RBE 502

Homework 3 Solution

Problem #1

ⓐ $\dot{x} = -y - x^3 \qquad \dot{y} = x - y^3$

at origin $x = y = 0$

at equilibrium $\dot{x} = \dot{y} = 0$

$$\boxed{\begin{aligned} \dot{x} &= -y - x^3 = -0 - 0^3 = 0 \\ \dot{y} &= x - y^3 = 0 - 0^3 = 0 \end{aligned}}$$

ⓑ $V(x,y) = \frac{x^2 + y^2}{2}$

equilibrium is asymptotically stable if $\frac{dV}{dt} < 0$ (decreasing) & $\dot{V}(0) = 0$

& $V(x,y) \geq 0$ (positive definite) $V(0,0) = 0$ only here $V(\infty, \infty) \to \infty$ as $(x,y) \to \infty$

$$\frac{dV}{dt} = \frac{\delta V}{\delta x}\frac{dx}{dt} + \frac{\delta V}{\delta y}\frac{dy}{dt}$$

$$= x\dot{x} + y\dot{y} = x(-y - x^3) + y(x - y^3)$$

$$= -(x^4 + y^4)$$ ✓ since $x^4 + y^4$ is positive, $-(x^4+y^4)$ is negative $\left\{ \begin{array}{l} \text{for } x \neq 0 \\ y \neq 0 \end{array} \right.$

✓ also $x^2 + y^2$ is positive when $x$ & $y$ are real

✓ and $\frac{0^2 + 0^2}{2} = 0$ ✓ $\frac{\infty^2 + \infty^2}{2} = \infty$

Problem #2

$$\dot{x} = -x^3 + 2y^3 \qquad \dot{y} = -2xy^2$$

$$V = \frac{1}{2}(x^2 + y^2)$$

$$\dot{V} = x\dot{x} + y\dot{y} = -x^4 + 2xy^3 + (-2xy^3) = -x^4$$

✓ $\dot{V} \leq 0$ only negative semi definite because it does not depend on y

& an observability type condition

for $\dot{V} = 0$ x always $= 0$ or $x \equiv 0$

$$\dot{y} = -2(0)y^2 = 0$$

$$\frac{dx}{dt} = \frac{d(0)}{dt} = 0 = \dot{x} = -x^3 + 2y = -0^3 + 2y$$

(0,0) is only invarient set in E = x axis

$$y = 0$$

So stable

Problem #3

Scalar system $\dot{x} = ax^3$

linearization does eigenvalues of $\left.\frac{\delta f}{\delta x}\right|_{x=0}$ have negative real parts?

$$\frac{\delta f}{\delta x} = \frac{\delta \dot{x}}{\delta x} = 3ax^2 \qquad 3ax^2\big|_{x=0} = 0$$

don't know eigenvalues

Problem #3 continued

$V(x) = x^4 \geq 0$ $\quad V(0) = 0^4 = 0 \quad V(\infty) = \infty^4 = \infty$

positive definite

$\dot{V}(x) = 4x^3 \cdot \dot{x} = 4ax^6$

$\dot{V}(x) = 0$ for $x = 0$

$\dot{V}(x) \leq 0$ if $a < 0$

$\dot{V}(x) \neq 0$ if $x \neq 0$ & $a \neq 0$ Stable

$\dot{V}(x) \geq 0$ if $a > 0$ unstable

at $a = 0$ $\quad \dot{V}(x) = 0$

Stable but not asymptotically stable

marginally stable

Problem #4


$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{K}{l} x_2$

$k = 0 \quad \dot{x}_2 = -\frac{g}{l} \sin x_1$

$x_1 = \theta$

$\dot{x}_1 = x_2 = \dot{\theta}$

$\dot{x}_2 = \ddot{\theta} = -\frac{g}{l} \sin\theta$

$PE = mlg(1 - \cos x_1)$

$KE = \frac{1}{2} m l^2 \dot{\theta}^2$

$V = mlg(1 - \cos x_1) + \frac{1}{2} m l^2 \dot{\theta}^2$ $\quad$ all terms positive so $V \geq 0$

$V(0,0) = mlg(1 - \cos 0) + \frac{1}{2} m l^2 0^2 = 0 \quad V(\infty, \infty) =$ something $+ \infty = \infty$

Problem #4 continued

$$\dot{V} = mlg \sin\theta (\dot{\theta}) + ml^2 \dot{\theta} \ddot{\theta}$$

$$= mlg\dot{\theta} \sin\theta + ml^2 \dot{\theta} \left(-\frac{g}{l} \sin\theta\right) = mlg\dot{\theta} \sin\theta - mlg\dot{\theta} \sin\theta$$

$$= 0$$

since $V =$ total energy $\dot{V} =$ change in energy $= 0$

energy is constant