Comparison between Free and Regular Languages

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GRAMMARS FOR REGULAR LANGUAGES

Regular languages are a special case of the free ones.

Regular languages can be generated by grammars that exhibit strongly restricted production rules.

The phrases of a regular language necessarily contain repeated factors as the length of the string grows.

It is possible to prove in a rigorous way that some (free) languages can not be generated by any regular expression.

REGULAR LANGUAGES:
Recognizing phrases requires only a finite amount of memory.

FREE LANGUAGES:
Recognizing phrases requires a unbounded amount of memory.

FROM THE REGULAR EXPRESSION TO THE FREE GRAMMAR

The iterative regular operators (star and cross) must be replaced by means of recursive production rules.

Decompose the regexp into subexpressions and denumerate them progressively. After the very definition of regexp, the following cases are obtained, which allow to design the corresponding production rules shown aside.

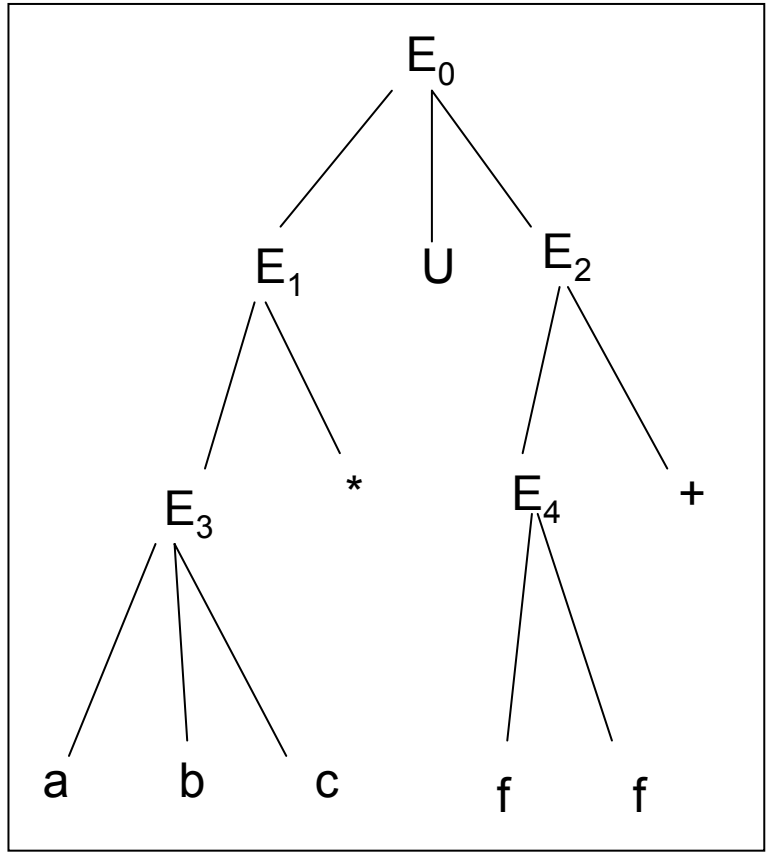
1. $r = r_1.r_2....r_k$	1. $E = E_1E_2...E_k$
2. $r = r_1 \cup r_2 \cup .... \cup r_k$	2. $E = E_1 \bigcup E_2 \bigcup ... \bigcup E_k$
3. $r = (r_1)^*$	3. $E = EE_1 \mid \varepsilon$ or $E = E_1E \mid \varepsilon$
4. $r = (r_1)^+$	4. $E = EE_1 \mid E_1$ or $E = E_1E \mid E_1$
5. $r = b \in \Sigma$	5. $E = b$
6. $r = \varepsilon$	6. $E = \varepsilon$

Uppercase letters indicate the non-terminal symbols. If E_i is a terminal symbol $a_i \in \Sigma$, the symbol a_i is written instead of E_i.

EXAMPLE

$$E = (abc)^* \cup (ff)^+$$

2.	$E_1 \cup E_2$	$E_0 \to E_1 \mid E_2$
3.	E_3^*	$E_1 \to E_1 E_3 \mid \varepsilon$
4.	E_4^+	$E_2 \to E_2 E_4 \mid E_4$
1.	abc	$E_3 \to abc$
1.	ff	$E_4 \to ff$



If the regexp to start from is ambiguous, the derived equivalent free grammar is ambiguous as well.

