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Analytical solutions to the Grad–Shafranov equation

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Two families of exact analytical solutions of the Grad–Shafranov equation are presented by specifying the highest polynomial dependence of the plasma current density on the flux function Ψ in such a way that the Grad–Shafranov equation becomes a linear inhomogeneous differential equation. Both the pressure profile and the poloidal current profile each have two free parameters. X-points can be represented by superposition of solutions. Examples of the exact equilibrium solution are given for both a D-shaped plasma and a toroidally diverted plasma. [DOI: 10.1063/1.1756167]

I. INTRODUCTION

For axially symmetric configurations, where φ is the ignorable angle in the cylindrical coordinate system (r,φ,z), Maxwell's equations together with the force balance equation reduce, for stationary and ideally conducting plasmas, to the scalar partial differential equation named the Grad–Shafranov equation (GSE) (Refs. 1 and 2):

$$\frac{\partial^2\Psi}{\partial r^2}-\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{\partial^2\Psi}{\partial z^2}=-4\pi^2\mu_0 r^2 p'(\Psi)-\frac{1}{2}\mu_0^2 F^{2'}(\Psi)$$
$$=-2\pi\mu_0 r j_\varphi, \tag{1}$$

where $\Psi=\Psi(r,z)$ is the poloidal stream function and represents the poloidal magnetic flux; $j_\varphi=2\pi r(p'+\mu_o/(8\pi^2r^2)F^{2'})$ is the toroidal current density; $F(\Psi)=2\pi r B_\varphi/\mu_0$ is the poloidal current, and $p=p(\Psi)$ is the plasma pressure. $p(\Psi)$ and $F(\Psi)$ are in the set of ideal magnetohydrodynamic (MHD) arbitrary functions of the Ψ function. The prime symbol denotes derivation with respect to Ψ. The SI system of units will be used throughout this paper.

Analytical solutions of the GSE are very useful for theoretical studies of plasma equilibrium, transport, and magnetohydrodynamic stability. These solutions can be used also as a benchmark of numerical codes, but existing exact solutions are very restricted in a variety of allowed current density profiles.

The simplest analytical solution to the inhomogeneous GSE is the well-known Solov'ev equilibrium,[3] and corresponds to source functions linear in Ψ. Equilibria of this type have been extensively used for equilibrium, transport, and stability studies. However, the Solov'ev equilibrium solutions are overconstrained in shape or in poloidal beta (plasma current).

For the same Solov'ev equilibrium case, by expanding the solution of the homogeneous equations in a polynomial form in r (of fourth degree) and z (of second degree), and assuming an up–down symmetry, it is possible to describe the plasma shape by four parameters.[4] The shape of the current profile is essentially flat, and the two existing free parameters allow one to choose the plasma current $I_{\rm pl}$ and the poloidal beta β_p (or the plasma pressure on the magnetic axis p_0).

By using source functions quadratic in Ψ for the GSE, Maschke[5] and Herrnegger[6] first found a class of exact analytical solutions. By using a rectangular plasma boundary with homogeneous boundary conditions, i.e., the classical Sturm–Liouville eigenvalue problem, Maschke obtained the $\Psi(r,z)$ solution as a product between the Coulomb wave functions (depending on r only) and the cosine of the z coordinate (looking for solutions which are symmetric with respect to the z plane). There are five free parameters in these solutions, but one of them is fixed by the boundary conditions on the rectangle, and two others as eigenvalues as a function of the plasma pressure. Hence there remain only two free parameters, determining together the magnitude of the pressure and of the current density.

An exact solution of the large-aspect-ratio approximation with an additional assumption of a simple relation between the magnetic flux and the current density was constructed by Greene.[7]

More recently, McCarthy[8] presented a new family of solutions where the plasma pressure is linear in Ψ, while the squared poloidal current has both, a quadratic and a linear Ψ term. Thus, the current density profile has three free parameters and the GSE looks like $\Delta^*\Psi=-Sr^2-T\Psi-U$. Show-

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ing that $\Delta^*(Sr^2+U)=0$, he proceeds to solve the more tractable homogeneous equation and eigenvalue problem $\Delta^*\Psi_h=T\Psi_h$. Once Ψ_h is found, by assuming separability in r and z, the solution to the full problem is obtained by setting $\Psi=\Psi_h-Sr^2/T-U/T$. By using Bessel functions and modified Bessel functions, he obtains eight real solutions.

In the present paper we present two new families of exact analytical solutions by considering that the current density profile has four free parameters, the most complicated dependence on the flux function Ψ which still maintains the linear character of the equation with partial derivatives.

In contrast to previous analytical solutions, the present family of solutions offers the possibility to independently specify the plasma current I_{pl}, the poloidal beta β_{pol}, the internal inductance l_i, and the safety factor at the magnetic axis q_{ax} or at the plasma boundary q_b.

The paper is organized as follows: In Sec. II we present two new families of exact analytical solutions. In Sec. III numerical results are given with examples from a D-shaped toroidal plasma and from the ASDEX Upgrade tokamak (Axially Symmetric Divertor EXperiment).[9] The last section contains a few concluding remarks and comments. Finally, in the Appendix the derivation of the particular inhomogeneous solutions is presented.

II. TWO FAMILIES OF EXACT EQUILIBRIUM SOLUTIONS

Considering the pressure and the poloidal current profiles dependencies on Ψ of the form

$$p(\Psi)=\overline{a}\Psi^2+\overline{b}\Psi, \tag{2}$$

$$F^2(\Psi)=\overline{\alpha}\Psi^2+\overline{\beta}\Psi+F_0^2, \tag{3}$$

the GSE and the plasma current density looks like

$$\frac{\partial^2\Psi}{\partial r^2}-\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{\partial^2\Psi}{\partial z^2}=-(ar^2+\alpha)\Psi-(br^2+\beta), \tag{4}$$

$$j_\varphi(r,z)=\frac{1}{2\pi\mu_0}\left[\left(ar+\frac{\alpha}{r}\right)\Psi+br+\frac{\beta}{r}\right], \tag{5}$$

where

$$a=8\pi^2\mu_0\overline{a},\quad b=4\pi^2\mu_0\overline{b},\quad \alpha=\mu_0^2\overline{\alpha},\quad \beta=1/2\mu_0^2\overline{\beta},$$

are the four free parameters allowing to independently specify the plasma current I_{pl}, the poloidal beta β_{pol}, the internal inductance l_i and the safety factor at the magnetic axis q_{ax} or at the plasma boundary q_b.

We have converted the original inhomogeneous partial differential equation (PDE) into two problems: a homogeneous PDE and an inhomogeneous ordinary differential equation, with the general solution

$$\Psi=\Psi_o+\Psi_{\mathrm{nop}}, \tag{6}$$

where

$$\Psi_o=R(r)Z(z), \tag{7}$$

represents the general homogeneous solution, while Ψ_{nop} is any particular inhomogeneous solution. Thus,

$$R''-\frac{1}{r}R'+(ar^2+\alpha-k^2)R=0, \tag{8}$$

$$Z''+k^2Z=0, \tag{9}$$

with k an arbitrary constant. The notation with prime for the derivative with respect to the independent variable will be used throughout this paper.

