The Mathematics of Tsunamis

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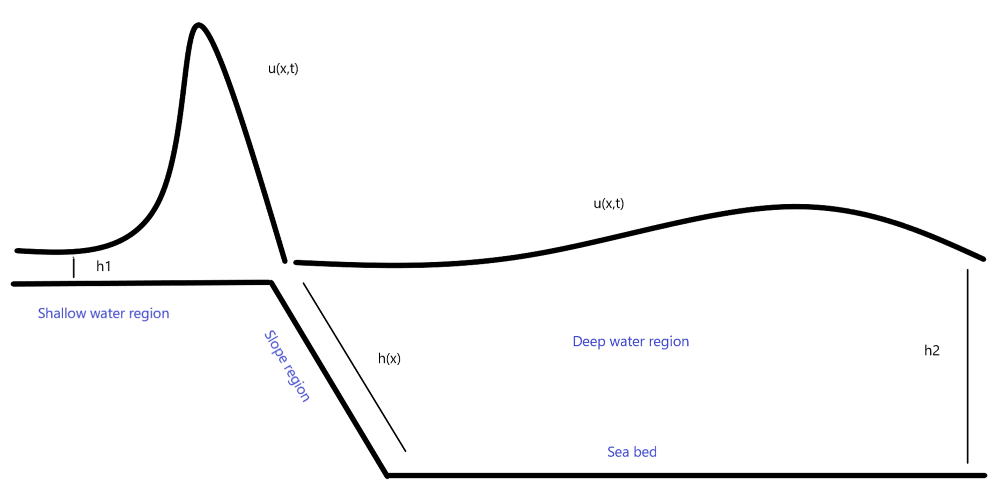
Background

Tsunamis are one of the most devastating natural disasters we experience and are caused by a disturbance or explosion, often times far from shore, resulting in a solitary wave traveling with great speed. The 2004 Indian Ocean earthquake and tsunami killed approximately 230,000 people and is known as the most devastating tsunamic event in modern history.



A solitary wave moves from right to left in a canal and approaches a slope, simulating a deep ocean region and a continental shelf. The speed of the wave is proportionate to the depth of the water, and once it reaches the shelf, most of the wave will be reflected back, but a narrow, tall peak will continue forward into the shallow water region with a reduced speed.

We can define a factor called the *depth ratio*. This will simply be the depth of the shallow water region divided by the deep water region. The slope of the shelf can be determined using the depth ratio and the distance over which the shelf resides. We will see later how altering this will affect the resulting wave.

Methods

A tsunami is a natural manifestation of a Korteweg-de Vries, or soliton, wave equation that takes the form of the following differential equation

$$\frac{\partial^2 u}{\partial t^2} = g\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) \qquad (1)$$

Where g is gravity and h represents the depth of the water. In the regions with constant h, the differential equation is solved using separation of variables with separation constant $-\omega^2$. We obtain two new differential equations

$$X''(x) + \frac{\omega^2}{gh}X(x) = 0 \qquad (2)$$

$$T''(t) + \omega^2 T(t) = 0 \qquad (3)$$

The solution of $u(x,t)$ in regions of constant h is

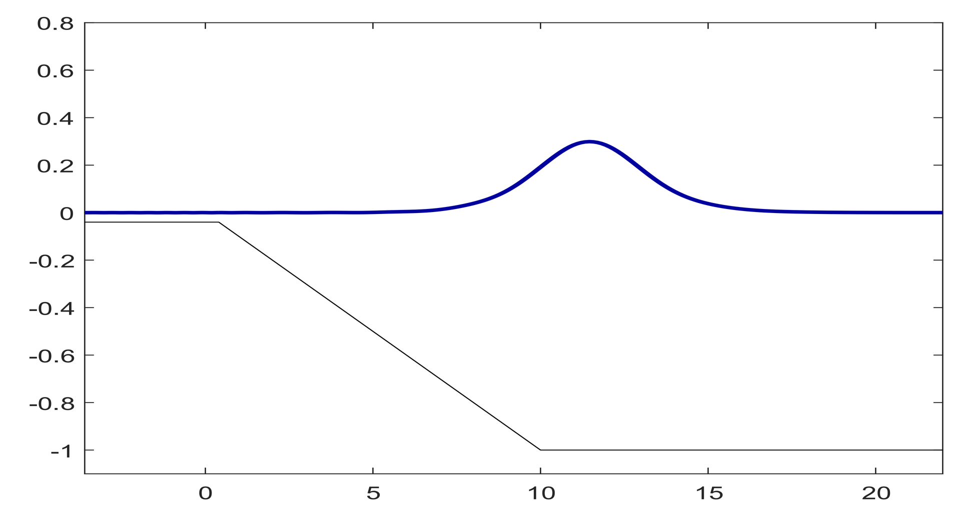
$$u(x,t) = C_1 e^{i\omega(t+x/c)} + C_2 e^{i\omega(t-x/c)} \quad \text{with} \quad c = \sqrt{gh} \qquad (4)$$

