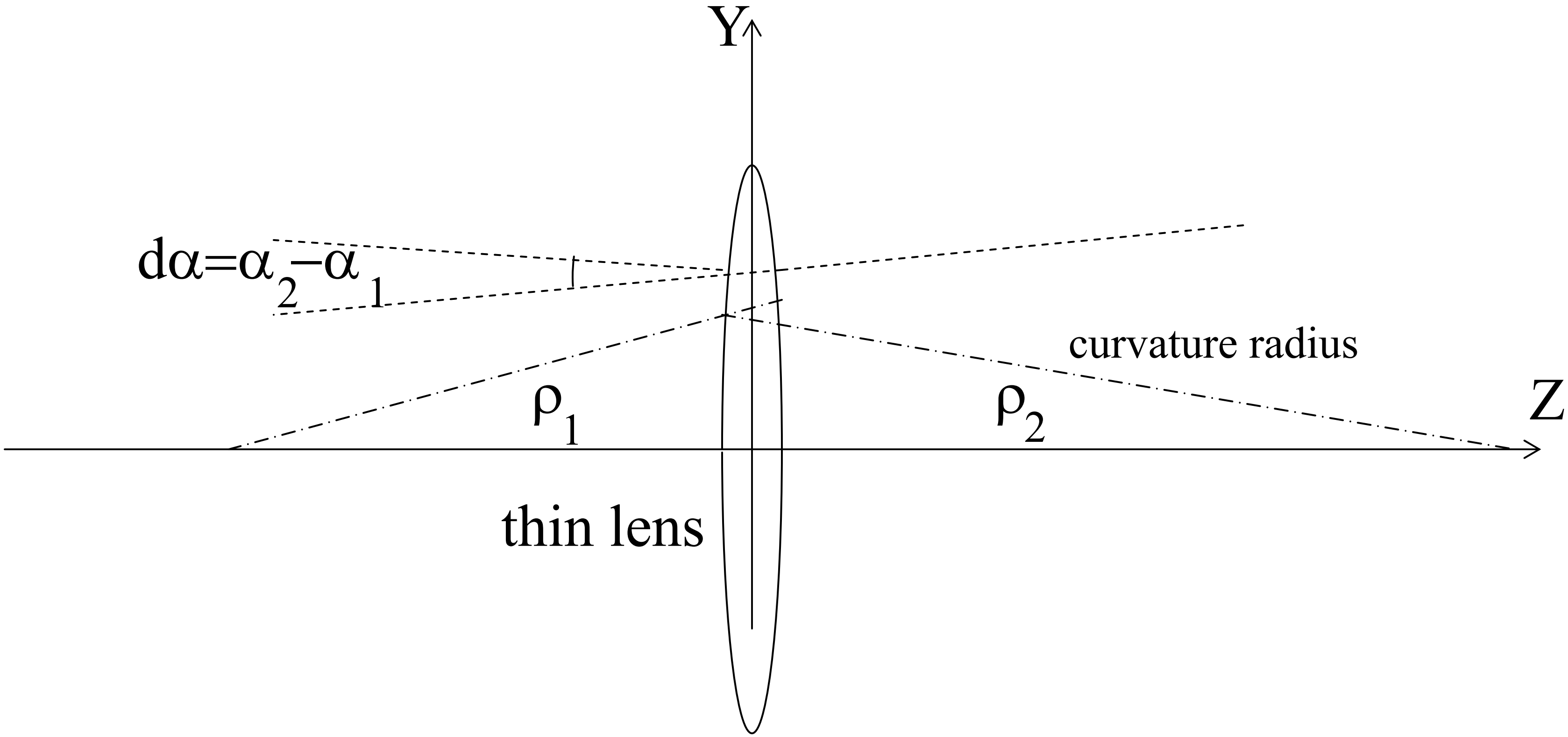
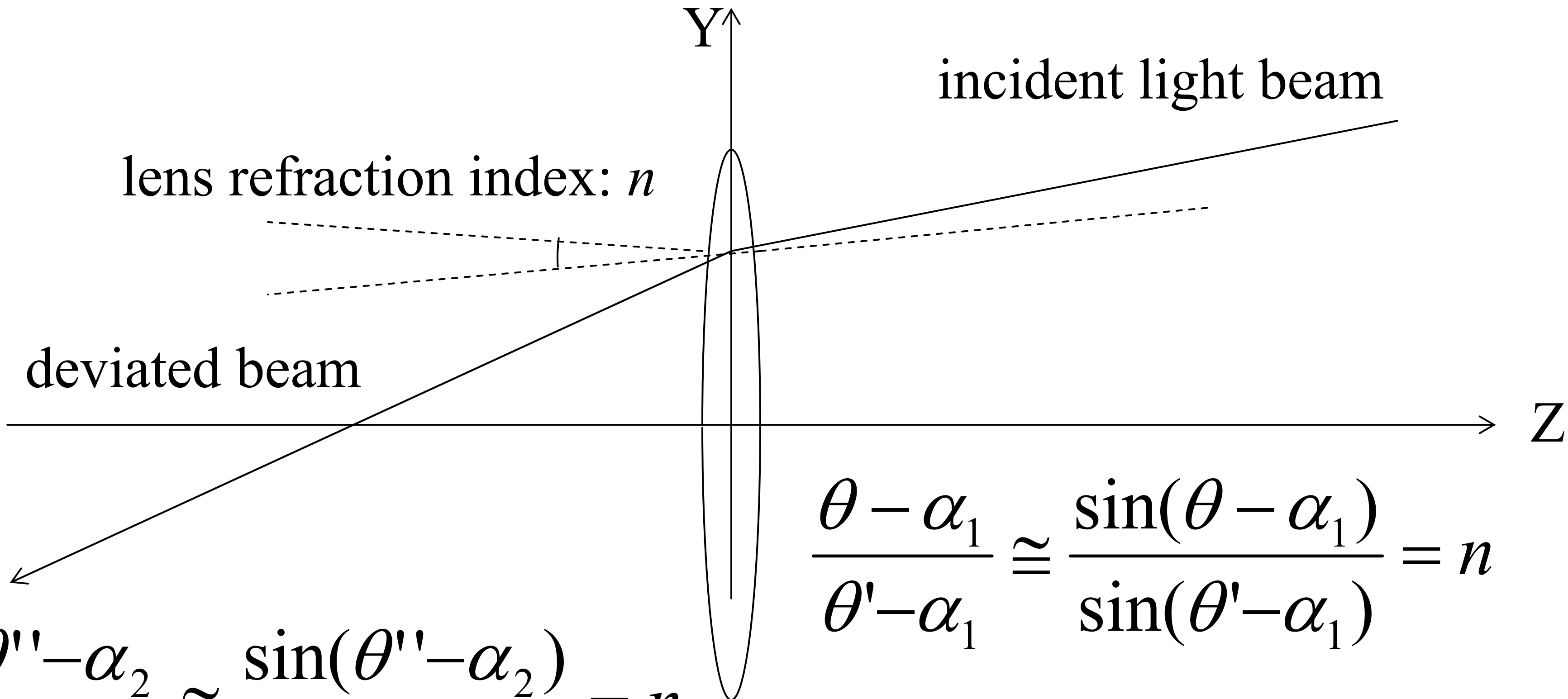
Camera: optical system



small angles: $$\alpha_1 \cong \frac{Y}{\rho_1} \qquad \alpha_2 \cong -\frac{Y}{\rho_2}$$



$$\frac{\theta-\alpha_1}{\theta'-\alpha_1} \cong \frac{\sin(\theta-\alpha_1)}{\sin(\theta'-\alpha_1)} = n$$

$$\frac{\theta''-\alpha_2}{\theta'-\alpha_2} \cong \frac{\sin(\theta''-\alpha_2)}{\sin(\theta'-\alpha_2)} = n$$

deviation angle ? $\Delta\theta = \theta'' - \theta$

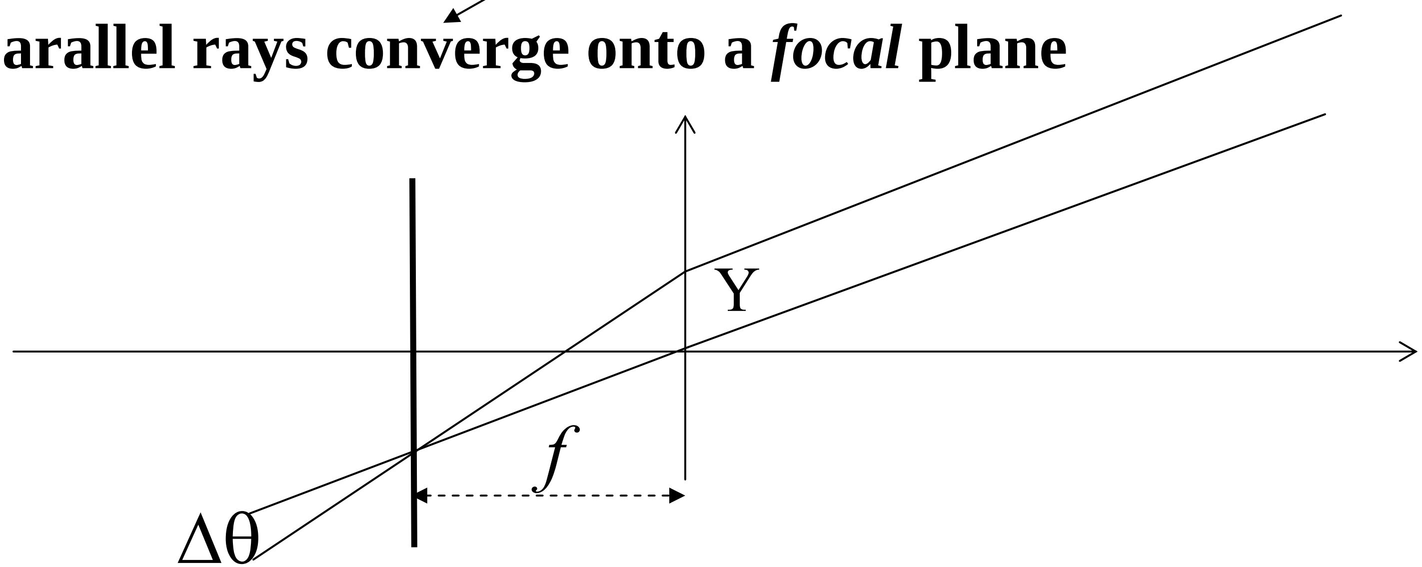
$$\Delta\theta \cong (n-1)Y\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)$$

Thin lens rules

a) Y=0 → $\Delta\theta = 0$ beams through lens center: undeviated

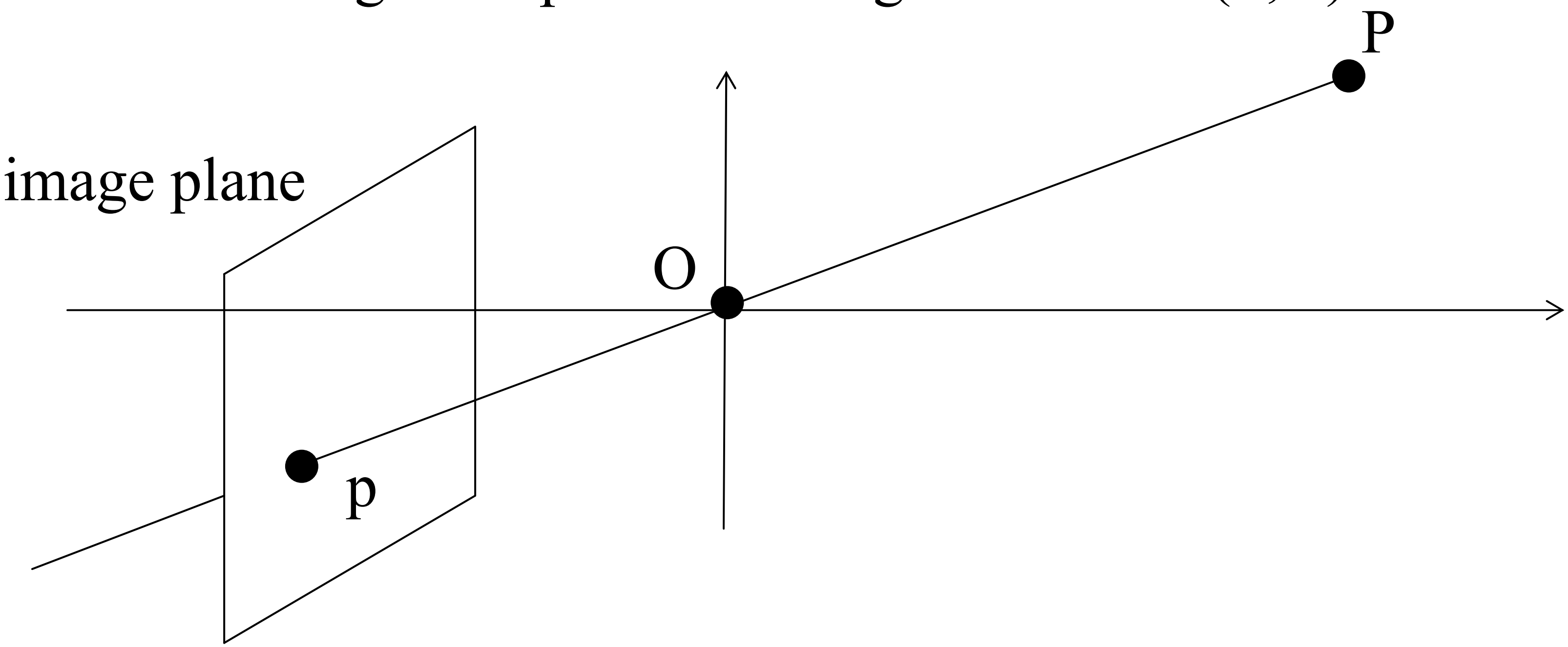
b) $f\,\Delta\theta = Y$ → $$\boxed{f = \frac{1}{(n-1)(\frac{1}{\rho_1}+\frac{1}{\rho_2})}}$$ independent of y

parallel rays converge onto a *focal* plane



Hp: **Z >> a** $r \rightarrow f$

the image of a point P belongs to the line (P,O)



p = image of P = image plane ∩ line(O,P)

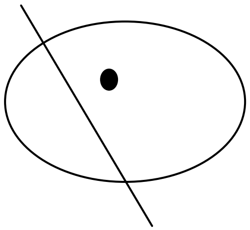
interpretation line of p: line(O,p) =
locus of the scene points projecting onto image point p

Projective 2D geometry

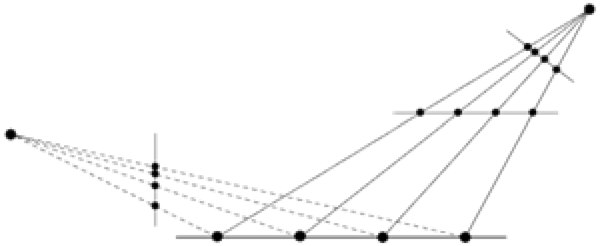
Notes based on
di R.Hartley e A.Zisserman “Multiple view geometry”

Projective 2D Geometry

- Points, lines & conics
- Transformations & invariants



- 1D projective geometry and
  the Cross-ratio



Homogeneous coordinates

Homogeneous representation of lines

$ax + by + c = 0 \qquad (a,b,c)^\mathsf{T}$

$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \qquad (a,b,c)^\mathsf{T} \sim k(a,b,c)^\mathsf{T}$

equivalence class of vectors, any vector is representative

Set of all equivalence classes in $\mathbf{R}^3 - (0,0,0)^\mathsf{T}$ forms $\mathbf{P}^2$

Homogeneous representation of points

$\mathrm{x} = (x, y)^\mathsf{T}$ on $\mathrm{l} = (a,b,c)^\mathsf{T}$ if and only if $ax + by + c = 0$

$(x,y,1)(a,b,c)^\mathsf{T} = (x,y,1)\mathrm{l} = 0 \qquad (x,y,1)^\mathsf{T} \sim k(x,y,1)^\mathsf{T}, \forall k \neq 0$

The point x lies on the line l if and only if $\mathrm{x}^\mathsf{T}\mathrm{l} = \mathrm{l}^\mathsf{T}\mathrm{x} = 0$

Homogeneous coordinates $(x_1, x_2, x_3)^\mathsf{T}$ but only 2DOF

Inhomogeneous coordinates $(x, y)^\mathsf{T}$

Points from lines and vice-versa

Intersections of lines

The intersection of two lines l and l' is $\mathrm{x} = \mathrm{l} \times \mathrm{l}'$

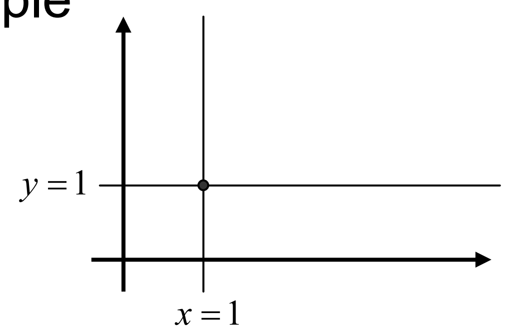
Line joining two points

The line through two points x and x' is $\mathrm{l} = \mathrm{x} \times \mathrm{x}'$

Line joining two points: parametric equation

A point on the line through two points x and x' is y = x + θ x'

Example

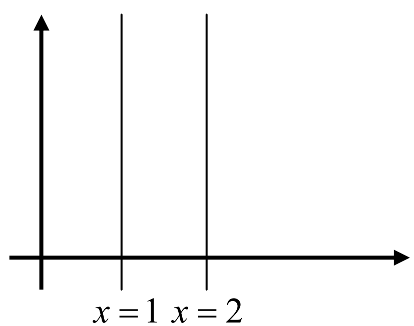


Ideal points and the line at infinity

Intersections of parallel lines

$\mathbf{l} = (a, b, c)^\mathsf{T}$ and $\mathbf{l}' = (a, b, c')^\mathsf{T}$ $\qquad$ $\mathbf{l} \times \mathbf{l}' = (b, -a, 0)^\mathsf{T}$

Example



$(b, -a)$ tangent vector

(a, b) normal direction

Ideal points $\qquad (x_1, x_2, 0)^\mathsf{T}$

Line at infinity $\qquad \mathbf{l}_\infty = (0, 0, 1)^\mathsf{T}$

$$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathbf{l}_\infty$$

Note that in $\mathbf{P}^2$ there is no distinction between ideal points and others

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \,\alpha\, {}^{x_1}\!/\!{}_{x_3},\ y \,\alpha\, {}^{x_2}\!/\!{}_{x_3}$

$$ax_1^{\,2} + bx_1x_2 + cx_2^{\,2} + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\,\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a : b : c : d : e : f\}$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

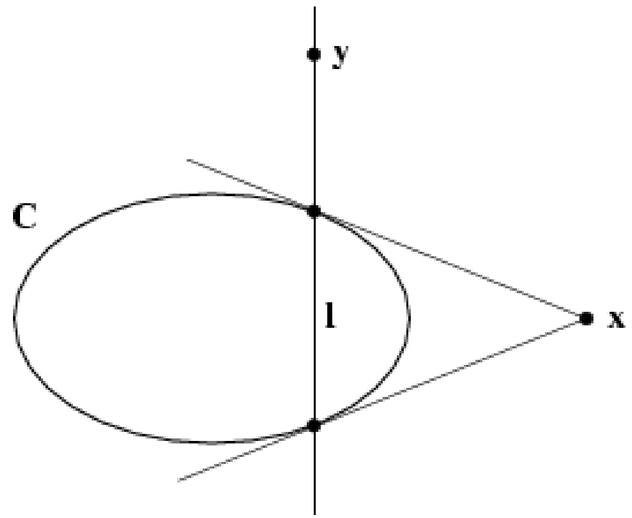
$$\left(x_i^2, x_iy_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\mathbf{c} = 0$$

Pole-polar relationship

The polar line $\mathrm{l}=\mathbf{C}\mathrm{x}$ of the point x with respect to the conic $\mathbf{C}$ intersects the conic in two points. The two lines tangent to $\mathbf{C}$ at these points intersect at x



Polarity: cross ratio

Cross ratio of 4 colinear points $y_i = x + \theta_i x'$ (with i=1,..,4)

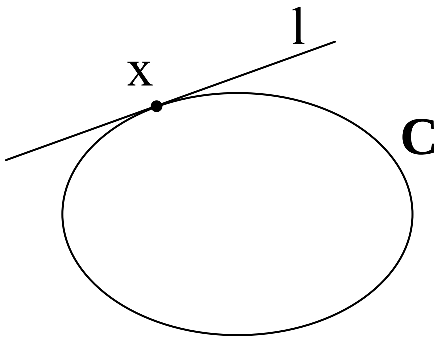
ratio of ratios

$$\frac{\theta_1 - \theta_3}{\theta_1 - \theta_4} \Big/ \frac{\theta_2 - \theta_3}{\theta_2 - \theta_4}$$

Harmonic 4-tuple of colinear points: such that CR=-1

Tangent lines to conics

The line $\mathbf{l}$ tangent to $\mathbf{C}$ at point $\mathbf{x}$ on $\mathbf{C}$ is given by $\mathbf{l}=\mathbf{C}\mathbf{x}$



Dual conics

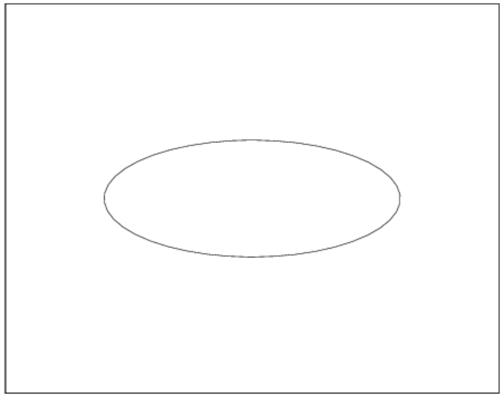
A line tangent to the conic $\mathbf{C}$ satisfies $\quad \mathbf{l}^{\mathsf{T}}\,\mathbf{C}^{*}\,\mathbf{l} = 0$

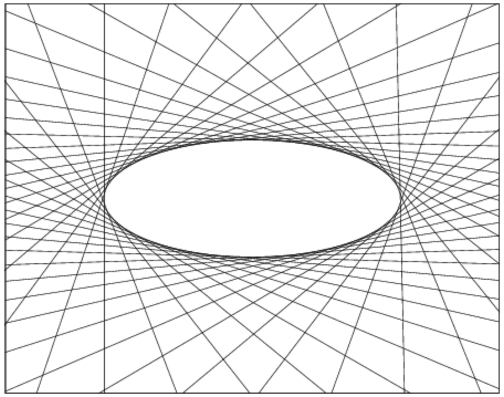
In general ($\mathbf{C}$ full rank): $\quad \mathbf{C}^{*} = \mathbf{C}^{-1} \quad$ in fact

Line $\mathbf{l}$ is the polar line of $\mathbf{y}$: $\mathbf{y} = \mathbf{C}^{-1}\mathbf{l}$, $\quad$ but since $\mathbf{y}^{\mathrm{T}}\mathbf{C}\mathbf{y} = 0$

$\rightarrow \mathbf{l}^{\mathrm{T}}\mathbf{C}^{-\mathrm{T}}\mathbf{C}\,\mathbf{C}^{-1}\mathbf{l} = 0 \;\rightarrow \mathbf{C}^{*} = \mathbf{C}^{-\mathrm{T}} = \mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes

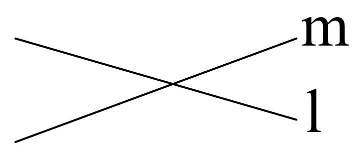




Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

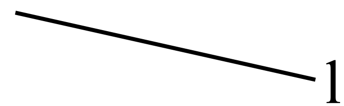
e.g. two lines (rank 2)



$$\mathbf{C} = \mathrm{lm}^{\mathsf{T}} + \mathrm{ml}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathrm{ll}^{\mathsf{T}}$$



Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics $\left(\mathbf{C}^{*}\right)^{*} \neq \mathbf{C}$

Projective transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

Theorem:

A mapping $h: P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 represented by a vector x it is true that $h(x) = \mathbf{H}x$

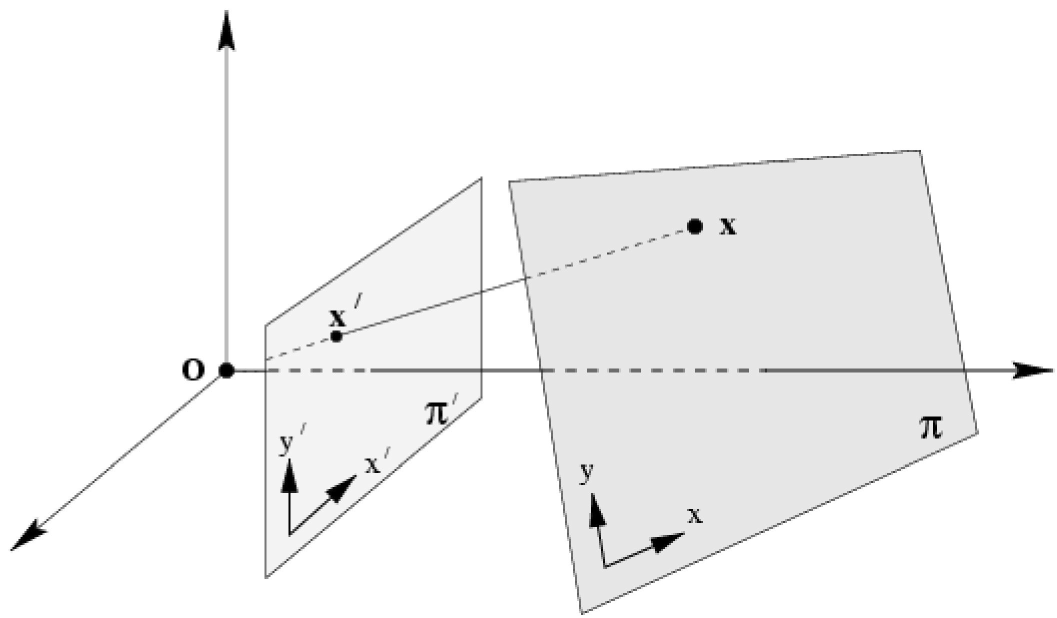
Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad x' = \mathbf{H}\,x$$

8DOF

projectivity=collineation=projective transformation=homography

Mapping between planes



central projection may be expressed by x'=Hx

(application of theorem)

Removing projective distortion

select four points in a plane with known coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

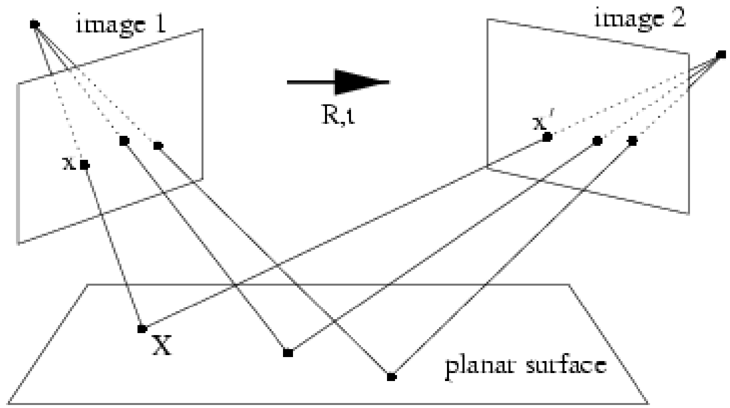
$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

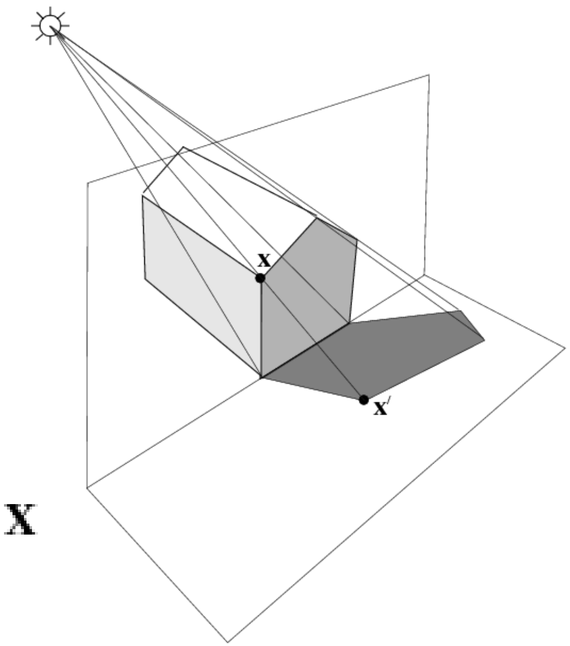
(linear in h_{ij})

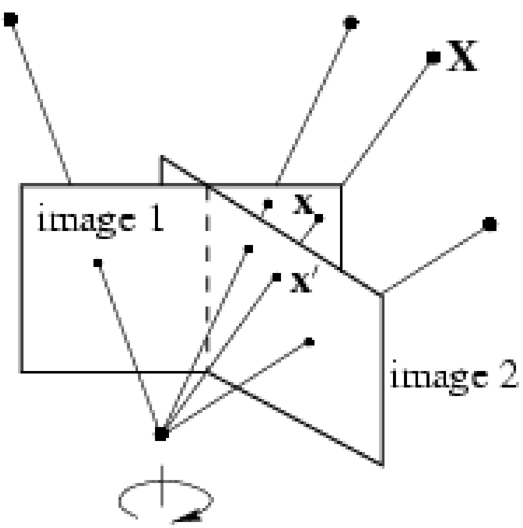
(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary,
better ways to compute (see later)

More examples







Transformation of lines and conics

For a point transformation

$$\mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

Transformation for lines

$$\mathrm{l}' = \mathbf{H}^{-\mathsf{T}}\,\mathrm{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

A hierarchy of transformations

Projective linear group

Affine group (last row (0,0,1))

Euclidean group (upper left 2x2 orthogonal)

Oriented Euclidean group (upper left 2x2 det 1)

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*

e.g. Euclidean transformations leave distances unchanged



Class I: Isometries

(*iso*=same, *metric*=measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$

orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\,\mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation)

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

(*isometry + scale*)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\,\mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)

also know as *equi-form* (shape preserving)

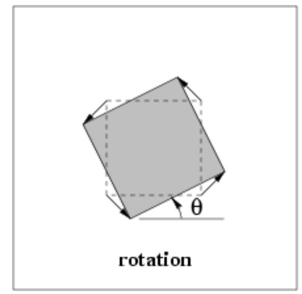
metric structure = structure up to similarity (in literature)

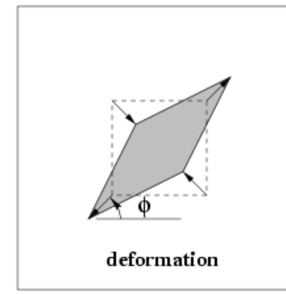
Invariants: ratios of length, angle, ratios of areas, parallel lines

Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_A\, \mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x}$$





$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\mathrm{T} = (\underline{\mathbf{U}\mathbf{V}}^\mathrm{T})(\mathbf{V}\mathbf{D}\mathbf{V}^\mathrm{T})$$

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \quad \text{where} \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel segment lengths, ratios of areas

Action of affinities and projectivities on line at infinity

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity,
but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite,
allows to observe vanishing points, horizon,

Class VI: Projective transformations

$$x' = \mathbf{H}_P\, x = \begin{bmatrix} \mathbf{A} & \mathsf{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} x \qquad \mathrm{v} = (v_1, v_2)^{\mathsf{T}}$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action: non-homogeneous over the plane

Invariants: cross-ratio of four points on a line
(ratio of ratios)

Projective geometry of 1D

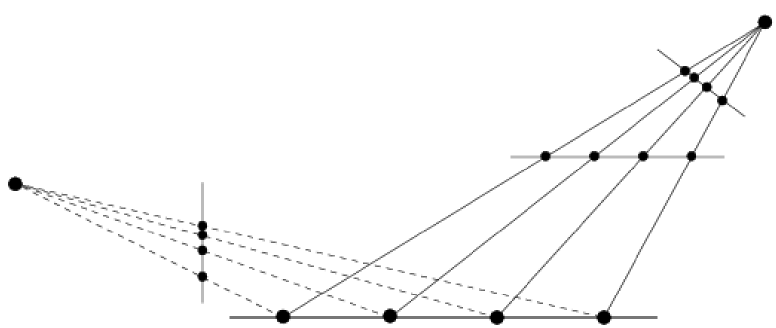
$$(x_1, x_2)^\mathsf{T} \qquad x_2 = 0$$

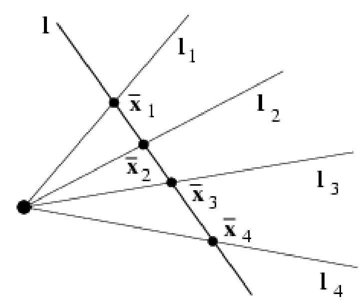
$$\overline{\mathrm{x}}' = \mathbf{H}_{2\times 2}\,\overline{\mathrm{x}}$$ 3DOF (2x2-1)

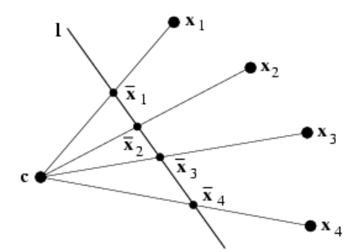
The cross ratio

$$Cross(\overline{\mathrm{x}}_1, \overline{\mathrm{x}}_2, \overline{\mathrm{x}}_3, \overline{\mathrm{x}}_4) = \frac{|\overline{\mathrm{x}}_1, \overline{\mathrm{x}}_2||\overline{\mathrm{x}}_3, \overline{\mathrm{x}}_4|}{|\overline{\mathrm{x}}_1, \overline{\mathrm{x}}_3||\overline{\mathrm{x}}_2, \overline{\mathrm{x}}_4|} \qquad |\overline{\mathrm{x}}_i, \overline{\mathrm{x}}_j| = \det\begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

Invariant under projective transformations



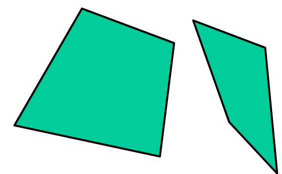




Overview transformations

Projective
8dof

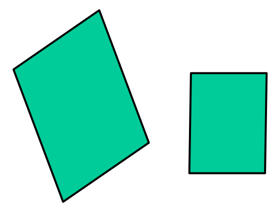
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

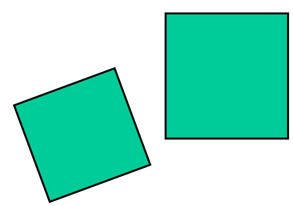
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity l_∞

Similarity
4dof

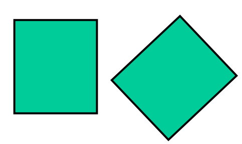
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

Number of invariants?