Depending on the sign of the a parameter, two possible families of analytical solutions have been deduced.

A. The $a<0$ case

1. The general homogeneous solution

By introducing in Eq. (8) the new variable $x\equiv\sqrt{-a}r^2$, we obtain

$$R''+\left(-\frac{1}{4}-\frac{\eta}{x}\right)R=0, \tag{10}$$

where

$$\eta\equiv\frac{k^2-\alpha}{4\sqrt{-a}}. \tag{11}$$

With

$$R(x)\equiv xe^{-(x/2)}Q(x), \tag{12}$$

we have

$$xQ''+(2-x)Q'-(1+\eta)Q=0, \tag{13}$$

i.e., the differential equation for the hypergeometric functions.[10,11] For $\eta\neq 0,1$, Q will be of the form

$$Q=C_1\,{}_1F_1(1+\eta;2;x)+C_2\Bigg[\frac{2}{x}+\eta\log x\,{}_1F_1(1+\eta;2;x)-\frac{1}{x}\,{}_1F_1(\eta;1;x)+\sum_{n=0}^{\infty}\frac{(\eta)_{n+1}x^n}{n!(n+1)!}\left(\sum_{r=1}^{n+1}\frac{1}{r+\eta-1}-2\sum_{r=1}^{n}\frac{1}{r}\right)\Bigg]. \tag{14}$$

C_1 and C_2 are arbitrary constants, while $(\alpha)_k$ designates the Pochhammer symbol

$$(\alpha)_0=1,\quad (\alpha)_k=\alpha(\alpha+1)\cdots(\alpha+k-1),\quad k=1,2,\ldots.$$

The hypergeometric functions ${}_1F_1(1+\eta;2;x)$, ${}_2F_2(1+\eta,1;2,2;x)$ as well as the double sums are convergent.

Thus, the homogeneous solution $\Psi_o(r,z)=R(r)Z(z)$ looks like

$$\Psi_o(r,z)=\sqrt{-a}r^2e^{-(\sqrt{-a}/2)\,r^2}\Bigg\{C_1\,{}_1F_1(1+\eta;2;\sqrt{-a}r^2)+C_2\Bigg[\frac{2}{\sqrt{-a}r^2}+\eta\log(\sqrt{-a}r^2)\,{}_1F_1(1+\eta;2;\sqrt{-a}r^2)$$
$$-\frac{1}{\sqrt{-a}r^2}\,{}_1F_1(\eta;1;\sqrt{-a}r^2)+\sum_{n=0}^{\infty}\frac{(\eta)_{n+1}(\sqrt{-a}r^2)^n}{n!(n+1)!}\left(\sum_{r=1}^{n+1}\frac{1}{r-1+\eta}-2\sum_{r=1}^{n}\frac{1}{r}\right)\Bigg]\Bigg\}[C_3\cos(kz)+C_4\sin(kz)], \tag{15}$$

with C_3 and C_4 arbitrary constants.

2. The particular solution of the inhomogeneous equation

We have to seek for a solution Ψ_{nop} satisfying the equation

$$\frac{\partial^2\Psi_{\text{nop}}}{\partial r^2}-\frac{1}{r}\frac{\partial\Psi_{\text{nop}}}{\partial r}+\frac{\partial^2\psi_{\text{nop}}}{\partial z^2}+(ar^2+\alpha)\Psi_{\text{nop}}=-(br^2+\beta). \tag{16}$$

We have the freedom to choose the particular solution of the form

$$\Psi_{\text{nop}}(r)=-\frac{\beta}{\alpha}+g_{\text{nop}}(r). \tag{17}$$

This solution has to satisfy the equation

$$\Psi''_{\text{nop}}-\frac{1}{r}\Psi'_{\text{nop}}+(ar^2+\alpha)\Psi_{\text{nop}}=-(br^2+\beta), \tag{18}$$

such that g_{nop} satisfies the equation

$$g''_{\text{nop}}-\frac{1}{r}g'_{\text{nop}}+(ar^2+\alpha)g_{\text{nop}}=\left(\frac{\beta}{\alpha}a-b\right)r^2. \tag{19}$$

With the new variable

$$x\equiv\sqrt{-a}r^2, \tag{20}$$

we obtain the equation

$$g''_{\text{nop}}+\left(-\frac{1}{4}+\frac{\kappa}{x}\right)g_{\text{nop}}=-\frac{a}{\alpha}\Lambda, \tag{21}$$

where

$$\Lambda\equiv-\frac{1}{4a}\left(\frac{b}{a}\alpha-\beta\right),\quad \kappa\equiv\frac{\alpha}{4\sqrt{-a}}. \tag{22}$$

Using the new variable

$$\zeta\equiv-\frac{\alpha}{a\Lambda}\frac{1}{x}e^{x/2}g_{\text{nop}}, \tag{23}$$

we obtain

$$x\zeta''+(2-x)\zeta'-\tilde{a}\zeta=e^{x/2}\equiv F(\zeta), \tag{24}$$

where $\tilde{a}=1-\kappa$.

But since the exponential can be written in the form

$$e^{x/2}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x}{2}\right)^n, \tag{25}$$

we are looking for a polynomial expression for ζ, hence

$$\zeta(x)=\sum_{n=0}^{\infty}a_nx^n. \tag{26}$$

After some tedious calculations given in the Appendix, we obtain finally

$$\zeta=\left[2\,{}_2F_1\left(1,1;\tilde{a};\frac{1}{2}\right)-2+a_0\right]{}_1F_1(\tilde{a};2;x)$$
$$-\sum_{n=0}^{\infty}\frac{1}{(n+\tilde{a})n!}\,{}_2F_1\left(n+2,1;n+\tilde{a}+1;\frac{1}{2}\right)\left(\frac{x}{2}\right)^2. \tag{27}$$

Thus, the inhomogeneous solution takes the form

$$g_{\text{nop}}=-\frac{a}{\alpha}\Lambda xe^{-x/2}\zeta. \tag{28}$$

But

$$[2\,{}_2F_1(1,1;\tilde{a};\tfrac{1}{2}1)-2+a_0]\,{}_1F_1(\tilde{a};2;x),$$

is a solution of the homogeneous equation, and can be arbitrarily absorbed into Eq. (15). The particular solution looks now like

$$g_{\text{nop}}=\frac{a}{\alpha}\Lambda xe^{-(x/2)}\sum_{n=0}^{\infty}\frac{1}{(n+\tilde{a})n!}\,{}_2F_1$$
$$\times\left(n+2,1;n+\tilde{a}+1;\frac{1}{2}\right)\left(\frac{x}{2}\right)^2. \tag{29}$$