Every regular language is free, but there are many free languages that are not regular, like for instance palindromes and the arithmetic expressions with parentheses.

$$REG \subset LIB$$

LINEAR GRAMMAR

A grammar is said to be LINEAR if every rule has at most one non-terminal symbol occurring in the right member of the rule.

$$A \to uBv \quad \text{where} \quad u, v \in \Sigma^*, B \in (V \bigcup \varepsilon)$$

The family of linear free languages is still by far larger than that of regular languages.

EXAMPLE: here follows a linear free language that is not regular

$$L_1 = \left\{ a^n b^n \mid n \geq 1 \right\} = \left\{ ac, aacc, \ldots \right\}$$

$$S \to aSc \mid ac$$

UNI-LINEAR GRAMMAR (or GRAMMAR of TYPE 3)

RIGHT UNI-LINEAR RULE (or simply RIGHT LINEAR):

$$A \to uB \quad \text{where} \quad u \in \Sigma^*, B \in (V \bigcup \varepsilon)$$

LEFT UNI-LINEAR RULE (or simply LEFT LINEAR):

$$A \to Bv \quad \text{where} \quad v \in \Sigma^*, B \in (V \bigcup \varepsilon)$$

The corresponding syntax trees grow in a heavily unbalanced way.

If a left or right uni-linear grammar contains a recursive derivation, then such a derivation is recursive on the left or right, respectively.

EXAMPLE: the phrases that contain the substring *aa* and terminate with *b* are generated by the following regexp (ambiguous):

$$(a \mid b)^* aa(a \mid b)^* b$$

1. right linear grammar G_d

$S \to aS \mid bS \mid aaA \quad A \to aA \mid bA \mid b$

2. left linear grammar G_s

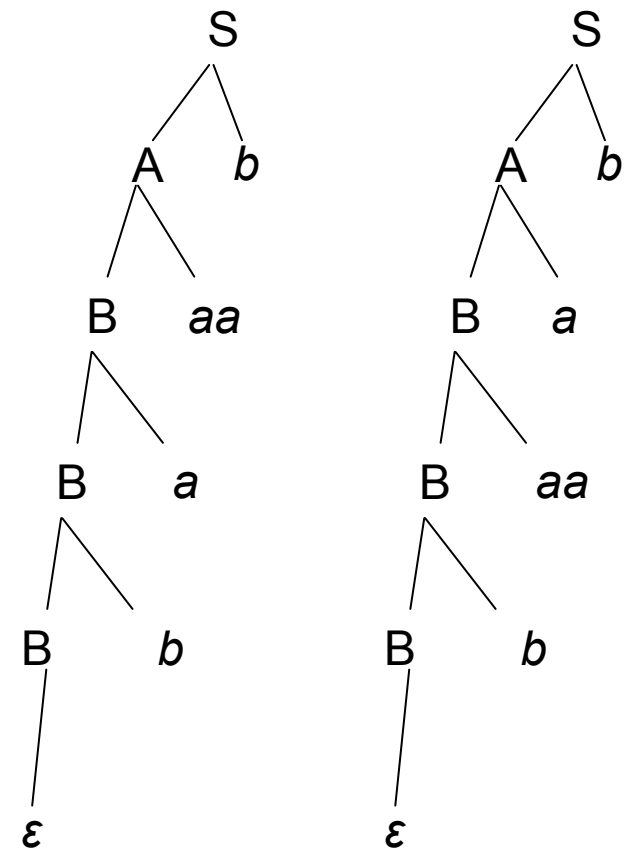
$S \to Ab \quad A \to Aa \mid Ab \mid Baa$

$B \to Ba \mid Bb \mid \varepsilon$

3. equivalent non-unilinear grammar

$E_1 \to E_2 aaE_2 b \quad E_2 \to E_2 a \mid E_2 b \mid \varepsilon$

G_s – syntax trees of the ambiguous phrase *baaab*



EXAMPLE : arithmetic expressions without parentheses

$$L = \{a, a+a, a*a, a+a*a, \ldots\}$$

$$G_d: \quad S \to a \mid a+S \mid a*S$$

$$G_s: \quad S \to a \mid S+a \mid S*a$$

G_s and G_d are not adequate to compilation, as their syntax structure does not reflect the usual precedence rules between arithmetic operators. It is convenient to transform a unilinear grammar into a strictly unilinear grammar, where every rule contains at most a single terminal character.

$$A \to aB \quad \text{or} \quad A \to Ba \quad \text{where} \quad a \in (\Sigma \cup \varepsilon), B \in (V \cup \varepsilon)$$

Without any loss of generality, one can impose to have only null terminal rules.

$$A \to aB \mid \varepsilon \quad \text{where} \quad a \in \Sigma$$

FROM UNILINEAR GRAMMAR TO REGULAR EXPRESSION: LINEAR EQUATIONS

UNILINEAR GRAMMAR / REGULAR LANGUAGE
First it is proved that the family of the languages generated by unilinear grammars strictly contains that of regular languages. Then it is proved that these two families actually coincide.

