# Analysis of MAC Protocols Using Markov Chains

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**Abstract**

Markov Chain modeling is a powerful method for statistical analysis of stochastic phenomena. In computer networks, medium access control (MAC) can be seen as a Markov Chain, where the MAC channel changes between being busy and idle whenever data transmission starts and ends. The paper serves as a review and discussion on modeling MAC protocols using Markov Chains and its comparison with other analytical methods. The properties of Markov Chains are explained from a mathematical approach and are used to create a model for analyzing MAC channels. The exploration and basis for stochastic analysis in computer networks are explained and shown with a comparative analysis on ALOHA and CSMA protocols.

## 1 Introduction

With new technologies on IoT, video game and 3D video streaming, and cloud computing, there is an ever-growing demand for faster, more efficient computer communication. Not to mention, the era of Big Data threatens to increase internet traffic exponentially, which calls for improvements of our network systems.

Designing efficient network protocols often requires thorough mathematical analysis to evaluate the expected behavior of a protocol. For this, the field of probability and statistics provides a powerful stochastic method for analyzing such systems: Markov Chains. This stochastic method allows us to consider certain attributes of network systems as variables dependent on the state in which the system finds itself in. Medium access control (MAC) involves communication through a wired, optical, or wireless transmission medium. In MAC, the communication channel shifts between idle and busy states. From this assumption, methods specific to Markov Chains can be used to further study the channel's behavior and characteristics.

## 2 Medium Access Control Communication

Before we delve into the exploration of a Markov Chain model of a networking system it is paramount to understand the characteristics, restraints, and unique properties of medium access control (MAC) communication.

The MAC sub-layer handles hardware interactions through a wired, optical, or wireless transmission medium. Both the MAC sub-layer and the Logical Link Control (LLC) sub-layer make up the data-link layer (transfer data between nodes on a network segment across a physical medium). The MAC sub-layer is responsible for flow control (manage rate of transmitted data) and multiplexing (handle multiple signals from different terminals) on a single transmission medium. The MAC sub-layer abstracts the physical layer so that all upper layers of the OSI network stack are "oblivious" to it, including the LLC [10].

Packets are transmitted through the channel from a sender to a receiver. Whenever a node transmits the channel is considered to be busy. Otherwise, the channel is considered to be idle. If two or more nodes transmit at the same time, the channel is busy but unable to transmit, causing a "collision" (multiple access interference - MAI).

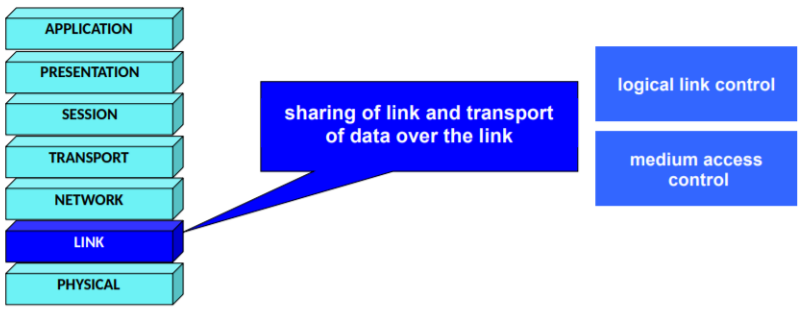


Figure 1: The Open Systems Interconnection model (OSI model) showing the Medium Access Control (MAC) and Logical Link Control (LLC) sub-layers.[8]

Development of a MAC protocol (and the mathematics linked to its analysis) can get quite complex. Thus, let us consider a minimalist and simple MAC system to which the Markov property applies. To recall, the Markov property requires a process which allows for predictions of a new state based solely on its previous state. This "memory-less" property can be expressed the following way

$$\begin{aligned} P[X(n+1) = j \mid X(n) = i,\, X(n-1) = i_{n-1}, \cdots,\ X(0) = i_0\ ] \\ = P[X(n+1) = j | X(n) = i],\ \forall n. \end{aligned} \tag{1}$$

Where $i$ is the previous $i$-th state and $j$ is the new $j$-th state, and $t$ is the time slot number in a Discrete Time Markov Chain (DTMC) [14].