One can prove that the series in g_{nop} is absolutely convergent. Therefore, the particular solution ψ_{nop} of the inhomogeneous equation will be

$$\Psi_{\text{nop}}(r)=-\frac{\beta}{\alpha}+\frac{\sqrt{-a}}{4}r^2e^{-(\sqrt{-a}/2)\,r^2}$$
$$\times\left(\frac{\beta}{\alpha}-\frac{b}{a}\right)\sum_{n=0}^{\infty}\frac{1}{(n+\tilde{a})n!}$$
$$\times{}_2F_1\left(n+2,1;n+\tilde{a}+1;\frac{1}{2}\right)\left(\frac{\sqrt{-a}r^2}{2}\right)^n. \tag{30}$$

Finally, the general solution is $\Psi(r,z)=\Psi_o(r,z)+\Psi_{\text{nop}}(r)$, with $\Psi_o(r,z)$ given by Eq. (15) and $\Psi_{\text{nop}}(r)$ given by Eq. (30). a, b, α, and β are free parameters, while C_1, C_2, C_3, C_4, and k ($k^2\neq\alpha$, $\alpha+4\sqrt{-a}$) are some arbitrary constants.

B. The $a>0$ case

1. The general homogeneous solution

By using the new variable

$$x\equiv\frac{\sqrt{a}}{2}r^2, \tag{31}$$

we obtain

$$R''+\left(1-\frac{2\eta}{x}\right)R=0, \tag{32}$$

where η is given by the same expression as in Eq. (11).

Equation (32) is the differential equation for the Coulomb wave functions[10] with the Coulomb wave functions $F_L(\eta,x)$ and $G_L(\eta,x)$ as solutions. In our case $L=0$.

Thus, the general solution of the homogeneous equation will be given by

$$\Psi_o(r,z)=\left[C_1F_0\left(\eta,\frac{\sqrt{a}}{2}r^2\right)+C_2G_0\left(\eta,\frac{\sqrt{a}}{2}r^2\right)\right]\times[C_3\cos(kz)+C_4\sin(kz)]. \tag{33}$$

2. The particular solution of the inhomogeneous equation

As for the $a<0$ case, we are seeking for a particular solution of the form

$$\Psi_{\text{nop}}(r)=-\frac{\beta}{\alpha}+g_{\text{nop}}(r), \tag{34}$$

and by using again the same new variable

$$x\equiv\frac{\sqrt{a}}{2}r^2$$

we obtain the equation

$$g''_{\text{nop}}+\left(1-\frac{2\gamma}{x}\right)g_{\text{nop}}=\frac{\beta}{\alpha}-\frac{b}{a}, \tag{35}$$

where

$$\gamma\equiv-\frac{\alpha}{4\sqrt{a}}.$$

Let us take the particular inhomogeneous solution of the form

$$g_{\text{nop}}(x)=xe^{-\mathrm{i}x}\zeta(x)\left(\frac{\beta}{\alpha}-\frac{b}{a}\right), \tag{36}$$

where $\zeta(x)$ is the new variable. Thus

$$x\zeta''+2(1-\mathrm{i}x)\zeta'-2(\mathrm{i}+\gamma)\zeta=e^{\mathrm{i}x}. \tag{37}$$

Therefore

$$\Psi_{\text{nop}}(x)=-\frac{\beta}{\alpha}+xe^{-\mathrm{i}x}\left(\frac{\beta}{\alpha}-\frac{b}{a}\right)\zeta(x). \tag{38}$$

We are seeking for $\zeta(x)$ a polynomial form

$$\zeta(x)=\sum_{n=0}^{\infty}a_nx^n. \tag{39}$$

After some calculations similar to those made for the $a<0$ case, we obtain

$$\zeta=\mathrm{i}\sum_{n=0}^{\infty}\frac{1}{2(n+1-\mathrm{i}\gamma)n!}\,{}_2F_1\left(n+2,1;n+2-\mathrm{i}\gamma;\frac{1}{2}\right)(\mathrm{i}x)^n, \tag{40}$$

with $\mathrm{i}\equiv\sqrt{-1}$. Since the hypergeometric function ${}_2F_1(n+2,1;n+2-\mathrm{i}\gamma;1/2)$ is convergent, and the series is as well, we obtain the following particular solution of the inhomogeneous equation:

$$\Psi_{\text{nop}}(r)=-\frac{\beta}{\alpha}+\mathrm{i}\frac{\sqrt{a}}{4}r^2e^{-\mathrm{i}\sqrt{a}/2\,r^2}\left(\frac{\beta}{\alpha}-\frac{b}{a}\right)\times\sum_{n=0}^{\infty}\frac{1}{\left(n+1+\mathrm{i}\frac{\alpha}{4\sqrt{a}}\right)n!}\times{}_2F_1\left(n+2,1;n+2+\mathrm{i}\frac{\alpha}{4\sqrt{a}};\frac{1}{2}\right)\left(\mathrm{i}\frac{\sqrt{a}}{2}r^2\right)^n. \tag{41}$$

The physical particular solution is given by the real part of the solution (41): $\Re(\Psi_{\text{nop}})$.

The general solution for this case is: $\Psi(r,z)=\Psi_o(r,z)+\Re(\Psi_{\text{nop}}(r))$, with $\Psi_o(r,z)$ given by Eq. (33) and $\Psi_{\text{nop}}(r)$ given by Eq. (41).

III. NUMERICAL RESULTS

A. Determination of the coefficients C_j

By using the notation $\omega\equiv kz$ for both cases, $a>0$ and $a<0$, respectively, the general solution can be put in the following compact form

$$\Psi(r,z)=[C_1F_1(\eta,r)+C_2F_2(\eta,r)][C_3\cos(\omega)+C_4\sin(\omega)]+\Psi_{\text{nop}}(r), \tag{42}$$

where

$$F_1(\eta,r)=\sqrt{-a}r^2e^{-\sqrt{-a}/2\,r^2}\,{}_1F_1(1+\eta;2;\sqrt{-a}r^2),$$

$$F_2(\eta,r)=\sqrt{-a}r^2e^{-\sqrt{-a}/2\,r^2}\left[\frac{2}{\sqrt{-a}r^2}+\eta\log(\sqrt{-a}r^2)\times{}_1F_1(1+\eta;2;\sqrt{-a}r^2)-\frac{1}{\sqrt{-a}r^2}\,{}_1F_1(\eta;1;\sqrt{-a}r^2)+\sum_{n=0}^{\infty}\frac{(\eta)_{n+1}(\sqrt{-a}r^2)^n}{n!(n+1)!}\times\left(\sum_{r=1}^{n+1}\frac{1}{r-1+\eta}-2\sum_{r=1}^{n}\frac{1}{r}\right)\right], \tag{43}$$

for the $a<0$ case, and

$$F_1(\eta,r)=F_0(\eta,r),\quad F_2(\eta,r)=G_0(\eta,r) \tag{44}$$

for the $a>0$ case. Equation (42) can be written in the form

$$\Psi(r,z)=D_1F_1\cos(\omega)+D_2F_2\cos(\omega)+D_3F_1\sin(\omega)+D_4F_2\sin(\omega)+\Psi_{\text{nop}}(r), \tag{45}$$

where $D_1\equiv C_1C_3$, $D_2\equiv C_2C_3$, $D_3\equiv C_1C_4$, $D_4\equiv C_2C_4$, and $\omega\in(0:2\pi)$.