In the shallow water region, C_2 is zero because the wave only moves in the positive x direction. In the deep water region, we use $C_2 = R(\omega)$, the reflection coefficient. The solution over the slope region, where h is a function of x, requires a numerical solution, which was computed using MatLab `dsolve`.

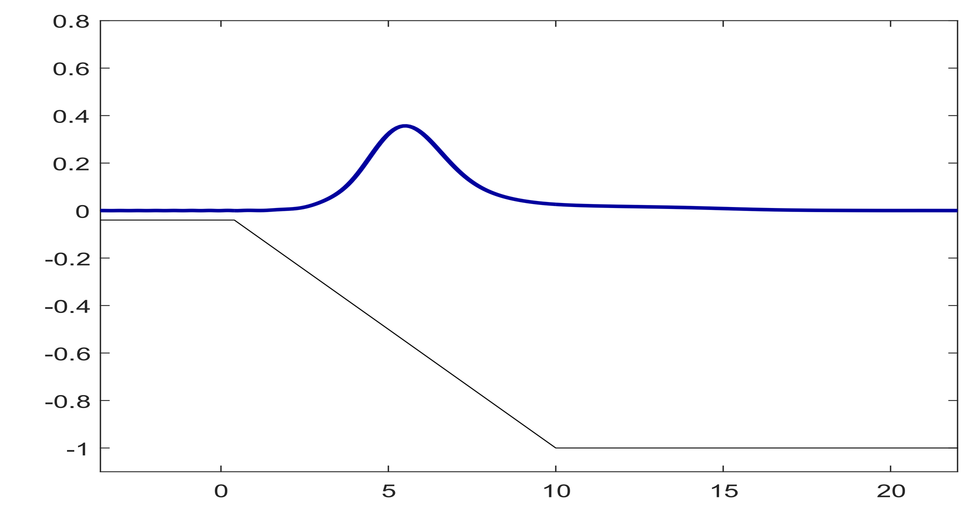
Results

We can use Fast Fourier Transform (FFT) and inverse FFT to generate points and "stitch" our solutions together. Once we have many points generated we can plot the results and see an accurate representation of a tsunami.

Wave Evolution over Time



t = 50



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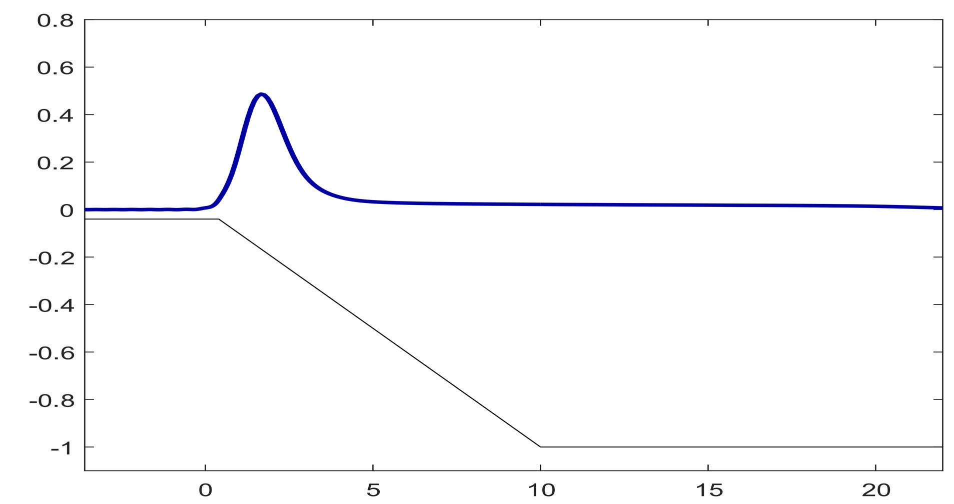

