Example

$$\dot{x} = -x + y + xy$$
$$\dot{y} = x - y - x^2 - y^3$$

Question: is $(0,0)^T$ a stable equilibrium?

$V(x,y) = x^2 + y^2$ $\quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \Rightarrow \nabla V = \begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{bmatrix}$

$\frac{\partial V}{\partial \vec{x}} f(\vec{x}) = 2x \cdot \dot{x} + 2y \cdot \dot{y}$ $\quad \dot{V} = \nabla V \cdot f(\vec{x})$

$= 2x \cdot (-x + y + xy) + 2y(x - y - x^2 - y^3)$ $\quad = \nabla V^T f(\vec{x})$

$= -2x^2 + 2xy + 2x^2y + 2xy - 2y^2 - 2x^2y - 2y^4$ $\quad = \begin{bmatrix} 2x \\ 2y \end{bmatrix}^T \begin{bmatrix} -x+y+xy \\ x-y-x^2-y^3 \end{bmatrix}$

$= -2x^2 + 4xy - 2y^2 - 2y^4$

$= -2(x^2 - 2xy + y^2) - 2y^4 = -2(x-y)^2 - 2y^4 \leq 0$

"=" holds only if $x = y$, $y = 0$

Example

consider the system

$$\dot{x}_1 = -x_1 + g(x_2) \tag{3}$$
$$\dot{x}_2 = -x_2 + h(x_1) \tag{4}$$

$\dot{x}_1 = -x_1 + x_2 = 0$

$\dot{x}_2 = -x_2 + x_1 = 0$

where $|g(z)| \leq |z|/2$ and $|h(z)| \leq |z|/2$.

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\dot{V} = \frac{\partial V}{\partial x} f(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= x_1(-x_1 + g(x_2)) + x_2 \cdot (-x_2 + h(x_1))$$

$$= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 \cdot h(x_1) \leq 0$$

$$\leq -x_1^2 - x_2^2 + |x_1 g(x_2)| + |x_2 h(x_1)| \quad *$$

$$|g(x_2)| \leq \frac{|x_2|}{2} \; ; \quad |h(x_1)| \leq \frac{|x_1|}{2} \; ;$$

$$\leq -x_1^2 - x_2^2 + \underbrace{\frac{|x_1 x_2|}{2} + \frac{|x_2 x_1|}{2}}_{|x_1 x_2|} \leq 0$$

$$x_1^2 + x_2^2 \geq 2|x_1 x_2|$$

$$\leq -(x_1^2 + x_2^2) + \frac{1}{2}(x_1^2 + x_2^2) = -\frac{1}{2}(x_1^2 + x_2^2)$$

$$\leq 0 \quad \text{and} \quad \dot{V} = 0 \Rightarrow x_1 = x_2 = 0$$

$$V(x) = (x_1^2 + x_2^2)\frac{1}{2} \; ; \quad \dot{V} \leq -\alpha V$$

$$\alpha = 1$$