The four constants C_1, C_2, C_3, and C_4 are independent but their products D_1, D_2, D_3, and D_4 are not:

$$C_2C_4=\frac{C_2C_3C_1C_4}{C_1C_3},\quad D_4=\frac{D_2D_3}{D_1}. \tag{46}$$

For a current point on the plasma boundary (r_i,z_i) we can write

$$\Psi(r_i,z_i)=\sum_k [D_{1k}F_1(\eta_k,r_i)\cos(\omega_{ki})+D_{2k}F_2(\eta_k,r_i)\cos(\omega_{ki})+D_{3k}F_1(\eta_k,r_i)\sin(\omega_{ki})+D_{4k}F_2(\eta_k,r_i)\sin(\omega_{ki})]+\Psi_{\text{nop}}(r_i). \tag{47}$$

Knowing the values of the a, b, α, and β parameters, the constants D_{1k}, D_{2k}, D_{3k}, and D_{4k} can be determined if both the value of the flux function Ψ on the plasma boundary and the plasma contour are given.

The numerical evaluation of the generalized hypergeometric series as a solution to the generalized hypergeometric functions has been made by using the routines written in Ref. 12, while the Coulomb wave functions have been calculated with the help of the routines reported in Ref. 13.

B. Determination of the parameters *a*, *b*, *α*, and *β*

The parameters a, b, α, and β are determined by using integral relations concerning the following global plasma parameters: the toroidal plasma current I_{pl}, the poloidal beta β_p, the internal inductance l_i and the safety factor at the plasma boundary q_b or at the magnetic axis q_{ax}.

Thus, using Eq. (5) one obtains

$$a\int\int r\Psi(r,z)drdz+\alpha\int\int\frac{1}{r}\Psi(r,z)drdz+b\int\int rdrdz+\beta\int\int\frac{1}{r}drdz=2\pi\mu_0I_{\text{pl}}. \tag{48}$$

Starting from the definition of the poloidal beta value β_p,

$$\beta_p\equiv\frac{2\mu_0\langle p\rangle_V}{\langle B_p\rangle_L^2}=2\mu_0\frac{\int\int prdrdz}{\overline{B}_p^2\int\int rdrdz},\quad \overline{B}_p=\frac{\mu_oI_{\text{pl}}}{\oint dl}, \tag{49}$$

and taking into account Eq. (2), we can write

$$\frac{a}{2}\int\int\Psi^2(r,z)rdrdz+b\int\int\Psi(r,z)rdrdz=2\pi^2\mu_o^2\beta_p\frac{\int\int rdrdz}{(\oint dl)^2}I_{\text{pl}}^2. \tag{50}$$

The internal inductance l_i can be determined from the definition relation[14,15]

$$l_i\equiv\frac{\langle B_p^2\rangle_V}{\langle B_p\rangle_L^2}=\frac{(\oint dl)^2}{2\pi\mu_0I_{\text{pl}}^2}\frac{\int\int\Psi j_\varphi drdz}{\int\int rdrdz}. \tag{51}$$

Thus,

$$a\int\int\Psi^2(r,z)rdrdz+\alpha\int\int\Psi^2(r,z)\frac{1}{r}drdz+b\int\int\Psi(r,z)rdrdz+\beta\int\int\Psi(r,z)\frac{1}{r}drdz=4\pi^2\mu_0^2l_iI_{\text{pl}}^2\frac{\int\int rdrdz}{(\oint dl)^2}. \tag{52}$$

All the double integrals in the above written relations have to be performed over the total cross-section area of the plasma.

The safety factor at any flux surface can be calculated with the known relation

$$q_\Psi=\frac{\mu_0}{4\pi^2}F(\Psi)\oint_{\Psi=\text{const}}\frac{dl}{r^2\sqrt{B_r^2+B_z^2}}, \tag{53}$$

where B_r and B_z are the components of the poloidal magnetic field. The line integral has to be performed around a flux surface at any cross section. The value of the safety factor on the magnetic axis q_{ax} can be calculated as the continuous limit of the relation (53).

Another way to obtain the expression of q_{ax}, is to start from the GSE written in a flux coordinate system (a,θ,ζ) and to perform an averaging of this equation between magnetic surfaces[16]

$$(K_0\Psi_a')_a'=-TL_0-PV_0, \tag{54}$$

where

$$T(\Psi)\equiv\frac{\mu_0^2}{2}\frac{dF^2}{d\Psi},\quad P(\Psi)\equiv4\pi^2\mu_0\frac{dp}{d\Psi},$$

$$K\equiv\frac{g_{\theta\theta}}{rD},\quad V\equiv Dr,\quad L\equiv\frac{D}{r}, \tag{55}$$

and

$$D=r_a'z_\theta'-r_\theta'z_a',\quad g_{\theta\theta}=r_\theta'^2+z_\theta'^2. \tag{56}$$

The subscripts 0 denote the zeroth harmonics of the corresponding values in poloidal angle θ of a Fourier development of the cylindrical coordinates. In this coordinate system the safety factor has the expression

$$q=-\frac{\Phi_a'}{\Psi_a'}=-\mu_0\frac{FL_0}{\Psi_a'}. \tag{57}$$

Near the magnetic axis we can use the following simple Fourier representation of the coordinates:

$$r=R_{\text{ax}}+a\cos(\theta),\quad z=Z_{\text{ax}}+a\lambda\sin(\theta), \tag{58}$$

where $R_{\text{ax}},Z_{\text{ax}}$ are the coordinates of the magnetic axis, while λ is the ellipticity of the magnetic surfaces. Thus, in the close vicinity of the magnetic axis we can write

