# Lecture notes: Introduction to Lyapunov Stability and position regulation for robot

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# Outline

This lecture note is based on

- Chapter 8 in M. Spong **Robot modeling and control.**

# Properties of robot manipulator dynamics

Given the model of n-link robot manipulator:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

- $M(q)$ is symmetric, positive definite.
- $\dot{M}(q) - 2C(q,\dot{q})$ is **skew symmetric**.

$A^T = -A$

property. for all $v \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$

$v^T A v = 0$

proof: $(v^T A v)^T = v^T A^T v = -v^T A v$

$(v^T A v)^T + v^T A v = 0$

$v^T A v = u$

$u + u = 0$

$u = 0$

# Passivity

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \underbrace{N(q)}_{\text{gravity}} = \tau$$

$$\int_0^T \dot{q}^T(\zeta)\tau(\zeta)d\zeta \geq -\beta, \quad \beta > 0, \forall T > 0$$

The energy dissipated from the system has a lower bound $-\beta$.

**proof**

Total Energy.

$$H(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q) \qquad \frac{\partial P(q)}{\partial q} = N(q)^T$$

$$\dot{H} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial \dot{q}}\ddot{q}$$

$$= \left(\frac{1}{2}\dot{q}^T \dot{M}(q) + \frac{\partial P}{\partial q}\right)\dot{q} + \dot{q}^T \underbrace{M(q)\ddot{q}}_{(\tau - N(q) - C(q,\dot{q})\dot{q})}$$

$$= \frac{1}{2}\dot{q}^T\underbrace{(\dot{M}(q) - 2C(q,\dot{q}))}_{0}\dot{q} + N(q)^T\dot{q} - \dot{q}^T N(q) + \dot{q}^T\tau$$

$$= \dot{q}^T\tau$$

$$\int_0^T \dot{q}\tau\, dt = \int_0^T \dot{H}(q,\dot{q})dt = \underbrace{H(T)}_{\geq 0} - H(0) \geq -H(0)$$

# Centralized control of robot manipulator

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$$

Special case: Plannar manipulator, $N(q) = 0$.
**Control objective**: Set point tracking.

- asymptotic stabilization (= regulation) of the closed-loop equilibrium state $q_d \in \mathbb{R}^n$.

$$q = q_d, \quad \dot{q} = 0.$$

PD CONTROL: proportional + derivative action on the error.

$$u = -K_P(q - q_d) \cancel{+} - K_D\dot{q}.$$

$\dot{q} = \dot{q} - \dot{q}_d$ error in velocity.

$\dot{q}_d = 0$

# Asymptotic convergence with PD

$N(q) = 0$, the decentralized control $\overset{\tau}{u} = -K_P e - K_D \dot{e}$ achieves asymptotic convergence for **set point tracking**.
Proof: Lyapunov function

err

$e = q - q_d$

$\dot{e} = \dot{q} - \dot{q}_d = \dot{q}$

$$V = \frac{1}{2}\dot{q}^T \underbrace{M(q)}_{\geq 0}\dot{q} + \frac{1}{2}e^T K_P e.$$

positive definite $\underbrace{K_P}_{\text{symmetric}}$ $\quad K_P \geq 0$

$$\dot{V} = \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}}\ddot{q}$$

$$= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + e^T K_p \dot{e}$$

$$= \dot{q}^T(\tau - C(q,\dot{q})\dot{q}) + (q - q_d)^T K_p \dot{q} + \frac{1}{2}\dot{q}^T \dot{M}\dot{q}$$

$$= \frac{1}{2}\dot{q}^T \underbrace{(\dot{M} - 2C(q,\dot{q}))}_{0}\dot{q} + \dot{q}^T\tau + (q - q_d)^T K_p \dot{q}$$

$$= \dot{q}^T(-K_p e - K_D \dot{e}) + \underbrace{(q - q_d)^T K_p \dot{q}}_{e^T K_p \dot{q}}$$

$$= -\dot{q}^T K_P e - \dot{q}^T K_D \dot{e} + e^T K_P \dot{q} = \underbrace{-\dot{q}^T K_D \dot{q}}_{\text{negative definite}} \leq 0$$

