

Problem 1

(20pt) Consider the system

$$\begin{aligned}\dot{x} &= -y - x^3 \\ \dot{y} &= x - y^3\end{aligned}$$

- 5pt : show that the origin is an equilibrium.
- 15pt: Take $V(x, y) = \frac{x^2+y^2}{2}$, and show the system asymptotically converge to the origin.

Problem 2

(20pt) Consider the system

$$\begin{aligned}\dot{x} &= -3y \\ \dot{y} &= x - \alpha(2y^3 - y),\end{aligned}$$

where $\alpha < 0$. Take $V(x, y) = \frac{x^2+3y^2}{2}$, and show the system asymptotically converge to the origin.

Problem 3

(30pt) Consider a scalar system

$$\dot{x} = ax^3$$

- 10pt: Show that linearization method fails to determine stability of the origin.
- 15pt: Use the Lyapunov function

$$V(x) = x^4$$

to show that the system is stable for $a < 0$ and unstable for $a > 0$.

- 5pt: What can you say about the system for $a = 0$?

Problem 4

(30pt) Consider the pendulum equation

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{l} x_2. \tag{2}$$

Assume zero friction, i.e. let $k = 0$, and show that the origin is stable. (Hint. Show that the energy of the pendulum is constant along all system trajectories.)