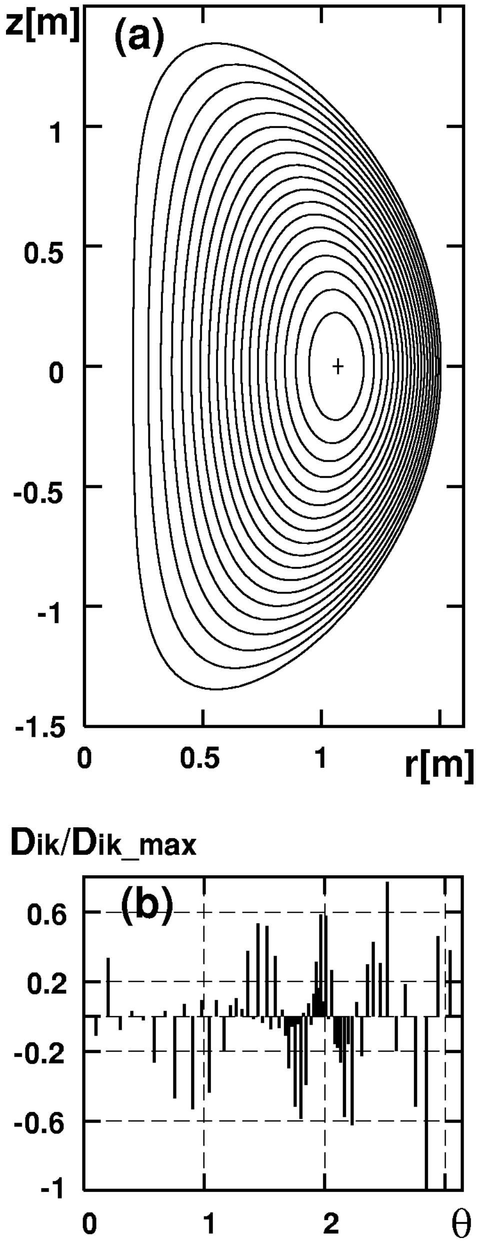

FIG. 1. Equilibrium parameters calculated analytically for a D-shaped plasma. (a) Constant poloidal flux lines; (b) distribution of the $D_{ik}/D_{ik}^{\max}$ coefficients ($i=1,2$, $k=1,\ldots,32$) along the upper plasma boundary. θ is the polar angle from Eq. (62) with its origin at the outer part of the $z=0$ axis. The plasma is characterized by the parameters: $I_{\rm pl}=1$ MA, $B_\varphi^{\rm vac}=1$ T at $R^{\rm vac}=0.8$ m, $\beta_p=1$, $l_i=0.6$, and $q_{\rm ax}=2.3$. The correspondent plasma model parameters [Eq. (4)] are: $a=3.037\,709\,033\,333\,2$, $\alpha=0.123\,918\,350\,760\,12$, $b=2.733\,938\,129\,999\,9$, and $\beta=0.111\,526\,515\,684\,11$.

$$K_0=\frac{a(\lambda^2+1)/2}{R_{\rm ax}\lambda},\quad L_0=\frac{a\lambda}{R_{\rm ax}},\quad V_0=R_{\rm ax}a\lambda, \tag{59}$$

and the expression of the safety factor at the magnetic axis becomes

$$q_{\rm ax}=-\frac{\Phi_a'}{\Psi_a'}=\frac{\mu_0 F}{R_{\rm ax}(T+PR_{\rm ax}^2)}\,\frac{\lambda^2+1}{\lambda}. \tag{60}$$

From this expression we obtain the following relation between the plasma parameters:

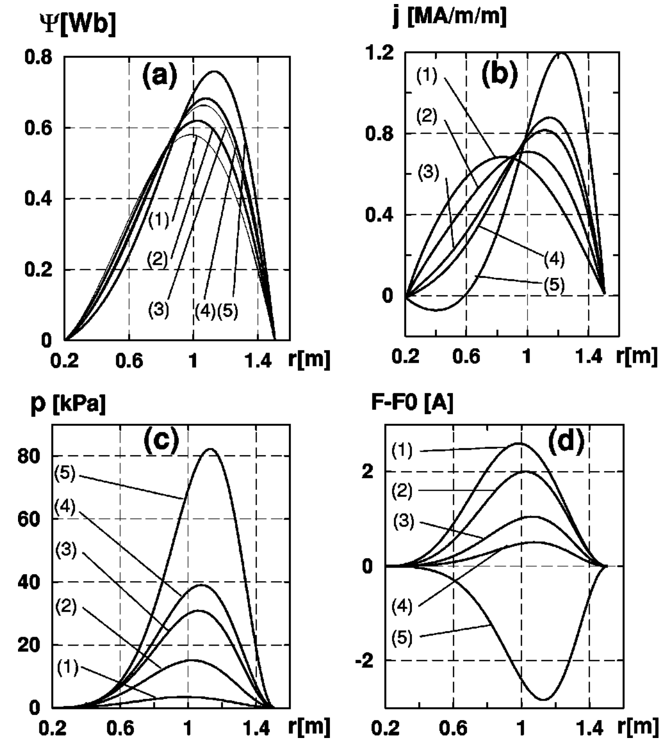

FIG. 2. Equilibrium parameters calculated analytically for a D-shaped plasma characterized by the parameters: $I_{\rm pl}=1$ MA, $B_\varphi^{\rm vac}=1$ T at $R^{\rm vac}=0.8$ m, and different poloidal beta coefficients. The plasma model parameters a and b are calculated, while the remainder two parameters $b=0.05$ and $\beta=-0.02$ are kept fixed. The internal inductance l_i and the safety factor on the axis are now calculated values. (1)-$\beta_p=0.1$, $l_i=0.891$, $q_{\rm ax}=2.29$, $a=0.861\,496\,1$, $\alpha=7.768\,225\,113\,362\,6$; (2)-$\beta_p=0.4$, $l_i=0.97$, $q_{\rm ax}=1.917$, $a=3.740\,037\,678\,774\,1$, $\alpha=5.275\,193\,054\,755\,4$; (3)-$\beta_p=0.8$, $l_i=1.07$, $q_{\rm ax}=1.539$, $a=6.817\,598\,191\,297$, $\alpha=2.440\,073\,216\,604\,4$; (4)-$\beta_p=1.$, $l_i=1.118$, $q_{\rm ax}=1.377$, $a=8.167\,334\,177\,333\,9$, $\alpha=1.141\,626\,626\,128\,2$; and (5)-$\beta_p=2.$, $l_i=1.343$, $q_{\rm ax}=0.943$, $a=14.024\,474\,794\,659$, $\alpha=-4.865\,918\,990\,410\,5$. (a) Poloidal flux $\Psi(r,0)$, (b) toroidal current density $j(r,0)$, (c) pressure $p(r,0)$, and (d) poloidal current $F(r,0)-F_0$ distributions in the $z=0$ plane.

$$aR_{\rm ax}^2\Psi_{\rm ax}+\alpha\Psi_{\rm ax}+bR_{\rm ax}^2+\beta =2\sqrt{\alpha\Psi_{\rm ax}^2+2\beta\Psi_{\rm ax}+(F_0\mu_0)^2}/R_{\rm ax}/q_{\rm ax}. \tag{61}$$

Equations (48), (50), (52), and (61) represent the four relations necessary to determine the four parameters a, α, b, and β.

As an exemplification of our approach two plasma boundaries have been considered, a D-shaped plasma contour and a toroidally diverted one. The considered D-shaped plasma boundary has been described by the following parametric equations:

$$r=a\cos(\theta)+\gamma\sin^2(\theta)+R_0,\quad z=a\lambda\sin(\theta)+Z_0, \tag{62}$$

with $R_0=0.854779$, $Z_0=0$, $a=0.65$, $\lambda=2.07$, $\gamma=-0.3$, and $\theta\in[0:2\pi]$.