One can think of the rules of a right (uni)linear grammar as of equations the unknown terms of which are the languages generated by each non-terminal symbol. Suppose that G is a strictly right linear grammar and that G contains only null terminal rules $A \to \varepsilon$.

$$G = (V, \Sigma, P, S)$$

$$L_A = \left\{ x \in \Sigma^* \mid A \overset{+}{\Rightarrow} x \right\} \quad L(G) \equiv L_S$$

replace the (recursive) production rule $A \rightarrow a\ A$ by the equation $L_A = L_a\ L_A$

replace the production rule $A \rightarrow b\ B$ by the equation $L_A = L_b\ L_B$

replace the production rule $A \rightarrow a$ by the equation $L_A = L_a$

unite terminal rules to non-terminal rules: $A \rightarrow b\ B$ and $A \rightarrow c$ so that $L_A = L_c \mid L_b\ L_B$

and so on when there are more terms

then solve the equations by substitution

in the case of a recursive equation $L_A = L_x \mid L_y\ L_A$, solve by means of Arden's identity: $L_A = L_y^*\ L_x$

EXAMPLE: set-theoretic equations

Here follows a grammar that generates a list of elements *e* (possibly empty) separated by the character *s*.

$$S \to sS \mid eA \quad A \to sS \mid \varepsilon$$

The grammar can be transformed into a set of equations, as aside:

$$\begin{cases} L_s = sL_S \cup eL_A \\ L_A = sL_S \cup \varepsilon \end{cases}$$

$$\begin{cases} L_s = sL_S \cup e(sL_S \cup \varepsilon) \\ L_A = sL_S \cup \varepsilon \end{cases}$$

$$\begin{cases} L_s = (s \cup es)L_S \cup e \\ L_A = sL_S \cup \varepsilon \end{cases}$$

and using Arden's identity one obtains

$$\begin{cases} L_s = (s \cup es)^* e \\ L_A = sL_S \cup \varepsilon \end{cases} \qquad L_A = s(s \cup es)^* e \cup \varepsilon$$

EVERY UNILINEAR LANGUAGE IS REGULAR

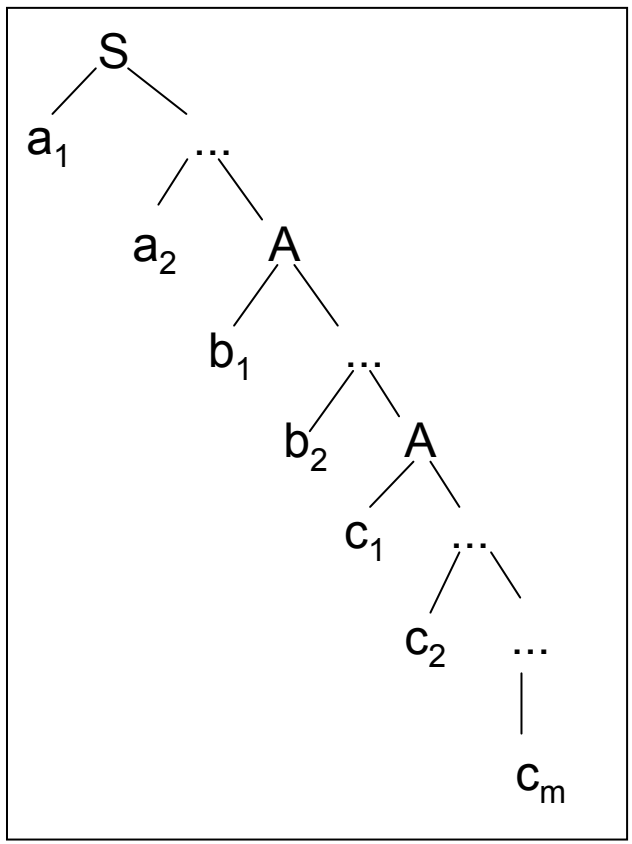
COMPARISON BETWEEN REGULAR AND FREE LANGUAGES:
UNAVOIDABLE REPETITIONS

THE LIMITATIONS OF THE FAMILY OF REGULAR LANGUAGES:
what linguistic structures can be modeled by means of regular expressions, and what ones instead need the larger generating power of free grammars ?