Let $p_{ij}$ denote the probability of the system changing to state $j$ given state $i$. Also, let us denote state Idle (0) and state Busy (1). It is important to clarify that for any MAC system,

$$p_{01} = p_{10} = 1. \tag{2}$$

This means that the probability of changing state is certain [11]. A way to understand this concept is by comparing it to a door. What is the probability of a door being opened when it is closed? The answer is 1, since closing a door when it is closed is impossible, thus it changes states with transition probability 1. Since the system is non-persistent, the channel must be Idle before it becomes Busy again. Thus the channel shifts back and forth from state 0 to 1 with probability 1.

# 3 Markov Chain Model

A Markov Chain model for a two state system is shown in **Figure 2**

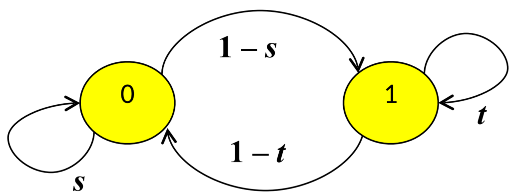


Figure 2: Transition diagram for MAC channel system.[8]

Moreover, the following transition matrix is inferred from the previous diagram

$$P = \overline{p_{ij}} = \begin{bmatrix} p_{00} = s & p_{01} = 1 - s \\ p_{10} = 1 - t & p_{11} = t \end{bmatrix} \tag{3}$$

where $1 \geq p_{nm} \geq 0$.

## 3.1 Properties of Homogeneous Markov Chains

It is important to explain the properties associated with Markov Chains before delving into analysis of our MAC system.

Let $X(t)$ be the state of the process at time $t$. Let us denote $p_{ji}$ as the probability of state $i$ to shift to state $j$. Additionally, let $\pi_i = P[X(t) = i]$ represent the stationary probability of $i$, which represents the probability that the system will find itself at state $i$ at time $t$. The following properties are true for all Markov Chain models that are homogeneous (referred to as "balance equations") [5]

$$\sum_{i=0}^{\infty} p_{ji} = 1 \tag{4}$$

- Any system must transition to some state starting from some state.

$$\sum_{j=0}^{\infty} \pi_j = 1 \tag{5}$$

- The probability that $X(t)$ assumes any value at time $t$ is certain.

$$\pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij} \tag{6}$$

- Given a Markov Chain with stationary probabilities, a system can be at a given state $k$ only if it reaches it from some state.

$$\pi_j \sum_{i=0}^{\infty} p_{ij} = \sum_{i=0}^{\infty} \pi_i p_{ij} \tag{7}$$

- The likelihood of transitions into a state must equal the likelihood of transitions away the state.

## 3.2 Timeless Property

Observe that the transition matrix $P$ represents the matrix that transforms a state vector at time $t$ to the next state at $t+1$. Therefore at time 0 starting at state $i$ we get

$$\begin{pmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_{ii} \\ p_{ji} \end{pmatrix}$$

Since we started in state $i$, both vectors (left and right of the equation) represent the probability of the state being $i$ (up) and $j$ (down) at times $t$ and $t+1$ respectively. Since $p_{mn}$ is a stochastic real number, the resulting is a probability vector.

This means we can create the sequence [3]

$$\mathbf{P}(t) = \begin{pmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{pmatrix} \cdot \mathbf{P}(t-1) = \prod_{t=0}^{\infty} P \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P^t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Where $\mathbf{P}$(t) represents the probability vector of the system at time $t$. This sequence converges to a vector $\mathbf{q}$ consisting of the stationary probabilities of the Markov Chain

$$\lim_{n \to \infty} \mathbf{P}(n) = \begin{pmatrix} \pi_i \\ \pi_j \end{pmatrix} = \mathbf{q} \tag{8}$$

In fact, the vector $\mathbf{q}$ is the eigenvector of the transition matrix for the system $M$

$$P \cdot \mathbf{q} = \mathbf{q} \tag{9}$$

***It is important to note that for the $2 \times 2$ case, it is true that the sequence converges to a steady-state vector $q$, which is also the eigenvector for matrix P [17]. For Markov Chains with more states we require the Markov Chain to be irreducible. A Markov Chain is considered irreducible if there exists a non-zero probability from every state to every other state [13]. In other words, there is a path from every state to every other state, thus the system can't get stuck in a small group of states. We will explain intuitively how the sequence $\mathbf{P}(n)$ converges in the MAC channel case, thus the proof for the general case will be omitted.**