The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation

e.g. configuration of 4 points in general position has 8 dof (2/pt)

and so 4 similarity, 2 affinity and zero projective invariants

Recovering metric and affine properties from images

- Parallelism
- Parallel length ratios

- Angles
- Length ratios

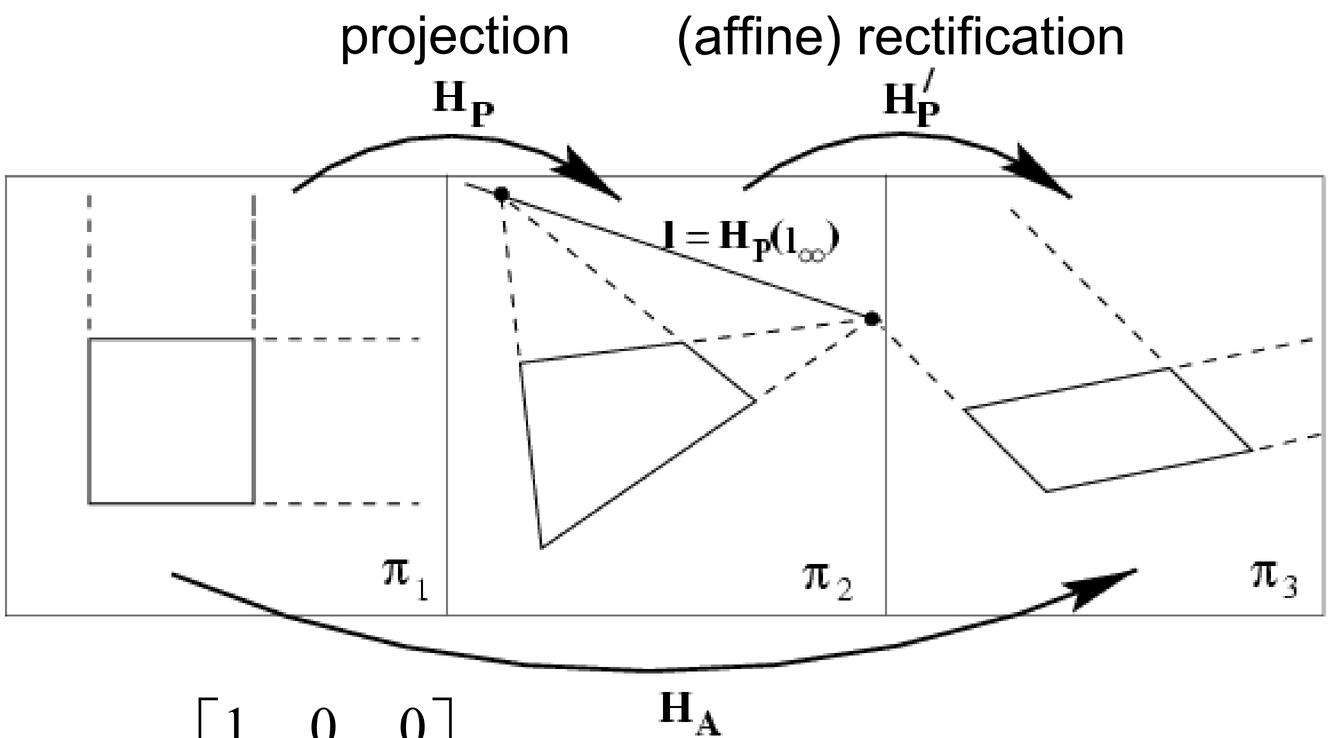
The line at infinity

$$\mathbf{l}'_\infty = \mathbf{H}_A^{-\mathsf{T}}\,\mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{t}^{\mathsf{T}}\mathbf{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty$$

The line at infinity l_∞ is a fixed line under a projective transformation H if and only if H is an affinity

Note: not fixed pointwise

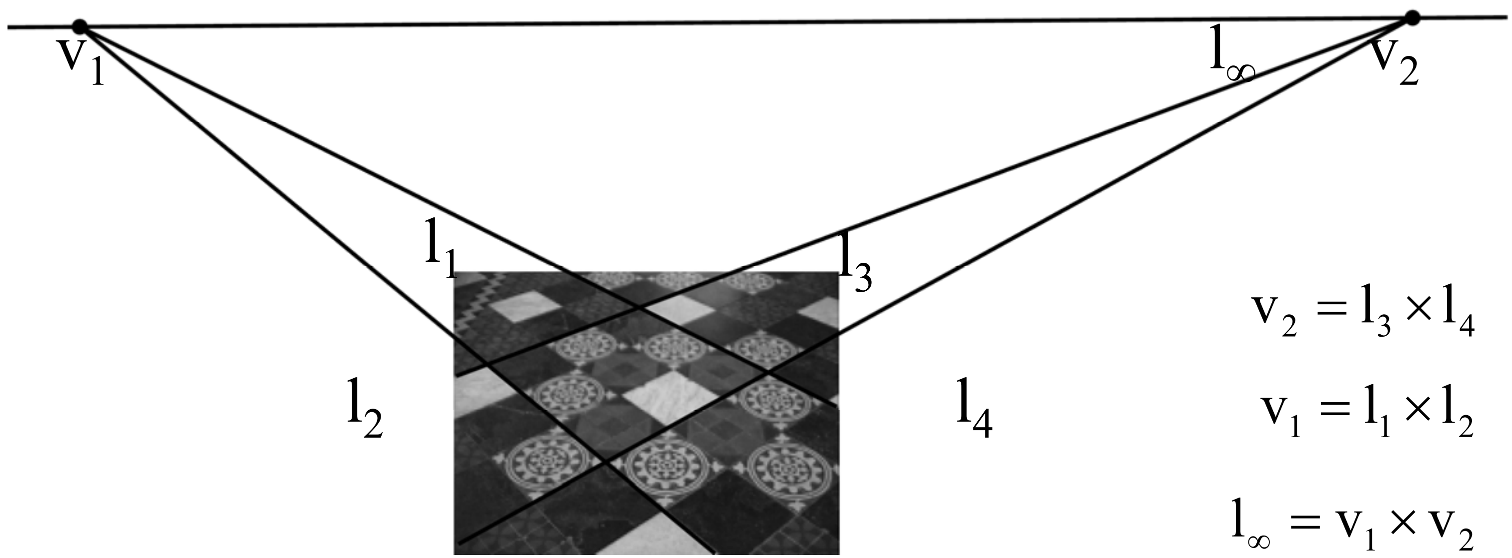
Affine properties from images



$$\mathbf{H'}_P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \mathbf{H}_A \qquad \mathrm{l'}_\infty = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\mathsf{T}, l_3 \neq 0$$

in fact, any point x on $\mathrm{l'}_\infty$ is mapped to a point at the ∞

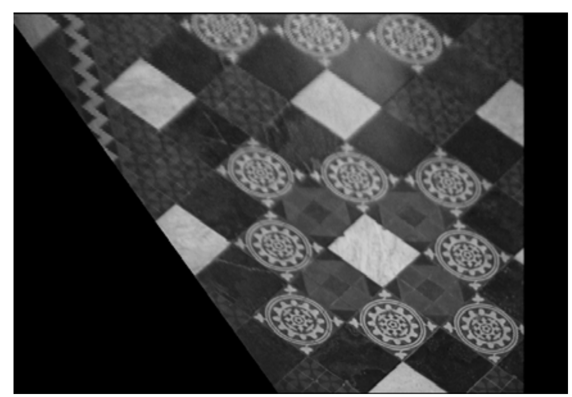
Affine rectification



$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

$$l_\infty = v_1 \times v_2$$



The circular points

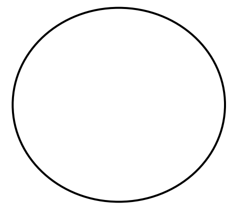
$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

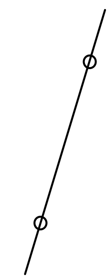
$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The circular points I, J are fixed points under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

The circular points

"circular points"





l_∞

$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$x_3 = 0$$

$$x_1^2 + x_2^2 = 0$$

Intersection points between any circle and l_∞ $\longrightarrow$ $\left| \begin{array}{l} \mathrm{I} = (1, i, 0)^\mathsf{T} \\ \mathrm{J} = (1, -i, 0)^\mathsf{T} \end{array} \right.$

Algebraically, encodes orthogonal directions

$$\mathrm{I} = (1,0,0)^\mathsf{T} + i(0,1,0)^\mathsf{T}$$

Circular points invariance

- $\{I,J\} = l_\infty \cap$ any circumference
- Similarity: circ’ → circ”
- Similarity: circ’ $\cap$ l_∞ → circ’’ $\cap$ l_∞
- Similarity: $\{I,J\}$ → $\{I,J\}$
- → circular points: invariant under similarity

Conic dual to the circular points

$$\mathbf{C}^*_\infty = \mathrm{IJ}^\mathsf{T} + \mathrm{JI}^\mathsf{T} \qquad \mathbf{C}^*_\infty = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{C}^*_\infty$: line conic = set of lines through any of the circular points

$$\mathbf{C}^*_\infty = \mathbf{H}_S \mathbf{C}^*_\infty \mathbf{H}_S^\mathsf{T}$$

The dual conic $\mathbf{C}^*_\infty$ is fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

Note: $\mathbf{C}^*_\infty$ has 4DOF

l_∞ is the null vector

Angles

Euclidean: $\mathrm{l} = (l_1, l_2, l_3)^\mathsf{T} \qquad \mathrm{m} = (m_1, m_2, m_3)^\mathsf{T}$

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{\left(l_1^2 + l_2^2\right)\left(m_1^2 + m_2^2\right)}}$$

Projective: $$\cos\theta = \frac{\mathrm{l}^\mathsf{T}\,\mathbf{C}_\infty^*\,\mathrm{m}}{\sqrt{\left(\mathrm{l}^\mathsf{T}\,\mathbf{C}_\infty^*\,\mathrm{l}\right)\left(\mathrm{m}^\mathsf{T}\,\mathbf{C}_\infty^*\,\mathrm{m}\right)}}$$

$\mathrm{l}^\mathsf{T}\,\mathbf{C}_\infty^*\,\mathrm{m} = 0$ (orthogonal)

Metric properties from images

$$\mathbf{C}_\infty^*{}' = (\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S)\mathbf{C}_\infty^*(\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S)^\mathsf{T}$$

$$= (\mathbf{H}_P\mathbf{H}_A)\mathbf{H}_S\mathbf{C}_\infty^*\mathbf{H}_S^\mathsf{T}(\mathbf{H}_P\mathbf{H}_A)^\mathsf{T}$$

$$= (\mathbf{H}_P\mathbf{H}_A)\mathbf{C}_\infty^*(\mathbf{H}_P\mathbf{H}_A)^\mathsf{T}$$

$$= \begin{bmatrix} \mathbf{K}\mathbf{K}^\mathsf{T} & \mathbf{K}^\mathsf{T}\mathrm{v} \\ \mathrm{v}^\mathsf{T}\mathbf{K} & \mathrm{v}^\mathsf{T}\mathrm{v} \end{bmatrix}$$

Rectifying transformation from SVD

$$\mathbf{C}_\infty^*{}' = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{U}^\mathsf{T} \qquad \mathbf{H} = \mathbf{U}$$

Why $\mathbf{C}_\infty^*{}' = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{U}^\mathsf{T}$?

Normally: SVD (Singular Value Decomposition)

$$\mathbf{C}_\infty^*{}' = \mathbf{V}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\mathbf{U}^\mathsf{T} \quad \text{with } \mathbf{U} \text{ and } \mathbf{V} \text{ orthogonal}$$

But $\mathbf{C}_\infty^*{}'$ is symmetric → $\mathbf{C}_\infty^*{}'^\mathsf{T} = \mathbf{UDV}^\mathsf{T} = \mathbf{VDU}^\mathsf{T} = \mathbf{C}_\infty^*{}'$

and SVD is unique → $\mathbf{U} = \mathbf{V}$

Observation : $\mathbf{H{=}U}$ orthogonal (3x3): not a $\mathbf{P}^2$ isometry

Metric from affine

Once the image has been affinely rectified

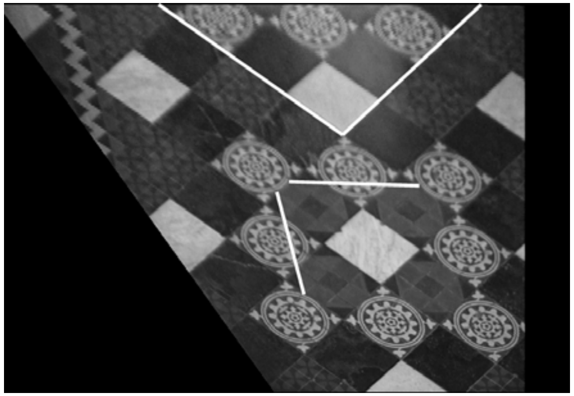
$$\mathbf{C}_\infty^*{}' = \mathbf{H_A} \mathbf{C}_\infty^* \mathbf{H_A}^\mathsf{T}$$

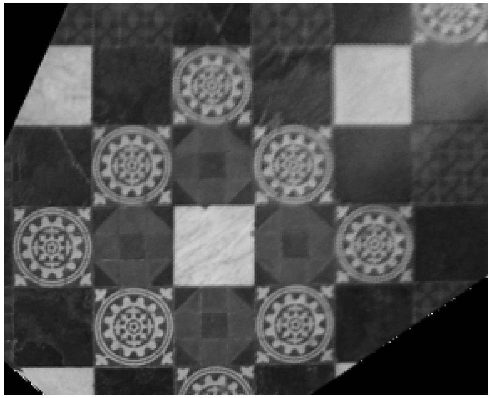
$$\mathbf{C}_\infty^*{}' = \begin{bmatrix} \mathbf{K} & \mathrm{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{K}^\mathsf{T} & \mathbf{0} \\ \mathrm{t}^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{KK}^\mathsf{T} & \mathbf{0} \\ \mathbf{0}^\mathsf{T} & 0 \end{bmatrix}$$

Metric from affine

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathbf{KK}^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

$$\left(l'_1 m'_1, l'_1 m'_2 + l'_2 m'_1, l'_2 m'_2\right)\left(k_{11}^2 + k_{12}^2, k_{11}k_{12}, k_{22}^2\right)^\mathsf{T} = 0$$





Metric from projective

$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \end{pmatrix} \begin{bmatrix} \mathbf{K}\mathbf{K}^{\mathsf{T}} & \mathbf{K}^{\mathsf{T}}\mathrm{v} \\ \mathrm{v}^{\mathsf{T}}\mathbf{K} & \mathrm{v}^{\mathsf{T}}\mathrm{v} \end{bmatrix} \begin{pmatrix} m'_1 \\ m'_2 \\ m'_3 \end{pmatrix} = 0$$

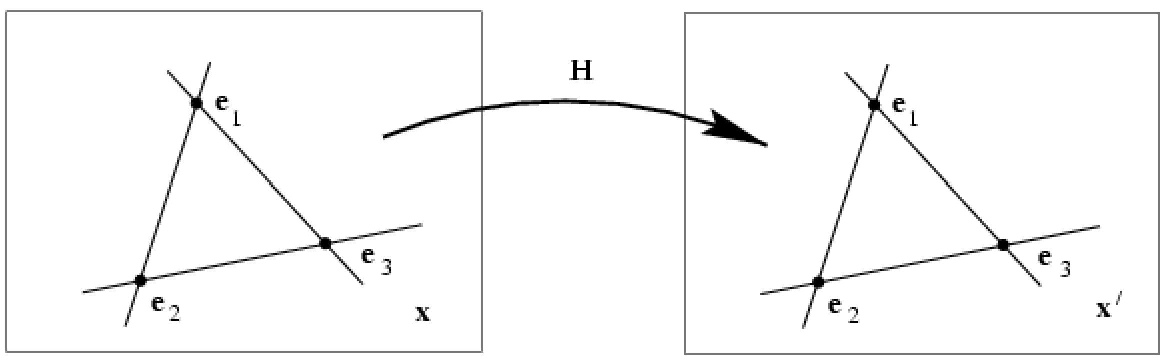
$$\left(l'_1 m'_1, 0.5\left(l'_1 m'_2 + l'_2 m'_1\right), l'_2 m'_2, 0.5\left(l'_1 m'_3 + l'_3 m'_1\right), 0.5\left(l'_2 m'_3 + l'_3 m'_2\right), l'_3 m'_3\right)\mathrm{c} = 0$$

Fixed points and lines

$\mathbf{H}\,\mathrm{e} = \lambda\,\mathrm{e}$ (eigenvectors $\mathbf{H}$ =fixed points)

(λ_1=λ_2 $\Rightarrow$ pointwise fixed line)

$\mathbf{H}^{-\mathsf{T}}\,\mathrm{l} = \lambda\,\mathrm{l}$ (eigenvectors $\mathbf{H}^{-\mathsf{T}}$ =fixed lines)



Projective 3D geometry

Singular Value Decomposition

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V^{\mathsf{T}}_{n\times n} \qquad m \geq n$$

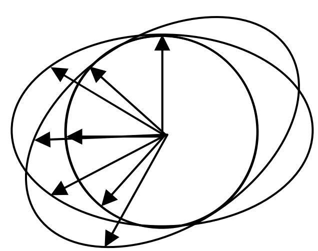
$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

$$U^{\mathsf{T}}U = I$$

$$V^{\mathsf{T}}V = I$$

$$A = U_1\,\sigma_1\,V_1^{\mathsf{T}} + U_2\,\sigma_2\,V_2^{\mathsf{T}} + \cdots + U_n\,\sigma_n\,V_n^{\mathsf{T}}$$



$$U\Sigma V^{\mathsf{T}}\mathbf{X}$$

Singular Value Decomposition

$$\mathrm{A} = \mathrm{U}\Sigma\,\mathrm{V}^{\mathsf{T}}$$

- Homogeneous least-squares

$$\min\|\mathrm{AX}\| \text{ subject to } \|\mathrm{X}\| = 1 \qquad \text{solution } \mathrm{X} = \mathrm{V}_n$$

- Span and null-space

$$\mathbf{S}_L = [\mathrm{U}_1\ \mathrm{U}_2];\ \mathbf{N}_L = [\mathrm{U}_3\mathrm{U}_4]$$
$$\mathbf{S}_R = [\mathrm{V}_1\mathrm{V}_2];\ \mathbf{N}_R = [\mathrm{V}_3\mathrm{V}_4]$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Closest rank r approximation

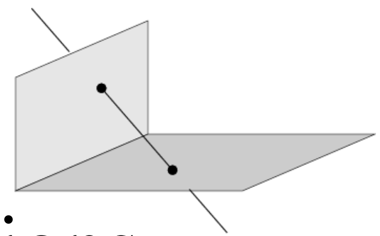
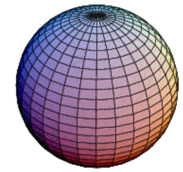
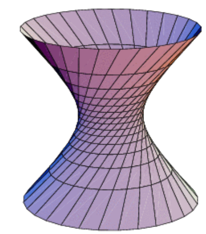
$$\widetilde{\mathrm{A}} = \mathrm{U}\widetilde{\Sigma}\,\mathrm{V}^{\mathsf{T}} \qquad \widetilde{\Sigma} = diag(\sigma_1, \sigma_2, \Lambda\ , \sigma_r,\ 0\ \ , \Lambda\ ,\ 0\ )$$

- Pseudo inverse

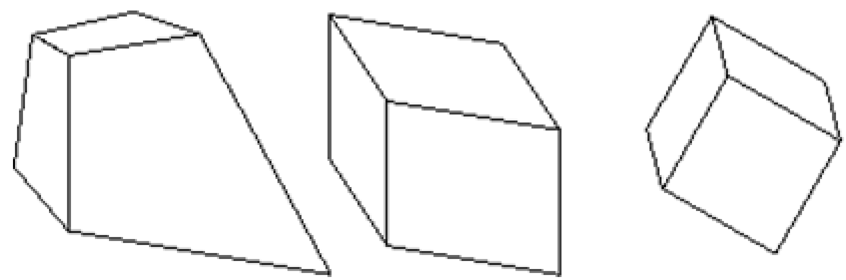
$$\mathrm{A}^{+} = \mathrm{V}\Sigma^{+}\,\mathrm{U}^{\mathsf{T}} \quad \Sigma^{+} = diag(\sigma_1^{-1}, \sigma_2^{-1}, \Lambda\ , \sigma_r^{-1},\ 0\ \ , \Lambda\ ,\ 0\ )$$

Projective 3D Geometry

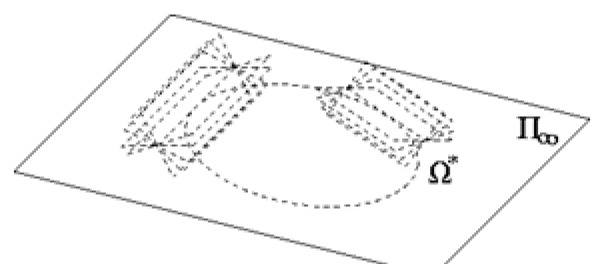
- Points, lines, planes and quadrics

- Transformations



- Π_∞, ω_∞ and Ω_∞



3D points

3D point

$(X, Y, Z)^{\mathsf{T}}$ in $\mathbf{R}^3$

$\mathrm{X} = (X_1, X_2, X_3, X_4)^{\mathsf{T}}$ in $\mathbf{P}^3$

$$\mathrm{X} = \left(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}, 1 \right)^{\mathsf{T}} = (X, Y, Z, 1)^{\mathsf{T}} \qquad (X_4 \neq 0)$$

projective transformation

$$\mathrm{X}' = \mathbf{H}\,\mathrm{X} \quad \text{(4x4-1=15 dof)}$$

Planes

3D plane

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

$$\pi^{\mathsf{T}} \mathrm{X} = 0$$

Transformation

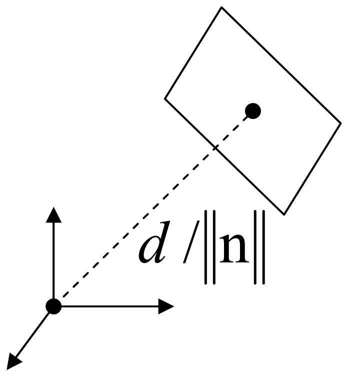
$$\mathrm{X}' = \mathbf{H}\,\mathrm{X}$$

$$\pi' = \mathbf{H}^{-\mathsf{T}}\,\pi$$

Euclidean representation

$$\mathrm{n}.\widetilde{\mathrm{X}} + d = 0 \qquad \mathrm{n} = (\pi_1, \pi_2, \pi_3)^{\mathsf{T}} \qquad \widetilde{\mathrm{X}} = (X, Y, Z)^{\mathsf{T}}$$

$$\pi_4 = d \qquad X_4 = 1$$



Dual: points ↔ planes, lines ↔ lines

Planes from points

Solve π from $X_1^T\pi = 0$, $X_2^T\pi = 0$ and $X_3^T\pi = 0$

$$\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \pi = 0 \quad \text{(solve } \pi \text{ as right nullspace of } \begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix}\text{)}$$

Or implicitly from coplanarity condition

$$\det\left[X\, X_1 X_2 X_3\right] = 0 \qquad \det\begin{bmatrix} \boxed{X_1} & \boxed{(X_1)_1 \quad (X_2)_1 \quad (X_3)_1} \\ \boxed{X_2} & \boxed{(X_1)_2 \quad (X_2)_2 \quad (X_3)_2} \\ \boxed{X_3} & \boxed{(X_1)_3 \quad (X_2)_3 \quad (X_3)_3} \\ \boxed{X_4} & \boxed{(X_1)_4 \quad (X_2)_4 \quad (X_3)_4} \end{bmatrix} = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = \left(D_{234}, -D_{134}, D_{124}, -D_{123}\right)^T$$

Points from planes

Solve X from $\pi_1^{\mathsf{T}}\mathrm{X}=0,\ \pi_2^{\mathsf{T}}\mathrm{X}=0$ and $\pi_3^{\mathsf{T}}\mathrm{X}=0$

$$\begin{bmatrix}\pi_1^{\mathsf{T}}\\ \pi_2^{\mathsf{T}}\\ \pi_3^{\mathsf{T}}\end{bmatrix}\mathrm{X}=0$$ (solve X as right nullspace of $\begin{bmatrix}\pi_1^{\mathsf{T}}\\ \pi_2^{\mathsf{T}}\\ \pi_3^{\mathsf{T}}\end{bmatrix}$)

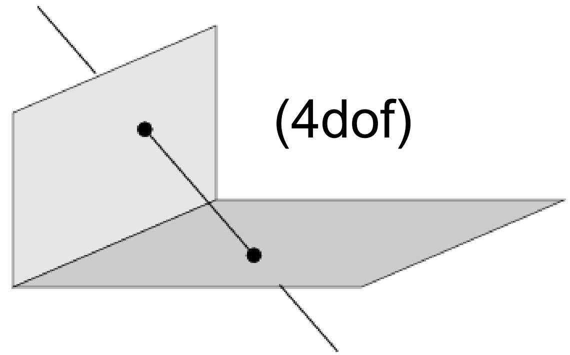
Representing a plane by its span

$$\mathrm{X}=\mathbf{M}\,\mathrm{x} \qquad \mathbf{M}=\left[\mathrm{X}_1\mathrm{X}_2\mathrm{X}_3\right] \qquad \pi=\left(a,b,c,d\right)^{\mathsf{T}}$$

$$\pi^{\mathsf{T}}\,\mathbf{M}=0 \qquad \mathbf{M}=\begin{bmatrix}\mathrm{p}\\ \mathrm{I}\end{bmatrix} \qquad p=\left(-\frac{b}{a},-\frac{c}{a},-\frac{d}{a}\right)^{\mathsf{T}}$$

Lines

$$W = \begin{bmatrix} A^{\mathsf{T}} \\ B^{\mathsf{T}} \end{bmatrix} \quad \lambda A + \mu B$$



$$W^{*} = \begin{bmatrix} P^{\mathsf{T}} \\ Q^{\mathsf{T}} \end{bmatrix} \quad \lambda P + \mu Q$$

two points $\mathbf{A}$ and $\mathbf{B}$

two planes $\mathbf{P}$ and $\mathbf{Q}$

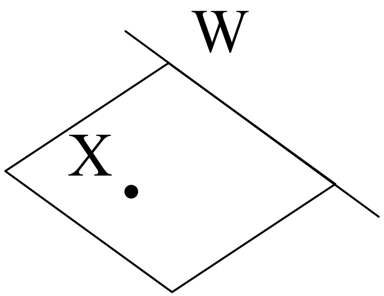
$$W^{*} W^{\mathsf{T}} = W W^{*\mathsf{T}} = 0_{2\times 2}$$

