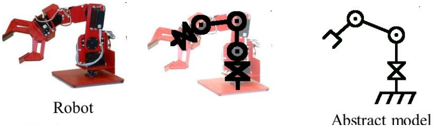
Lecture notes: Dynamic model of robots: Lagrangian approach

Jie Fu

Department of Electrical and Computer Engineering
Robotics Engineering Program
Worcester Polytechnic Institute

RBE502, 2017



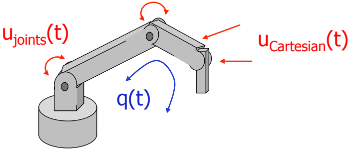
This lecture is based on

- Chapter 4 of Murray, Richard M., et al. A mathematical introduction to robotic manipulation. CRC press, 1994.

Dynamic model

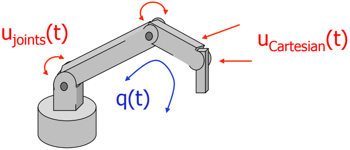
To analyze robotic systems and control them, need to have

- The relation between **generalized forces** $u(t)$ acting on the robot, and **robot motion**, i.e., configurations $q(t)$ over time



a system of equations: $\Phi(q, \dot{q}, \ddot{q}) = u$.

Direct dynamics



direct relation from **input** to **state**.

- Experimentally, given the input trajectory $u(t)$ (torque or force), can **measure the joint variables** with sensors $q(t)$. Note: both u and q can be high-dimensional.
- Simulation: Given a second order model $\Phi(q, \dot{q}, \ddot{q}) = u$, can **integrate numerically** the differential equations to obtain $\hat{q}(t)$.

If the model is **perfect**, then $q(t) \approx \hat{q}(t)$.
In reality, model is not perfect and there can be external noises too: Need closed-loop control.

Euler-Lagrange Method

An **energy-based** method to construct the dynamic model.

- Symbolic/closed form solutions of dynamic equations.
- Used commonly for control design.

Assumption: **rigid body dynamics.**

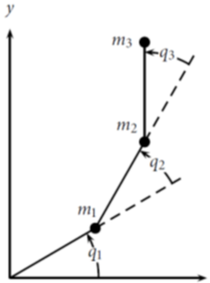
example: rigid robotic arm, bipedal robots.

not applicable for: soft robotics.

Euler-Lagrange Modeling

Generalized coordinates:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \in \mathbb{R}^n$$



Generalized input:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n$$

Euler-Lagrange Modeling

$\phi(q, \dot{q}, \ddot{q}) = u$

The Lagrangian

$$L(q, \dot{q}) = K(q, \dot{q}) - P(q)$$

- $K(q, \dot{q})$ — the total kinetic energy.
- $P(q)$ — the potential energy.

Euler-Lagrange equation relates $q, \dot{q}$ with the input u:

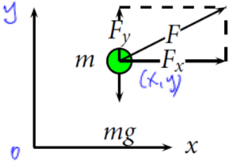
$$\phi(q, \dot{q}, \ddot{q}) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i. \quad i = 1, \ldots, n.$$

u_i — generalized forces performing work on q_i.

- Based on the **principle of virtual work**

Euler-Lagrange Modeling

A simple example:



- Generalized coordinates: x, y
- Potential energy: $P(x, y) = mgy$.
- Kinetic energy: $K(x, y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$\mathcal{L}(x,y) = K - P$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy.$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F_x$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

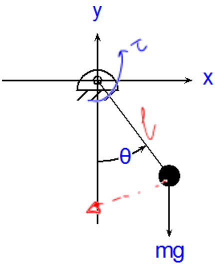
$$\frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} = F_x$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} \; ; \; \frac{\partial L}{\partial y} = -mg \Rightarrow m\ddot{y} - (-mg) = F_y$$

$$m\ddot{y} = F_y - mg$$

Example: Pendulum



$$\mathcal{L} = K - P$$

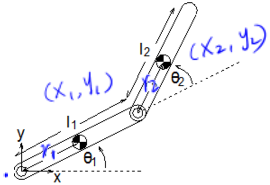
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i$$

$i = 1, \quad q_1 = \theta. \quad u_1 = \tau$

- Generalized coordinates: θ
- Potential energy: $P(x, y) = mg\ell(1 - \cos\theta)$.
- Kinetic energy: $K(x, y) = \frac{1}{2}m\ell^2\dot{\theta}^2$.