### 3.3 Markov Chains in a MAC Channel

As explained in **Section 2**, the channel shifts back and forth from Busy state to Idle state with probability 1. Thus the transition matrix for this Markov Chain is

$$P_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{10}$$

It follows that [3]

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \pi_i \\ \pi_j \end{pmatrix} = \begin{pmatrix} \pi_j \\ \pi_i \end{pmatrix}$$

Observe that the transition matrix $P$ represents a transformation that "flips" the values $\pi_i$ and $\pi_j$. The steady-state vector $\mathbf{q}$ for the sequence $\mathbf{P}(t)$ has the characteristic $\pi_i = \pi_j$ [17]. From **Equation 5** we get $\pi_i + \pi_j = 1$. Therefore

$$\mathbf{q} = \begin{pmatrix} \pi_i \\ \pi_j \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

Not to mention, vector $\mathbf{q}$ is an eigenvector for matrix $P$.

# 4 Throughput Analysis

First of all, let $S$ be the throughput of a MAC system $M$, where throughput represents the rate of data transfer over a medium. Additionally, let us denote $T_0 = \overline{I}$ to be the average idle time of the channel, and $T_1 = \overline{B}$ to be the average busy time of the channel. Let $\overline{U}$ be the utilization period, which is the portion of time used to send user data. For any random event $X$, the portion of time $t$ where $X$ happens with probability $P[X = x] = p_x$ is equal to $tp_x$. Since $\pi_1 = \pi_0$, throughput is defined as

$$S = \frac{\textbf{Data Transfer Time}}{\textbf{Idle Time} + \textbf{Busy Time}} = \frac{\pi_1 \overline{U}}{\pi_0 \overline{I} + \pi_1 \overline{B}} = \frac{\overline{U}}{\overline{I} + \overline{B}}, \tag{11}$$

Lets assume the same model introduced by Kleinrock and Tobagi [11] for analyzing MAC protocols. It is important to note that this model is only an approximation of the real case. Regardless, this analysis provides a good baseline for the analysis of protocols such as ALOHA, CSMA, CSMA/CD, and CSMA/CA [7]. According to the model, there is a large number of nodes that constitute a Poisson source transmiting data packets through the channel with an aggregate mean generation rate of $\lambda$ packets per unit time [6]. Thus, we can assume that packet arrivals follow a Poisson distribution. With Poisson arrivals, the inter-arrival times (time between packet arrivals) are exponentially distributed [11].

For simplicity of calculations, let us perform the following assumptions on the MAC system $M$ (all variables mentioned below are expressed in unit time) [6]

- The channel is assumed to introduce no errors, so multiple access interference (MAI) is the only source of errors (refered as collisions).

- Data packets are sent with packet size $\delta$.
- ACKs have a packet size $\alpha$.
- Nodes have a transmission period $\tau$. Transmission period is the time it takes for a packet to arrive at the receiver after it has being sent.
- There is a processing time $\omega$ in which a receiver reads a packet and transmits an ACK. This processing time is constant among all nodes.

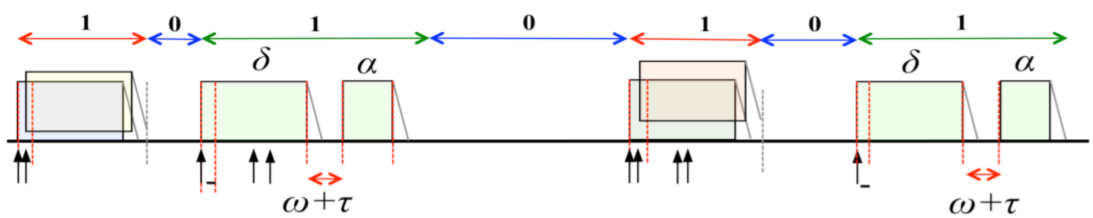


Figure 3: MAC channel shifting between Busy (1) and Idle (0) states [8].

The following methods are taken from the analysis of professor Jose Joaquin Garcia-Luna-Acevez at University of California Santa Cruz [6] [7] [8] [9].