Example: X-axis

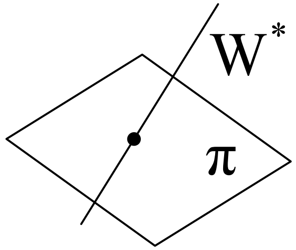
$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad W^{*} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Points, lines and planes

$$\mathbf{M} = \begin{bmatrix} W \\ X^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{M}\,\pi = 0$$



$$\mathbf{M} = \begin{bmatrix} W^{*} \\ \pi^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{M}\,X = 0$$



Quadrics and dual quadrics

$X^{\mathsf{T}}QX = 0$ (Q : 4x4 symmetric matrix)

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ o & \bullet & \bullet & \bullet \\ o & o & \bullet & \bullet \\ o & o & o & \bullet \end{bmatrix}$$

1. 9 d.o.f.
2. in general 9 points define quadric
3. det Q=0 ↔ degenerate quadric
4. pole – polar $\pi = QX$
5. (plane ∩ quadric)=conic $C = M^{\mathsf{T}}QM$ $\pi : X = Mx$
6. transformation $Q' = H^{-\mathsf{T}}QH^{-1}$

$\pi^{\mathsf{T}}Q^{*}\pi = 0$

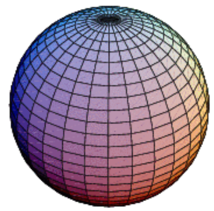
1. relation to quadric $Q^{*} = Q^{-1}$ (non-degenerate)
2. transformation $Q'^{*} = HQ^{*}H^{\mathsf{T}}$

Quadric classification

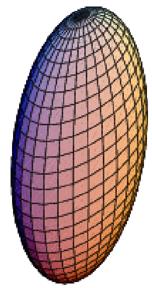
Rank	Sign.	Diagonal	Equation	Realization
4	4	(1,1,1,1)	$X^2+ Y^2+ Z^2+1=0$	No real points
	2	(1,1,1,-1)	$X^2+ Y^2+ Z^2=1$	Sphere
	0	(1,1,-1,-1)	$X^2+ Y^2= Z^2+1$	Hyperboloid (1S)
3	3	(1,1,1,0)	$X^2+ Y^2+ Z^2=0$	Single point
	1	(1,1,-1,0)	$X^2+ Y^2= Z^2$	Cone
2	2	(1,1,0,0)	$X^2+ Y^2= 0$	Single line
	0	(1,-1,0,0)	$X^2= Y^2$	Two planes
1	1	(1,0,0,0)	$X^2=0$	Single plane

Quadric classification

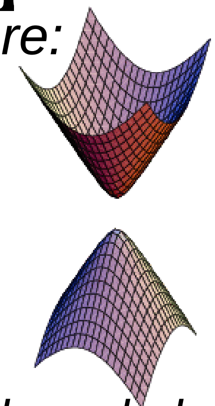
Projectively equivalent to *sphere:*



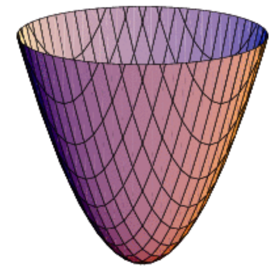
sphere



ellipsoid

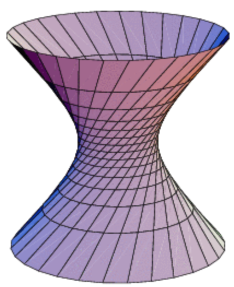


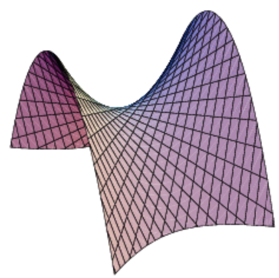
hyperboloid of two sheets



paraboloid

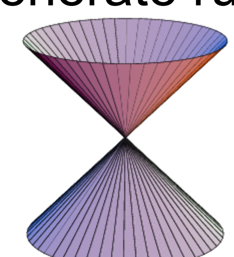
Ruled quadrics:



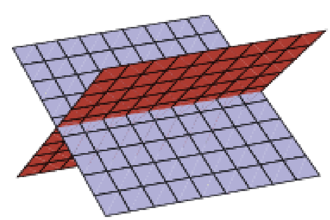


hyperboloids of one sheet

Degenerate ruled quadrics:

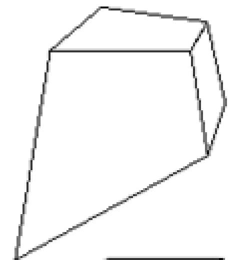
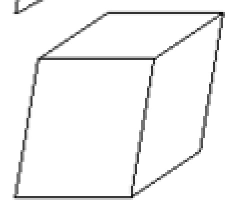
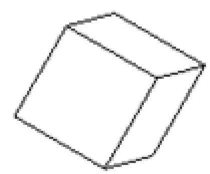
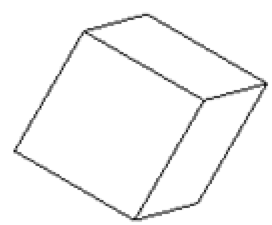


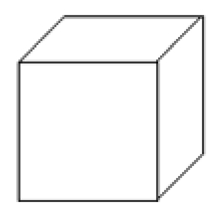
cone



two planes

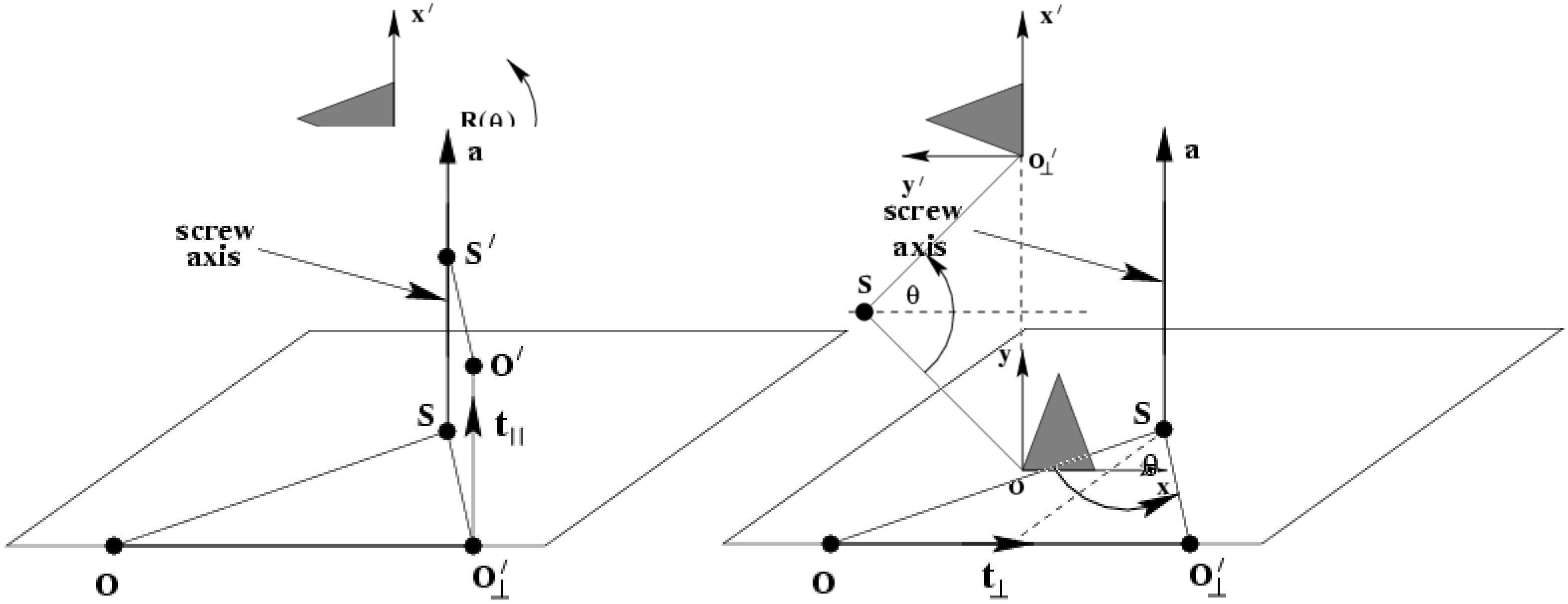
Hierarchy of transformations

Projective 15dof	$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency
Affine 12dof	$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		Parallellism of planes, Volume ratios, centroids, **The plane at infinity π_∞**
Similarity 7dof	$\begin{bmatrix} s\mathrm{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		**The absolute conic Ω_∞**
Euclidean 6dof	$\begin{bmatrix} \mathrm{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		Volume



Screw decomposition

Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.



screw axis // rotation axis

$$t = t_{//} + t_{\perp}$$

The plane at infinity

$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}} \pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{A}\,\mathrm{t} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canical position $\pi_\infty = (0,0,0,1)^\mathsf{T}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^\mathsf{T}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞

The absolute conic

The absolute conic Ω_∞ is a (point) conic on π_∞.

In a metric frame:

$$\left.\begin{array}{c} X_1^2+X_2^2+X_3^2 \\ X_4 \end{array}\right\} = 0$$

or conic for directions: $(X_1, X_2, X_3)\mathrm{I}(X_1, X_2, X_3)^\mathsf{T}$

(with no real points)

The absolute conic Ω_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. Ω_∞ is only fixed as a set
2. Circle intersect Ω_∞ in two points
3. Spheres intersect π_∞ in Ω_∞

Absolute conic invariance

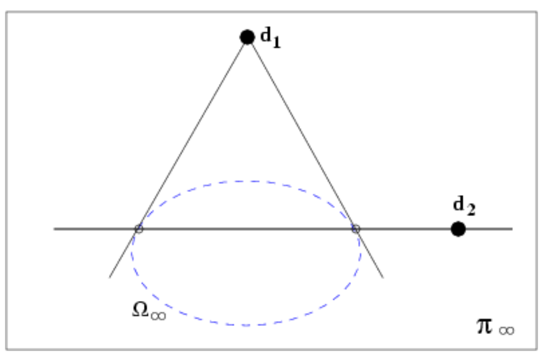
- $\Omega_\infty = \pi_\infty$ ⇯ any sphere
- Similarity: sphere’ → sphere”
- Similarity: sphere’ ⇯ π_∞ → sphere’’ ⇯ π_∞
- Similarity: Ω_∞ → Ω_∞
- → Ω_∞ : invariant under similarity

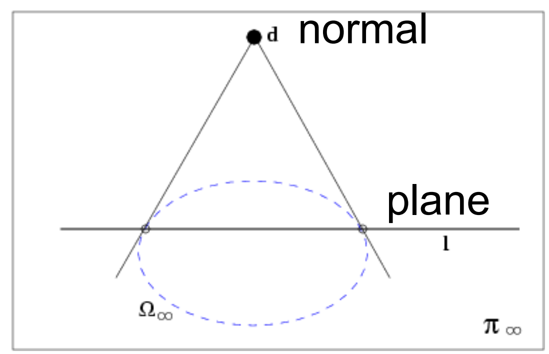
The absolute conic

Euclidean:
$$\cos\theta = \frac{\left(\mathrm{d}_1^{\mathsf{T}}\mathrm{d}_2\right)}{\sqrt{\left(\mathrm{d}_1^{\mathsf{T}}\mathrm{d}_1\right)\left(\mathrm{d}_2^{\mathsf{T}}\mathrm{d}_2\right)}}$$

Projective:
$$\cos\theta = \frac{\left(\mathrm{d}_1^{\mathsf{T}}\Omega_\infty \mathrm{d}_2\right)}{\sqrt{\left(\mathrm{d}_1^{\mathsf{T}}\Omega_\infty\mathrm{d}_1\right)\left(\mathrm{d}_2^{\mathsf{T}}\Omega_\infty\mathrm{d}_2\right)}}$$

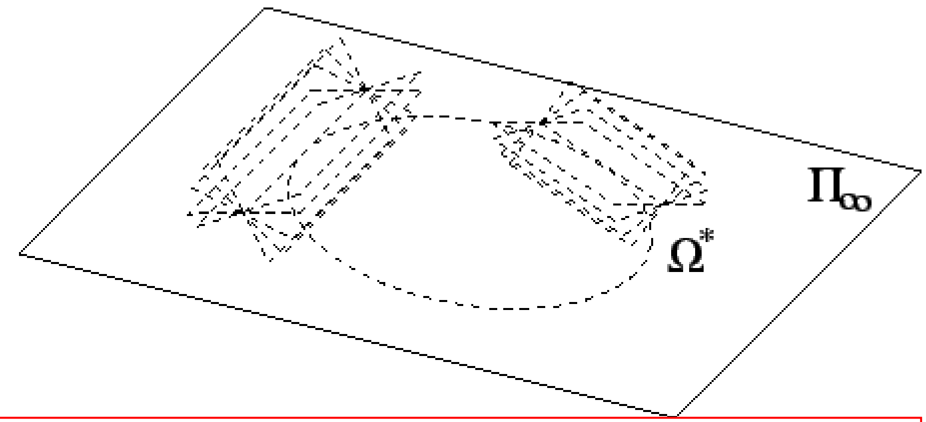
$$\mathrm{d}_1^{\mathsf{T}}\Omega_\infty \mathrm{d}_2 = 0$$ (orthogonality=conjugacy)





The absolute dual quadric

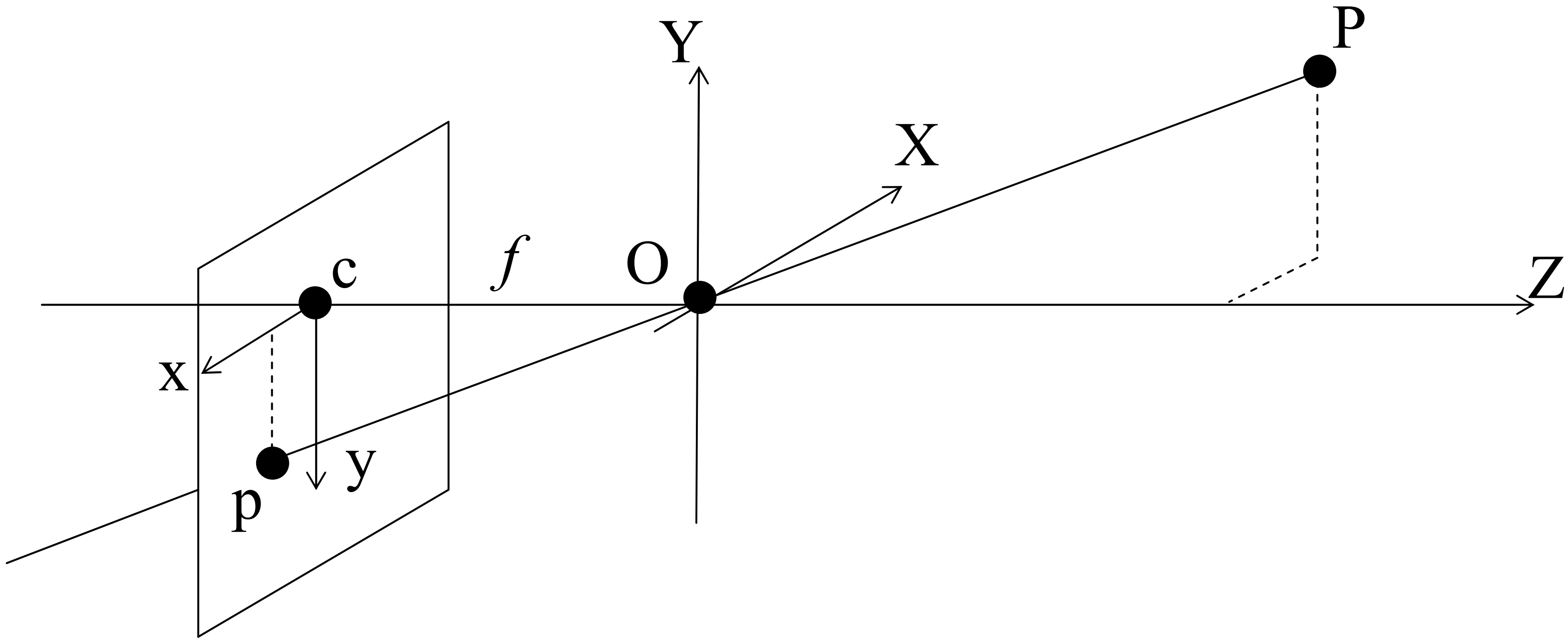
$$\Omega^*_\infty = \begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix}$$



The absolute conic Ω^*_∞ is a fixed conic under the projective transformation $\mathbf{H}$ iff $\mathbf{H}$ is a similarity

1. 8 dof
2. plane at infinity π_∞ is the nullvector of Ω_∞
3. Angles:

$$\cos\theta = \frac{\pi_1^T \Omega^*_\infty \pi_2}{\sqrt{\left(\pi_1^T \Omega^*_\infty \pi_1\right)\left(\pi_2^T \Omega^*_\infty \pi_2\right)}}$$



$$x = f\frac{X}{Z}$$

$$y = f\frac{Y}{Z}$$

perspective projection

-nonlinear
-not shape-preserving
-not length-ratio preserving

Homogeneous coordinates

- In 2D: add a third coordinate, w

•Point $[x,y]^T$ expanded to $[u,v,w]^T$

•Any two sets of points $[u_1,v_1,w_1]^T$ and $[u_2,v_2,w_2]^T$ represent the same point if one is multiple of the other

•$[u,v,w]^T \rightarrow [x,y]$ with $x=u/w$, and $y=v/w$

•$[u,v,0]^T$ is the point at the infinite along direction (u,v)

Transformations

translation by vector $[d_x, d_y]^T$

$$T = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

scaling (by different factors in x and y)

$$S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation by angle θ

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous coordinates

In 3D: add a fourth coordinate, t

•Point $[X,Y,Z]^T$ expanded to $[x,y,z,t]^T$

•Any two sets of points $[x_1,y_1,z_1,t_1]^T$ and $[x_2,y_2,z_2,t_2]^T$ represent the same point if one is multiple of the other

•$[x,y,z,t]^T \rightarrow [X,Y,Z]$ with $X=x/t$, $Y=y/t$, and $Z=z/t$

•$[x,y,z,0]^T$ is the point at the infinite along direction (x,y,z)

Transformations

translation

$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

scaling

$$S = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

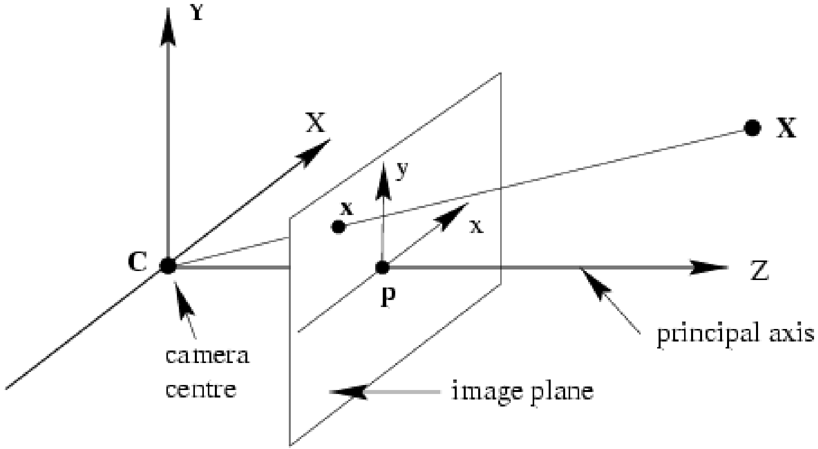
rotation

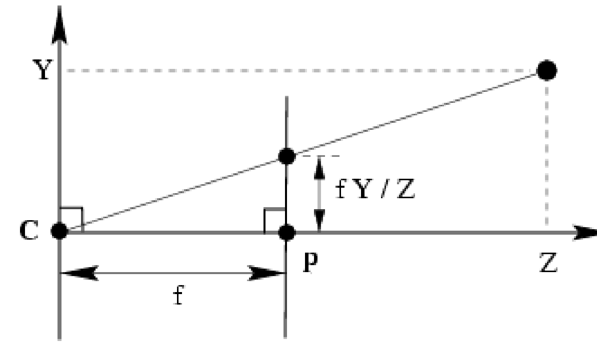
$$R = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ \sin\phi\cos\psi\sin\theta - \cos\phi\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \sin\phi\cos\theta \\ \cos\phi\cos\psi\sin\theta + \sin\phi\sin\psi & \cos\phi\sin\psi\sin\theta - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

Obs: rotation matrix is an orthogonal matrix

i.e.: $R^{-1} = R^T$

Pinhole camera model





$$(X, Y, Z)^T \ \alpha \ (fX/Z, fY/Z)^T$$

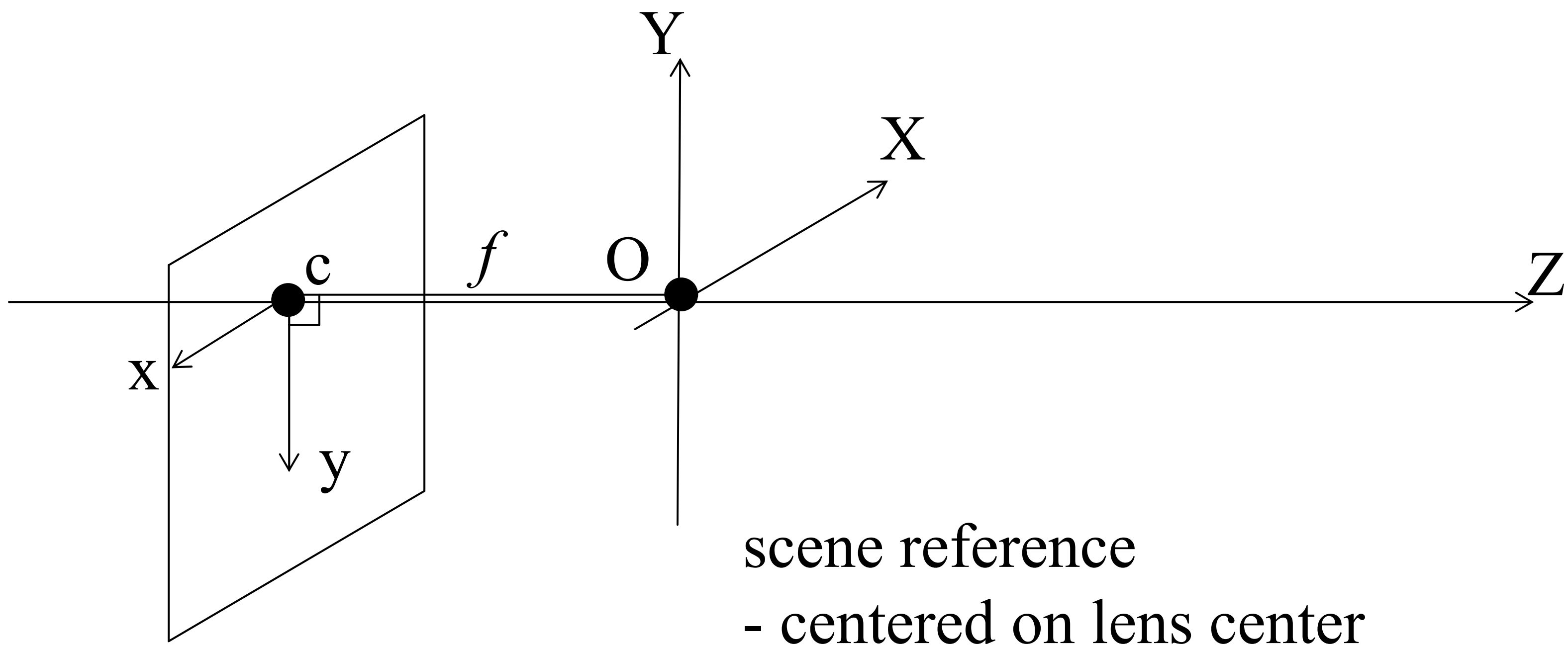
$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \alpha \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Scene->Image mapping: perspective transformation

$$\begin{vmatrix} u \\ v \\ w \end{vmatrix} = \begin{vmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} X \\ Y \\ Z \\ 1 \end{vmatrix} \quad \text{with} \quad x = \frac{u}{w} \quad y = \frac{v}{w}$$

With "ad hoc" reference frames, for both image and scene

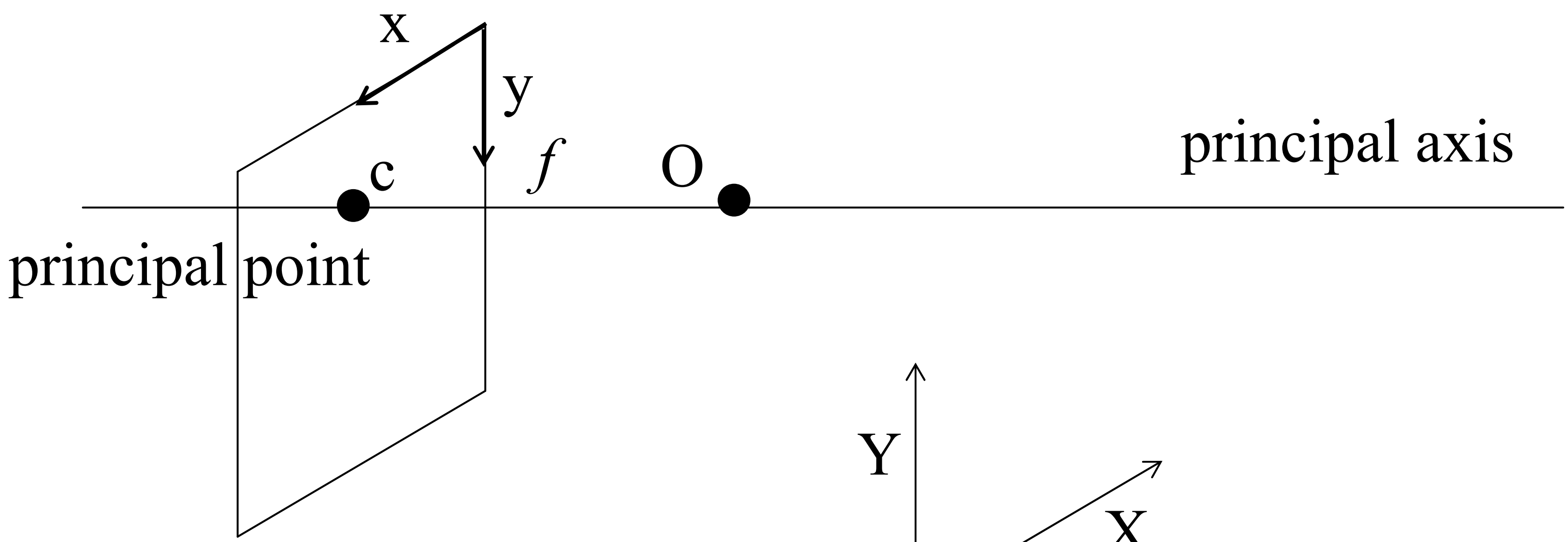
Let us recall them



scene reference
- centered on lens center
- Z-axis orthogonal to image plane
- X- and Y-axes opposite to
  image x- and y-axes

image reference
- centered on principal point
- x- and y-axes parallel to
  the sensor rows and columns
- Euclidean reference

Actual references are generic



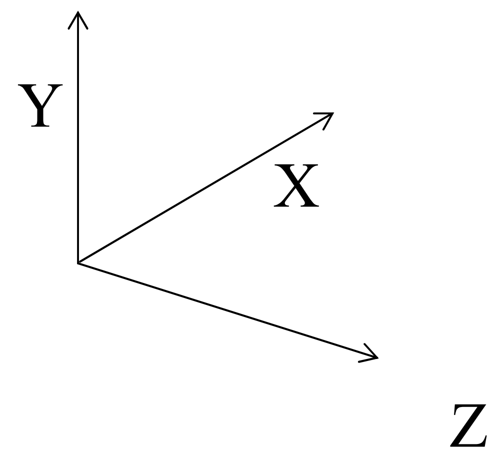
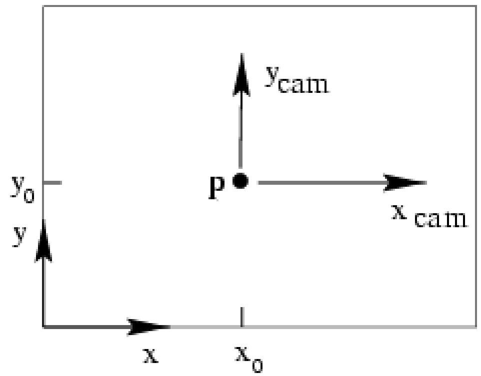


image reference
- centered on upper left corner
- nonsquare pixels (aspect ratio)

→ noneuclidean reference

scene reference
- not attached to the camera

Principal point offset

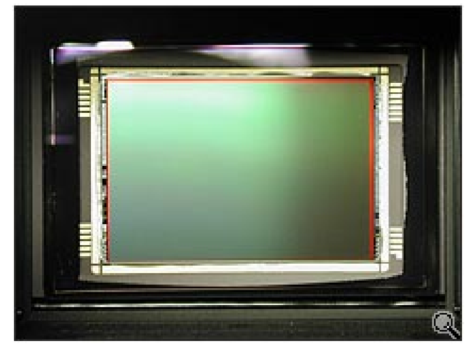


$$(X,Y,Z)^T \mapsto (fX/Z+u_o, fY/Z+v_o)^T$$

$(u_o, v_o)^T$ principal point

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX+Zu_o \\ fY+Zv_o \\ Z \end{pmatrix} = \begin{bmatrix} f & & u_o & 0 \\ & f & v_o & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

CCD camera



$$K = \begin{bmatrix} f_x & & u_o \\ & f_y & v_o \\ & & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & & \\ & a & \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & u_o \\ & f & v_o \\ & & 1 \end{bmatrix}$$

