \bar{A} & \mathbf{0} \\ \hdashline [1\ 0] & 0 \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix} + \left[\begin{array}{c} \bar{B} \\ \hdashline 0 \end{array}\right] u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r.$$

Note the dynamics of z is not influenced by the input.

$\Rightarrow$ as long as $x_1 = r$. z reach its equilibrium. $\dot{z}_e = 0$.

let $u = -Kx - K_i z + K_r r$

$$\dot{\xi} = \left[\begin{array}{c:c} \bar{A} & 0 \\ \hdashline [1\ 0] & 0 \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix} + \left[\begin{array}{c} \bar{B} \\ \hdashline 0 \end{array}\right] [-Kx - K_i z + K_r r]$$

$$+ \begin{bmatrix} 0_{2\times1} \\ -1 \end{bmatrix} r$$

$$= \left[\begin{array}{c:c} \bar{A} - \bar{B}K & -\bar{B}K_i \\ \hdashline [1\ 0] & 0 \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix} + \left[\begin{array}{c} \bar{B}K_r \\ \hdashline -1 \end{array}\right] r$$

The equilibrium (x_e, z_e) satisfies $\dot{\xi}_e = 0$

$$\begin{cases} (\bar{A}-\bar{B}K)x_e - \bar{B}K_i z_e + \bar{B}K_r \cdot r = 0 & ① \\ [1 \quad 0]x_e - r = 0 & ② \end{cases}$$

Thus: $x_e = -(\bar{A}-\bar{B}K)^{-1}[-\bar{B}K_i z_e + \bar{B}K_r \cdot r]$.

Consider the closed-loop system.

$$\dot{x} = (\bar{A}-\bar{B}K)x - \bar{B}K_i z + \bar{B}K_r \cdot r$$

When z_e reaches its equilibrium. $\dot{z} = y - r = 0$

z_e will stay a constant as long as $y = r$.

When z_e is a constant,

$$\dot{x} = (\bar{A}-\bar{B}K)x - \underbrace{\bar{B}K_i z + \bar{B}K_r \cdot r}_{\text{constant.}}$$

is a stable system if $\bar{A}-\bar{B}K$ is a stable matrix

① if x_e is stable $\Rightarrow$ z_e constant $\Rightarrow$ $y = r$.

Thus: select K to make $A-BK$ stable.

② After selecting K, select K_i and K_r to make

$$(A) \begin{cases} Cx_e = r \\ -C(\bar{A}-\bar{B}K)^{-1}[-\bar{B}K_i z_e + \bar{B}K_r r] = r. \end{cases}$$

and (B): select K, K_i to make the closed-loop dynamics behaves as we design.

$$\det\left[sI - \begin{bmatrix} \bar{A}-\bar{B}K & \vdots & -\bar{B}K_i \\ [I \;\; 0] & \vdots & 0 \end{bmatrix} \right)$$

with desired ~~eigenvalues~~ roots.

* Actually, K_r can be

$$K_r = \frac{-1}{C(\bar{A}-\bar{B}K)^{-1}\bar{B}}$$

or even $K_r = 0$.

Because when the system reaches steady states, by the choice of K and K_i,

Even with $K_r = 0$. as $d_t = 0$.

$\boxed{y = r}$ is ensured.