$$m\ell^2\ddot{\theta} + mg\ell\sin(\theta) = \tau$$

Example: 2-DOF planner robot



1) $K_T := \frac{1}{2} m_1 \|V_1\|^2 + \frac{1}{2} m_2 \|V_2\|^2$

$x_1 = r_1 \cos\theta_1 = r_1 C_1$

$y_1 = r_1 \sin\theta_1 = r_1 S_1$

$x_2 = l_1 \cos\theta_1 + r_2 \cos(\theta_1+\theta_2) = l_1 C_1 + r_2 C_{12}$

$y_2 = l_1 \sin\theta_1 + r_2 \sin(\theta_1+\theta_2) = l_1 S_1 + r_2 S_{12}$

$v_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -r_1 S_1 \dot{\theta}_1 \\ r_1 C_1 \dot{\theta}_1 \end{bmatrix}$ $\quad V_2 =$

- Generalized coordinates: $q = [\theta_1, \theta_2]^\intercal$.
- Potential energy: 0 (moving in horizontal plane).
- Kinetic energy: $K(x, y) = K_T + K_R$ (translational kinetic energy and rotational kinetic energy.) $= \frac{1}{2} \dot{q}^T \underbrace{M(q)}_{2\times 2} \dot{q}$ inertia matrix

$\dot{x}_1 = \frac{dx_1}{dt} = \frac{d(r_1 \cos\theta_1)}{dt} = -r_1 \sin\theta_1 \frac{d\theta_1}{dt}$

$$K_R = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

inertia I_i: moment of inertia about the axis through the COM of link i, parallel to the z-axis

$$\mathcal{L} = K - \underbrace{P}_{0}$$

Summarizing ...

- Kinetic energy: $K(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$ where $M(q)$ is the inertia matrix of the rigid body.
- Potential energy: $P(q)$.
- Lagrangian: $L = K(q,\dot{q}) - P(q)$.
- Euler-Lagrangian Equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i. \quad i = 1, \ldots, n.$$

where u_i is the generalized force performing work on the q_i-th coordinate.

state : $q, \dot{q}, \ddot{q}$

joint variables $\theta_1, \theta_2 \cdots = q_1, q_2 \cdots$

Summarizing ...

Applying Euler-Lagrangian equation: Note that $L(q, \dot{q}) = \frac{1}{2} \sum_{ij} m_{ij}(q) \dot{q}_i \dot{q}_j - P(q)$.

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}(q) \dot{q}_j \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}(q) \ddot{q}_j + \sum_j \sum_i \frac{\partial m_{kj}(q)}{\partial q_i} \dot{q}_i \dot{q}_j$$

and

$$\frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{ij} \frac{\partial m_{ij}(q)}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial P}{\partial q_k}$$

- Linear in term $\ddot{q}$.
- Quadratic in $\dot{q}$.
- Nonlinear terms in q.

k-th dynamics

$$\sum_j m_{kj}(q)\ddot{q}_j + \sum_{i,j} \underline{\left(\frac{\partial m_{kj}(q)}{\partial q_i} - \frac{1}{2}\frac{\partial m_{ij}(q)}{\partial q_k}\right)}\dot{q}_i\dot{q}_j + \frac{\partial P(q)}{\partial q_k} = u_k$$

denote $c_{kij}(q) = 1/2(\frac{\partial m_{kj}(q)}{\partial q_i} + \frac{\partial m_{ki}(q)}{\partial q_j} - \frac{\partial m_{ij}(q)}{\partial q_k})$, note $c_{kij}(q) = c_{kji}(q)$, called called the **Christoffel** symbols corresponding to the inertia matrix.

Finally,

$$\underline{\sum_j m_{kj}(q)\ddot{q}_j} + \underline{\sum_{i,j} c_{kij}(q)}\dot{q}_i\dot{q}_j + \frac{\partial P(q)}{\partial q_k} = u_k$$

Summarizing...

$$\sum_j m_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{kij}(q)\dot{q}_i\dot{q}_j + \frac{\partial P(q)}{\partial q_k} = u_k$$

- $m_{kj}(q)$ — inertia at joint k when joint j accelerates.
- $c_{kij}(q)$ — coefficient of Coriolis force at joint k when both joint i and joint j are moving.

Summarizing...

$$\sum_j m_{kj}\ddot{q}_j + \sum_{i,j} c_{kij}(q)\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = u_k \quad k=1,\cdots N$$

is equivalently expressed as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

- M is symmetric and positive definite.
- $\dot{M} - 2C$ is a skew-symmetric matrix (matrix A is skew-symmetric iff $A^T = -A$.)

These properties are important for control design.