Scene-image relationship wrt actual reference frames

image

$$\mathbf{u} = \mathbf{A} \cdot \begin{vmatrix} u \\ v \\ w \end{vmatrix} = \begin{vmatrix} 1 & (s \cdot a) & u_o \\ 0 & a & v_o \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} u \\ v \\ w \end{vmatrix}$$

normally, s=0

scene

$$\begin{vmatrix} X \\ Y \\ Z \\ 1 \end{vmatrix} = \begin{vmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{vmatrix} \cdot \mathbf{X}$$

$$\mathbf{u} = \mathbf{P} \cdot \mathbf{X} = \begin{vmatrix} 1 & 0 & u_o \\ 0 & a & v_o \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \cdot \left|\begin{array}{c:c} \mathbf{R} & \mathbf{t} \\ \hdashline 0 & 1 \end{array}\right| \cdot \mathbf{X}$$

scene-camera tranformation

orthogonal (3D rotation) matrix

$$\mathbf{P} = \begin{vmatrix} f & 0 & u_o \\ 0 & af & v_o \\ 0 & 0 & 1 \end{vmatrix} \cdot \left|\begin{array}{c:c} \mathbf{I} & 0 \end{array}\right| \cdot \left|\begin{array}{c:c} \mathbf{R} & \mathbf{t} \\ \hdashline 0 & 1 \end{array}\right| = \left|\begin{array}{c:c} \mathbf{K} \cdot \mathbf{R} & \mathbf{K} \cdot \mathbf{t} \end{array}\right|$$

extrinsic camera parameters

K upper triangular: intrinsic camera parameters

P: 10-11 degrees of freedom (10 if s=0)

$$\mathbf{P} = \left|\ \mathbf{K}\cdot\mathbf{R} \ \vdots\ \mathbf{K}\cdot\mathbf{t}\ \right| = \left|\ \mathbf{M} \ \vdots\ \mathbf{m}\ \right| = \left|\ \mathbf{M} \ \vdots\ -\mathbf{M}\cdot\mathbf{o}\ \right| = \mathbf{M}\cdot\left|\ \mathbf{I} \ \vdots\ -\mathbf{o}\ \right|$$

with $\mathbf{M} = \mathbf{K}\cdot\mathbf{R}$ and $\boxed{\mathbf{o} = -\mathbf{M}^{-1}\cdot\mathbf{m}}$

$$\mathbf{u} = \mathbf{M}\cdot\left|\ \mathbf{I} \ \vdots\ -\mathbf{o}\ \right|\cdot\mathbf{X} = \mathbf{M}\cdot\left|\ \mathbf{I} \ \vdots\ -\mathbf{o}\ \right|\cdot\begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

i.e., defining $\mathbf{x} = [x, y, z]^\mathsf{T}$

$$\mathbf{u} = \mathbf{M}\cdot\left|\ \mathbf{I} \ \vdots\ -\mathbf{o}\ \right|\cdot\begin{vmatrix} \mathbf{x} \\ \hdashline 1 \end{vmatrix} = \mathbf{M}\cdot(\mathbf{x}-\mathbf{o})$$

Interpretation of o:

u is image of **x** if $(\mathbf{x}-\mathbf{o})=\lambda\mathbf{M}^{-1}\cdot\mathbf{u}$

i.e., if $\boxed{\mathbf{x}=\mathbf{o}+\lambda\mathbf{M}^{-1}\cdot\mathbf{u}}$

The locus of the points **x** *whose image is* **u**
is a straight line through **o** *having direction* $\mathbf{d}=\mathbf{M}^{-1}\cdot\mathbf{u}$

$\mathbf{o}=-\mathbf{M}^{-1}\cdot\mathbf{m}$ is independent of **u**

o is the camera viewpoint (perspective projection center)

line(**o**, **d**) = *Interpretation line of image point* **u**

Intrinsic and extrinsic parameters from P

M → **K** and **R**

$$\mathbf{M} = \mathbf{K} \cdot \mathbf{R} \longrightarrow \mathbf{M}^{-1} = \mathbf{R}^{-1} \cdot \mathbf{K}^{-1} = \mathbf{R}^{T} \cdot \mathbf{K}^{-1}$$

RQ-decomposition of a matrix: as the product between an orthogonal matrix and an upper triangular matrix

M and **m** → **t**

$$\mathbf{o} = -\mathbf{M}^{-1} \cdot \mathbf{m} = -\underbrace{(\mathbf{KR})^{-1}\mathbf{Kt}}_{\downarrow} = -\mathbf{R}^{\mathbf{T}}\mathbf{K}^{-1}\mathbf{Kt} = -\mathbf{R}^{T}\mathbf{t}$$

$$\mathbf{t} = -\mathbf{Ro}$$

Camera center

null-space camera projection matrix

$$\mathrm{P}\mathbf{O} = 0$$

$$\mathrm{X} = \lambda \mathrm{A} + (1-\lambda)\mathbf{O}$$

$$\mathrm{x} = \mathrm{PX} = \lambda \mathrm{PA} + (1-\lambda)\mathrm{P}\mathbf{O}$$

For all A all points on AO project on image of A,

therefore O is camera center

Image of camera center is $(0,0,0)^{\mathsf{T}}$, i.e. undefined

Finite cameras: $\mathbf{O} = \begin{pmatrix} \mathbf{o} \\ 1 \end{pmatrix} = \begin{pmatrix} -\mathrm{M}^{-1}\mathbf{m} \\ 1 \end{pmatrix}$

Infinite cameras: $\mathbf{o} = \begin{pmatrix} \mathrm{d} \\ 0 \end{pmatrix}, \mathrm{Md} = 0$

Action of projective camera on point

Forward projection

$$x = PX$$

$$x = PD = [M \mid m]D = Md$$

Back-projection

$$P\mathbf{O} = 0$$

$$X = P^{+}x \qquad P^{+} = P^{\mathsf{T}}\left(PP^{\mathsf{T}}\right)^{-1} \qquad PP^{+} = I$$

(pseudo-inverse)

$$X(\lambda) = P^{+}x + \lambda\mathbf{O}$$

$$d = M^{-1}x$$

$$X(\lambda) = \mu\underbrace{\begin{pmatrix} M^{-1}x \\ 0 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} -M^{-1}m \\ 1 \end{pmatrix}}_{O} = \begin{pmatrix} M^{-1}(\mu x - m) \\ 1 \end{pmatrix}$$

Camera matrix decomposition

Finding the camera center

$$\mathrm{P}\mathbf{O} = 0$$ (use SVD to find null-space)

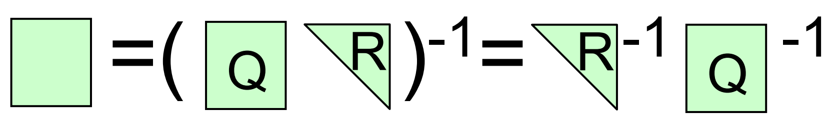
$$X = \det([\mathrm{p}_2, \mathrm{p}_3, \mathbf{m}]) \quad Y = -\det([\mathrm{p}_1, \mathrm{p}_3, \mathbf{m}])$$

$$Z = \det([\mathrm{p}_1, \mathrm{p}_2, \mathbf{m}]) \quad T = -\det([\mathrm{p}_1, \mathrm{p}_2, \mathbf{m}])$$

Finding the camera orientation and internal parameters

$$\mathrm{M} = \mathrm{KR}$$ (use RQ decomposition ~QR)

(if only QR, invert)

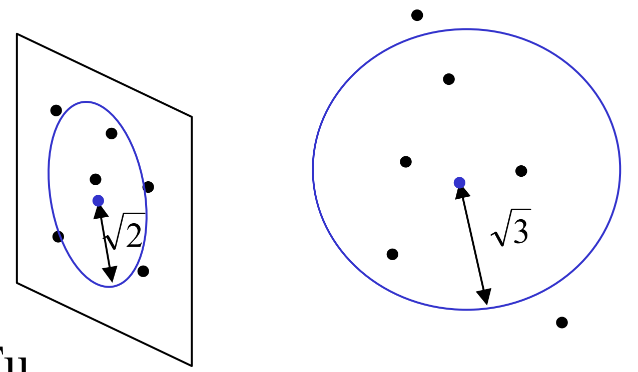


Radial distortion

$$x = x_o + (x_o - c_x)(K_1 r^2 + K_2 r^4 + \ldots)$$
$$y = y_o + (y_o - c_y)(K_1 r^2 + K_2 r^4 + \ldots)$$

$$r = (x_o - c_x)^2 + (y_o - c_y)^2 \ .$$

Data normalization



$\tilde{\mathrm{X}}_i = \mathrm{UX}_i$

$\tilde{\mathrm{u}}_i = \mathrm{Tu}_i$

(i) translate origin to gravity center
(ii) (an)isotropic scaling

Exterior orientation

Calibrated camera, position and orientation unkown

$\rightarrow$ Pose estimation

6 dof $\Rightarrow$ 3 points minimal (4 solutions in general)

Properties of perspective transformations

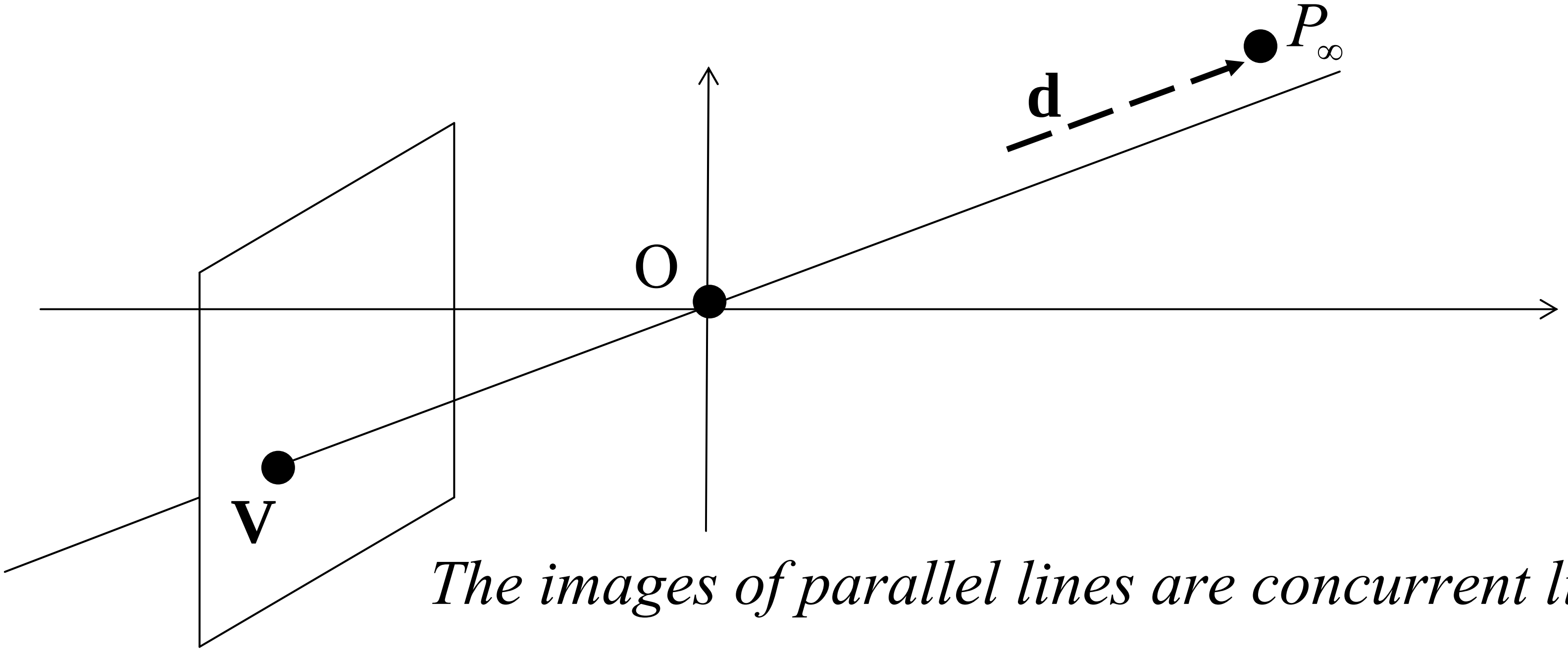
1) vanishing points

V image of the point at the ∞ along direction **d**

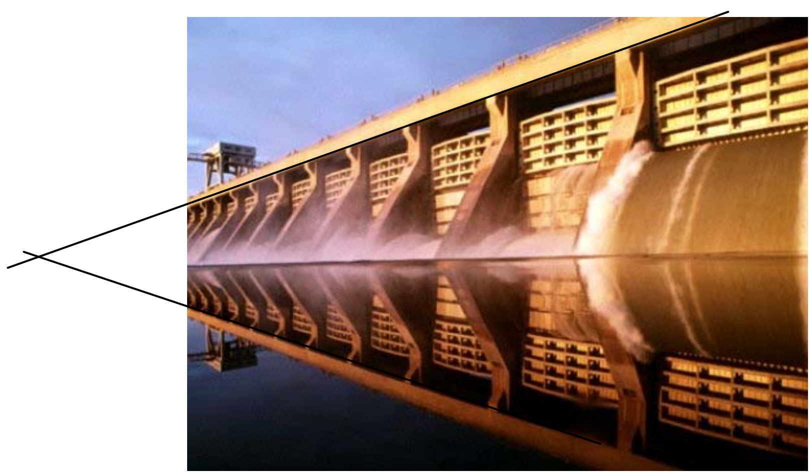
$$\mathbf{u}_{\mathbf{V}} = \left| \; \mathbf{M} \;\vdots\; \mathbf{m} \; \right| \cdot \left| \begin{array}{c} \mathbf{d} \\ \hdashline 0 \end{array} \right| = \mathbf{M} \cdot \mathbf{d}$$

$$\mathbf{d} = \mathbf{M}^{-1} \cdot \mathbf{u}_{\mathbf{V}}$$

$$\boxed{\textit{the interpretation line of } \mathbf{V} \textit{ is parallel to } \mathbf{d}}$$



The images of parallel lines are concurrent lines



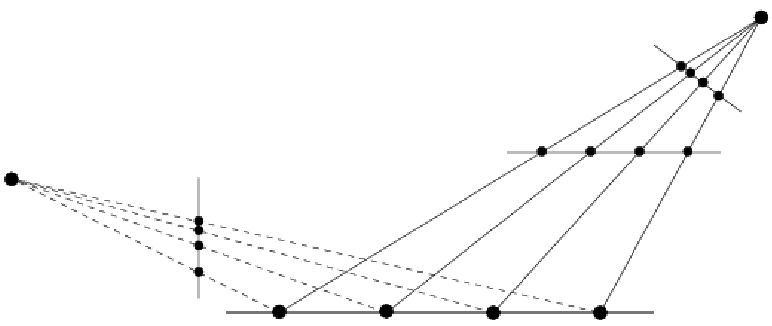
Properties of perspective transformations ctd.

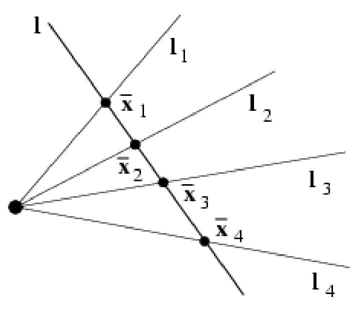
2) cross ratio invariance

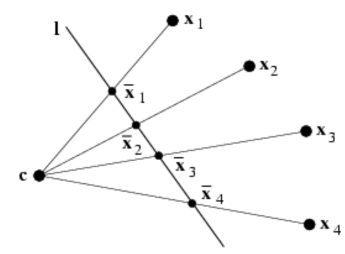
Given four colinear points $(\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4)$

let (x_1, x_2, x_3, x_4) be their abscissae

$$CR(\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4) = \frac{\dfrac{x_1 - x_3}{x_1 - x_4}}{\dfrac{x_2 - x_3}{x_2 - x_4}}$$







Cross ratio invariance under perspective transformation

a point on the line y=0=z $\mathbf{X} = [x, y, z, t]^\mathsf{T} = [x, 0, 0, t]^\mathsf{T}$

its image $\mathbf{u} = [u, v, w]^\mathsf{T} = \mathbf{P} \cdot \mathbf{X}$ belongs to a line

its coordinate u $\underline{\mathsf{u}} = [u, w]^\mathsf{T} = \begin{vmatrix} \mathbf{p}_1 \\ \mathbf{p}_4 \end{vmatrix} \cdot \begin{vmatrix} x \\ t \end{vmatrix} = \mathsf{P}_{14} \cdot \underline{\mathsf{x}}$

$$CR(\underline{\mathsf{u}}_1, \underline{\mathsf{u}}_2, \underline{\mathsf{u}}_3, \underline{\mathsf{u}}_4) = \frac{\det|\underline{\mathsf{u}}_1, \underline{\mathsf{u}}_2| \det|\underline{\mathsf{u}}_3, \underline{\mathsf{u}}_4|}{\det|\underline{\mathsf{u}}_1, \underline{\mathsf{u}}_3| \det|\underline{\mathsf{u}}_2, \underline{\mathsf{u}}_4|} = \frac{\det \mathsf{P}_{14} \det|\underline{\mathsf{x}}_1, \underline{\mathsf{x}}_2| \det \mathsf{P}_{14} \det|\underline{\mathsf{x}}_3, \underline{\mathsf{x}}_4|}{\det \mathsf{P}_{14} \det|\underline{\mathsf{x}}_1, \underline{\mathsf{x}}_3| \det \mathsf{P}_{14} \det|\underline{\mathsf{x}}_2, \underline{\mathsf{x}}_4|}$$

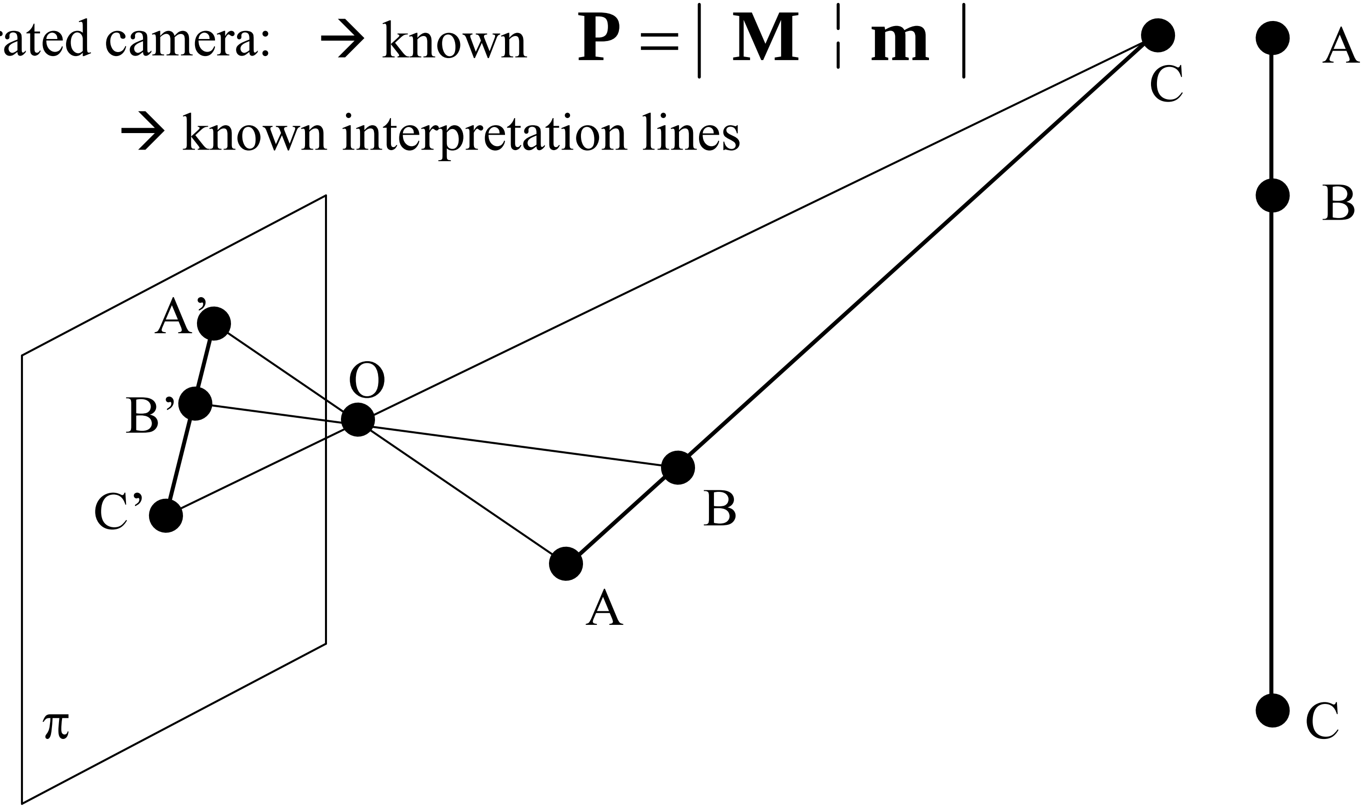
$$= \frac{\det|\underline{\mathsf{x}}_1, \underline{\mathsf{x}}_2| \det|\underline{\mathsf{x}}_3, \underline{\mathsf{x}}_4|}{\det|\underline{\mathsf{x}}_1, \underline{\mathsf{x}}_3| \det|\underline{\mathsf{x}}_2, \underline{\mathsf{x}}_4|} = CR(\underline{\mathsf{x}}_1, \underline{\mathsf{x}}_2, \underline{\mathsf{x}}_3, \underline{\mathsf{x}}_4)$$

Object localization 1: three colinear points

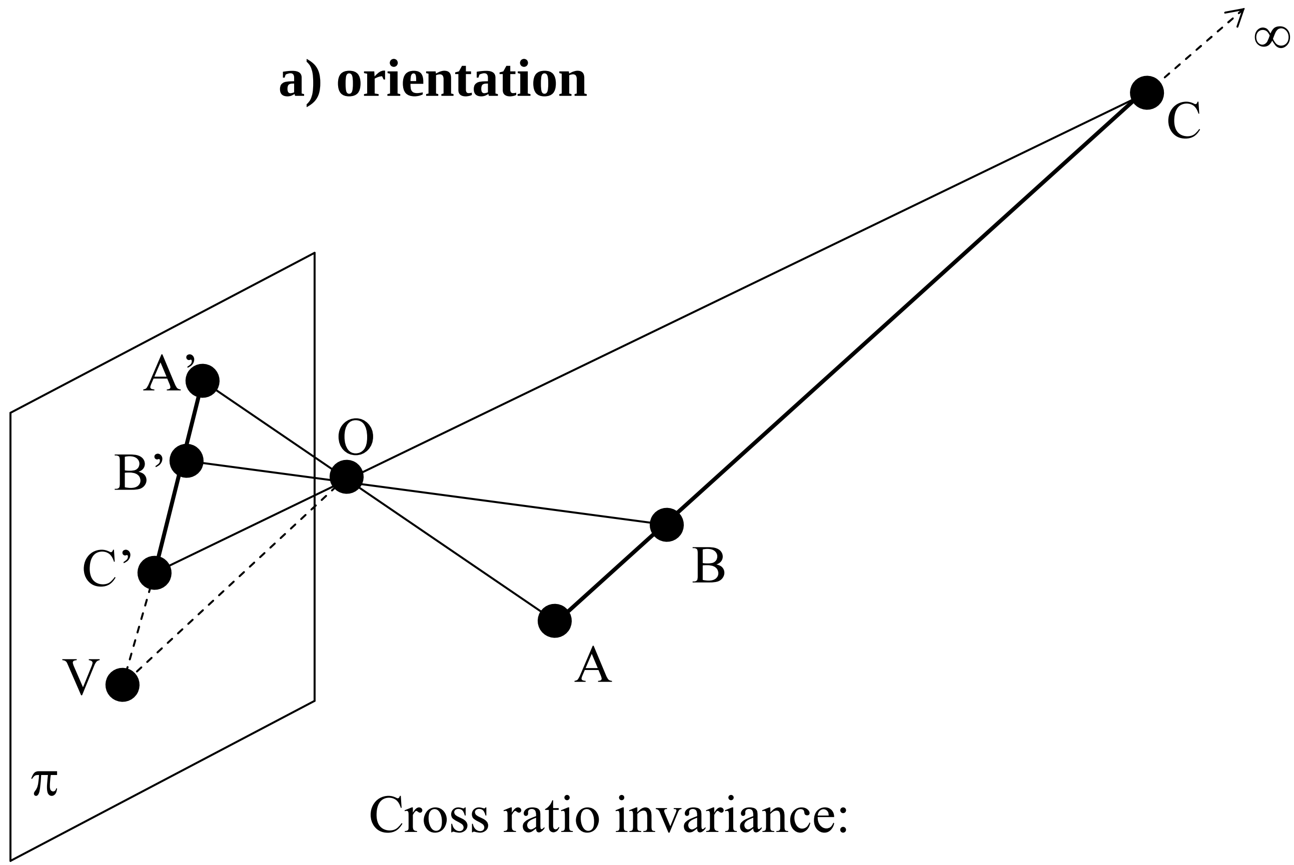
geometric model of an object
a perspective image of the object ⟶ position and orientation of the object **?**

calibrated camera: → known $\mathbf{P} = \left| \ \mathbf{M} \ \vdots \ \mathbf{m} \ \right|$

→ known interpretation lines



a) orientation



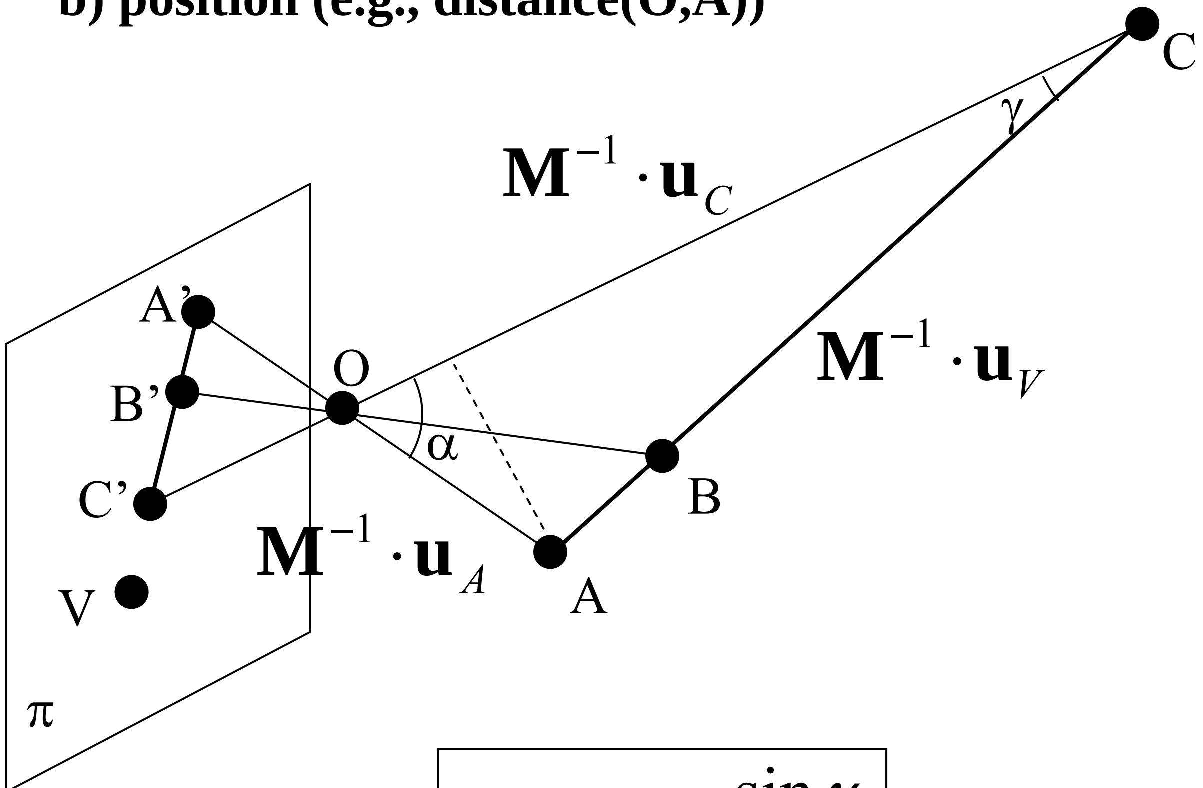
Cross ratio invariance:

solve $CR(A', B', C', V) = CR(A, B, C, \infty) = \dfrac{a-c}{b-c}$ for V (image of ∞)

V: vanishing point of the direction of (A,B,C) $\longrightarrow$

interpretation line of V *parallel* to (A,B,C) direction $\mathbf{M}^{-1} \cdot \mathbf{u}_V$

b) position (e.g., distance(O,A))