$K_P$ symmetric

$K_D$ positive definite

and $=0$ only if $\boxed{\dot{q}=0}$

A.S: Lyapunov candidate $V \geq 0$

$\dot{V} \leq 0$ and only $=0$ at equilibrium

Equilibrium: $q_d$, $\dot{q}_d = 0$

$$\left\{ \begin{pmatrix} q_d \\ 0 \end{pmatrix} \right\} = \underbrace{\{x \mid \dot{V} \equiv 0\}}$$

$$\dot{V} = -\dot{q}^T K_D \dot{q}$$

$\dot{q} = 0$ & $\ddot{q} = 0$

$$\underbrace{M\ddot{q}}_{0} + \underbrace{C(q,\dot{q})\dot{q}}_{0} = -K_P(q-q_d) - \underbrace{K_D(\dot{q})}_{0}$$

$$K_P(q-q_d) = 0 \Rightarrow q = q_d$$

# Asymptotic convergence with PD

$$u = -K_p(q - q_d) - K_D\dot{q}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -\begin{bmatrix} K_{p_1} & & \\ & K_{p_2} & \\ & & K_{p_3} \end{bmatrix}\begin{bmatrix} q_1 - q_{1d} \\ q_2 - q_{2d} \\ q_3 - q_{3d} \end{bmatrix} - \begin{bmatrix} K_{D_1} & & \\ & K_{D_2} & \\ & & K_{D_3} \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$\boxed{u_i = -K_{p_i}(q_i - q_{id}) - K_{D_i}\dot{q}_i \quad \forall i = 1, 2, 3}$$

# Asymptotic convergence with PD

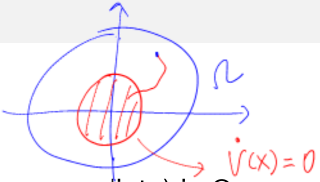
Challenge: $\dot{V} = 0$ when $\dot{q} = 0$, but does not show $q = q_d$.

The closed-loop system is asymptotically stable, witnessed by LaSalle's invariance principle.

Preliminaries: Consider an unforced system $\dot{x} = f(x)$, $f(0) = 0$.
**Defn:** A set $X$ is

- an <u>invariant set</u>: if $x(0) \in X$, then for all $t \in R$, $x(t) \in X$.
- a <u>positively invariant set</u>: if $x(0) \in X$, then for all $t > 0$, $x(t) \in X$.

# LaSalle's invariance principle




**Thm: (LaSalle's theorem)**

- Let $\Omega$ be a postively invariant set.
- If : $\exists V$ function (not need to be a Lyapunov candiate) in $\Omega$: $\dot{V}(x) \leq 0$ along the trajectory of $\dot{x} = f(x)$
- Then: system trajectories starting within $\Omega$ asymptotically converge to the **the largest invariant set**

$\dot{V} \equiv 0$

$$\underset{\sim}{M} \subseteq S = \{x \in \mathbb{R}^n : \underline{\dot{V}(x) = 0}\}.$$

Unlike Lyapunov theorems, LaSalles theorem does not require the function $V$ to be positive definite.

Corrollary:

$$M = \{0\} : \text{ asymptotic stability.}$$

# LaSalle's invariance principle

Revisit the PD controller:

# PD with gravity compensation

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \tau$

PD without gravity compensation:

$$u = -K_P e - K_D \dot{e}$$

$M\ddot{q} + C(\cdot)\dot{q} = \boxed{\tau - N(q)}$ $u$

$\tau'' = u$

with gravity compensation

$$u = -K_P e - K_D \dot{e} + N(q)$$

The **same** Lyapunov function

$$V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2} e^T K_P e.$$

and verify that

$$\dot{V} = \dot{q}^T (u - N(q) + K_P e)$$

$= \dot{q}^T(-K_P e - K_D \dot{e} + K_P e) = -\dot{q}^T K_D \dot{e} = -\dot{q}^T K_D \dot{q}$

# Comments on PD control

- " choice of control gains affects robot evolution during transients and practical settling times.
- full $K_P$ and $K_D$ gain matrices allow to assign desired eigenvalues to the linear approximation of the robot dynamics around the final desired state $(q_d, 0)$.
- when (joint) viscous friction $-F_v\dot{q}$ is present, the derivative term $K_D\dot{q}$ in the control law is not strictly necessary. But having $K_D\dot{q}$ allows more flexible modulation.
- in the absence of tachometers, the actual realization of the derivative term in the feedback law requires some processing of the position data measured at the joints by encoders or resolvers.