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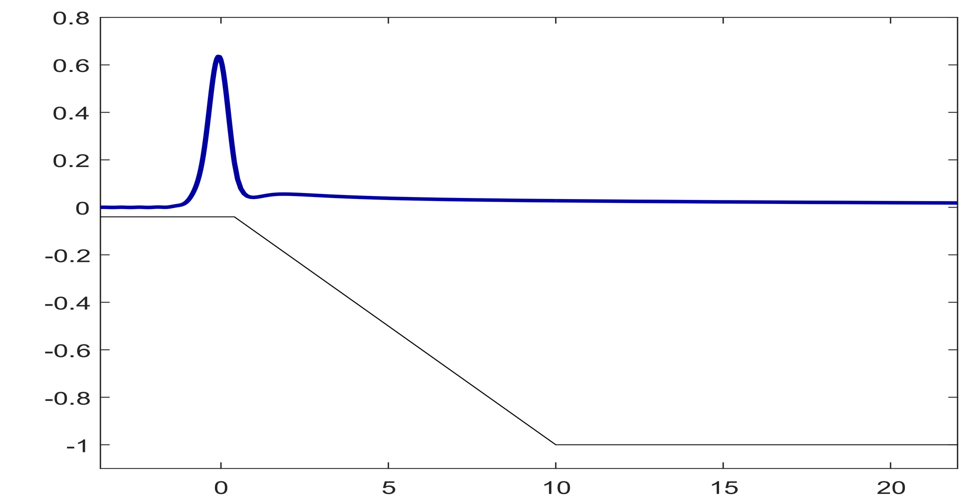
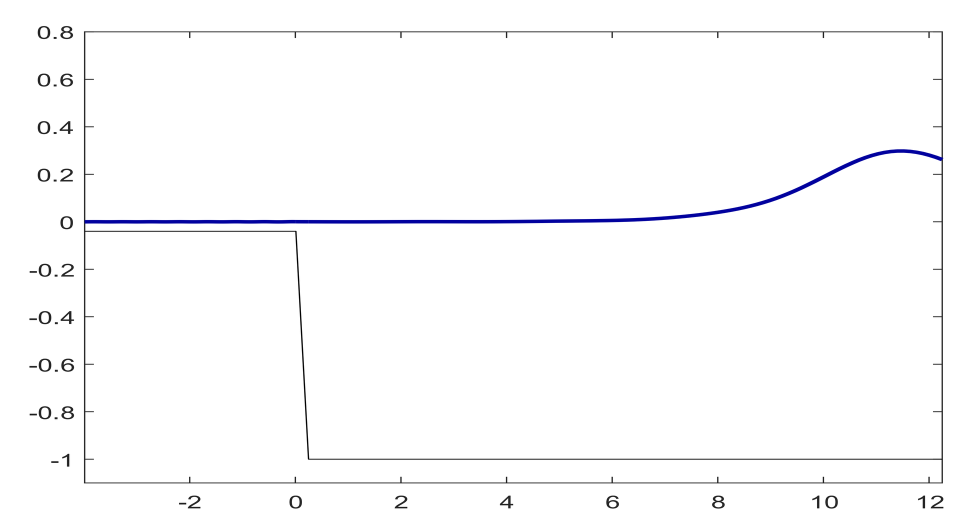


t = 200

If we decrease the distance over which the slope region resides, effectively increasing the slope of the shelf, we will observe the amplitude of the wave on shore decreases.



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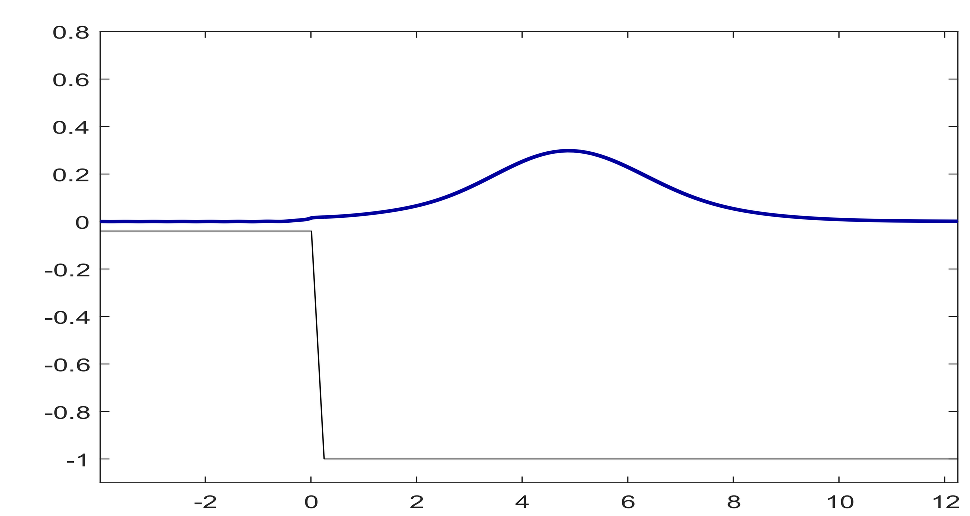

