Research Statement

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My research interests, in the broadest sense, are in the algebraic and geometric structures underlying physical theories. My work during my PhD studies at the University of Edinburgh under the supervision of José Figueroa-O'Farrill (JMF) focussed on applications of Lie (super)algebra cohomology in mathematical physics, in particular in supergravity and symplectic geometry. In my PhD thesis, *Spencer cohomology, supersymmetry and the structure of Killing superalgebras* [1], I expanded upon a body of work studying the supersymmetry algebras of supersymmetric supergravity backgrounds via Spencer cohomology [2–8], an approach which not only provides fresh insights into the structure of these superalgebras but also gives one a notion of supersymmetric backgrounds which goes beyond supergravity, the consequences of which we have only begun to explore. My contributions to this ongoing project include the development of an overarching theoretical framework, enabling applications in physics and geometry beyond supergravity theory, and extensive calculations in across a large class of particular examples. I have also worked on aspects of symplectic geometry, proving a (often stated but never fully justified) version of a classic theorem on homogeneous symplectic spaces, and have more recently considered how this might generalised to the homothetic symplectic setting.

In this statement I will provide an overview of my research output and future directions I intend to pursue, which encompass problems in geometry, supersymmetry and gravitational theory across various regimes, and also discuss some other areas I would be interested in working on.

Spencer cohomology and supersymmetry

Background theory A basic problem in any supergravity theory is the determination of its bosonic backgrounds; that is, the solutions to the classical theory in which the fermions vanish. Such a background is said to be *supersymmetric* if is preserved by at least one of the supersymmetries of the theory. Mathematically, this means that for some choice of local supersymmetry parameter ϵ, interpreted as a spinor field on the background, the supersymmetry variations of all fermionic fields vanish. The vanishing of the gravitino variation[1] is interpreted as a first-order linear differential equation called the *Killing spinor equation* which can generally be written in the form $\nabla\epsilon = \beta\epsilon$, where ∇ is the Levi-Civita connection and β is a 1-form with values in spinor endomorphisms (which is parametrised by the bosonic background fields). The solutions to this equation are known as *Killing spinors*.

Together with Killing vectors which preserve the bosonic data of a background, the Killing spinors form a Lie superalgebra known as a *Killing superalgebra* (KSA), which one interprets as a supersymmetry algebra of the background. This superalgebra has a particular algebraic structure: it is a *filtered subdeformation* of a subalgebra of the Poincaré superalgebra [1, 4, 9]. Although somewhat obscure[2], there is a simpler version of this result present in Riemannian geometry: the isometry algebra of a Riemannian manifold is a filtered subdeformation of the Euclidean algebra, and an analogous result holds in any signature. Filtered deformations of graded Lie superalgebras are governed their *Spencer cohomology* [11], a refinement of the Chevalley–Eilenberg cohomology with values in the adjoint representation. While this perspective is somewhat abstract, the Spencer cohomology calculation actually yields a very concrete result, namely a spinor endomorphism-valued 1-form β which can be interpreted either as an "infinitesimal" deformation of the Poincaré superalgebra (which one may be able to "integrate" to full filtered deformation), or alternatively as the β appearing in the Killing spinor equation. While a priori performing this calculation for the whole Poincaré superalgebra only tells one about the *maximally* supersymmetric backgrounds, in particular cases it actually completely recovers the Killing spinor equation, and it is sometimes possible to recover the bosonic equations of motion by imposing integrability conditions [3, 5–7]. That is to say, using only the representation theory of the Poincaré superalgebra, one can, at least in certain cases, recover the bosonic sector of the supergravity theory without any prior knowledge of the field content, field equations or supersymmetry transformations.

My work In my first published work [7], which built upon my MSc thesis, we applied the methods described above in the case of minimal $D = 5$ supergravity and found that the Spencer cohomology calculation (supplemented by an integrability condition for Killing spinors) recovers both the Killing spinor equation and bosonic

[1] For the present discussion, I ignore the (algebraic) equations arising from the variations of any other fermions which may be present.
[2] I am not aware of this result being explicitly stated in the literature, but it is a corollary of a result in Cartan geometry [10].

equations of motion, but we also found that the form of the Killing spinor equation suggested by the Spencer calculation is more general than that of the supergravity theory, with additional terms proportional to a bosonic field not present in that theory. The spinors satisfying this equation nonetheless generate a KSA, and the class of geometries which support them is larger than that of the supergravity Killing spinors. The notion of "supersymmetry" for gravitational backgrounds – the admissibility of Killing spinors – is thus more general than the one given by supergravity.[3] We classified the maximally supersymmetric backgrounds under this more general notion of supersymmetry, finding three previously unknown supersymmetric geometries on top of those present in supergravity, and described the KSAs.