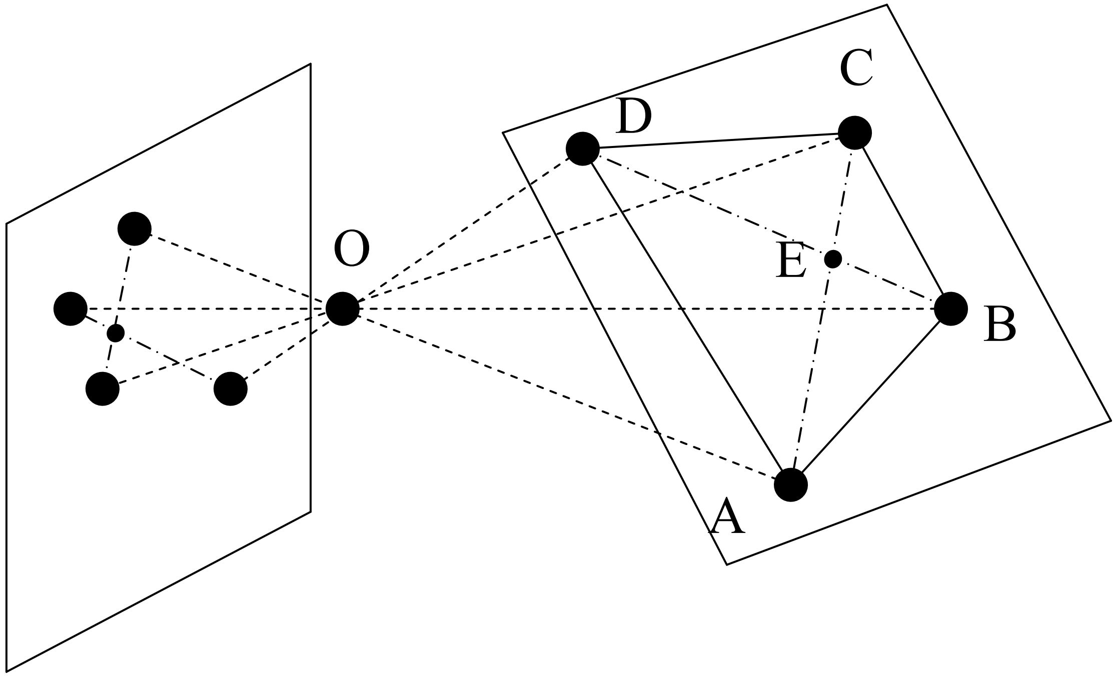


$$\overline{OA} = \overline{AC} \frac{\sin\gamma}{\sin\alpha}$$

interpretation lines

angles α and γ

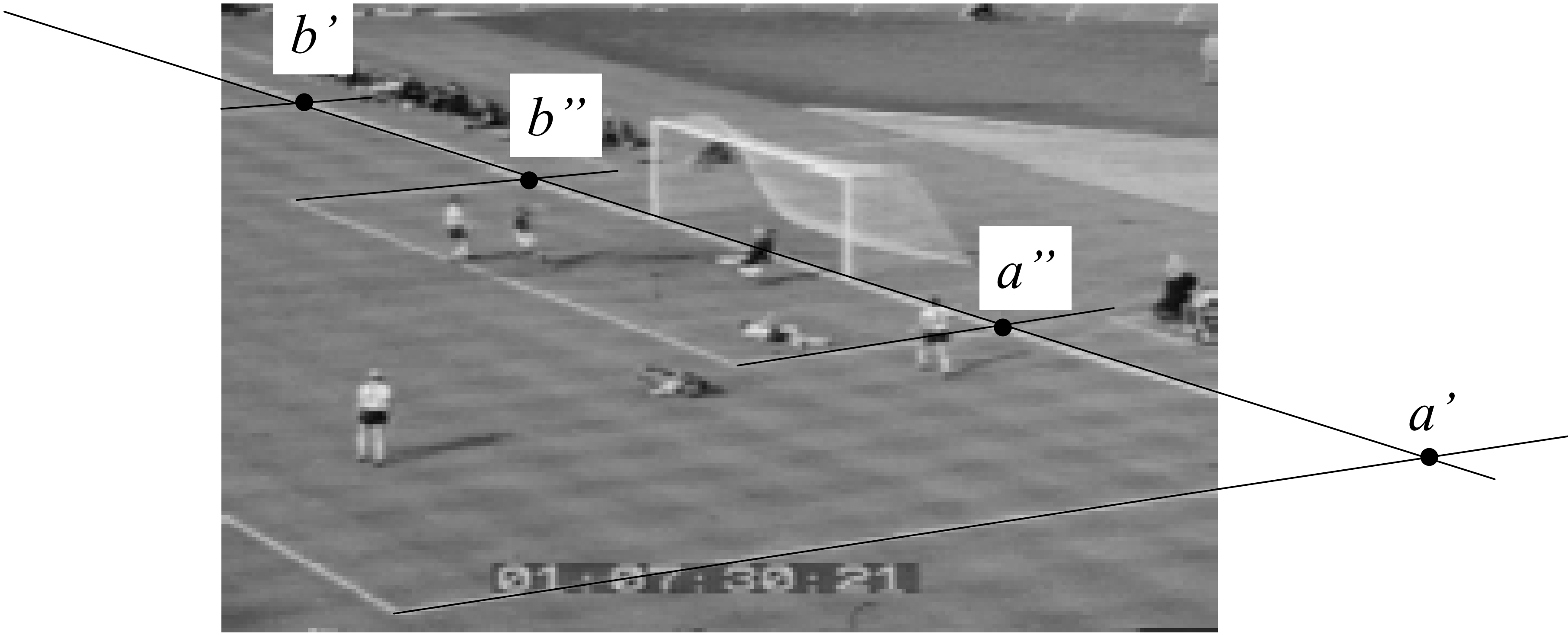
Object localization 2: four coplanar points



(i) orientation of (A,E,C)
(ii) orientation of (B,E,D)
(iii) distance (O,A)

Off-side

Find vanishing point of the field-bottom line direction



images of symmetric segments

a and b: abscissae of the endpoints of a segment

$c=(a+b)/2$: abscissa of segment midpoint,

$d=\infty$: point at the infinite along the segment direction

$$CR(a,b,c,d)=\frac{\dfrac{a-c}{b-c}}{\dfrac{a-d}{b-d}}=\frac{a-c}{b-c}=-1$$

Harmonic 4-tuple (a,b,c,d)

(a',b') and (a'', b'') are image of symmetric segments

→ same image of the midpoint c', same vanishing point d'

→ solve $\begin{cases} CR(a',b',c',d') = \dfrac{\dfrac{a'-c'}{b'-c'}}{\dfrac{a'-d'}{b'-d'}} = -1 \\ CR(a'',b'',c',d') = \dfrac{\dfrac{a''-c'}{b''-c'}}{\dfrac{a''-d'}{b''-d'}} = -1 \end{cases}$ for c', d'

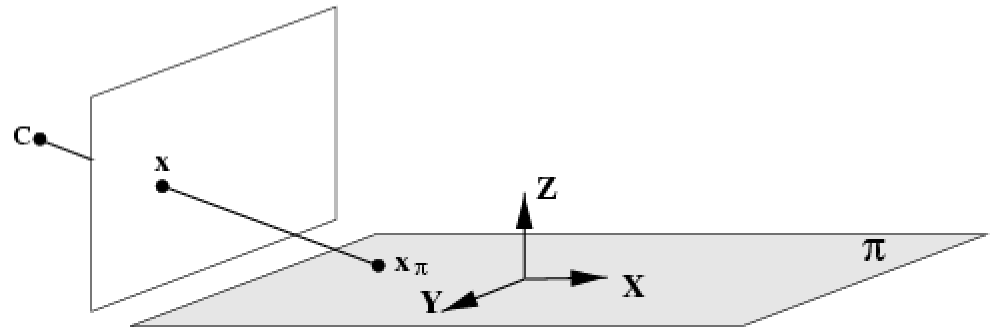
→ system of two linear equations in ($c'd'$) and ($c'+d'$)

→ two degree equation, whose solutions are c' and d'

among the two solutions, the one for d' is the value external to the range $[a',b']$

What can be told from a single image?

Action of projective camera on planes



$$\mathrm{x} = \mathrm{P}\mathrm{X} = \left[\mathrm{p}_1 \mathrm{p}_2 \mathrm{p}_3 \mathrm{p}_4\right] \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = \left[\mathrm{p}_1 \mathrm{p}_2 \mathrm{p}_4\right] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

The most general transformation that can occur between a scene plane and an image plane under perspective imaging is a plane projective transformation

Action of projective camera on lines

forward projection

$$X(\mu) = P(A + \mu B) = PA + \mu PB = a + \mu b$$

back-projection

$$\left. \begin{array}{l} l^T x = 0 \\ x = PX \end{array} \right| \longrightarrow \quad l^T PX = 0 = \Pi^T X$$

with

$$\boxed{\Pi = P^T l}$$

Interpretation plane of line l

Image of a conic

$$\mathrm{x} = \mathrm{PX} = \left[\mathrm{p}_1 \mathrm{p}_2 \mathrm{p}_3 \mathrm{p}_4\right] \begin{bmatrix} X \\ Y \\ 0 \\ T \end{bmatrix} = \left[\mathrm{p}_1 \mathrm{p}_2 \mathrm{p}_4\right] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = \left[\mathrm{p}_1 \mathrm{p}_2 \mathrm{p}_4\right] x = Px$$

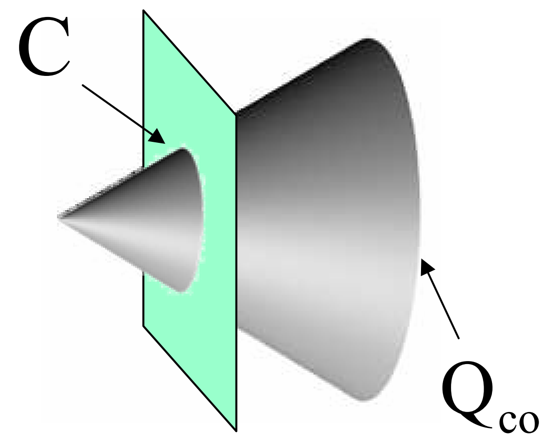
$$x^{\mathrm{T}} \mathrm{C} x = 0 = \mathrm{x}^{\mathrm{T}} P^{-\mathrm{T}} \mathrm{C} P^{-1} \mathrm{x}$$

therefore

$$\boxed{\mathrm{C}' = P^{-\mathrm{T}} \mathrm{C} P^{-1}}$$

Action of projective camera on conics

back-projection of a conic C to cone Q_{co}



back-projection of a conic C to cone Q_{co}

$$\left.\begin{array}{r}\mathrm{x}^{\mathrm{T}}\mathrm{C}\mathrm{x}=0\\ \mathrm{x}=\mathrm{P}\mathrm{X}\end{array}\right| \longrightarrow \mathrm{x}^{\mathrm{T}}\mathrm{C}\mathrm{x}=\mathrm{X}^{\mathrm{T}}\mathrm{P}^{\mathrm{T}}\mathrm{C}\mathrm{P}\mathrm{X}=0=\mathrm{X}^{\mathrm{T}}\mathrm{Q}_{co}\mathrm{X}$$

with

Interpretation cone of a conic C

$$\boxed{\mathrm{Q}_{co}=\mathrm{P}^{\mathrm{T}}\mathrm{C}\mathrm{P}}$$

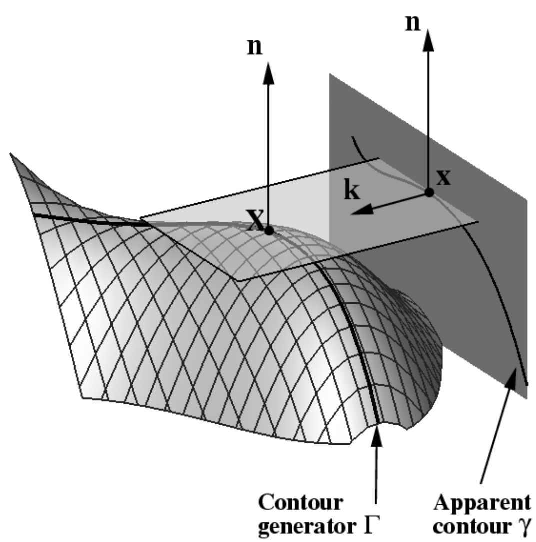
example:

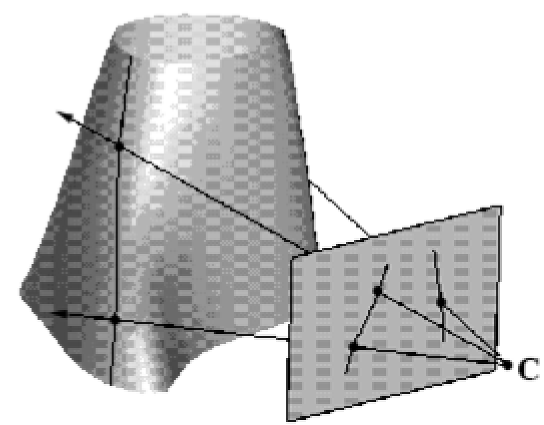
$$\mathrm{Q}_{co}=\begin{bmatrix}\mathrm{K}^{\mathrm{T}}\\ 0\end{bmatrix}\mathrm{C}^{\mathrm{T}}[\mathrm{K}\,|\,0]=\begin{bmatrix}\mathrm{K}^{\mathrm{T}}\mathrm{C}\mathrm{K} & 0\\ 0 & 0\end{bmatrix}$$

Images of smooth surfaces

The contour generator Γ is the set of points X on S at which rays are tangent to the surface. The corresponding apparent contour γ is the set of points x which are the image of X, i.e. γ is the image of Γ

The contour generator Γ depends only on position of projection center, γ depends also on rest of P





Action of projective camera on quadrics

apparent contour of a quadric Q

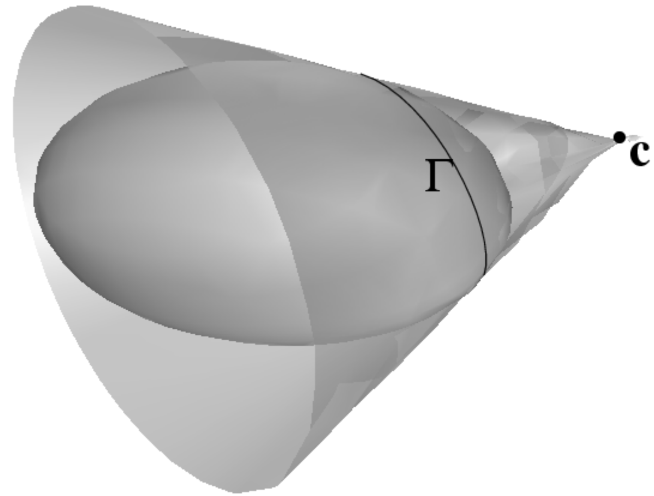
dual quadric $\mathrm{Q}^* = \mathrm{Q}^{-1}$ is a plane quadric:

the set of planes tangent to Q $\Pi^{\mathrm{T}}\mathrm{Q}^*\Pi = 0$

Let us consider only those planes that are backprojection of image lines

$$\Pi = \mathrm{P}^{\mathrm{T}}\mathrm{l} \qquad \Pi^{\mathrm{T}}\mathrm{Q}^*\Pi = \mathrm{l}^{\mathrm{T}}\mathrm{P}\mathrm{Q}^*\mathrm{P}^{\mathrm{T}}\mathrm{l} = 0$$

with $\boxed{\mathrm{C}^* = \mathrm{P}\mathrm{Q}^*\mathrm{P}^T}$ its dual is $\boxed{\mathrm{C} = \mathrm{C}^{*-1}}$



The plane containing the apparent contour Γ of a quadric Q from a camera center O follows from pole-polar relationship

$$\Pi = Q\mathbf{O}$$

The cone with vertex V and tangent to the quadric Q is

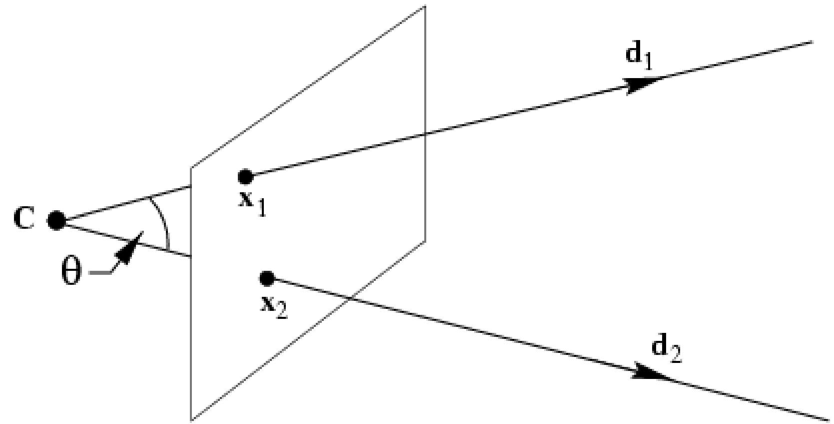
$$Q_{CO} = (V^T QV)Q - (QV)(QV)^T \qquad Q_{CO} V = 0$$

back-projection to cone

What does calibration give?

$$x = K[I\,|\,0]\begin{bmatrix} d \\ 0 \end{bmatrix}$$

$$d = K^{-1}x$$



$$\cos\theta = \frac{d_1^{\,T} d_2}{\sqrt{(d_1^{\,T} d_1)(d_2^{\,T} d_2)}} = \frac{x_1^{\,T}(K^{-T}K^{-1})x_2}{\sqrt{(x_1^{\,T}(K^{-T}K^{-1})x_1)(x_2^{\,T}(K^{-T}K^{-1})x_2)}}$$

An image line $\mathbf{l}$ defines a plane through the camera center with normal $\mathbf{n}=\mathbf{K}^T\mathbf{l}$ measured in the camera's Euclidean frame. In fact the backprojection of $\mathbf{l}$ is $\mathbf{P}^T\mathbf{l}$ → $\mathbf{n}=\mathbf{K}^T\mathbf{l}$

The image of the absolute conic Ω_∞

$$\mathrm{x} = \mathrm{P}\mathrm{X}_\infty = \mathrm{KR}[\mathrm{I} \mid -\mathrm{O}]\begin{pmatrix} \mathrm{d} \\ 0 \end{pmatrix} = \mathrm{KRd}$$

mapping between π_∞ to an image is given by the planar homogaphy $\mathrm{x}=\mathrm{Hd}$, with $\mathrm{H}=\mathrm{KR}$

absolute conic (IAC), represented by I_3 within π_∞ $(\mathrm{w}=0)$

its image (IAC) $\quad \omega = \left(\mathrm{KK}^{\mathrm{T}}\right)^{-1} = \mathrm{K}^{-\mathrm{T}}\mathrm{K}^{-1} \qquad \left(\mathrm{C}\ \alpha\ \ \mathrm{H}^{-\mathrm{T}}\mathrm{C}\mathrm{H}^{-1}\right)$

(i) IAC depends only on intrinsics

(ii) angle between two rays $\quad \cos\theta = \dfrac{\mathrm{x}_1^{\mathrm{T}}\omega\mathrm{x}_2}{\sqrt{\left(\mathrm{x}_1^{\mathrm{T}}\omega\mathrm{x}_1\right)\left(\mathrm{x}_2^{\mathrm{T}}\omega\mathrm{x}_2\right)}}$

(iii) DIAC=ω^*=KK$^\mathrm{T}$

(iv) $\omega \Leftrightarrow \mathrm{K}$ (Cholesky factorization)

(v) image of circular points belong to ω (image of absolute conic)

A simple calibration device

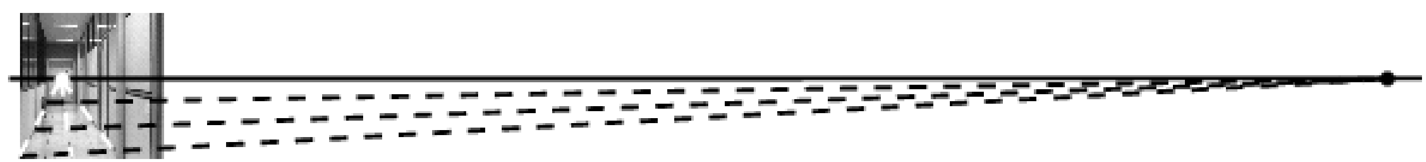
(i) compute $\mathbf{H_i}$ for each square
(corners ✂ (0,0),(1,0),(0,1),(1,1))

(ii) compute the imaged circular points $\mathbf{H_i}\ [1,\pm i,0]^T$

(iii) fit a conic ω to 6 imaged circular points

(iv) compute $\mathbf{K}$ from $\omega=\mathbf{K}^{-T}\ \mathbf{K}^{-1}$ through Cholesky factorization

(= Zhang's calibration method)



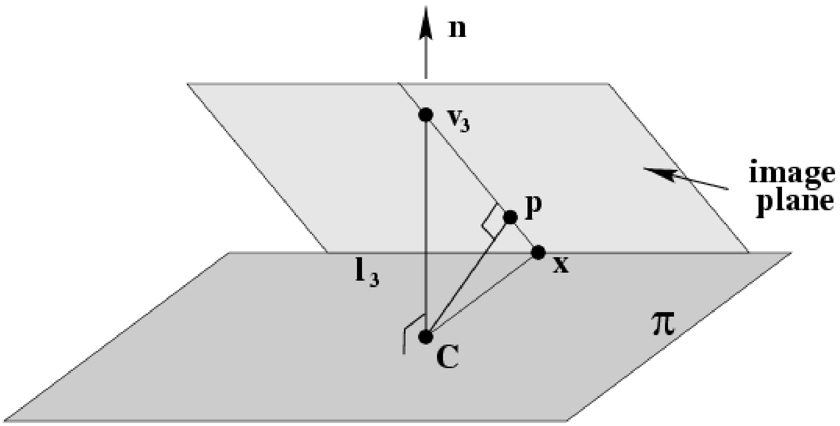
Orthogonality relation

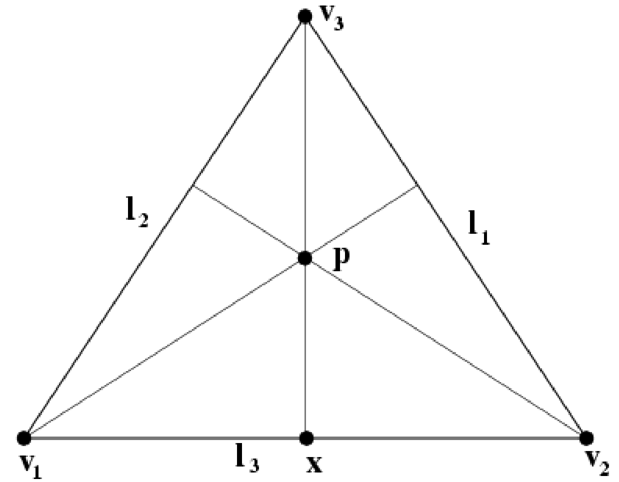
$$\cos\theta = \frac{\mathrm{v}_1{}^{\mathrm{T}}\omega\mathrm{v}_2}{\sqrt{\left(\mathrm{v}_1{}^{\mathrm{T}}\omega\mathrm{v}_1\right)\left(\mathrm{v}_2{}^{\mathrm{T}}\omega\mathrm{v}_2\right)}}$$

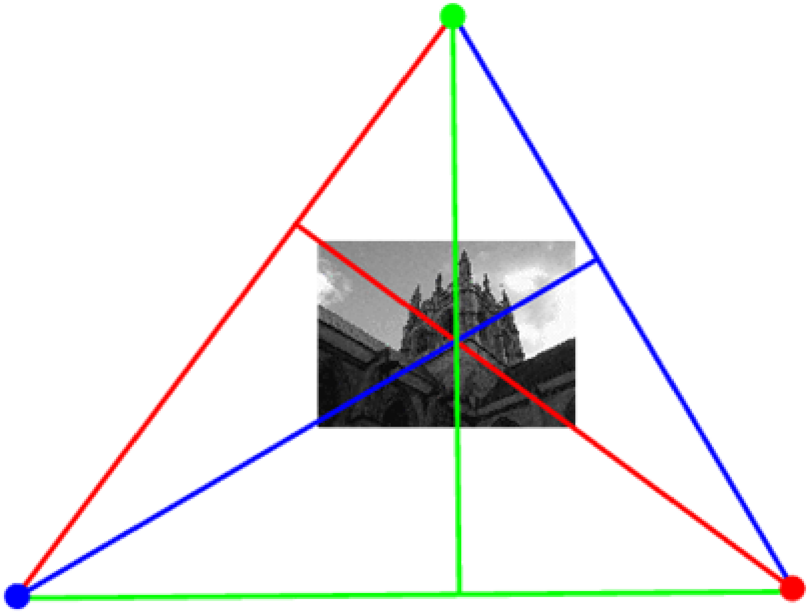
$$\mathrm{v}_1{}^{\mathrm{T}}\omega\mathrm{v}_2 = 0$$

$$\mathrm{l}_1{}^{\mathrm{T}}\omega^{*}\mathrm{l}_2 = 0$$

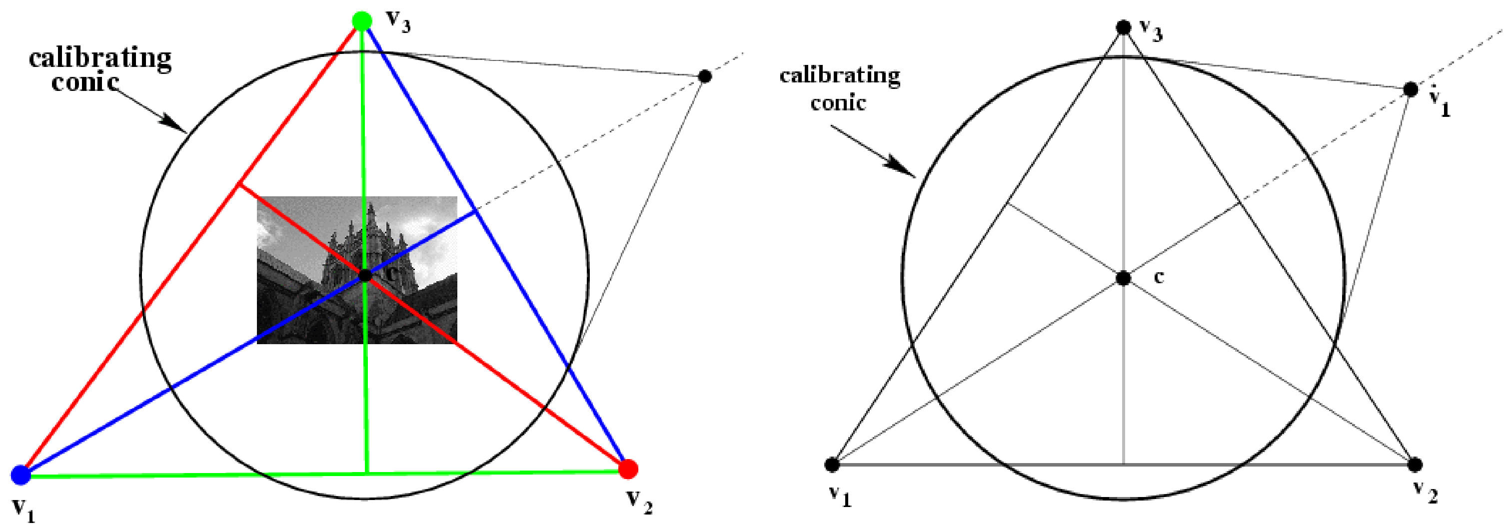
Calibration from vanishing points and lines







Calibration from vanishing points and lines



Two-view geometry

Epipolar geometry

F-matrix comp.

3D reconstruction

Structure comp.

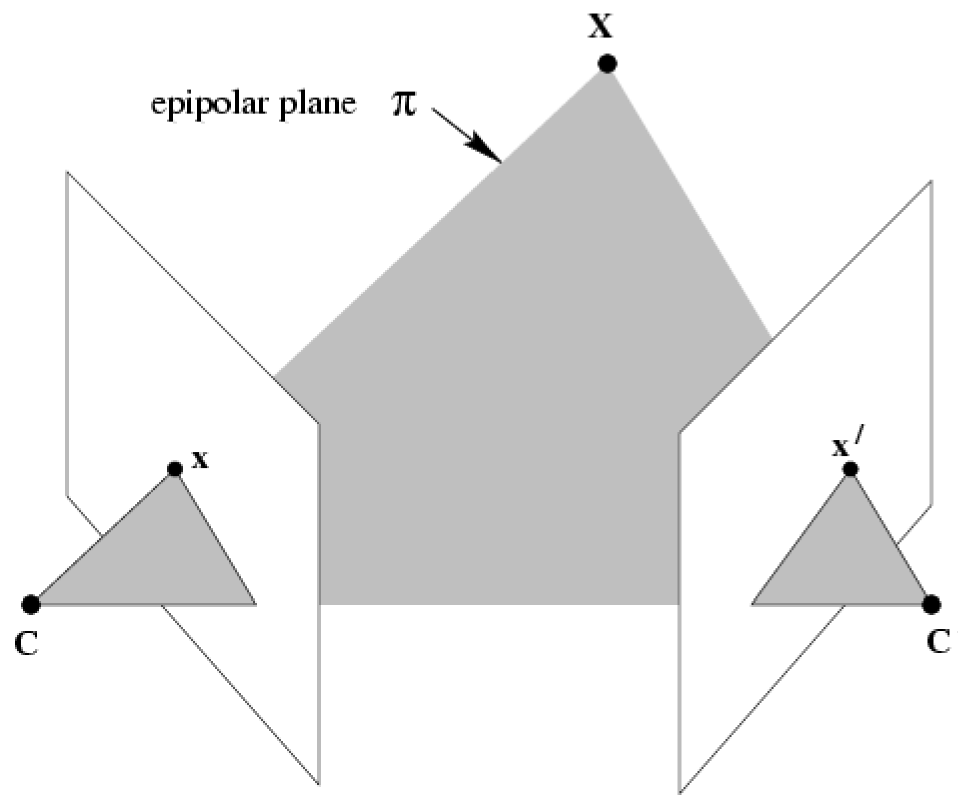
Three questions:

(i) **Correspondence geometry:** Given an image point $\mathbf{x}$ in the first view, how does this constrain the position of the corresponding point $\mathbf{x}'$ in the second image?

(ii) **Camera geometry (motion):** Given a set of corresponding image points $\{x_i \leftrightarrow x'_i\}$, $i=1,\ldots,n$, what are the cameras P and P' for the two views?

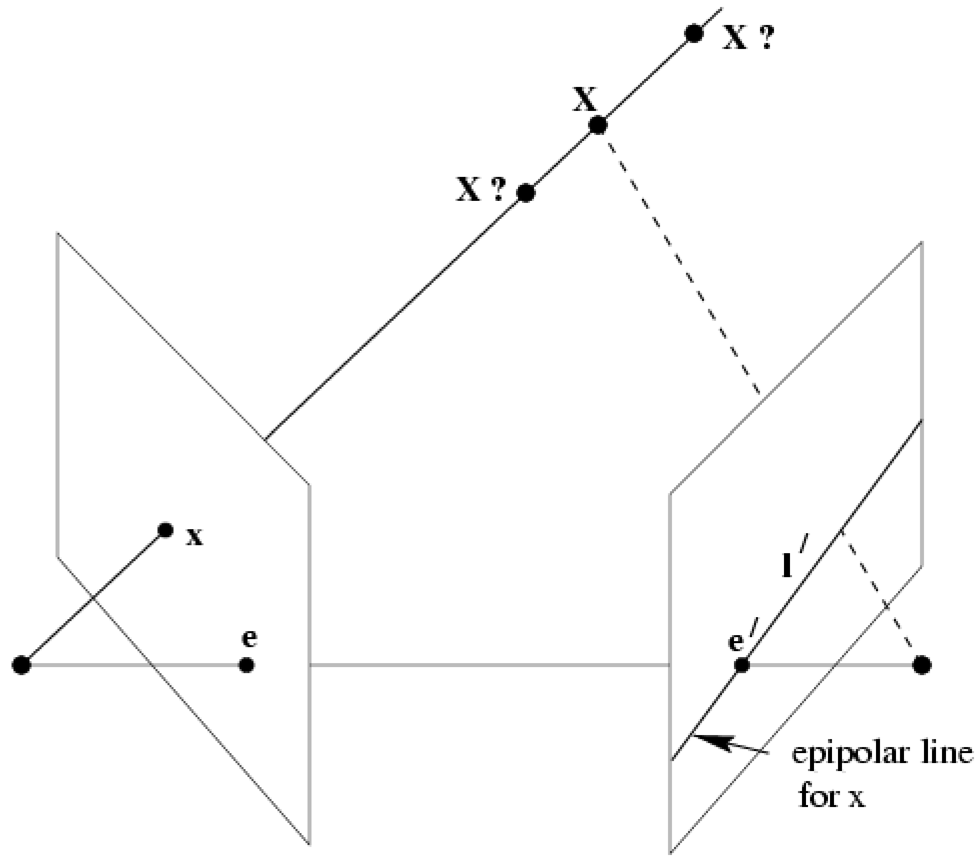
(iii) **Scene geometry (structure):** Given corresponding image points $x_i \leftrightarrow x'_i$ and cameras P, P', what is the position of (their pre-image) X in space?