Only thanks to recursion, a grammar will generate an infinite language. Here follows a property that is useful to prove that certain free languages are not regular.

PROPERTY: Let G be a unilinear grammar. Every sufficiently long phrase x (that is, $| x | > k$ for a constant k that depends only on G), generated by G, can be factored as $x = t\ u\ v$, where u is not empty, in such a way that for every integer $n \geq 1$, the string $t\ u^n\ v$ is generated by G as well (one says that the phrase x can be pumped by inserting in the middle new occurrences of the factor u, as many times as one wants).

PROOF: Consider a strictly right linear grammar, that has exactly *k* non-terminal symbols. In the derivation of a long phrase *x*, of *k* or more characters, there necessarily is a non-terminal symbol A that will occurr twice or more times.



$$t = a_1 a_2 ..., \quad u = b_1 b_2 ... \quad v = c_1 c_2 ... c_m$$

$$S \overset{+}{\Rightarrow} tA \overset{+}{\Rightarrow} tuA \overset{+}{\Rightarrow} tuv$$

allows to derive also $tuuv \quad tv$

TRY TO USE THE ABOVE PROPETY TO PROVE THAT L_1 IS NOT REGULAR

$$L_1 = \left\{ a^n c^n \mid n \geq 1 \right\}$$

EXAMPLE: language with two identical exponents

$$L_1 = \left\{ a^n c^n \mid n \geq 1 \right\}$$

Suppose by refutation that the language is regular:

$x = a^k c^k = tuv$ where *u* is not empty

	t	u	v	iterated string
1.	$t = a^h$	$u = a^i$	$v = a^j c^k$	$a^h a^i a^i a^j c^k$
		$1 \leq i \leq k$	$h + i + j = k$	$h + 2i + j > k$
2.	$t = a^h$	$u = a^i c^j$	$v = c^m$	$a^h a^i c^j a^i c^j c^m \notin a^+ c^+$
		$1 \leq i, j \leq k$	$h + i = j + m = k$	
3.	$t = a^k c^h$	$u = c^i$	$v = c^j$	$a^k c^h c^i c^i c^j$
		$1 \leq i \leq k$	$h + i + j = k$	$h + 2i + j > k$

In all the possible cases the iterated string (that should belong to the language if this were regular), does not exhibit the required form $a^n c^n$

THE ROLE OF SELF-INCLUSIVE DERIVATIONS

$$A \overset{+}{\Rightarrow} uAv \quad u \neq \varepsilon \wedge v \neq \varepsilon$$

A self-inclusive derivation may not occur in the unilinear grammar of regular languages. This does not happen in the free grammars that generate free non-regular languages (palindromes, Dyck, etc).

The absence of self-inclusive derivations allows to solve the set-theoretic equations associated with unilinear grammars, and gives rise to the simple structure of the languages these grammars generate.

A GRAMMR THAT DOES NOT ADMIT ANY SELF-INCLUSIVE DERIVATION ALWAYS GENERATES A REGULAR LANGUAGE.

EXAMPLE: a non-self-inclusive grammar

$$G: \quad S \to AS \mid bA \quad A \to aA \mid \varepsilon$$

$$\begin{cases} L_s = L_A L_S \cup bL_A \\ L_A = aL_A \cup \varepsilon \end{cases}$$

$$\begin{cases} L_s = L_A L_S \cup bL_A \\ L_A = a^* \end{cases}$$

$$L_s = a^* L_S \cup ba^*$$

$$L_S = (a^*)^* ba^* \qquad L(G) = a^* ba^*$$

ALTHOUGH THE GRAMMAR CONTAINS SELF-INCLUSIVE DERIVATIONS, THE GENERATED LANGUAGE MAY BE REGULAR, IN SOME SPECIAL CASES.

↓

ONE-LETTER LANGUAGES ENJOY A CURIOUS PROPERTY.

↓

THE LANGUAGE DEFINED BY A FREE GRAMMAR OVER A ONE-LETTER TERMINAL ALPHABET (|Σ| = 1) IS REGULAR.

$$G = \{S \to aSa \mid \varepsilon\}$$

$$S \Rightarrow aSa \qquad L(G) = (aa)^*$$

can be defined as

$$G_d = \{S \to aaS \mid \varepsilon\}$$

THE LIMITATIONS OF FREE LANGUAGES

In the free languages, the sufficiently long phrases necessarily contain some substrings that can be repeated as many times as one wants, and hence that give rise to nested structures. Such a constraint prevents free grammars of generating phrase structures where three or more parts are repeated for the same number of times.