Starting from the plasma current, the poloidal beta, the safety factor at the magnetic axis and the internal inductance, as input data, we have determined the four parameters of the considered plasma model and we have calculated the equilibrium values. In Fig. 1 constant poloidal flux lines (a) and

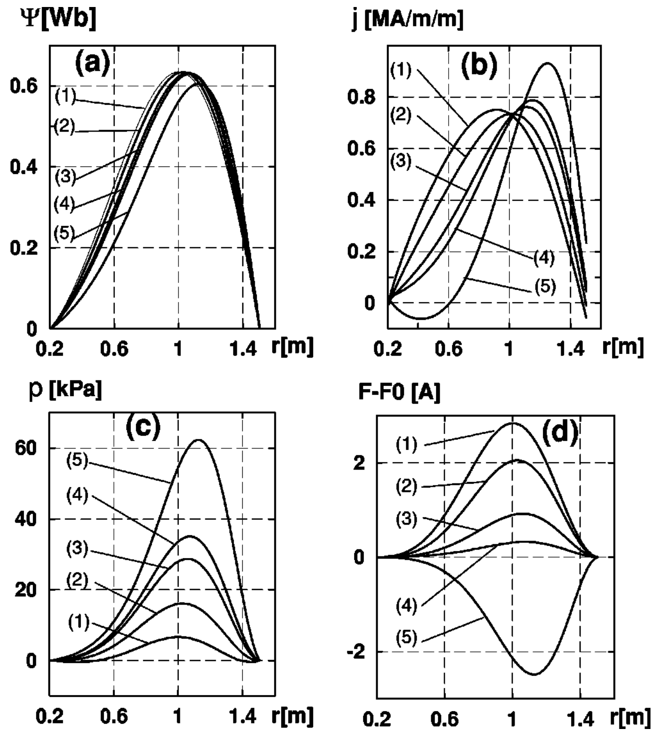

FIG. 3. Equilibrium parameters calculated analytically for a D-shaped plasma characterized by the parameters: $I_{\rm pl}=1$ MA, $R^{\rm vac}=0.8$ m, $l_i=1.$, $q_{\rm ax}=1.6$ and different poloidal beta coefficients. The four plasma model parameters are calculated. (1)-$\beta_p=0.1$, $a=2.632\,872\,894\,361\,8$, $\alpha=6.980\,286\,744\,763\,9$, $b=-0.318\,365\,791\,620\,92$, beta$=0.019\,876\,534\,2$; (2)-$\beta_p=0.4$, $a=4.139\,628\,865\,006$, $\alpha=5.127\,824\,615\,869$, $b=-0.052\,806\,103\,278\,491$, $\beta=-0.009\,917\,412\,3$; (3)-$\beta_p=0.8$, $a=6.447\,564\,959\,085\,8$, $\alpha=2.301\,009\,910\,134\,8$, $b=0.217\,106\,754\,842\,49$, $\beta=0.000\,055\,964\,3$; (4)-$\beta_p=1.$, $a=7.701\,877\,016\,023\,5$, $\alpha=0.746\,557\,451\,828\,34$, $b=0.328\,869\,133\,481\,84$, $\beta=0.020\,312\,502\,6$; and (5)-$\beta_p=2.$, $a=12.846\,764\,096\,776$, $\alpha=-6.709\,106\,656\,438\,4$, $b=1.220\,881\,506\,009\,6$, $\beta=-0.022\,645\,512\,8$. (a) Poloidal flux $\Psi(r,0)$, (b) toroidal current density $j(r,0)$, (c) pressure $p(r,0)$, and (d) poloidal current $F(r,0)-F_0$ distributions in the $z=0$ plane.

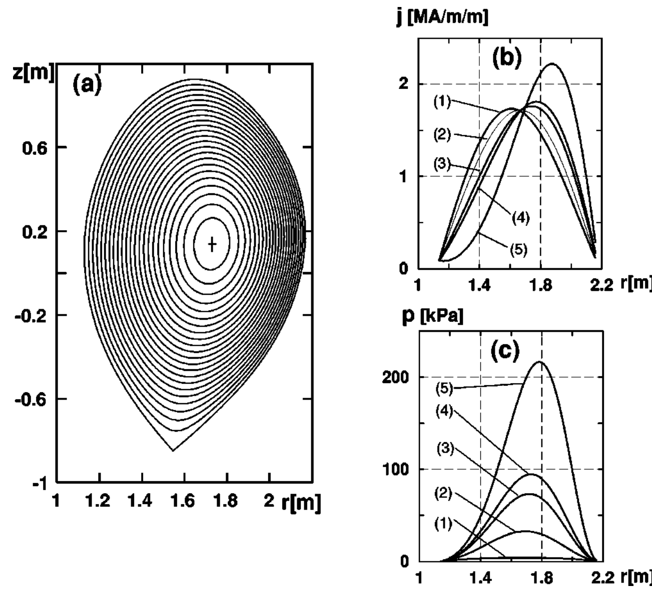

FIG. 4. Equilibrium parameters calculated analytically for the diverted plasma corresponding to the discharge No. 5000 at 1.55 s of the ASDEX Upgrade tokamak (Ref. 9), characterized by the parameters: $I_{\rm pl}=1$ MA, $\beta_p=1.$, $B_\varphi^{\rm vac}=1.687$ T at $R^{\rm vac}=1.65$ m, $l_i=0.68$, and $q_{\rm ax}=1.2$. The correspondent plasma model parameters are: $a=2.520\,013\,516\,172\,1$, $\alpha=0.240\,261\,447\,995\,81$, $b=2.268\,012\,164\,554\,9$, and $\beta=0.216\,235\,303\,196\,23$. In (a) constant poloidal flux lines for this configuration are represented. For the following plasma parameters: $I_{\rm pl}=1$ MA, $R^{\rm vac}=1.65$ m, $l_i=1.$, $q_{\rm ax}=0.7$, and different poloidal beta coefficients, we have obtained: (1)-$\beta_p=0.1$, $a=-0.167\,860\,399\,051\,09$, $\alpha=12.514\,244\,023\,85$, $b=0.253\,806\,324\,643\,45$, $\beta=0.029\,814\,802\,110\,195$; (2)-$\beta_p=0.4$, $a=1.112\,769\,392\,920\,2$, $\alpha=8.903\,258\,994\,428\,2$, $b=0.302\,538\,199\,097\,18$, $\beta=-0.014\,876\,117\,929\,816$; (3)-$\beta_p=0.8$, $a=2.865\,538\,138\,725\,5$, $\alpha=3.872\,998\,042\,573\,6$, $b=0.345\,748\,866\,006\,19$, $\beta=0.000\,083\,946\,442\,85$; (4)-$\beta_p=1.$, $a=3.764\,070\,018\,548\,5$, $\alpha=1.256\,055\,412\,51$, $b=0.361\,276\,854\,577\,02$, $\beta=0.003\,046\,875\,440\,348$; and (5)-$\beta_p=2.$, $a=8.268\,985\,456\,180\,3$, $\alpha=-12.266\,924\,197\,253$, $b=0.541\,895\,002\,672\,79$, $\beta=-0.003\,396\,826\,982\,49$. (b) Toroidal current density $j(r,0)$, and (c) pressure $p(r,0)$ distributions along the $z=Z_{\rm ax}$ axis.

