🎯 FORMAL LANGUAGES FINAL — ULTIMATE CHEAT SHEET
Exam: Thursday, Dec 5, 2025 @ 2:00 PM | OPEN NOTES | Target: 49/70 points

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⚡ TURING MACHINE FAST TEMPLATES

Basic TM Structure
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• States: $q_0$ (start→), qaccept (⊙), qreject (○)
• Transitions: symbol → symbol, Direction (R/L)
• Alphabet: Σ = {0, 1}, Blank = ␣
• ALWAYS handle 0, 1, and ␣ from EVERY state!
```

Template 1: Simple Recognition (e.g., A = {0, 00})
Pattern: DFA → TM (just add R movements)
```

$q_0$: 0→R $q_1$, 1→R qrej, ␣→R qrej
$q_1$: 0→R $q_2$, 1→R qrej, ␣→R qacc  ← accepts "0"
$q_2$: 0→R qrej, 1→R qrej, ␣→R qacc  ← accepts "00"
```

Template 2: Ends-With Pattern (e.g., B ends with "010")
Pattern: States = "memory" of last N symbols
```

$q_0$: 0→R $q_1$, 1→R $q_0$, ␣→R qrej
$q_1$: 0→R $q_1$, 1→R $q_2$, ␣→R qrej
$q_2$: 0→R $q_3$, 1→R $q_0$, ␣→R qrej
$q_3$: 0→R $q_1$, 1→R $q_0$, ␣→R qacc  ← ends with 010!
```

Template 3: Binary Decrement (w − 1)
Algorithm:
1. Scan RIGHT to end (find ␣)
2. Move LEFT, look at rightmost digit:
   - If **1**: write 0, ACCEPT ✓
   - If **0**: write 1, continue left (borrow)
3. Keep borrowing until find a 1
4. Edge case: reject if input = "0"

```

qstart: 0→R qstart, 1→R qstart, ␣→L qdec

```
qdec: 1→0,L qacc, 0→1,L qborrow, ⊔→R qrej
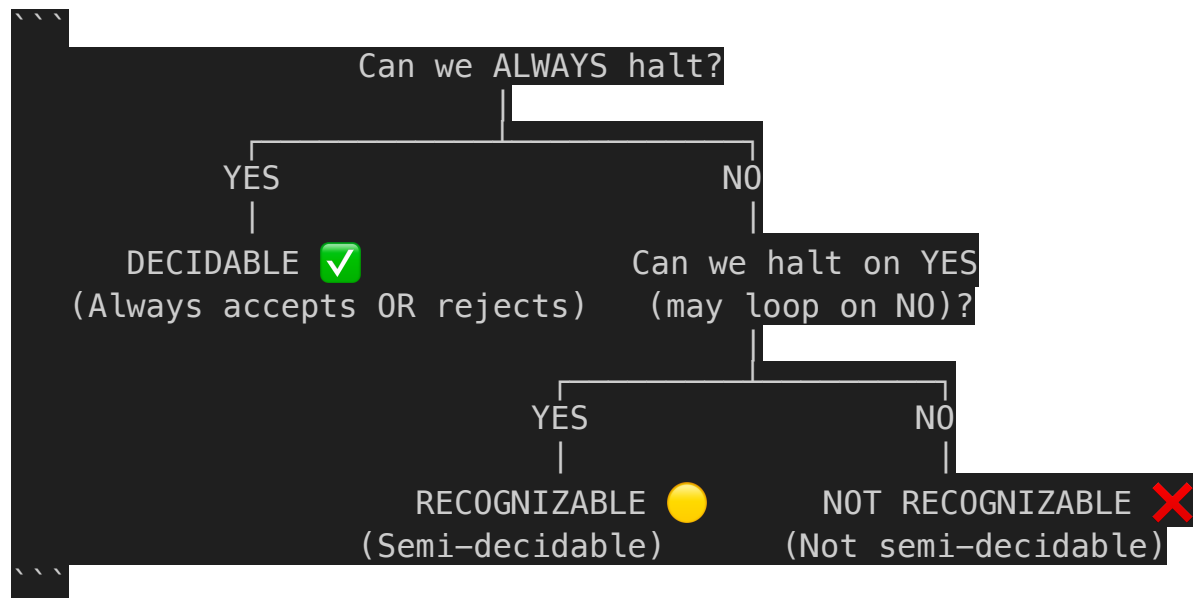qborrow: 0→0,L qborrow, 1→0,L qacc, ⊔→R qrej
```

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## 📊 COUNTABILITY INSTANT ANSWERS

| Set | Countable? | Method |
|-----|------------|---------|
| ℕ | ✅ YES | By definition |
| ℤ | ✅ YES | 0,1,−1,2,−2,3,−3,... |
| ℚ | ✅ YES | Enumerate by sum of numerator + denominator |
| ℝ | ❌ NO | Cantor's diagonal |
| Σ* | ✅ YES | ε, then by length: 0,1,00,01,10,11,... |
| ℕ×ℕ | ✅ YES | Enumerate by sum: (0,0),(0,1),(1,0),(0,2),(1,1),(2,0),... |

**For pairs (i,j):** Enumerate by i+j, then within each sum level.

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## 🎓 DECIDABILITY DECISION TREE

```
                Can we ALWAYS halt?
                        |
            ┌───────────┴───────────┐
          YES                       NO
            |                        |
       DECIDABLE ✅          Can we halt on YES
  (Always accepts OR rejects)   (may loop on NO)?
                                        |
                              ┌─────────┴─────────┐
                            YES                   NO
                              |                    |
                      RECOGNIZABLE 🟡     NOT RECOGNIZABLE ❌
                      (Semi-decidable)    (Not semi-decidable)
```