Later in my PhD studies, my attention turned to more theoretical questions: setting aside supergravity, for what choice of β can the solutions to the equation $\nabla\epsilon = \beta\epsilon$ on an arbitrary Lorentzian spin manifold (and perhaps also with arbitrary signature) be used to define a KSA? And how much of the relationship described above between Killing superalgebras, filtered subdeformations of the Poincaré superalgebra and Spencer cohomology holds in general? Indeed, up to that point this relationship had only been fully described in the context of $D = 11$ supergravity [4]. To answer this question, I developed a theoretical framework, valid (for the most part) for general signature, arbitrary $\mathcal{N}$-extension and other choices, and replicated almost all of the major results of loc. cit., in particular showing that any KSA is a filtered deformations of the Poincaré superalgebra (or an analogue thereof) and characterising, at least in Lorentzian signature, the *geometrically realisable* highly supersymmetric subdeformations of the Poincaré superalgebra, i.e. those which can be realised as (subalgebras of) KSAs. This framework also subsumes an analogue of the KSA which has been constructed from "geometric" Killing spinors on some higher-dimensional spheres [12]. These results recently appeared as preprint [9], as did a set of 2-dimensional examples [13] exhibiting the general theory and connections to geometric Killing spinors and their generalisations.

Pushing this top-down theoretical point of view further, I went on to consider how the framework might be adapted to include "gauged" supergravity theories, here meaning that the R-symmetry group is gauged (so that the gravitini carry colour charge). Killing spinors and KSAs for these theories were much less well understood; while earlier work applying Spencer cohomology methods to gauged $D = 6$ [6] described the algebraic side of the issue clearly (the objects of study being filtered subdeformations of the Poincaré superalgebra extended by the R-symmetry algebra), two separate descriptions of the KSA and the geometric setting in which it is defined were given, and it was not entirely clear how they might be unified. I clarified this situation in my own treatment, developing a natural definition of the KSA with gauged R-symmetry using spinors associated to a so-called "generalised spin" or "spin-G" structure [14, 15], a generalisation of spin-$\mathbb{C}$ structures.

Parallel to this theoretical treatment, I continued to work on particular examples, calculating the relevant Spencer cohomology groups for $\mathcal{N}$-extended Poincaré superalgebras for the first time, for arbitrary $\mathcal{N}$ in $D = 5$ and $D = 6$ (including gauging) and for Type IIA in $D = 10$. I was able to use the minimal gauged $D = 5$ case to demonstrate that the definition of the KSA I had developed was indeed the appropriate structure for gauged supergravity, and that, similarly to the ungauged case, the notion of "supersymmetry" suggested by the Spencer calculation is more general than the one arising from supergravity theory. I was also able to show, by describing some families of maximally supersymmetric backgrounds, that supersymmetric geometry is much richer under this regime than under supergravity. One intriguing consequence of this, discussed in more detail below, is that the well-studied class of supersymmetric 5-dimensional black holes might also be greatly enriched.

What to say about IIA? Should try to make future directions more concise, and add some other ideas.

Future directions I have identified a number of potentially fruitful avenues for further research which follow on from the work described above. I detail them below, starting with those with the clearest path to publication.

1. **Supersymmetric black holes.** An ansatz of supersymmetry on gravitational backgrounds, understood as the existence of Killing spinors, is often used for finding exotic black hole solutions, in particular in $D = 5$ [16]. It is an interesting question whether new "supersymmetric" black hole solutions can be found using the more general notion of Killing spinors that we have developed in $D = 5$. More generally, one could attempt to classify *all* supersymmetric geometries under this definition, as was already done in minimal $D = 5$ supergravity [17], and similar methods (suitably modified) could be applied in this generalised

[3] A similar phenomenon had already been observed in $D = 6$ [6].

setting. I have already carried out many of the relevant calculations, such as determining relations among the spinor bilinears, in my PhD thesis.