calculated D_{ik} coefficients along the upper half of the D-shaped plasma boundary (b) are represented. Note that due to the up–down symmetry, only terms in cosine have been retained. To see how the plasma parameters can be varied, two scenarios have been investigated for five different poloidal beta values. In the first one, the plasma model parameters a and α have been calculated, while the remainder two parameters b and β are given and kept constant. The such obtained flux, current density, pressure and poloidal current distributions are presented in Fig. 2. The calculated plasma model parameters a and α and the resulted plasma parameters l_i and $q_{\rm ax}$ are also reported. In the second scenario, the influence of the same different poloidal beta values on the same plasma profiles has been investigated at given plasma current, internal inductance and safety factor at the axis. These dependencies as well as the obtained plasma model parameters are reported in Fig. 3.

For the diverted plasma contours, corresponding to the discharge No. 5000 at 1.55 s of the ASDEX Upgrade tokamak,[9] similar results, corresponding to the second scenario are presented in Fig. 4. For this case, both set of terms, in $\cos(\omega_k)$ and $\sin(\omega_k)$ have been retained.

IV. CONCLUSIONS

We have presented two families of exact solutions to the Grad–Shafranov equation which for the first time, to our knowledge, have a current density parametrisation with four degrees of freedom. Thus, an independent choice of the plasma current $I_{\rm pl}$, the poloidal beta β_p, the internal inductance l_i, and the safety factor q (at the boundary or at the magnetic axis) can be made. In Sec. II we describe the two new families of analytical solutions resulting from the conversion of the original inhomogeneous partial differential equation into an homogeneous partial differential equation and an inhomogeneous ordinary differential equation, the solution being a linear combination between the general homogeneous solution and any particular inhomogeneous solution. Some details on drawing the inhomogeneous particular solution are given in the Appendix. The determination of the four independent plasma parameters and some numerical results are presented in Sec. III.

We intend to use these exact solutions as benchmark of numerical equilibrium codes,[17] especially in the vicinity of the magnetic axis and of the X point of a toroidally diverted configuration, by considering as input data for numerical codes plasma density profiles with given plasma model parameters a, α, b, and β, at given I_{pl}.

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APPENDIX

Introducing the polynomial expression for ζ given by Eq. (26) in Eq. (24), we obtain

$$F(\zeta)=x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+(2-x)\sum_{n=1}^{\infty} n a_n x^{n-1} -\tilde{a}\sum_{n=0}^{\infty} a_n x^n=\sum_{n=1}^{\infty}[(n+1)(n+2)a_{n+1}-(n +\tilde{a})a_n]x^n+2a_1-\tilde{a}a_0=1+\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x}{2}\right)^n. \quad \text{(A1)}$$

By choosing the following particular relationship

$$2a_1-\tilde{a}a_0=1, \quad \text{(A2)}$$

we obtain

$$a_n=\frac{n+\tilde{a}-1}{n(n+1)}a_{n-1}+\frac{1}{2^{n-1}(n+1)!}. \quad \text{(A3)}$$

Therefore, following a recurrent approach, we obtain

$$\begin{aligned}a_n&=\frac{n+\tilde{a}-1}{n(n+1)}\left[\frac{n+\tilde{a}-2}{n(n-1)}a_{n-2}+\frac{1}{2^{n-2}n!}\right]+\frac{1}{2^{n-1}(n+1)!}\\&=\frac{(n+\tilde{a}-1)(n+\tilde{a}-2)(n+\tilde{a}-3)}{(n-2)(n-1)^2n^2(n+1)}a_{n-3}\\&\quad+\frac{(n+\tilde{a}-1)(n+\tilde{a}-2)}{(n-1)n^2(n+1)}\frac{1}{2^{n-3}(n-1)!}\\&\quad+\frac{n+\tilde{a}-1}{n(n+1)}\frac{1}{2^{n-2}n!}+\frac{1}{2^{n-1}(n+1)!}\\&=\cdots=2\frac{(\tilde{a}+1)_{n-1}}{n!(n+1)!}a_1\\&\quad+\frac{1}{(n+1)!}\left[\frac{(n+\tilde{a}-1)\times\cdots\times(\tilde{a}+2)}{n\times\cdots\times 3}\frac{1}{2^1}\right.\\&\quad+\frac{(n+\tilde{a}-1)\times\cdots\times(\tilde{a}+3)}{n\times\cdots\times 4}\frac{1}{2^2}+\cdots\\&\quad+\frac{(n+\tilde{a}-1)(n+\tilde{a}-2)}{(n-1)n}\frac{1}{2^{n-3}}+\frac{n+\tilde{a}-1}{n}\frac{1}{2^{n-2}}\\&\quad\left.+\frac{1}{2^{n-1}}\right]\\&=\frac{(\tilde{a}+1)_{n-1}}{n!(n+1)!}\left[2a_1+\sum_{m=1}^{n-1}\frac{1}{(\tilde{a}+1)_m}\frac{(m+1)!}{2^m}\right],\\&\quad n=2,3,\ldots. \end{aligned}\quad \text{(A4)}$$

Thus,

$$\zeta=a_0+\frac{1+\tilde{a}a_0}{2}x+\sum_{n=2}^{\infty}\frac{(\tilde{a}+1)_{n-1}}{n!(n+1)!}\left[2a_1+\sum_{m=1}^{n-1}\frac{1}{(\tilde{a}+1)_m}\frac{(m+1)!}{2^m}\right]x^n \quad \text{(A5)}$$

or

$$\begin{aligned}\zeta&=a_0+\frac{1+\tilde{a}a_0}{2}x+\frac{2a_1}{a_0}\left[{}_1F_1(\tilde{a};2;x)-1-\frac{\tilde{a}}{2}x\right]\\&\quad+\sum_{n=2}^{\infty}\frac{(\tilde{a}+1)_{n-1}}{n!(n+1)!}\sum_{m=1}^{n-1}\frac{1}{(\tilde{a}+1)_m}\frac{(m+1)!}{2^m}x^n\\&=-\frac{1}{\tilde{a}}+\frac{1+\tilde{a}a_0}{\tilde{a}}{}_1F_1(\tilde{a};2;x)\\&\quad+2\tilde{a}\sum_{n=2}^{\infty}\frac{(\tilde{a}+1)_{n-1}}{n!(n+1)!}\sum_{m=2}^{n}\frac{1}{(\tilde{a})_m}\frac{(m)!}{2^m}x^n. \end{aligned}\quad \text{(A6)}$$