The epipolar geometry



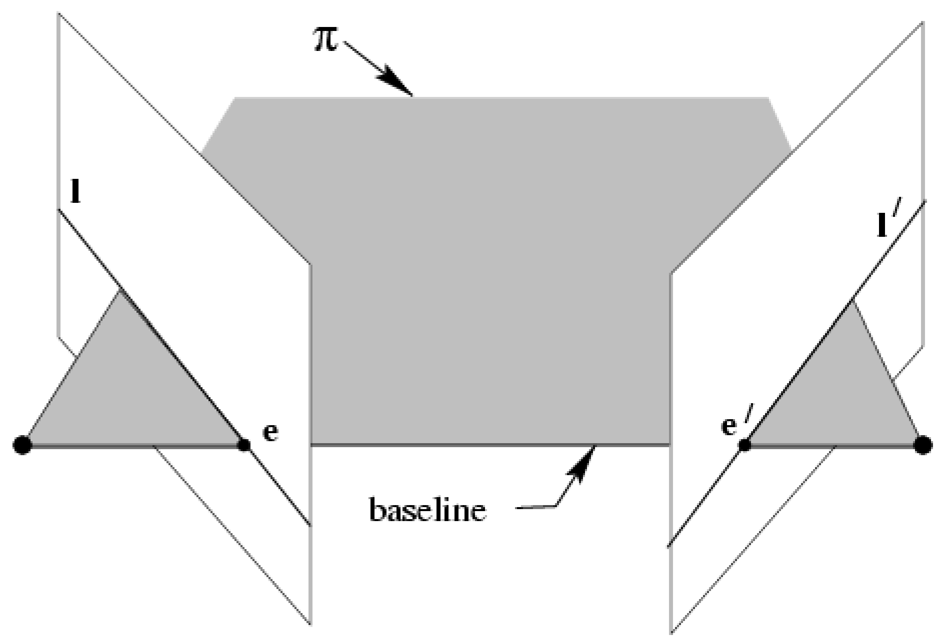
C,C',x,x' and X are coplanar

The epipolar geometry



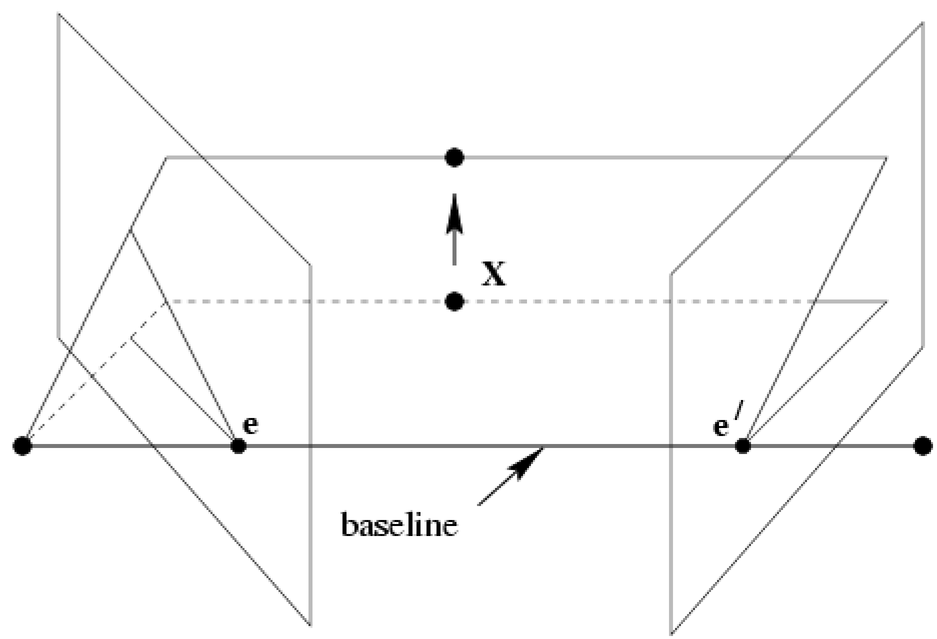
What if only C,C’,x are known?

The epipolar geometry



All points on π project on l and l’

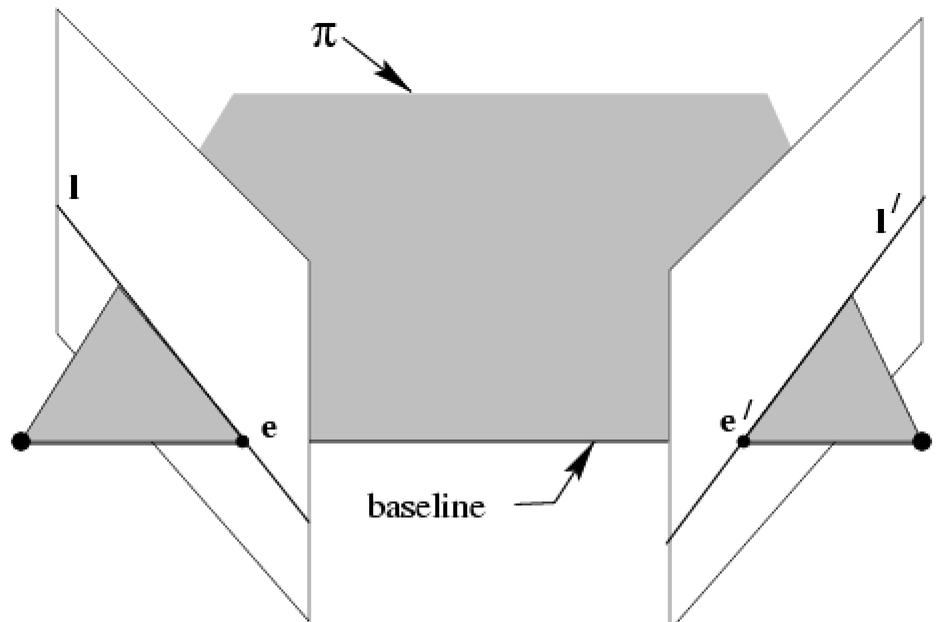
The epipolar geometry



Family of planes π and lines l and l'
Intersection in e and e'

The epipolar geometry

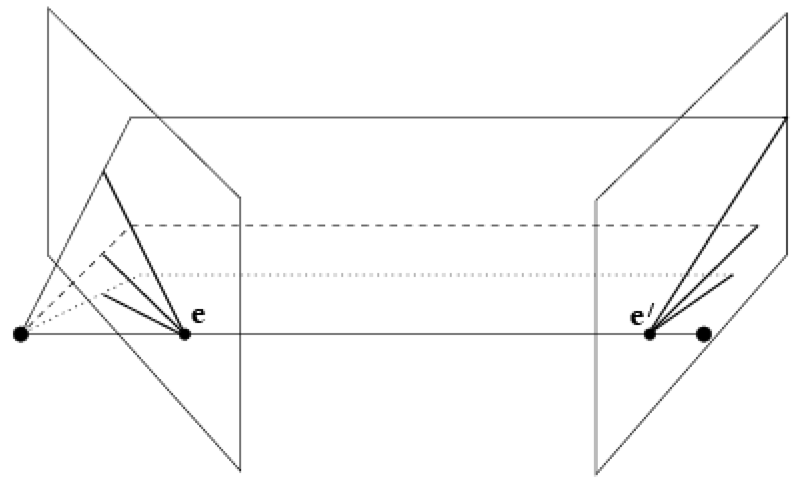
epipoles e,e'
= intersection of baseline with image plane
= projection of projection center in other image
= vanishing point of camera motion direction

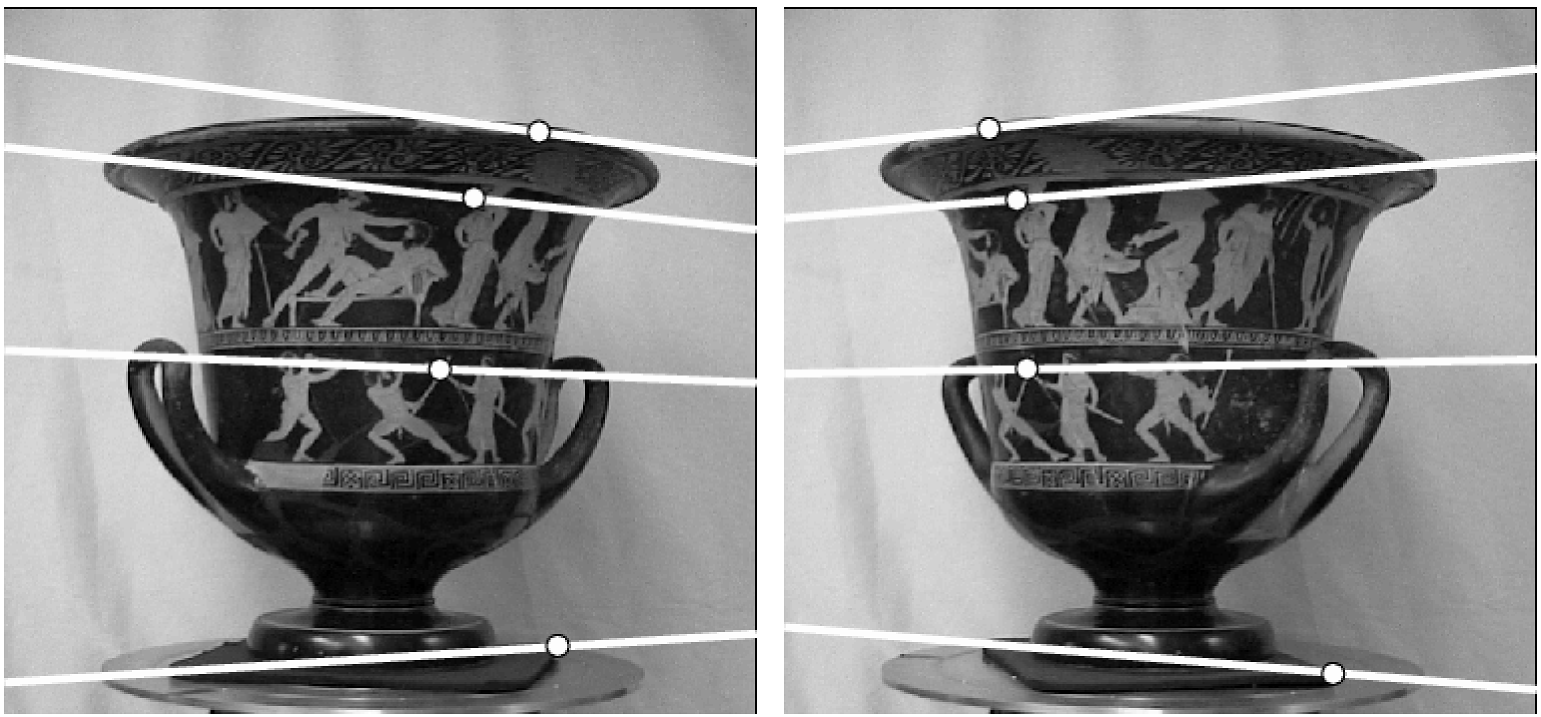


an epipolar plane = plane containing baseline (1-D family)

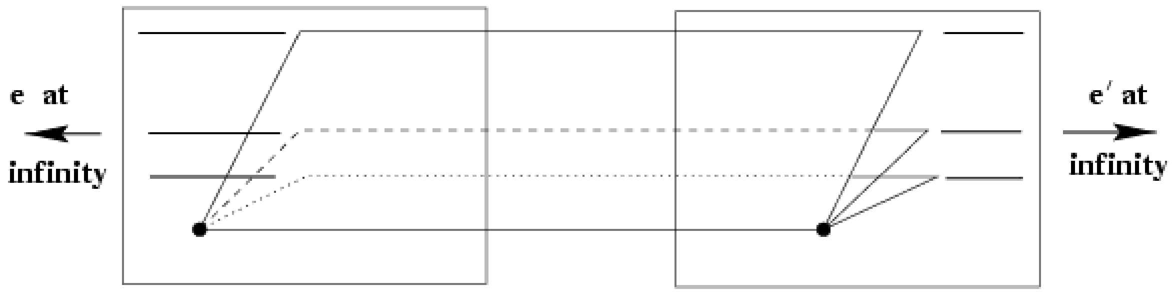
an epipolar line = intersection of epipolar plane with image
(always come in corresponding pairs)

Example: converging cameras

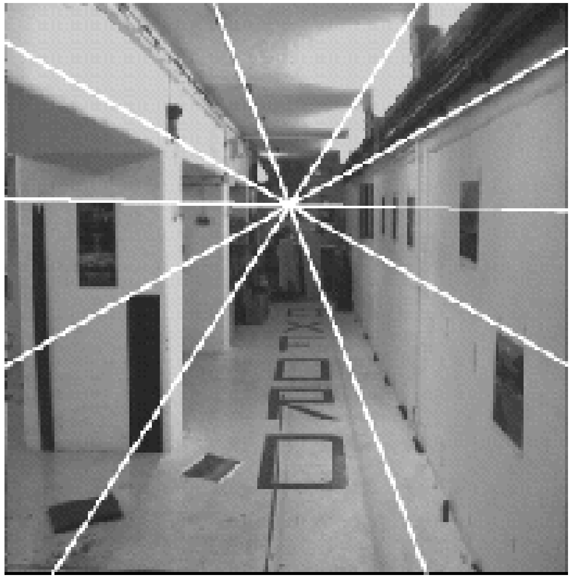


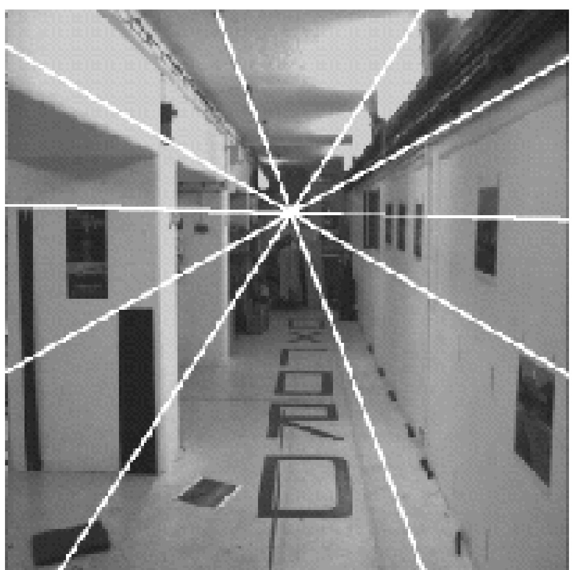


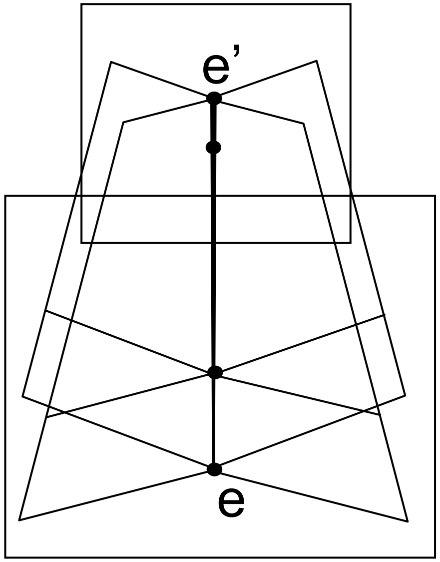
Example: motion parallel with image plane



Example: forward motion







The fundamental matrix F

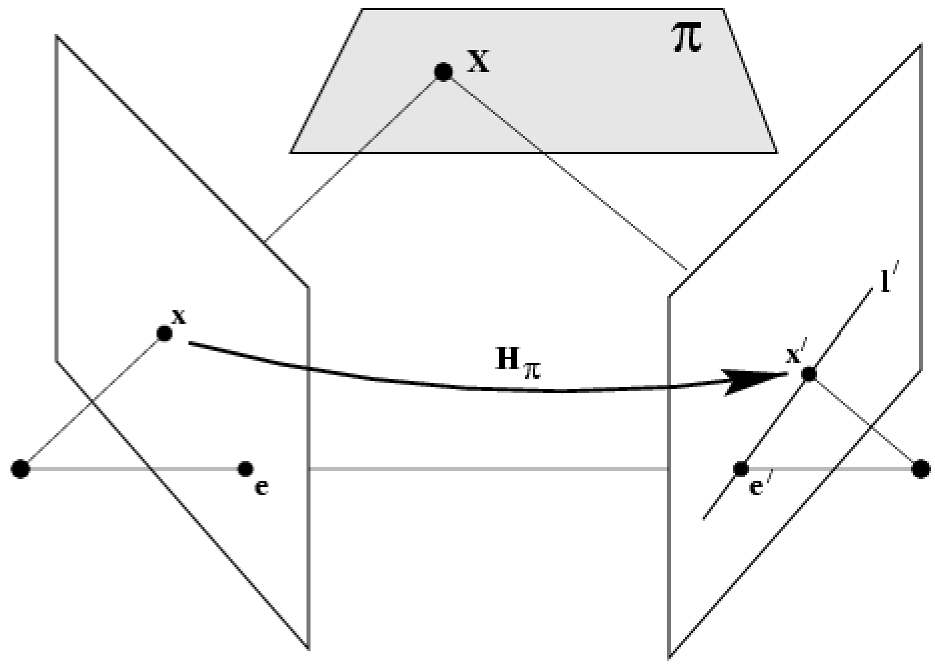
algebraic representation of epipolar geometry

$$x \alpha \quad l'$$

we will see that mapping is (singular) correlation
(i.e. projective mapping from points to lines)
represented by the fundamental matrix F

The fundamental matrix F

geometric derivation



$$x' = H_\pi x$$

$$l' = e' \times x' = [e']_\times H_\pi x = Fx$$

mapping from 2-D to 1-D family (rank 2)

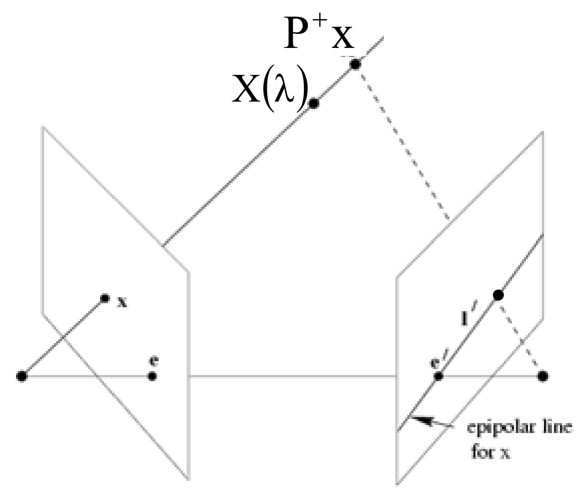
The fundamental matrix F

algebraic derivation

$$X(\lambda) = P^{+}x + \lambda C \qquad \left(P^{+}P = I\right)$$

$$l = P'C \times P'P^{+}x$$

$$F = [e']_{\times} P'P^{+}$$



(note: doesn’t work for C=C’ $\Rightarrow$ F=0)

The fundamental matrix F

correspondence condition

The fundamental matrix satisfies the condition that for any pair of corresponding points $x \leftrightarrow x'$ in the two images

$$x'^T F x = 0 \qquad \left(x'^T l' = 0\right)$$

The fundamental matrix F

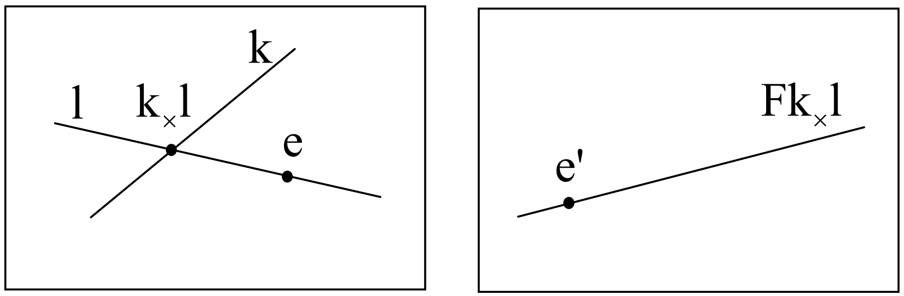
F is the unique 3x3 rank 2 matrix that satisfies $x'^T F x=0$ for all $x \leftrightarrow x'$

(i) **Transpose:** if F is fundamental matrix for (P,P'), then F^T is fundamental matrix for (P',P)
(ii) **Epipolar lines:** $l'=Fx$ & $l=F^T x'$
(iii) **Epipoles:** on all epipolar lines, thus $e'^T F x=0, \forall x$ $\Rightarrow e'^T F=0$, similarly $Fe=0$
(iv) **F** has 7 d.o.f. , i.e. 3x3-1(homogeneous)-1(rank2)
(v) **F** is a correlation, projective mapping from a point x to a line $l'=Fx$ (not a proper correlation, i.e. not invertible)

The epipolar line geometry

l,l' epipolar lines, k line not through e

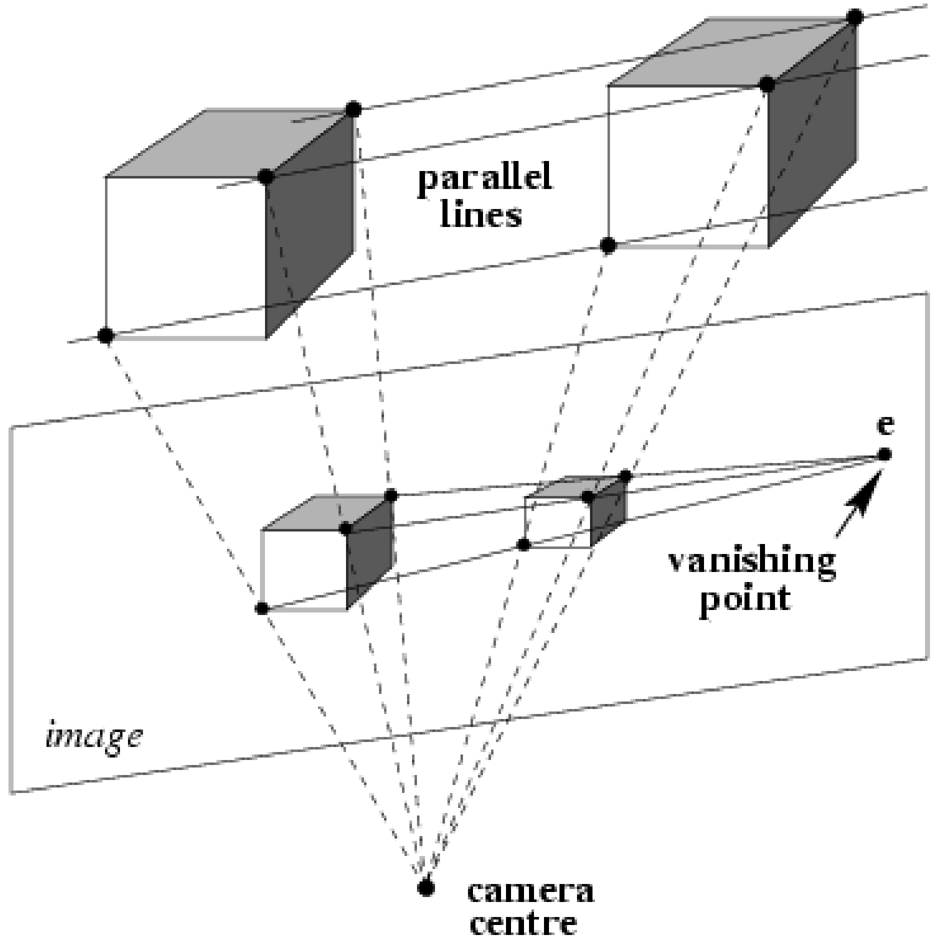
$\Rightarrow$ $l' = F[k]_x l$ and symmetrically $l = F^T[k']_x l'$



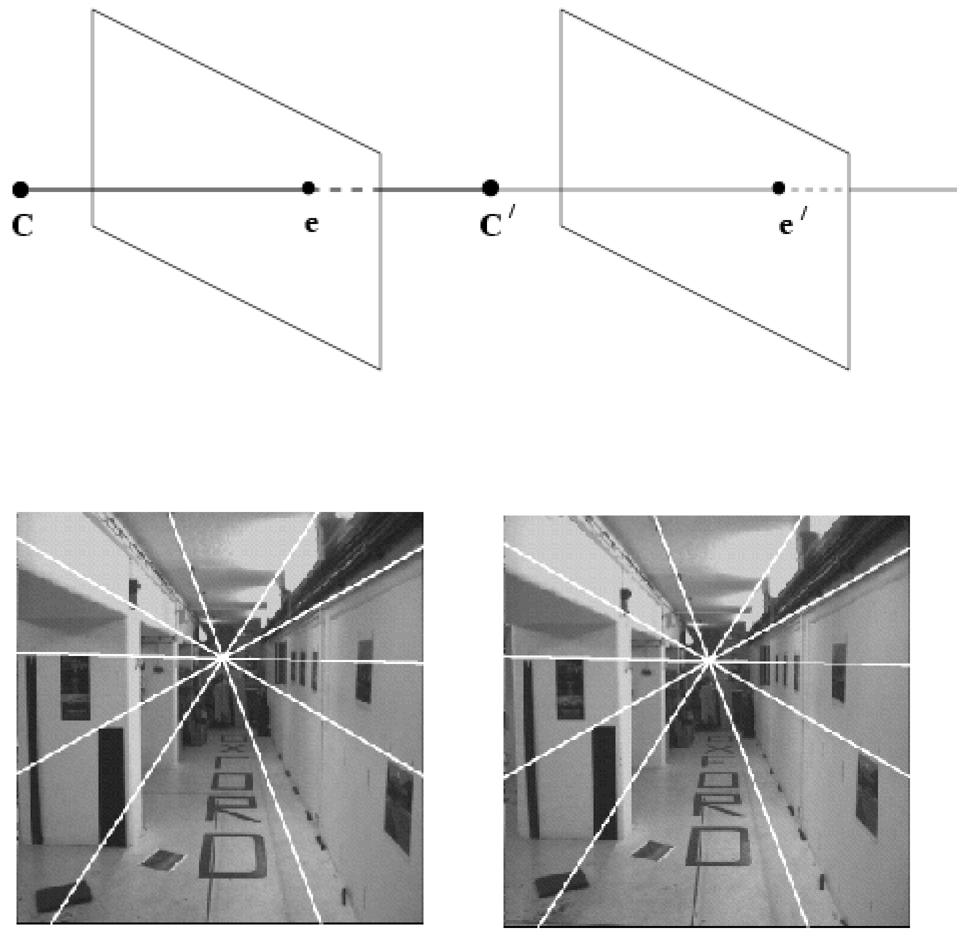
(pick k=e, since $e^T e \neq 0$)

$$l' = F[e]_\times l \qquad l = F^T[e']_\times l'$$

Fundamental matrix for pure translation



Fundamental matrix for pure translation



Fundamental matrix for pure translation

$$\mathrm{F} = [\mathrm{e}']_\times \mathrm{H}_\infty = [\mathrm{e}']_\times \qquad \left(\mathrm{H}_\infty = \mathrm{K}^{-1}\mathrm{RK}\right)$$

<u>example</u>:

$$\mathrm{e}' = (1,0,0)^\mathsf{T} \qquad \mathrm{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}_\times$$

$$\mathrm{x}'^\mathrm{T}\,\mathrm{Fx} = 0 \Leftrightarrow y = y'$$

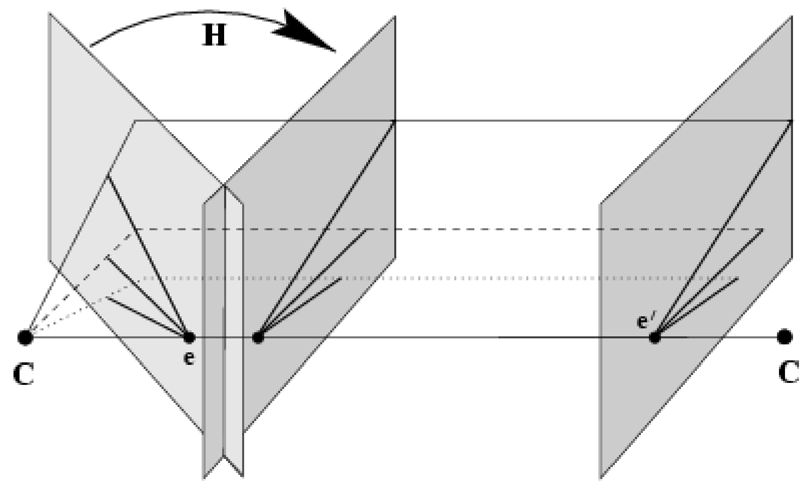
$$\mathrm{x} = \mathrm{PX} = \mathrm{K}[\mathrm{I}\,|\,0]\mathrm{X} \qquad (X,Y,Z)^\mathsf{T} = \mathrm{K}^{-1}\mathrm{x}/Z$$

$$\mathrm{x}' = \mathrm{P}'\mathrm{X} = \mathrm{K}[\mathrm{I}\,|\,\mathrm{t}]\begin{bmatrix} \mathrm{K}^{-1}\mathrm{x} \\ Z \end{bmatrix} \qquad \mathrm{x}' = \mathrm{x} + \mathrm{Kt}/Z$$

motion starts at x and moves towards e, faster depending on Z

pure translation: F only 2 d.o.f., $\mathrm{x}^\mathsf{T}[\mathrm{e}]_\mathrm{x}\mathrm{x}=0 \Rightarrow$ auto-epipolar

General motion



$$x'^{T} [e']_{\times} Hx = 0$$

$$x'^{T} [e']_{\times} \hat{x} = 0$$

$$x' = K'RK^{-1}x + K't/Z$$

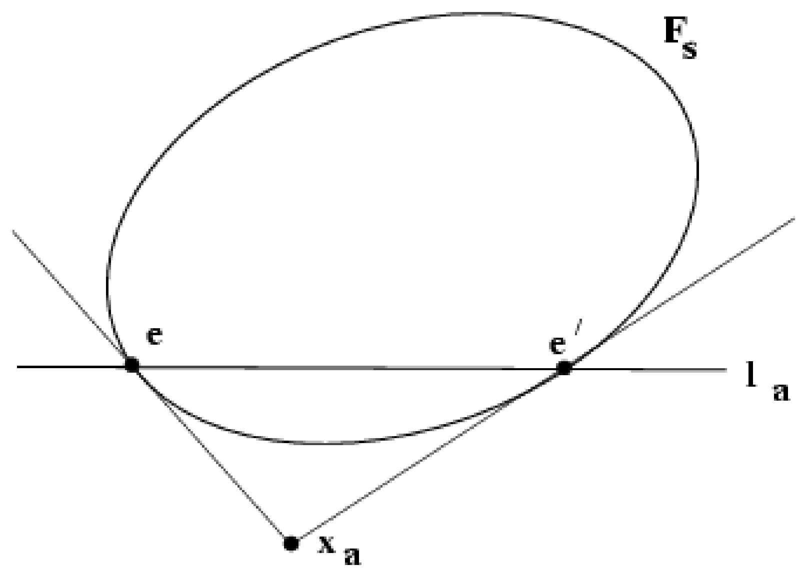
Geometric representation of F

$$F_S = \left(F + F^T\right)/2 \qquad F_A = \left(F - F^T\right)/2 \qquad \left(F = F_S + F_A\right)$$

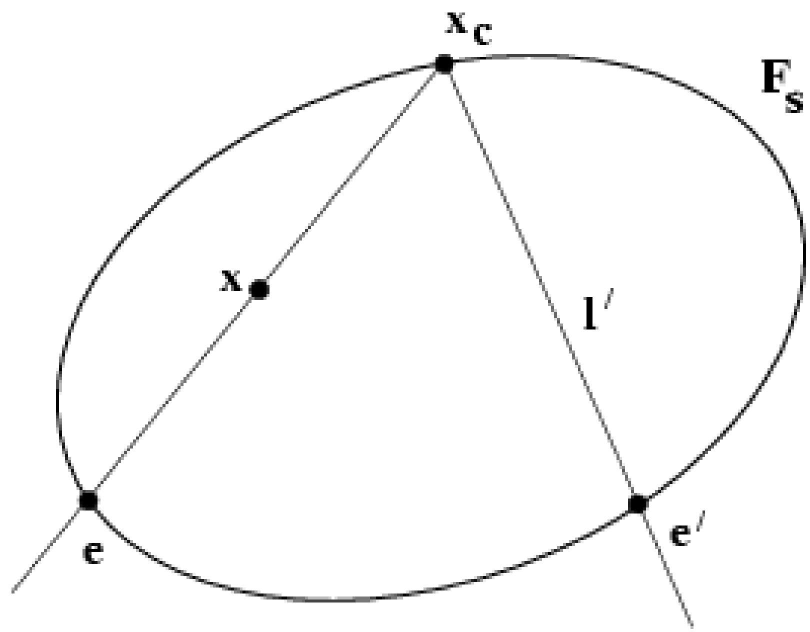
$$x \leftrightarrow x \qquad x^T F x = 0 \qquad \left(x^T F_A x \equiv 0\right)$$

$$x^T F_S x = 0$$

F_s: Steiner conic, 5 d.o.f.
F_a=$[x_a]_x$: pole of line ee' w.r.t. F_s, 2 d.o.f.