Key Rules
- **Decidable → Recognizable** (always true!)
- **If L and L̄ both recognizable → L is decidable**
- **If L decidable → L̄ is decidable** (flip accept/reject)

🧪 DECIDABILITY QUICK REFERENCE

Problem	Decidable?	Recognizable?	Why?
A_DFA	✅ YES	✅ YES	Can simulate DFA and halt
E_DFA	✅ YES	✅ YES	Check if accept state reachable
EQ_DFA	✅ YES	✅ YES	Minimize both, compare states
ALL_DFA	✅ YES	✅ YES	Check if complement is empty
A_CFG	✅ YES	✅ YES	CYK algorithm always terminates
E_CFG	✅ YES	✅ YES	Check if any derivation exists
EQ_CFG	❌ NO	❌ NO	Can't compare ∞ languages; complement IS recognizable
ALL_CFG	❌ NO	✅ YES	Can enumerate Σ* and test each
A_TM	❌ NO	✅ YES	Halting problem!
HALT_TM	❌ NO	✅ YES	Reduces from A_TM
E_TM	❌ NO	❌ NO	Can't test if TM accepts nothing

Pro Tips for Part 4
- **DFA problems:** Almost all DECIDABLE ✅
- **CFG problems:** Some decidable (A, E), NOT equality/all
- **TM problems:** Almost all UNDECIDABLE ❌
- **EQ_CFG special:** NOT decidable, NOT recognizable (complement IS recognizable!)

✅ EXAM DAY BATTLE TACTICS

Problem 1: TM Design (20 pts) — 15-20 min

- [ ] Draw arrow → into start state q_0
- [ ] Double circle ⊙ for qaccept
- [ ] Single circle ○ for qreject (can omit)
- [ ] Label EVERY transition: `symbol → symbol, Direction`
- [ ] Handle 0, 1, AND ␣ from every state
- [ ] Test with examples: trace through the TM!

Problem 2: Binary Decrement (15 pts) — 20–25 min
- [ ] **Part (a) — 5 pts:** Write clear English algorithm
  - "Scan right to end, move left, check digit..."
  - Mention edge case: reject "0"
- [ ] **Part (b) — 10 pts:** Complete state diagram
  - Show scan-right phase
  - Show decrement logic
  - Show borrow chain
  - Label ALL transitions!

Problem 3: Countability (10 pts) — 10 min
- [ ] Answer **YES** or **NO** clearly
- [ ] Show enumeration: (0,0),(0,1),(1,0),(0,2),(1,1),(2,0),...
- [ ] OR show bijection: $f(i,j) = 2^i \times 3^j$
- [ ] State: "Countable because we can list all elements systematically"

Problem 4: Decidability (20 pts) — 15–20 min
- [ ] Answer **BOTH**: (a) Is it decidable? (b) Is it recognizable?
- [ ] Give reasoning for each
- [ ] Remember: decidable → recognizable (one direction)
- [ ] Use complement arguments when helpful
- [ ] Reference known results (e.g., "Similar to EQ_DFA which IS decidable")

🔥 DANGER ZONES — DON'T LOSE POINTS!

❌ Common Error	✅ Fix
Forgot ␣ transitions	Add ␣ from EVERY state!
Didn't mark start state	Draw → arrow into q_0
Missing accept state circle	Double circle ⊙ on qaccept
Said "EQ_CFG is decidable"	NO! It's NOT decidable OR recognizable

Confused recognizable ↔ decidable	Decidable = ALWAYS halts; Recognizable = may loop
No justification for decidability	Always explain WHY (simulate, reduce, etc.)
Forgot edge cases in TM	Test with: ε, "0", "1", "00", "01", etc.

🎯 POINT MAXIMIZATION STRATEGY

70 points available, need 49 (70%)

Priority Tiers:
1. **Must-Get (35 pts):** Problem 1 (15/20), Problem 3 (10/10), Problem 4 (10/20)
2. **Push-For (20 pts):** Problem 2 part (a) (5/5), Problem 1 full (20/20), Problem 4 (15/20)
3. **Bonus (15 pts):** Problem 2 part (b) diagram (10/15), Problem 4 perfect (20/20)

Time Budget:
- **0-5 min:** Skim all problems, plan attack order
- **5-25 min:** Problem 1 (TM designs) – 20 pts
- **25-50 min:** Problem 2 (Binary decrement) – 15 pts
- **50-60 min:** Problem 3 (Countability) – 10 pts
- **60-80 min:** Problem 4 (Decidability) – 20 pts
- **80-90 min:** Review, check answers, trace examples

💡 LAST-MINUTE REMINDERS

1. **TM Design = DFA + Tape movements** (usually just add R to everything)
2. **Countability:** Show the enumeration explicitly! (0,0),(0,1),(1,0),...
3. **Decidability:** DFA ✅, most CFG ❌, all TM ❌
4. **Special:** EQ_CFG is NOT recognizable (complement IS recognizable)
5. **Trace examples:** Walk through your TM with "0", "00", "010"
6. **Open notes:** Use this sheet + detailed reference guide!

🚀 YOU'VE GOT THIS!

Remember:
— Clear diagrams > perfect notation
— Partial credit is generous
— Show your reasoning
— 49/70 is totally achievable!

Final check before exam:
— [ ] Print this cheat sheet
— [ ] Print detailed reference guide
— [ ] Practice tracing TMs
— [ ] Review countability enumerations
— [ ] Memorize decidability table

Go ace that exam! 🎯🔥