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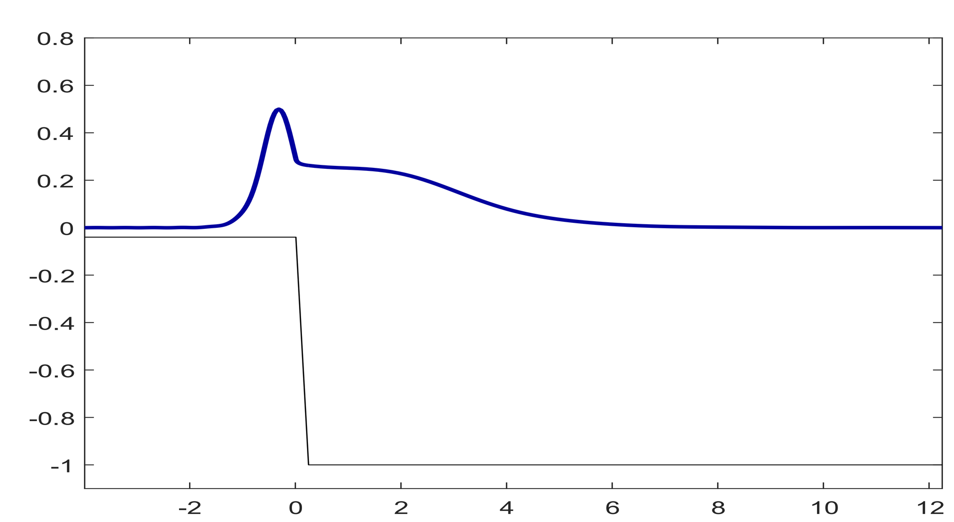

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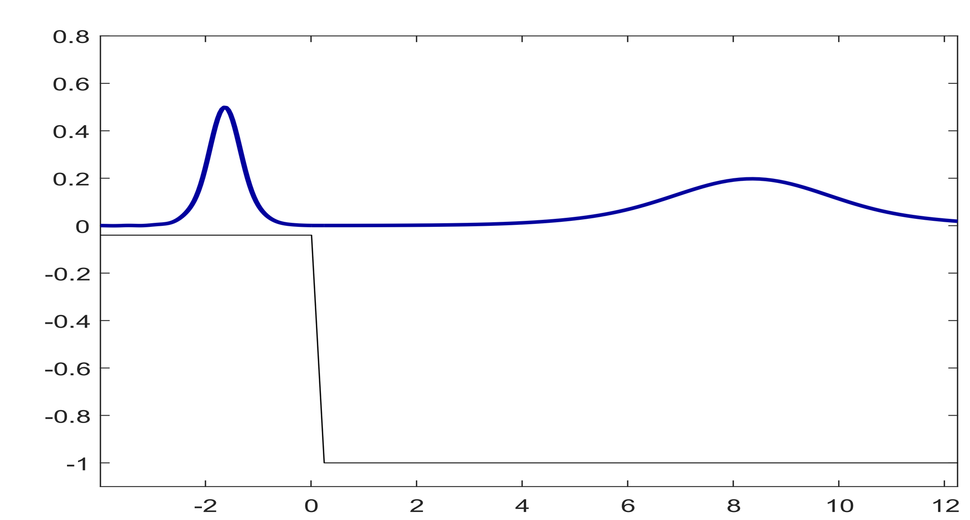

t = 200

Conclusion

This tells us that there is a direct correlation between slope of sea shelf and impact of a tsunami. A shelf with a steeper slope will reflect a higher proportion of the wave back into the deep water region. We can use this fact to identify coastal regions that are more susceptible to catastrophic tsunamis.

References

Derek G. Goring and F. Raichlen, Tsunamis - The Propagation of Long Waves onto a Shelf, Journal of Waterway, Port, Coastal and Ocean Engineering 118(1), 1992, pp. 41 - 63.

H. Lamb, Hydrodynamics, Dover, 1932.