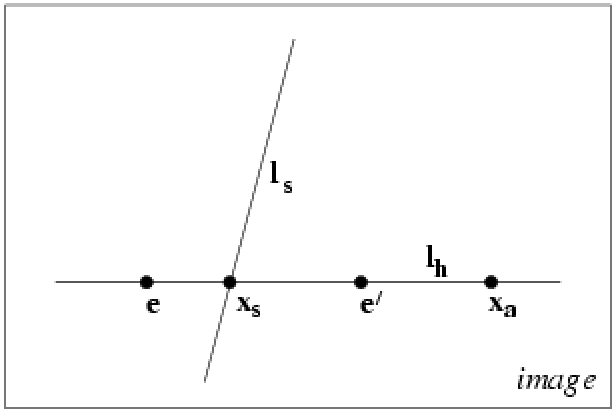


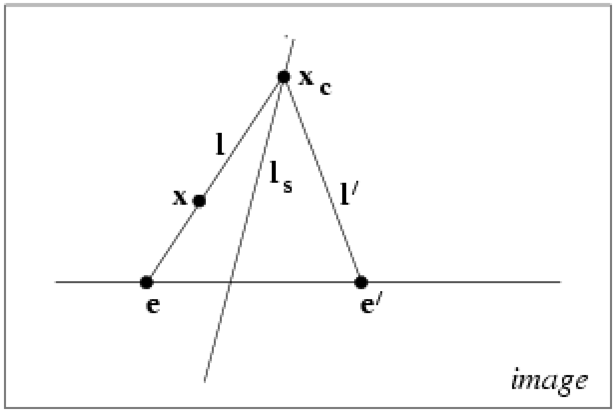
Geometric representation of F



Pure planar motion

Steiner conic F_s is degenerate (two lines)





Projective transformation and invariance

Derivation based purely on projective concepts

$$\hat{x} = Hx,\ \hat{x}' = H'x' \Rightarrow \hat{F} = H'^{-T} F H^{-1}$$

F invariant to transformations of projective 3-space

$$x = PX = (PH)(H^{-1}X) = \hat{P}\hat{X}$$
$$x' = P'X = (P'H)(H^{-1}X) = \hat{P}'\hat{X}$$

$(P, P')\ \alpha\ \ F$ unique

$F\ \alpha\ \ (P, P')$ not unique

canonical form

$$P = [I \mid 0] \qquad F = [m]_{\times} M$$
$$P' = [M \mid m]$$

Projective ambiguity of cameras given F

previous slide: at least projective ambiguity
this slide: not more!

Show that if F is same for (P,P') and ($\tilde{P}$,$\tilde{P}$'),
there exists a projective transformation H so that
$\tilde{P}$=HP and $\tilde{P}$'=HP'

$$\mathrm{P} = [\mathrm{I} \mid 0] \quad \mathrm{P}' = [\mathrm{A} \mid \mathrm{a}] \quad \tilde{\mathrm{P}} = [\mathrm{I} \mid 0] \quad \tilde{\mathrm{P}}' = [\tilde{\mathrm{A}} \mid \tilde{\mathrm{a}}]$$

$$\mathrm{F} = [\mathrm{a}]_\times \mathrm{A} = [\tilde{\mathrm{a}}]_\times \tilde{\mathrm{A}}$$

<u>lemma:</u> $\tilde{\mathrm{a}} = \mathrm{ka} \quad \tilde{\mathrm{A}} = k^{-1}\left(\mathrm{A} + \mathrm{av}^\mathrm{T}\right)$

$$\mathrm{aF} = \mathrm{a}[\mathrm{a}]_\times \mathrm{A} = 0 = \tilde{\mathrm{a}}\mathrm{F} \overset{\text{rank 2}}{\Longrightarrow} \tilde{\mathrm{a}} = \mathrm{ka}$$

$$[\mathrm{a}]_\times \mathrm{A} = [\tilde{\mathrm{a}}]_\times \tilde{\mathrm{A}} \Rightarrow [\mathrm{a}]_\times \left(\mathrm{k}\tilde{\mathrm{A}} - \mathrm{A}\right) = 0 \Rightarrow \left(\mathrm{k}\tilde{\mathrm{A}} - \mathrm{A}\right) = \mathrm{av}^\mathrm{T}$$

$$H = \begin{bmatrix} k^{-1}I & 0 \\ k^{-1}\mathrm{v}^\mathrm{T} & k \end{bmatrix}$$

$$\mathrm{P}'\mathrm{H} = [\mathrm{A} \mid \mathrm{a}] \begin{bmatrix} k^{-1}I & 0 \\ k^{-1}\mathrm{v}^\mathrm{T} & k \end{bmatrix} = [k^{-1}\left(\mathrm{A} - \mathrm{av}^\mathrm{T}\right) \mid k\mathrm{a}] = \tilde{\mathrm{P}}'$$

(22-15=7, ok)

Canonical cameras given F

F matrix corresponds to P,P' iff $P'^T F P$ is skew-symmetric

$$\left(X^T P'^T F P X = 0, \forall X\right)$$

F matrix, S skew-symmetric matrix

$$P = [I \mid 0] \quad P' = [SF \mid e'] \quad \text{(fund.matrix=F)}$$

$$\left([SF \mid e']^T F [I \mid 0] = \begin{bmatrix} F^T S^T F & 0 \\ e'^T F & 0 \end{bmatrix} = \begin{bmatrix} F^T S^T F & 0 \\ 0 & 0 \end{bmatrix} \right)$$

Possible choice:

$$P = [I \mid 0] \quad P' = [[e']_\times F \mid e']$$

Canonical representation:

$$P = [I \mid 0] \quad P' = [[e']_\times F + e' v^T \mid \lambda e']$$

The essential matrix

~fundamental matrix for calibrated cameras (remove K)

$$E = [t]_\times R = R[R^T t]_\times$$

$$\hat{x}'^T E \hat{x} = 0 \qquad \left(\hat{x} = K^{-1}x;\ \hat{x}' = K^{-1}x'\right)$$

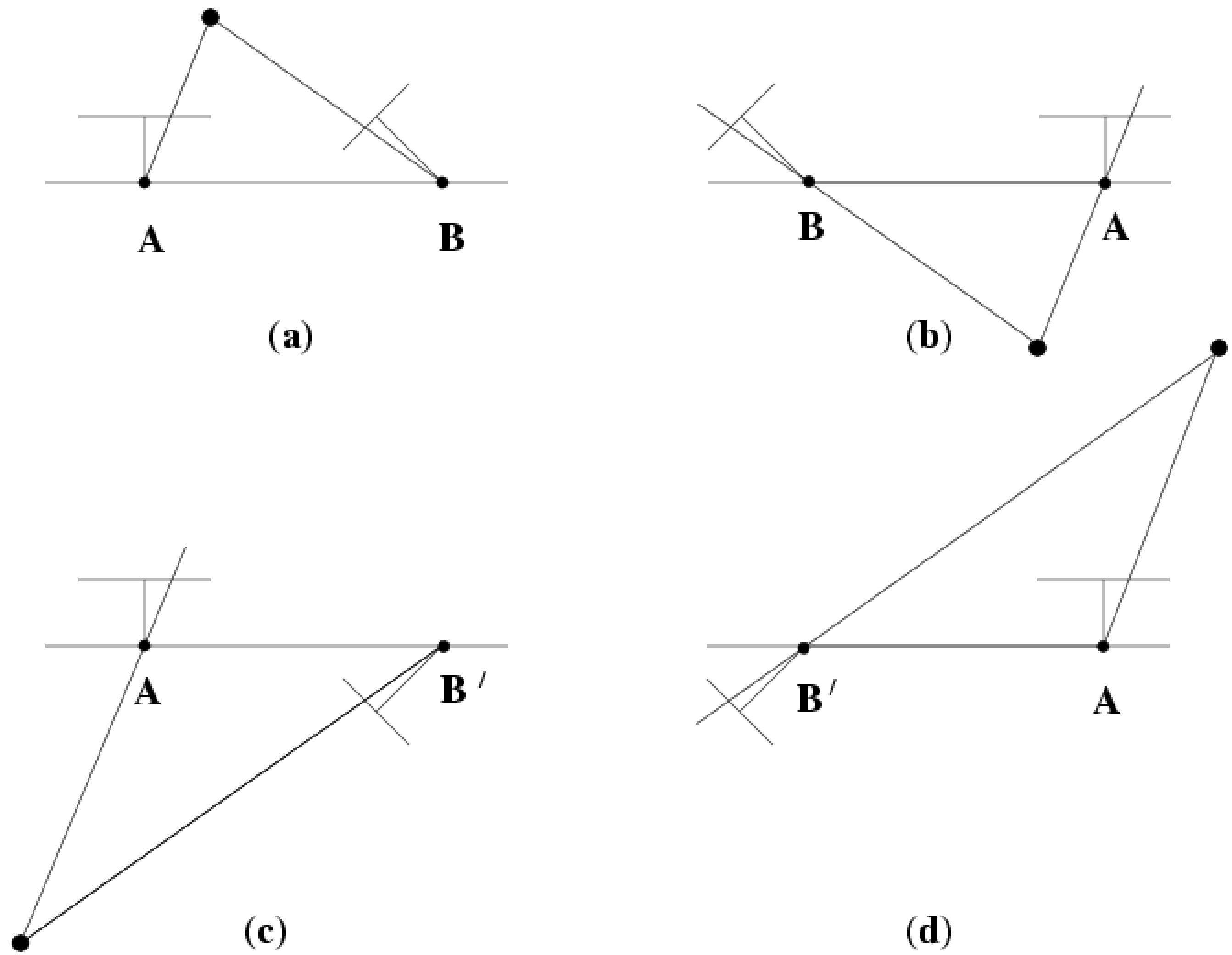
$$E = K'^T F K$$

5 d.o.f. (3 for R; 2 for t up to scale)

E is essential matrix if and only if
two singularvalues are equal (and third=0)

$$E = U\mathrm{diag}(1,1,0)V^T =$$

Four possible reconstructions from E



(only one solution where points is in front of both cameras)

Two-view geometry

- Epipolar geometry
- F-matrix comp.
- 3D reconstruction
- Structure comp.

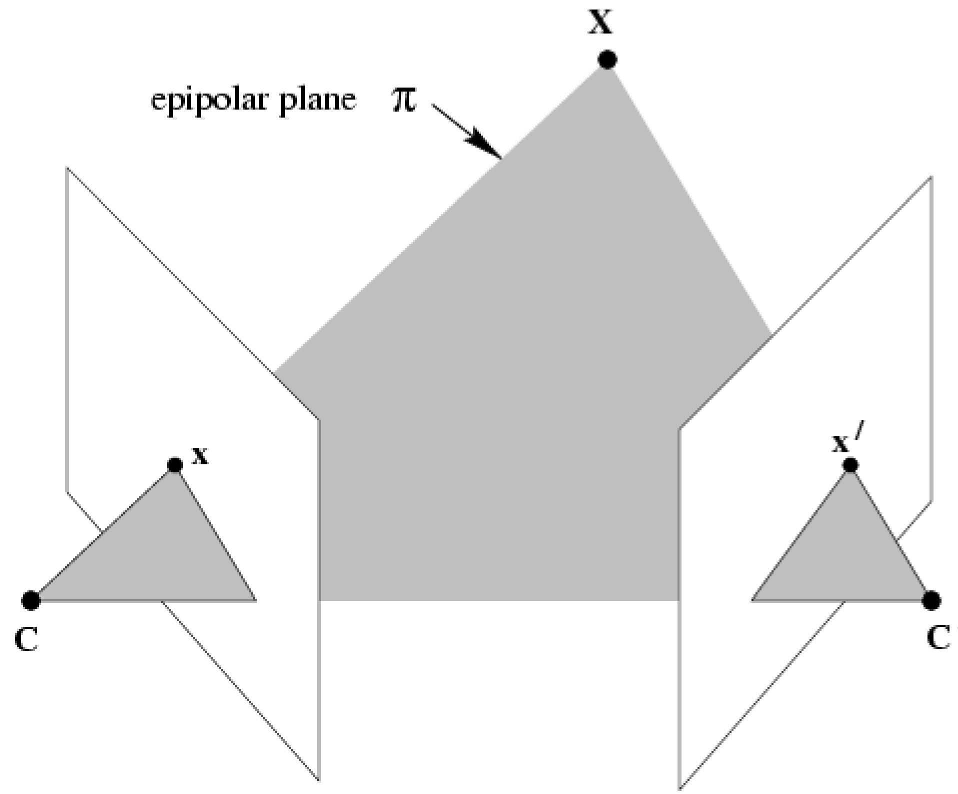
Three questions:

(i) **Correspondence geometry:** Given an image point x in the first view, how does this constrain the position of the corresponding point x' in the second image?

(ii) **Camera geometry (motion):** Given a set of corresponding image points $\{x_i \leftrightarrow x'_i\}$, $i=1,\ldots,n$, what are the cameras P and P' for the two views?

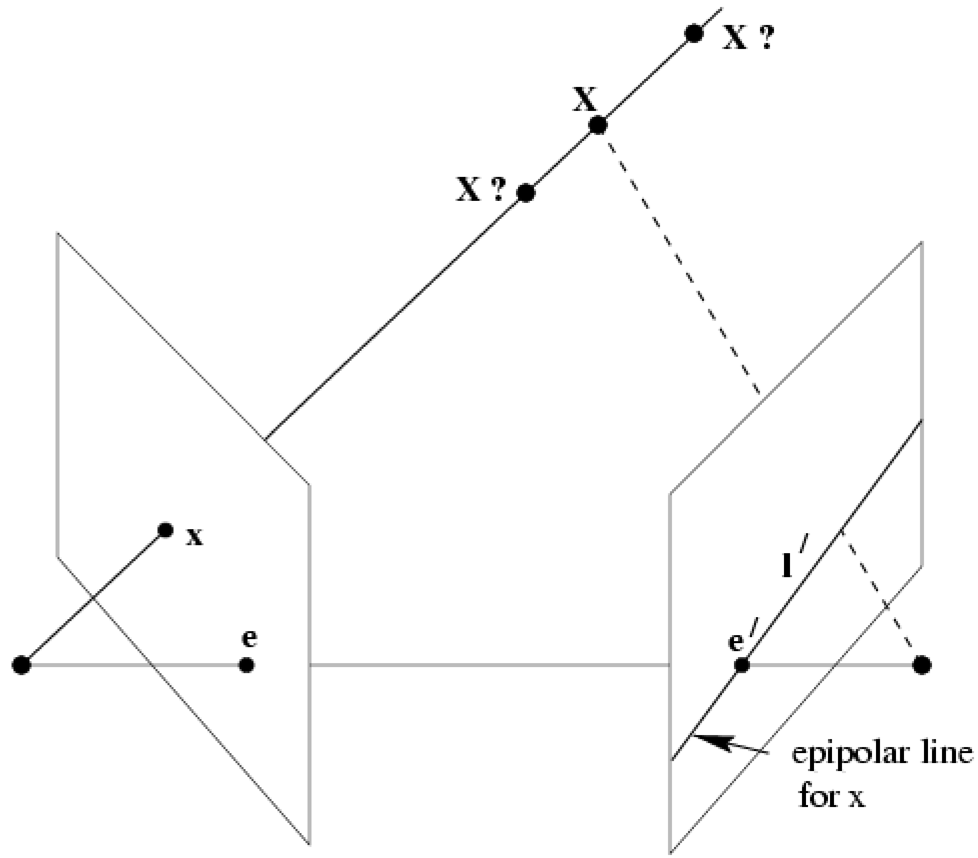
(iii) **Scene geometry (structure):** Given corresponding image points $x_i \leftrightarrow x'_i$ and cameras P, P', what is the position of (their pre-image) X in space?

The epipolar geometry



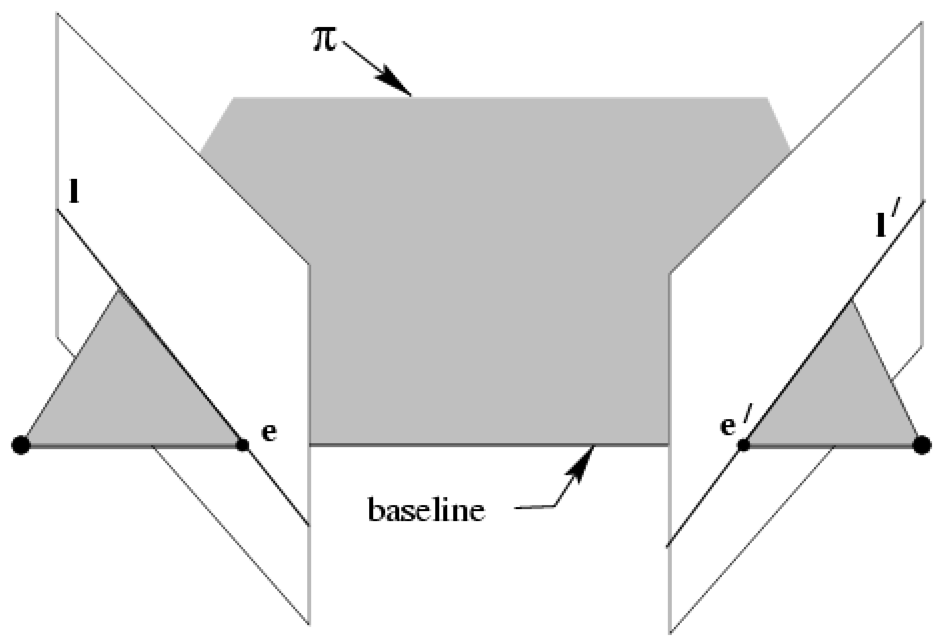
C,C',x,x' and X are coplanar

The epipolar geometry



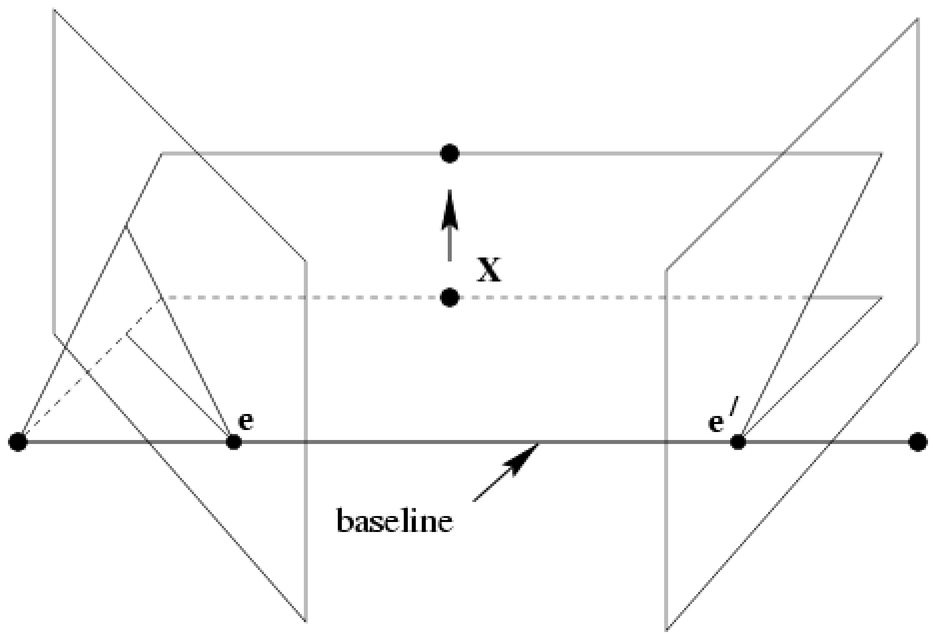
What if only C,C’,x are known?

The epipolar geometry



All points on π project on l and l'

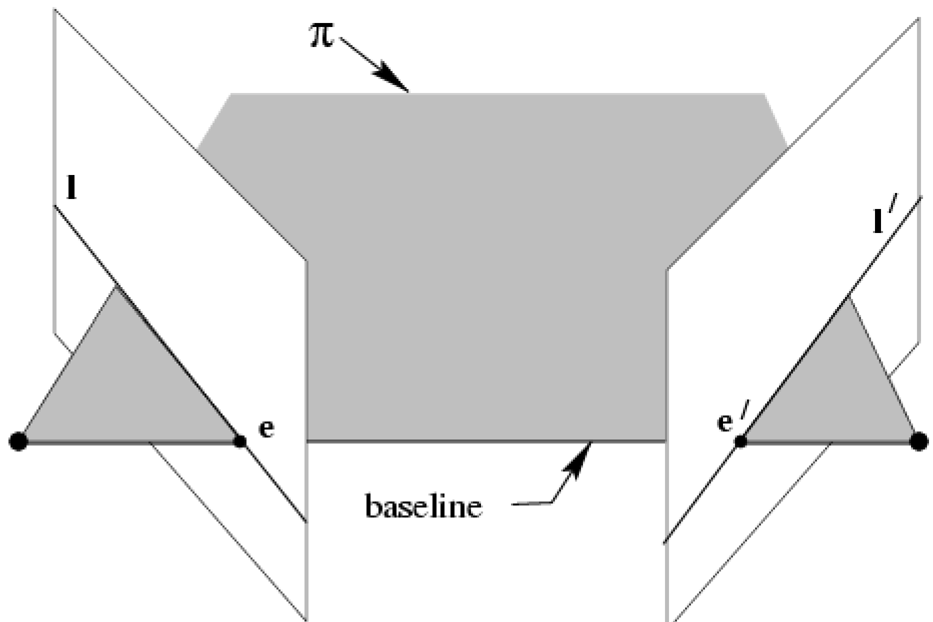
The epipolar geometry



Family of planes π and lines l and l'
Intersection in e and e'

The epipolar geometry

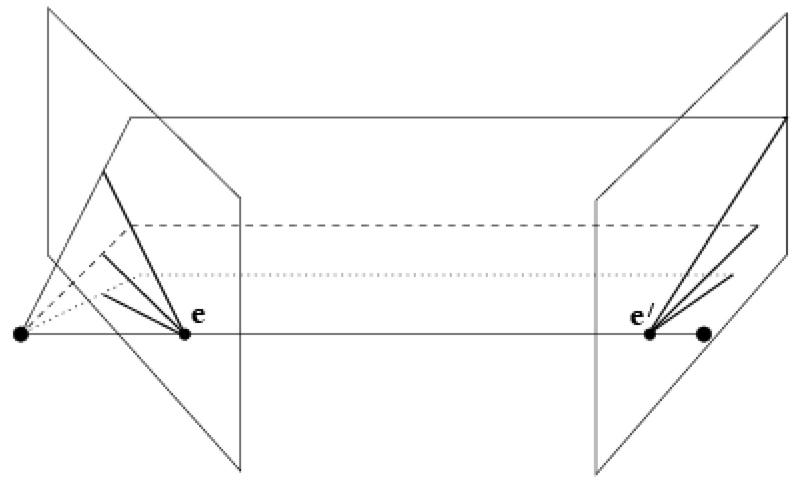
epipoles e,e’
= intersection of baseline with image plane
= projection of projection center in other image
= vanishing point of camera motion direction

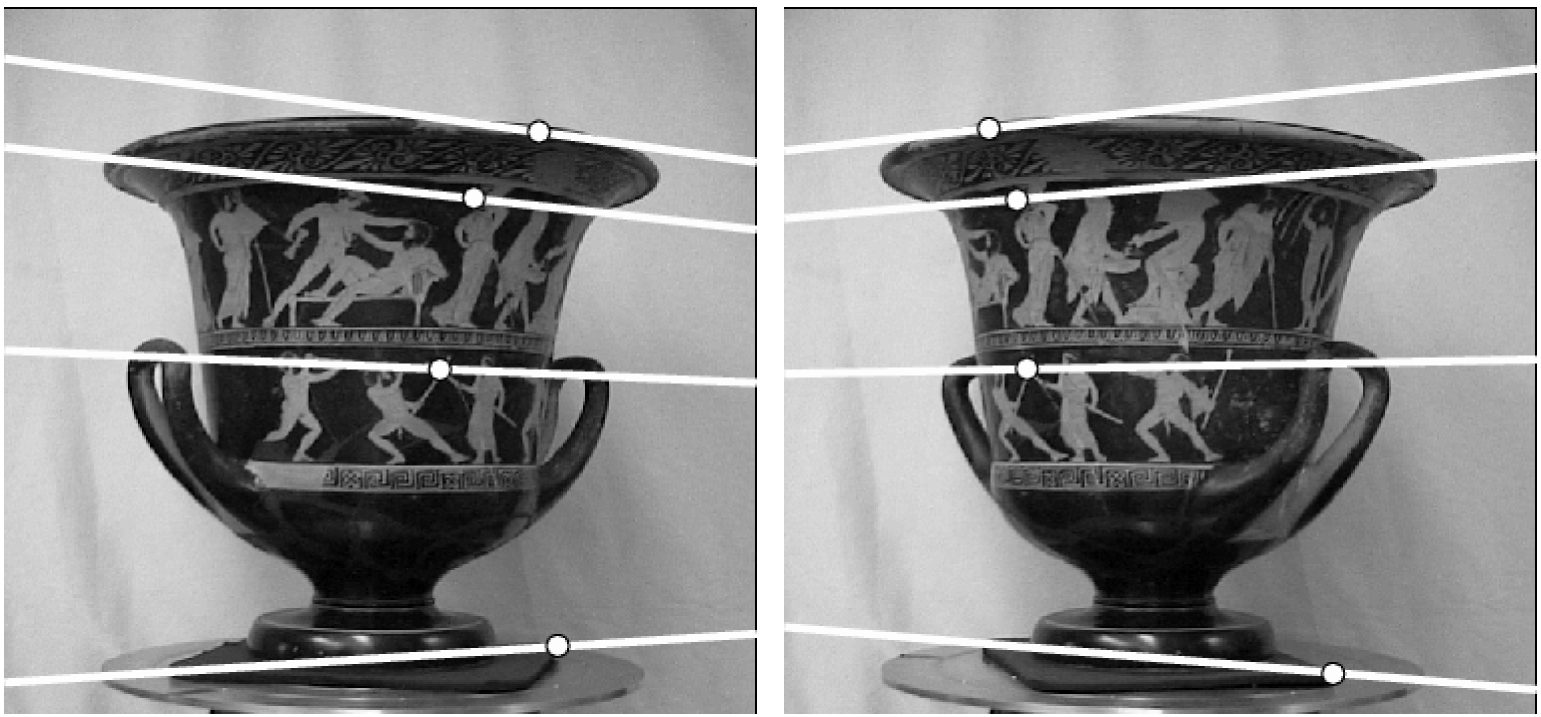


an epipolar plane = plane containing baseline (1-D family)

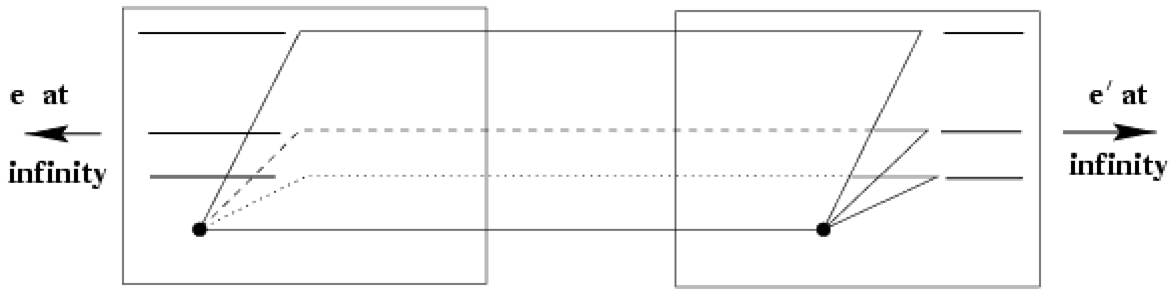
an epipolar line = intersection of epipolar plane with image
(always come in corresponding pairs)

