Assignment 3

1) Given system: $\dot{x} = -y - x^3 = f_1$
$\dot{y} = x - y^3 = f_2$

- The equilibrium point (x, y) are at the point where we have $\dot{x} = \dot{y} = 0$ (velocity is zero at equilibrium)

$$\therefore \quad x^3 + y = 0 \quad \text{or} \quad y = -x^3$$
$$-x + y^3 = 0 \quad \text{or} \quad y^3 = x$$

Based on both the equation $(0, 0)$ is a solution and therefore a equilibrium point.

- Linearizing Non linear equations about the equilibrium

$$\dot{X} = Ax + Bu$$

where,

$$A = \left.\frac{\partial f}{\partial x}\right|_{x = x_e} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{bmatrix}_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}^T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{x = x_e} = 0$$

- Stability at the equilibrium, by characteristic equation

$$\det[(\lambda I - A)] = 0$$

$$\det\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right] = (\lambda)^2 + 1 = 0$$

or $\lambda = \pm i = \pm\sqrt{-1}$

or $\lambda_1 = i$, $\lambda_2 = -i$

- Considering Lyapunov's Function candidate to be

$$V(x,y) = \frac{x^2+y^2}{2}$$

which satisfies : $V(0) = 0$

$\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ are continous

- For Lyapunov function candidate to be Lyapunov function

① $V(x,y) = \frac{x^2+y^2}{2} \geq 0$ is positive definite

② $$\dot{V}(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= x\dot{x} + y\dot{y}$$

$$= x(-y-x^3) + y(x-y^3)$$

$$= -\cancel{xy} - x^4 + \cancel{xy} - y^4$$

$$= -x^4 - y^4 \leq 0$$ is negative definite

($\dot{V}(0,0) = 0$ only at origin)

As the lyapunov function's derivative acts as a damping factor which converges to origin, hence the system is asymptotically stable.

2 new] Given system : $\dot{x} = -x^3 + 2y^3$
$\dot{y} = -2xy^2$

- Equilibrium positions are when velocity $= 0$ $\therefore \dot{x} = \dot{y} = 0$

$\therefore \dot{x} = 0 \Rightarrow x^3 = 2y^3$
$\dot{y} = 0 \Rightarrow 0 = -2xy^2$ $\therefore y = 0$ (zero factor principle)
$\therefore x^3 = 2y^3$ $\therefore x = 0$

Hence,
origin is a equilibrium point

- Given energy equation $V = \frac{1}{2}(x^2+y^2)$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$
$$= x\dot{x} + y\dot{y}$$
$$= x(-x^3 + 2y^3) + y(-2xy^2)$$
$$= -x^4 + \cancel{2xy^3} - \cancel{2xy^3}$$
$$= -x^4 \leq 0$$

($\dot{v} = 0$ only at $x = 0$)

Thus equilibrium point i.e. origin is stable.

- $E := \{x \mid \dot{V}(x) = 0\}$
$\therefore -x^4 = 0 \Rightarrow x = 0$

- M = subset of E ($x(t) \in E$, $x(t) \equiv 0$ $\therefore \dot{x} = 0$)
Largest invariant set
i.e. Substituting $x = 0$ in the give DE
$\dot{x} = -x^3 + 2y^3$
$\dot{y} = -2xy^2 \Rightarrow y = 0$

Thus $\dot{x} = 0 = -x^3 + 2(0)^3$ $\therefore x = 0$

M is just the point $(0,0)$

By Lasalle's Invariance Priciple $x(t) \rightarrow (0,0)$
Origin is asymptotically stable.

3] Given system:

$$\dot{x} = ax^3$$

The equilibrium point is when $\dot{x} = 0$

$$\therefore \quad 0 = ax^3 \Rightarrow x = 0$$

linearization: $\dot{x} = Ax + Bu$.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_e} = \begin{bmatrix} 3ax^2 \end{bmatrix}_{x=0} = \begin{bmatrix} 0 \end{bmatrix}$$

$$B = 0$$

$\therefore$ The system cannot be linearized, and if we use characteristic equation $\det|\lambda I - A| = 0$, $(\lambda = 0)$ we will get 0 as the only eigenvalue, so the principle of linearized stability is of no use.

- Given Lyapunov function candidate

$V(x) = x^4$ which satisfy $V(0) = 0$

$\frac{\partial V}{\partial x}$ is continous.

For Lyapunov function to be stable

① $v(x) = x^4 \geq 0$ i.e it is positive definite

② $\dot{v}(x) = \frac{\partial v}{\partial x} \dot{x}$

$= 4x^3 \dot{x}$

$= 4x^3 (ax^3)$

$= 4a x^6$

- $\therefore$ If $a < 0$ $\dot{v}(x) < 0$, the system is asymptotically stable system
- $\therefore$ If $a > a$, $\dot{v}(x) > 0$, $\therefore$ having $v(w) < V(0)$ or proposing new Lyapunov function shows that the system is unstable based on the Instability criterion
- $\therefore$ If $a = 0$, $\dot{v}(x) = 0$, the system is stable system but not asymptotically (no friction)

4] Given system:

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{l} x_2$

Equilibrium points when velocity $= 0$; $\dot{x}_1 = \dot{x}_2 = 0$

$\therefore$ $\dot{x}_1 = 0 \Rightarrow x_2 = 0$

$\dot{x}_2 = 0 \Rightarrow -\frac{g}{l} \sin x_1 = 0$ or $\sin x_1 = 0$

$\therefore x_1 = 0, \pi \cdots$

Hence $(0,0)$ & $(2n\pi, 0)$ are equilibrium

$\therefore$ origin is an equilibrium point.

Linearizing : $\dot{x} = Ax + Bu$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x = x_e} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{l} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -g/l & -k/l \end{bmatrix}$$

$$B = 0$$

$\therefore$ Stability at equilibrium can be found out using characteristic equation $\det(\lambda I - A) = 0$

$$\det\left[\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -g/l & -k/l \end{bmatrix}\right] = 0$$

$$\det\begin{bmatrix} \lambda & -1 \\ g/l & \lambda + k/l \end{bmatrix} = 0$$

$$\lambda(\lambda + k/l) + g/l = 0$$

$$\lambda^2 + \lambda\frac{k}{l} + \frac{g}{l} = 0$$

For no friction $k = 0$

$$\therefore \lambda = \pm\sqrt{\frac{g}{l}}$$

Considering Total Energy equation

$$V(x_1, x_2) = gl(1-\cos(x_1)) + l^2 \frac{x_2^2}{2} \geq 0$$ is positive definite

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2$$

$$= gl \sin(x_1)\dot{x}_1 + l^2 x_2 \dot{x}_2$$

$$= gl\, x_2 \sin(x_1) + l^2 x_2\left(-\frac{g}{l}\sin(x_1) - \frac{K}{l}x_2\right)$$

$$= -Kl x_2^2 \leq 0$$ is negative

for zero friction $K = 0$

$$\therefore \dot{V}(x_1, x_2) = 0$$

Hence the system is stable but not asymptotically stable by Lyupnov's function stability.