By using a new variable

$$\theta\equiv\sum_{n=2}^{\infty}\frac{(\tilde{a}+1)_{n-1}}{n!(n+1)!}\sum_{m=2}^{n}\frac{1}{(\tilde{a})_m}\frac{m!}{2^m}x^n, \quad \text{(A7)}$$

we have

$$\begin{aligned}\theta&=\frac{1}{\tilde{a}}\sum_{n=2}^{\infty}\frac{(\tilde{a})_n}{n!(2)_n}\left[\sum_{m=0}^{n}\frac{m!}{(\tilde{a})_n 2^m}-1-\frac{1}{2\tilde{a}}\right]x^n\\&=\frac{1}{\tilde{a}}\sum_{n=0}^{\infty}\frac{(\tilde{a})_n}{n!(2)_n}\sum_{m=0}^{n}\frac{m!}{(\tilde{a})_n 2^m}x^n-\frac{1}{\tilde{a}}\left[1+\frac{\tilde{a}}{2}\left(1+\frac{1}{2\tilde{a}}\right)x\right]\\&\quad-\frac{1}{\tilde{a}}\left(1+\frac{1}{2\tilde{a}}\right)\left[{}_1F_1(\tilde{a};2;x)-1-\frac{\tilde{a}}{2}x\right]\\&=\frac{1}{\tilde{a}}\sum_{n=0}^{\infty}\frac{(\tilde{a})_n}{n!(2)_n}\sum_{m=0}^{\infty}\frac{m!}{(\tilde{a})_n 2^m}x^n\\&\quad-\frac{1}{\tilde{a}}\sum_{n=0}^{\infty}\frac{(\tilde{a})_n}{n!(2)_n}\sum_{m=n+1}^{\infty}\frac{m!}{(\tilde{a})_n 2^m}x^n-\frac{1}{\tilde{a}}\\&\quad-\frac{1}{2}\left(1+\frac{1}{2\tilde{a}}\right)x-\frac{1}{\tilde{a}}\left(1+\frac{1}{2\tilde{a}}\right){}_1F_1(\tilde{a};2;x)\\&\quad+\frac{1}{\tilde{a}}\left(1+\frac{1}{2\tilde{a}}\right)+\frac{1}{2}\left(1+\frac{1}{2\tilde{a}}\right)x\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\tilde{a}}\, {}_2F_1\left(1,1;\tilde{a};\frac{1}{2}\right)\, {}_1F_1(\tilde{a};2;x) \\
&\quad - \frac{1}{\tilde{a}}\left(1+\frac{1}{2\tilde{a}}\right)\, {}_1F_1(\tilde{a};2;x) + \frac{1}{2\tilde{a}^2} \\
&\quad - \frac{1}{\tilde{a}} \sum_{n=0}^{\infty} \frac{(\tilde{a})_n}{n!(2)_n} \sum_{k=0}^{\infty} \frac{(k+n+1)!}{(\tilde{a})_{k+n+1} 2^{k+n+1}} x^n. \qquad \text{(A8)}
\end{aligned}
$$

But

$$(k+n+1)! = (n+2)_k (n+1)!,$$

$$(\tilde{a})_{n+k+1} = (\tilde{a})_{n+1} (n+\tilde{a}+1)_k,$$

thus,

$$
\begin{aligned}
\theta &= \frac{1}{2\tilde{a}} + \frac{1}{\tilde{a}}\left[{}_2F_1\left(1,1;\tilde{a};\frac{1}{2}\right) - 1 - \frac{1}{2\tilde{a}} \right] {}_1F_1(\tilde{a};2;x) \\
&\quad - \frac{1}{\tilde{a}} \sum_{n=0}^{\infty} \frac{1}{2(n+\tilde{a})n!}\, {}_2F_1\left(n+2,1;n+\tilde{a}+1;\frac{1}{2}\right)\left(\frac{x}{2}\right)^2,
\end{aligned}
$$

and the final expression for ζ is that given in Eq. (27)

$$
\begin{aligned}
\zeta &= \left[2\, {}_2F_1\left(1,1;\tilde{a};\frac{1}{2}\right) - 2 + a_0 \right] {}_1F_1(\tilde{a};2;x) \\
&\quad - \sum_{n=0}^{\infty} \frac{1}{(n+\tilde{a})n!}\, {}_2F_1\left(n+2,1;n+\tilde{a}+1;\frac{1}{2}\right)\left(\frac{x}{2}\right)^2.
\end{aligned}
$$

[1] V. D. Shafranov, Sov. Phys. JETP **6**, 545 (1958); Zh. Eksp. Teor. Fiz. **33**, 710 (1957).
[2] H. Grad and H. Rubin, in *Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 21, p. 190.
[3] L. S. Solov'ev, Sov. Phys. JETP **26**, 400 (1968); Zh. Eksp. Teor. Fiz. **53**, 626 (1976).
[4] S. B. Zheng, A. J. Wootton, and E. R. Solano, Phys. Plasmas **3**, 1176 (1996).
[5] E. K. Maschke, Plasma Phys. **15**, 535 (1972).
[6] F. Herrnegger, in *Proceedings of the 5th European Conference on Controlled Fusion and Plasma Physics*, Grenoble, August, 1972 (European Physical Society, Petit Lancy, 1972), Vol. I, p. 26.
[7] J. M. Greene, Plasma Phys. Controlled Fusion **30**, 327 (1988).
[8] P. J. McCarthy, Phys. Plasmas **6**, 3554 (1999).
[9] W. Schneider, P. J. McCarthy, K. Lackner, O. Behler, K. Martin, and R. Merkel, Fusion Eng. Des. **48**, 127 (2000).
[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972), pp. 537, 555.
[11] L. J. Slater, *Confluent Hypergeometric Functions* (Cambridge University Press, Cambridge, 1960).
[12] W. F. Perger, M. Nardin, and A. Bhalla, Comput. Phys. Commun. **77**, 249 (1993).
[13] A. R. Barnett, Comput. Phys. Commun. **21**, 297 (1981).
[14] L. E. Zakharov and V. D. Shafranov, *Reviews of Plasma Physics* (Consultants Bureau, New York, 1986), Vol. 11, p. 153.
[15] H. P. Zehrfeld (private communication, 2003).
[16] L. E. Zakharov and A. Pletzer, Phys. Plasmas **6**, 4693 (1999).
[17] C. V. Atanasiu, S. Günter, K. Lackner, A. Moraru, and A. A. Subbotin, in *Proceedings of the 28th EPS Conference on Controlled Fusion and Plasma Physics*, Funchal, June 2001 (European Physical Society, Petit Lancy, 2001), Vol. 25A, p. 1105.