PROPERTY: Let G be a free grammar. Every sufficiently long phrase x (that is, $| x | > k$ for a constant k that depends only on G), generated by G, can be factored as $x = y\ u\ w\ v\ z$, where $u,\ v$ are not empty, in such a way that for every integer $n \geq 1$, the string $y\ u^n\ w\ v^n\ z$ is generated by G as well (one says that the phrase x can be pumped by inserting in the middle equal numbers of new occurrences of the factors $u,\ v$, as many times as one wants).

THE LIMITATIONS OF FREE LANGUAGES

EXAMPLE: THE LANGUAGE WITH THREE EXPONENTS IS NOT FREE.
(the proof exploits the possibility of pumping a long phrase by repeating two independent factors, which is incompatible with the definition of such a language).

$$L = \left\{ a^n b^n c^n \mid n \geq 1 \right\}$$

EXAMPLE: THE COPY LANGUAGE

This sub-language is frequent in the programming languages, whenever two lists need have identical matching elements, like for instance in the lists of the formal and actual parameters of a procedure of function declaration and call.

$$L_{replic} = \left\{ uu \mid u \in \Sigma^+ \right\}$$

$$\Sigma = \{a, b\} \quad x = abbbabbb$$

THE CLOSURE PROPERTIES OF THE REG AND LIB FAMILIES

Here the closure properties of the REG and LIB families of languages are examined, with respect to the most common language operators.

Let L and R be two languages belonging to LIB and REG, respectively.

$R^R \in REG$	$L^R \in LIB$
$R^* \in REG$	$L^* \in LIB$
$\neg R \in REG$	$\neg L \notin LIB$
$R_1 \oplus R_2 \in REG$	$L_1 \oplus L_2 \in LIB$
$R_1 \bigcap R_2 \in REG$	$L_1 \bigcap L_2 \notin LIB$
	$L \bigcap R \in LIB$

denotes both union and concatenation

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COMMENTS AND EXAMPLES

1) The mirror image of L(G) is generated by the mirror grammar, obtained from G by mirroring the right sides of the productions (a right linear G becomes left linear).

2) The star, union and concatenation of free languages are themselves free.

G_1 G_2, L_1 $L_2,$	axioms	S_1 $S_2,$ $V_1 \bigcap V_2 = \varnothing$
star:		$S \to SS_1 \mid \varepsilon$
union:		$S \to S_1 \mid S_2$
concatenation:		$S \to S_1 S_2$

3) If the two grammars to be united share some non-terminal names, their union will generate a superset language of the true union of the two languages:

$$G_1 = \{S_1 \to aA, A \to bA \mid b\}, \quad G_2 = \{S_2 \to aA, A \to cA \mid c\}$$

$$G_{1\cup 2} = \{S \to S_1 \mid S_2, S_1 \to aA, A \to bA \mid b, S_2 \to aA, A \to cA \mid c\}$$

$$a(b \mid c)^+ \supset L(G_1) \bigcup L(G_2) = ab^+ \bigcup ac^+$$

4) The intersection of two free languages, in general, is not a free language.

$$\left\{a^n b^n c^n \mid n \geq 1\right\} = \left\{a^n b^n c^+ \mid n \geq 1\right\} \bigcap \left\{a^+ b^n c^n \mid n \geq 1\right\}$$

5) There exist free languages the complements of which (w.r.t. the universal language over the same alphabet) are not free languages.

CAUTION: of course, there exist also free languages the complements of which are still free languages (and similarly for intersection).

INTERSECTION OF FREE LANGUAGES WITH REGULAR LANGUAGES

One way to make a free grammar more selective, is to restrict the generated free language through a regular filter.

EXAMPLE: regular filter over the Dyck language - $\Sigma = \{ a, c \}$ (*a* open *c* close).

$$L_1 = L_D \bigcap \neg(\Sigma^* cc\Sigma^*) = (ac)^*$$

$$L_2 = L_D \bigcap \neg(\Sigma^* ca\Sigma^*) = \left\{ a^n c^n \mid n \geq 0 \right\}$$

The former intersection filters the Dyck language by removing the phrases that contain the factor *cc*, that is the phrases containing nested structures.