Example: converging cameras

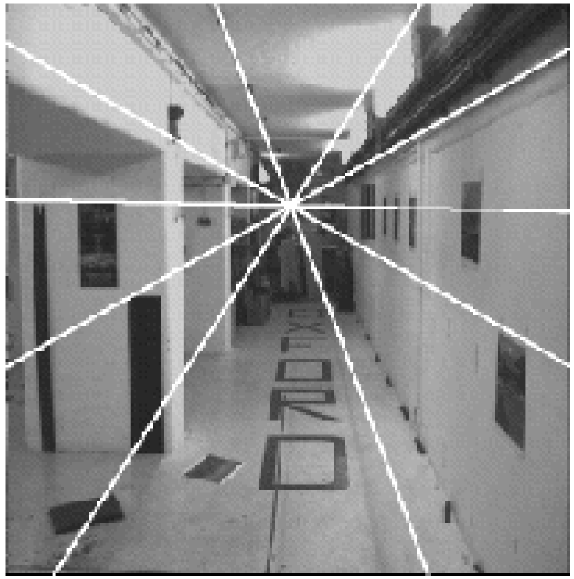


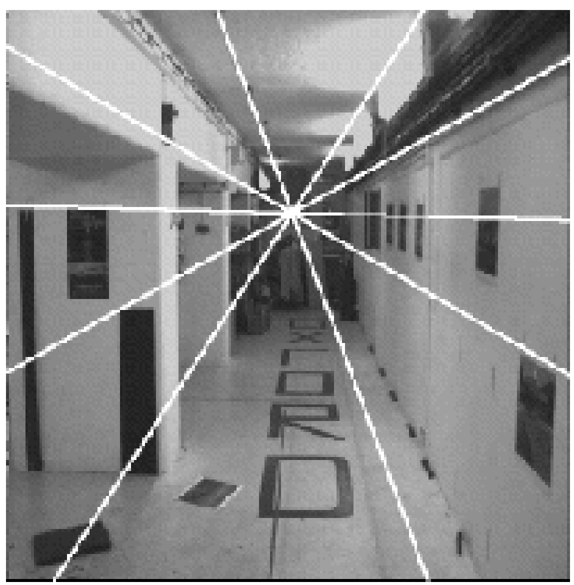


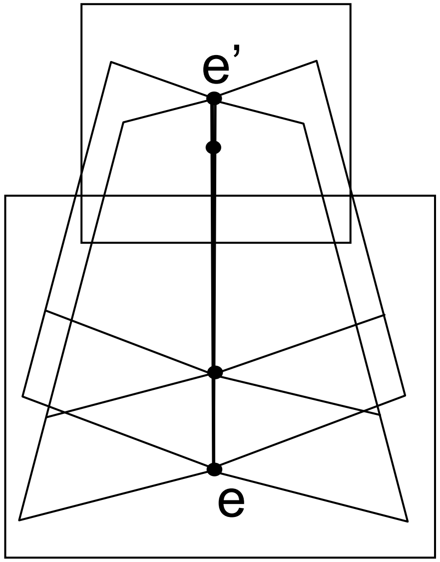
Example: motion parallel with image plane



Example: forward motion







The fundamental matrix F

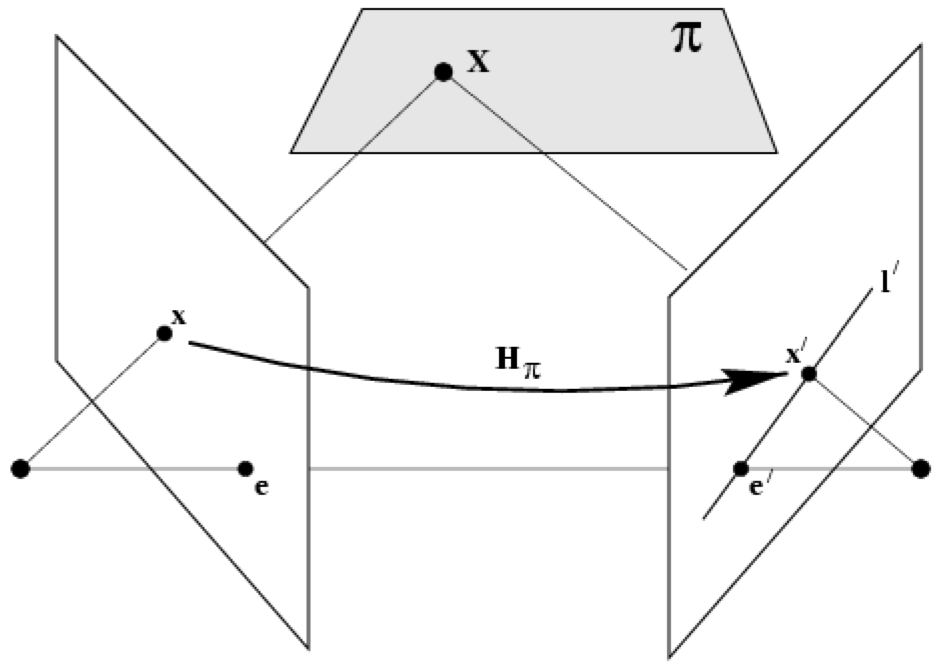
algebraic representation of epipolar geometry

$$x \alpha \quad l'$$

we will see that mapping is (singular) correlation
(i.e. projective mapping from points to lines)
represented by the fundamental matrix F

The fundamental matrix F

geometric derivation



$$x' = H_\pi x$$

$$l' = e' \times x' = [e']_\times H_\pi x = Fx$$

mapping from 2-D to 1-D family (rank 2)

The fundamental matrix F

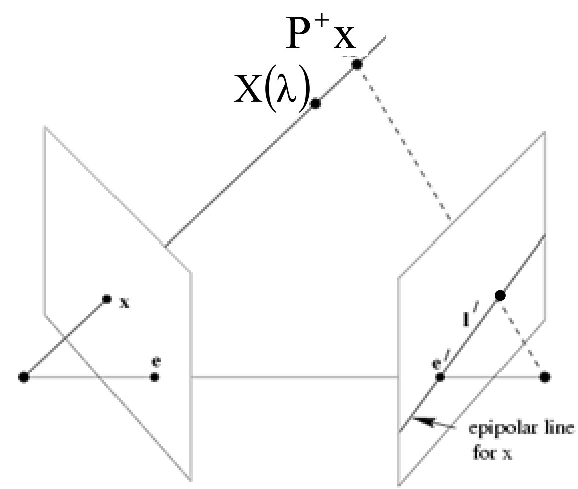
algebraic derivation

$$X(\lambda) = \begin{bmatrix} M^{-1}x \\ 0 \end{bmatrix} + \lambda C$$

$$l' = \begin{bmatrix} M' & m' \end{bmatrix} C \times \begin{bmatrix} M' & m' \end{bmatrix} \begin{bmatrix} M^{-1}x \\ 0 \end{bmatrix} = \begin{bmatrix} M' & m' \end{bmatrix} C \times M' M^{-1} x$$

$$x' \in l' \rightarrow x'^{T} l' = 0 \rightarrow x' F x = 0$$

$$F = [e']_{\times} M' M^{-1}$$



(note: doesn't work for C=C' $\Rightarrow$ F=0)

The fundamental matrix F

correspondence condition

The fundamental matrix satisfies the condition that for any pair of corresponding points $x \leftrightarrow x'$ in the two images

$$x'^{T} F x = 0 \qquad \left(x'^{T} l' = 0 \right)$$

The fundamental matrix F

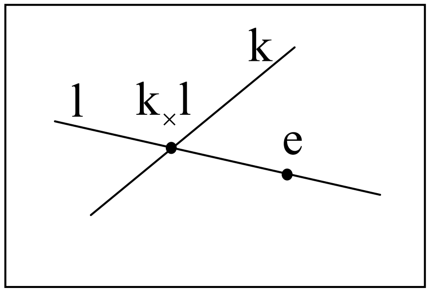
F is the unique 3x3 rank 2 matrix that satisfies $x'^{T}Fx=0$ for all $x \leftrightarrow x'$

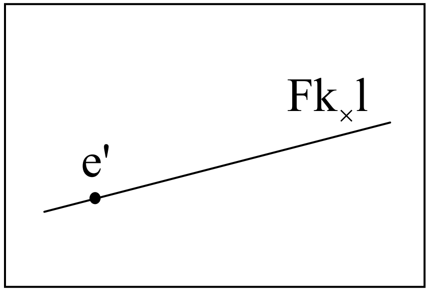
- **(i) Transpose:** if F is fundamental matrix for (P,P'), then F^T is fundamental matrix for (P',P)
- **(ii) Epipolar lines:** $l'=Fx$ & $l=F^T x'$
- **(iii) Epipoles:** on all epipolar lines, thus $e'^{T}Fx=0, \forall x$ $\Rightarrow e'^{T}F=0$, similarly $Fe=0$
- **(iv) F** has 7 d.o.f. , i.e. 3x3-1(homogeneous)-1(rank2)
- **(v) F** is a correlation, projective mapping from a point x to a line $l'=Fx$ (not a proper correlation, i.e. not invertible)

The epipolar line geometry

l,l' epipolar lines, k line not through e

$\Rightarrow$ $l'=F[k]_x l$ and symmetrically $l=F^T[k']_x l'$

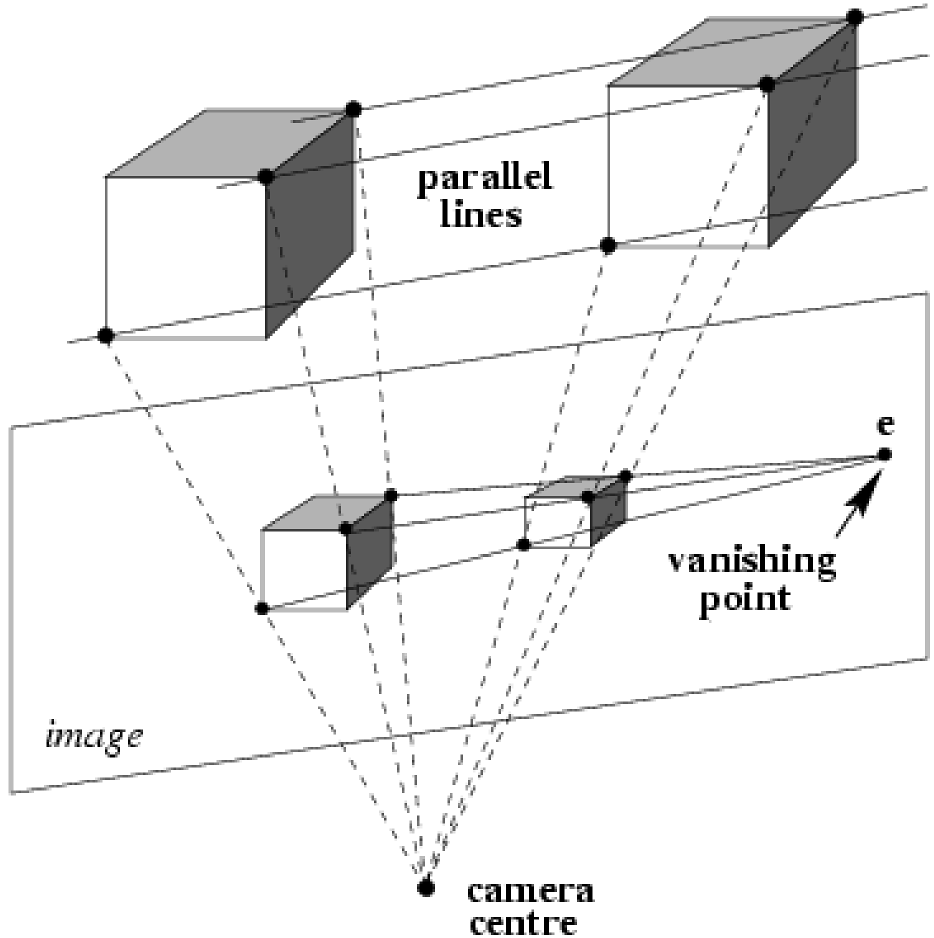




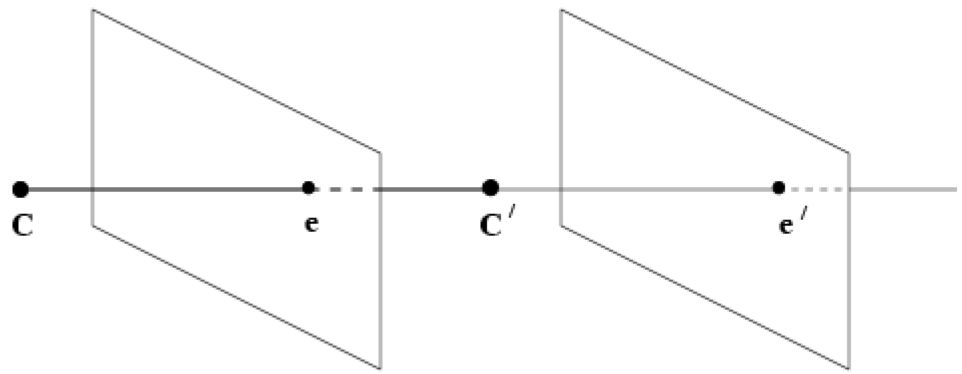
(pick k=e, since $e^Te \neq 0$)

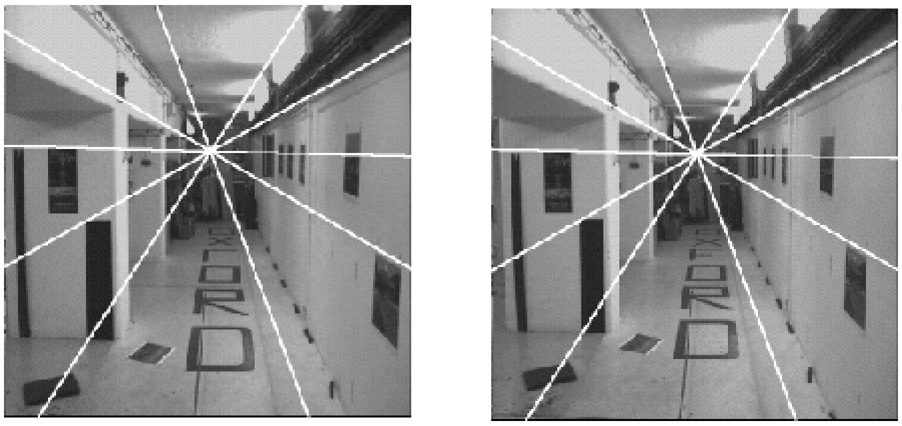
$$l' = F[e]_\times l \qquad l = F^T[e']_\times l'$$

Fundamental matrix for pure translation



Fundamental matrix for pure translation





Fundamental matrix for pure translation

$$F = [e']_\times H_\infty = [e']_\times \qquad \left(H_\infty = K^{-1}RK\right)$$

example:

$$e' = (1,0,0)^\mathsf{T} \qquad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}_\times$$

$$x'^\mathsf{T} Fx = 0 \Leftrightarrow y = y'$$

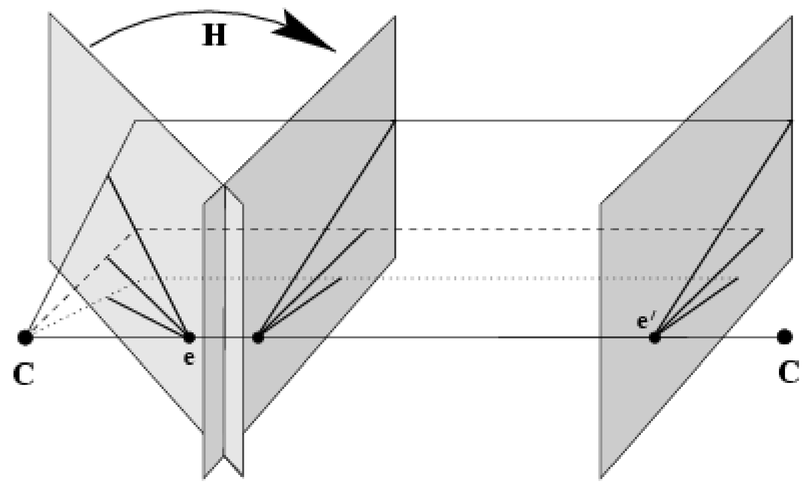
$$x = PX = K[I \mid 0]X \qquad (X,Y,Z)^\mathsf{T} = K^{-1}x/Z$$

$$x' = P'X = K[I \mid t]\begin{bmatrix} K^{-1}x \\ Z \end{bmatrix} \qquad x' = x + Kt/Z$$

motion starts at x and moves towards e, faster depending on *Z*

pure translation: F only 2 d.o.f., $x^\mathsf{T}[e]_x x=0 \Rightarrow$ auto-epipolar

General motion



$$x'^{T}[e']_{\times}Hx = 0$$

$$x'^{T}[e']_{\times}\hat{x} = 0$$

$$x' = K'RK^{-1}x + K't/Z$$

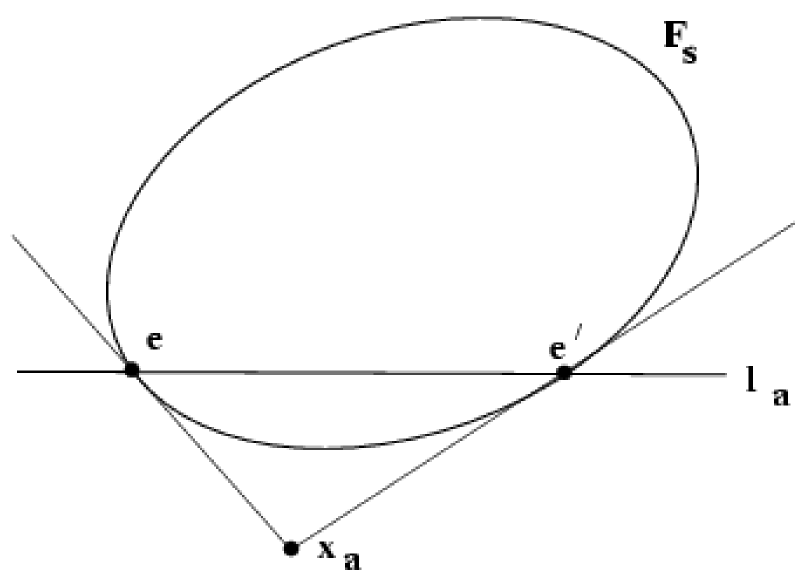
Geometric representation of F

$$F_S = \left(F + F^T\right)/2 \qquad F_A = \left(F - F^T\right)/2 \qquad \left(F = F_S + F_A\right)$$

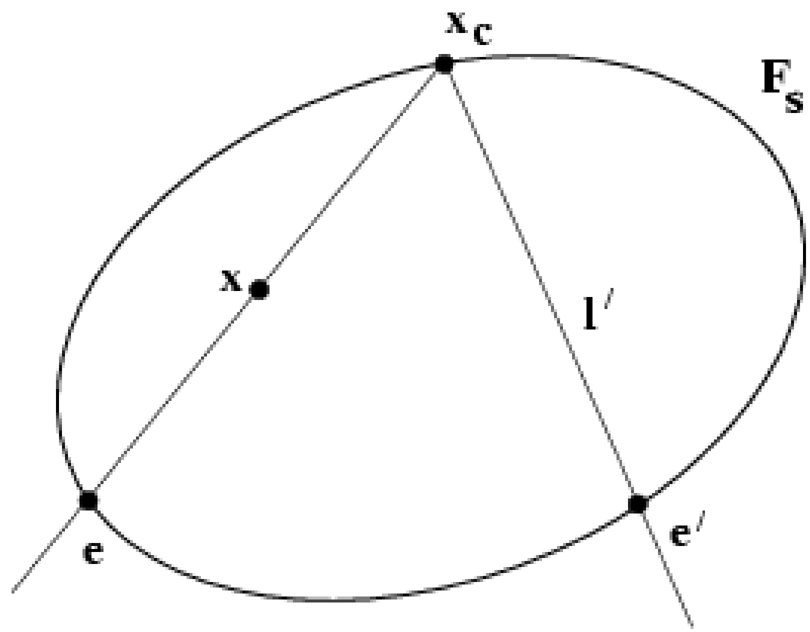
$$x \leftrightarrow x \qquad x^T F x = 0 \qquad \left(x^T F_A x \equiv 0\right)$$

$$x^T F_S x = 0$$

F_s: Steiner conic, 5 d.o.f.
$F_a=[x_a]_x$: pole of line ee' w.r.t. F_s, 2 d.o.f.

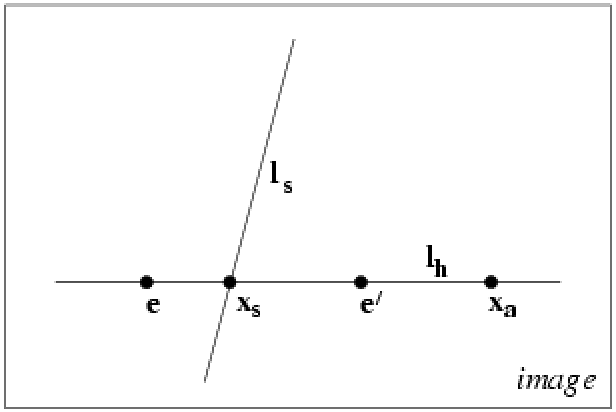


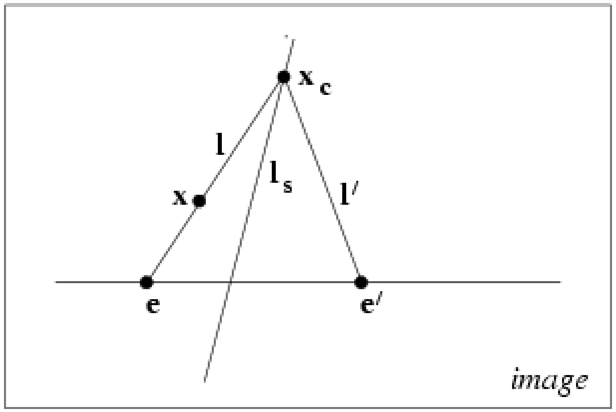
Geometric representation of F



Pure planar motion

Steiner conic F_s is degenerate (two lines)





Projective transformation and invariance

Derivation based purely on projective concepts

$$\hat{x} = Hx,\ \hat{x}' = H'x' \Rightarrow \hat{F} = H'^{-T} F H^{-1}$$

F invariant to transformations of projective 3-space

$$x = PX = (PH)(H^{-1}X) = \hat{P}\hat{X}$$

$$x' = P'X = (P'H)(H^{-1}X) = \hat{P}'\hat{X}$$

$(P, P')\ \alpha\ \ F$ unique

$F\ \alpha\ \ (P, P')$ not unique

canonical form

$$P = [I \mid 0] \qquad\qquad F = [m]_\times M$$
$$P' = [M \mid m]$$

Projective ambiguity of cameras given F

previous slide: at least projective ambiguity
this slide: not more!

Show that if F is same for (P,P') and ($\tilde{P}$,$\tilde{P}$'),
there exists a projective transformation H so that
$\tilde{P}$=HP and $\tilde{P}$'=HP'

$$\mathrm{P} = [\mathrm{I} \mid 0] \; \mathrm{P}' = [\mathrm{A} \mid \mathrm{a}] \;\; \tilde{\mathrm{P}} = [\mathrm{I} \mid 0] \; \tilde{\mathrm{P}}' = [\tilde{\mathrm{A}} \mid \tilde{\mathrm{a}}]$$

$$\mathrm{F} = [\mathrm{a}]_\times \mathrm{A} = [\tilde{\mathrm{a}}]_\times \tilde{\mathrm{A}}$$

<u>lemma:</u> $\tilde{\mathrm{a}} = \mathrm{ka} \;\; \tilde{\mathrm{A}} = k^{-1}\left(\mathrm{A} + \mathrm{av}^\mathrm{T}\right)$

$$\mathrm{aF} = \mathrm{a}[\mathrm{a}]_\times \mathrm{A} = 0 = \tilde{\mathrm{a}}\mathrm{F} \overset{\text{rank 2}}{\Longrightarrow} \tilde{\mathrm{a}} = \mathrm{ka}$$

$$[\mathrm{a}]_\times \mathrm{A} = [\tilde{\mathrm{a}}]_\times \tilde{\mathrm{A}} \Rightarrow [\mathrm{a}]_\times \left(\mathrm{k}\tilde{\mathrm{A}} - \mathrm{A}\right) = 0 \Rightarrow \left(\mathrm{k}\tilde{\mathrm{A}} - \mathrm{A}\right) = \mathrm{av}^\mathrm{T}$$

$$H = \begin{bmatrix} k^{-1}I & 0 \\ k^{-1}\mathrm{v}^\mathrm{T} & k \end{bmatrix}$$

$$\mathrm{P'H} = [\mathrm{A} \mid \mathrm{a}] \begin{bmatrix} k^{-1}I & 0 \\ k^{-1}\mathrm{v}^\mathrm{T} & k \end{bmatrix} = [k^{-1}\left(\mathrm{A} - \mathrm{av}^\mathrm{T}\right) \mid k\mathrm{a}] = \tilde{\mathrm{P}}'$$

(22-15=7, ok)

Canonical cameras given F

F matrix corresponds to P,P' iff $P'^{T}FP$ is skew-symmetric

$$\left(X^{T}P'^{T}FPX = 0, \forall X\right)$$

F matrix, S skew-symmetric matrix

$$P = [I \mid 0] \quad P' = [SF \mid e']$$ (fund.matrix=F)

$$\left([SF \mid e']^{T} F [I \mid 0] = \begin{bmatrix} F^{T}S^{T}F & 0 \\ e'^{T}F & 0 \end{bmatrix} = \begin{bmatrix} F^{T}S^{T}F & 0 \\ 0 & 0 \end{bmatrix}\right)$$

Possible choice:

$$P = [I \mid 0] \quad P' = [[e']_{\times} F \mid e']$$

Canonical representation:

$$P = [I \mid 0] \quad P' = [[e']_{\times} F + e' v^{T} \mid \lambda e']$$

The essential matrix

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$$E = K'^T F K$$

5 d.o.f. (3 for R; 2 for t up to scale)

E is essential matrix if and only if
two singularvalues are equal (and third=0)

SVD $$E = U\mathrm{diag}(1,1,0)V^T$$

Motion from E

Given

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

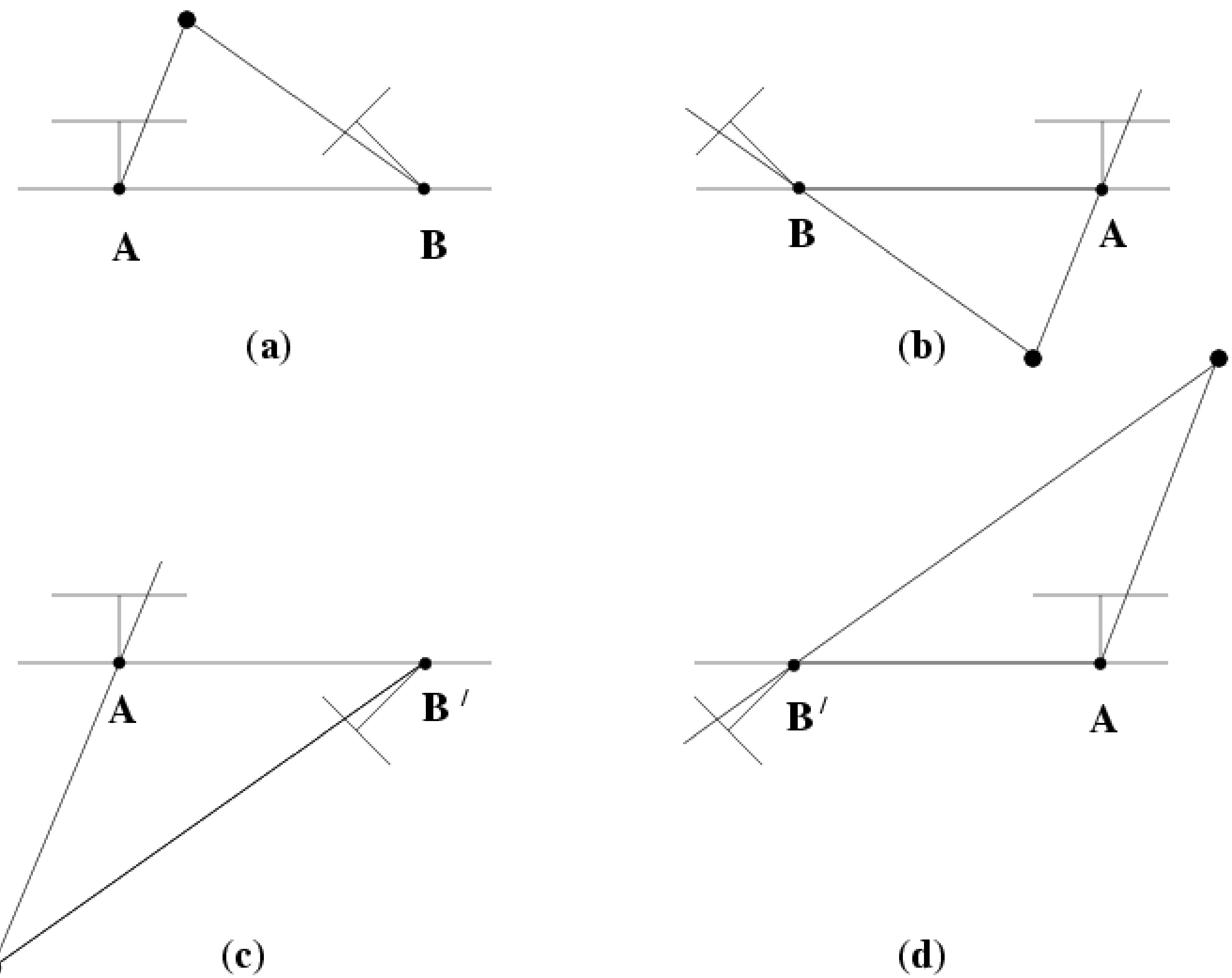
Four solutions

$$R' = UW^T V^T \qquad t' = U_3$$

and

$$R'' = UWV^T \qquad t'' = -U_3$$

Four possible reconstructions from E



(only one solution where points is in front of both cameras)