Problem 1

(45pt) Consider the double integrator.

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = u. \tag{2}$$

- (5pt) Write the system in the form $\dot{x} = Ax + Bu$ with state vector $x = (x_1, x_2)^T$ and input u. Verify that the system is reachable.
- (5 pt) Derive the condition on the feedback gain K such that $u = -Kx$ such that the closed-loop system is stable at the origin. Since the system is second order, choose a gain K such that the closed-loop characteristic polynomial $s^2 + 2\zeta_0 w_0 s + w_0^2$ has $w_0 = 1$ and $\zeta_0 = 0.7$.
- (10pt) Given a desired final state $x_f = (1, 1)$ and the initial state $x_0 = (-5, 0)$, let $x_d(t)$ be the desired trajectory such that $\dot{x}_d = Ax_d + Bu_d$ and $x_d(0) = x_0$ and $x_d(5) = x_f$. Assume the desired trajectory of state x_1 is given by $x_{1d}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ (and thus $x_{2d}(t) = \dot{x}_{1d}(t) = a_1 + 2a_2 t + 3a_3 t^2$). Find a set of parameters a_0, a_1, a_2, a_3 such that x_{1d} satisfies the constraints on the desired trajectory given the boundary condition. Compute the trajectory $u_d(t)$ from $x_d(t)$ using the dynamic model of the system.
- (5pt) let $e(t) = x(t) - x_d(t)$ be the error between the current state of the system at time t and the desired state x_d, derive the dynamic model by using e as the new state.
- (10pt)Design the trajectory tracking controller for the computed trajectory.
- (10pt) Implement the trajectory tracking controller in matlab. and test the following performance given initial error: Instead of starting from $x_0 = (-5, 0)$, how the closed-loop system performs given $x_0' = (-3, 1)$?

Problem 2

(55pt) Integral feedback for rejecting constant disturbances) Read chapter 6.4 of the book “feedback systems” : Integral Action. Use the following exercise to help your understanding.
Consider the double chain of integer

$$\dot{x}_1 = x_2 + d \tag{3}$$
$$\dot{x}_2 = u + d \tag{4}$$

and the output $y = x_1$, where d is a bounded noise/disturbance signal. To ensure zero steady state error under bounded disturbance d, we introduce an integral action using the following steps:

- (10pt) Introduce a new state variable $\frac{dz}{dt} = y - r$ where r is a reference input (desired output of the system). Write down the dynamic state-space model of the system with new state variable $\zeta = [x_1, x_2, z]$, i.e., find the A, B, F and H matrices in the following state-space model $\dot{\zeta} = A\zeta + Bu + Fr + Hd$.
- (15pt) Assume $d = 0$, design a feedback controller u for the new augmented system $\dot{\zeta} = A\zeta + Bu + Fr$ for tracking a constant reference $r \in \mathbb{R}$. Hint: Use the feedback controller $u = -\mathcal{K}\zeta + K_r r = -K[x1, x_2]^T - K_i z + K_r r$. Write one gain matrix $\mathcal{K}$ with your choice that makes the system stable. Give the poles of the closed-loop system characteristic polynomial.

Problem 2 continued on next page. . .

- (15pt) Now show that with any constant disturbance d, we can achieve zero steady state error of the system for stabilizing to the reference signal. That is, $\lim_{t\to\infty} y(t) - r = 0$. Hint: First, compute the equilibrium of the system at the augmented state space under zero disturbances. Pay attention to the augmented closed-loop system and understand which part of the state space disturbance affects. You need to reason with the stability of the equilibrium of the closed-loop system with the augmented state space.
- (15pt) Implement the controller in matlab. Test the controller under $d = 5, 10, 20$ and $r = 1$. Print and submit the pdf