2. **Kaluza–Klein reduction.** Another interesting question is whether the Spencer cohomology of the Poincaré superalgebra admits "Kaluza–Klein reduction", in other words, whether the relevant Spencer cohomology data in a given number of spacetime dimensions can be obtained from that in higher dimensions. While previous preliminary work on this problem by myself, JMF and others stalled due to technical difficulties, completing it could provide a more efficient way of performing Spencer cohomology computations. My own calculations (along with other unpublished calculations) in various dimensions could provide a fresh perspective on this problem. A related project would be a closer analysis of the 10-dimensional theories in the framework I have developed and a comparison with the 11-dimensional case.

3. **Rigid supersymmetry.** Supersymmetric bosonic backgrounds of supergravity theories are known to support rigid supersymmetric field theories whose symmetry superalgebras are the Killing superalgebras (via Festuccia–Seiberg [18]). The existence of more general supersymmetric backgrounds with KSAs opens the possibility that there may be novel rigid supersymmetric field theories on curved backgrounds waiting to be discovered which do not come from supergravity. It may be possible to write down such new theories by studying the representation theory of the KSAs.

4. **Geometric supergravity.** An aspect of Spencer cohomology which I have not yet emphasised is the natural connection to Cartan geometry [10] which would open up links to the so-called rheonomic or geometric supergravity formalism of Castellani–D'Auria–Fré [19] on which I organised a reading group in 2020, and I am interested in exploring how this perspective can complement my previous approaches to provide further insight into the structure and classification of supersymmetric backgrounds. This also connects with recent work on nonrelativistic theories of supergravity and strings (see eg. [20, 21]) as well as loop quantum supergravity [22].

Homogeneous symplectic geometry

Background Theory Souriau's seminal work on symplectic dynamics [23] introduced the notion of the moment map, which can be interpreted as encoding Noether's Theorem in his formalism, as well as a certain group cocycle which measures the failure of the moment map to be equivariant under the action of the symmetry group. A result by Kostant–Kirillov–Souriau (KKS) [23–25] says that the homogeneous symplectic spaces of a compact connected Lie group G are locally isomorphic to coadjoint orbits of a one-dimensional central extension of G which is constructed using the group cocycle associated to a certain moment map. The classic references for this theorem all assume (either implicitly or explicitly) that G is simply connected, but this result is often stated in the literature without this assumption.

My Work In 2021, motivated by an application to dynamical systems with Lifshitz symmetry [26], JMF and I conducted a search of the literature and identified that there was no published proof of the KKS result which do not assume that G is simply connected.[4] and identified that the difficulty in removing this assumption is showing that one can construct the central extension of G from the moment map cocycle data. Using results of Neeb [28] (and earlier Tuynman–Wiegerinck [29]) on the existence of such extensions, I was able to prove that this is indeed possible. Our proof can be found in the preprint [30] and is also summarised in a conference paper [31]. In ongoing work with other collaborators, we have been considering how to generalise this result to the "homothetic symplectic" case, where the group action only preserves the symplectic form up to rescaling, and this work has touched on the areas of Poisson, contact and Jacobi geometry. My own contribution here has been to identify the relevant geometric structures on coadjoint orbits and develop the cohomological framework, complementing analysis of particular examples by collaborators.

Future directions We hope to complete this generalisation to the homothetic case in the near future, however the putative analogue of the KKS result and its proof remain open problems. I would also be interested in continuing to work on related problems in symplectic geometry and dynamics.

Other interests

[4] We have, however, more recently been made aware of an obscure proof, somewhat different to our own, in a PhD thesis [27].

In addition to the topics detailed above, I am interested in exploring topics across mathematical physics at the nexus of supersymmetry, gravity, homological algebra, differential geometry and beyond. Throughout my PhD, I have been a very active participant in various seminars and reading groups, including some which I have organised myself. A particular highlight has been the GRIFT seminar series on interactions between geometric representation theory and quantum field theory, and also a related reading group on QFT for PhD students in mathematics, which together have reignited my interest and appreciation for QFT. In particular, I would be interested in working on mathematical and non-perturbative aspects of supersymmetric field theories and quantum gravity. My interest has also been drawn to mathematical general relativity and non-relativistic physics (in particular to Newton-Cartan and Carrolian (super)gravity, for instance) by a number of different seminars, and I would also be very open to working in these areas. **This part in particular should be adapted to the particular application.**

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