The latter intersection filters the Dyck language by removing the phrases that contain two or more concatenated nested structures, leaving only the phrases with one nested structure.

Both filtered languages are free, and the former one is regular as well.

ALPHABETIC TRANSFORMATIONS

Very similar languages may differ only as for a few terminal symbols. TRANSLITTERATION (or ALPHEBETIC HOMOMORPHISM) is a linguistic operation that replaces some terminal characters by some other terminal characters, over the same or a different alphabet. Some characters can also be canceled (that is, replaced by the empty string ε).

TRANSLITTERATION (ALPHABETIC HOMOMORPHISM) is a function:

$$h: \quad \Sigma \to \Delta \bigcup \{\varepsilon\}$$

ε-free translitteration (or alpha. hom.): for every $c \in \Sigma$ $\quad h(c) \neq \varepsilon$

Translitteration of a source string:

$$a_1 a_2 ... a_n \in \Sigma^*$$

$$h(a_1)h(a_2)...h(a_n)$$

$$h(v.w) = h(v).h(w)$$

EXAMPLE: printer

$h(c) = c$ if $c \in \{a, b, ..., z, 0, 1, ..., 9\}$;

$h(c) = c$ if c is a punctuation character

$h(c) = \pi$ if $c \in \{\alpha, \beta, ..., \omega\}$;

$h(\text{start-text}) = h(\text{end-text}) = \varepsilon$

$h(\text{start-text} \quad \textit{cost } \pi \textit{ is } 3.14 \quad \text{end-text}) = \textit{cost } \pi \textit{ is } 3.14$.

WORD TRANSLITTERATION (of ARBITRARY HOMOMORPHISM):
the transformed image of a character may be a string of length greater than one.

$h(\leftarrow) =:= \quad h(c) = c$ for every other $c \in \Sigma$

SUBSTITUTION: each character of the source string can be replaced by an entire string, out of a given language; the language to take the string from depends on the character to be replaced.

For instance, given the following source alphabet: $\Sigma = a, b, \ldots$

a substitution *h* associates with each letter *a* a specific replacement language over the destination alphabet Δ.

$$\boxed{h(a) = L_a, \quad h(b) = L_b}$$

Applying the substitution *h* to a source string *x*

$$x = a_1 a_2 \ldots a_n \in \Sigma^*$$

yields the following set of strings: $h(x) = \left\{ y_1 y_2 \ldots y_n \mid y_i \in L_{a_i} \right\}$

A word translitteration is a special case of substitution: each replacement language contains only one string. A translitteration is a special case of substitution: each replacement language reduces to one character (or the empty string).

CLOSURE WITH RESPECT TO ALPHABETIC TRANSFORMATIONS

PROPERTY: Both families REG and LIB (of regular and free languages, resp.) are closed with respect to word translitteration.

PROOF: Let G be a grammar that generates L, and *h* a word translitteration from the source alphabet Σ to the destination alphabet Δ. Apply the transformation *h* to every rule A → α of G, and leave unaltered the arrow metasymbol → and every non-terminal symbol.

$$h(A \rightarrow \alpha) \equiv A \rightarrow h(\alpha)$$

The production rules that are obtained in this way, are still of the free type and resemble closely those they derive from (only the terminal symbols are changed).

The obtained grammar generates the transformed language *h*(L(G)).

CLOSURE WITH RESPECT TO ALPHABETIC TRANSFORMATIONS

PROPERTY: Both families REG and LIB (of regular and free languages, resp.) are closed with respect to the substitution with languages of the same family.

PROOF: Let G be a grammar that generates the language L, and let *h* be a substitution such that, for every character $c \in \Sigma$, the replacement language L_c is free and is generated by a grammar G_c with axiom S_c; moreover, assume that the non-terminal alphabets of the grammars G, G_a, G_b, ... are disjoint.

In order to obtain a grammar G' that generates the transformed language *h*(L), apply the alphabetic translitteration *f* to the rules of G, where *f* is defined as $f(c) = S_c$ for every terminal charecter $c \in \Sigma$; and leave $f(A) = A$ for every other symbol of G.

To obtain the grammar G':
° to every rule $A \rightarrow \alpha$ of G apply the translitteration *f* that replaces each terminal character *c* of G by means of the axiom S_c of the corresponding grammar
° to the obtained rules, add those of the grammars G_a, G_b, ...
The obtained final grammar generates the transformed language *h*(L(G))…

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