## 4.1 ALOHAnet

Assume the system $M$ performs the ALOHAnet[1] (otherwise known as ALOHA) protocol across all nodes. For this analysis assume the protocol ALOHA has the following characteristics

- Stationary: The system cannot be silent forever or collapse.
- Non-Persistent: After sending a packet, the sender waits for an acknowledgement message (ACK) from the receiver indicating a successful delivery. The waiting time usually equals a round-trip time (RTT) which is the expected amount of time it takes to transmit a packet and receive an ACK back. If the sender times out, then it "backs-off" for a random duration before trying again.
- Channel-Access: A system where more than two terminals can be connected to the same network system (i.e. WiFi).

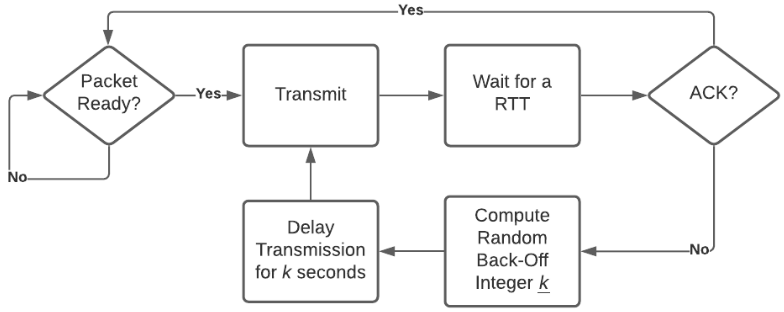


Figure 4: ALOHA protocol [8].

Let $C$ be the time of a collision interval, and $x_i$ be the time between transmissions over the same channel. Given the assumptions for the ALOHA protocol model, **Figure 5** shows the different types of intervals in our MAC channel.

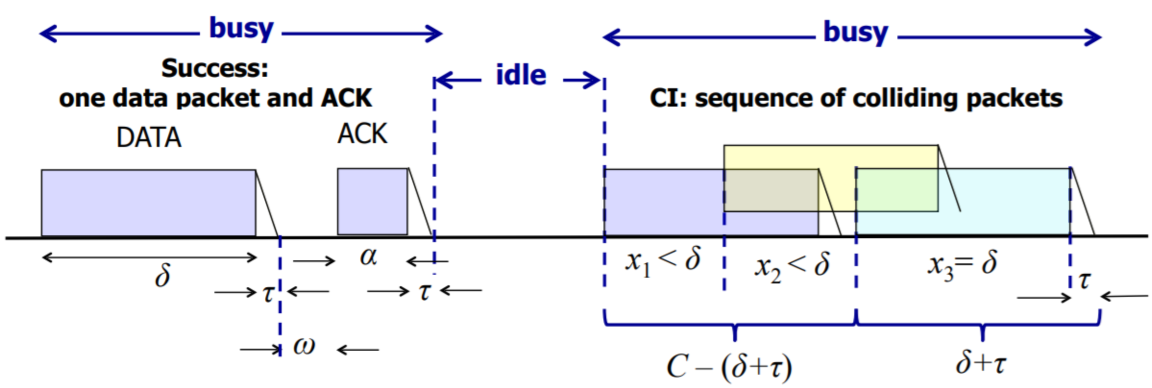


Figure 5: The channel shifts from Busy to Idle states and back. The channel has successful packets and some times it has collision intervals (CI) [8].

Since the inter-arrival times are exponentially distributed, the average length of an idle period is the expected value of an exponential distribution with parameter $\lambda$, thus

$$\overline{I} = E[X] = \frac{1}{\lambda} \tag{12}$$

*Proof.* An exponential distribution follows $f(x) = \lambda e^{-\lambda x}$. It follows that [16]

$$\begin{aligned} E[X] &= \int_0^\infty x f(x) dx = \int_0^\infty x \lambda e^{-\lambda x} dx = \int_0^\infty \lambda e^{-\lambda x} dx + [-x e^{-\lambda x}]_0^\infty \\ &= [-\frac{1}{\lambda} e^{-\lambda x}]_0^\infty + 0 - 0 = 0 + \frac{1}{\lambda} = \frac{1}{\lambda} \end{aligned}$$

□

Moreover, consider that a packet is successful if no other packet arrives during its transmission period, which lasts $\delta$ seconds. A channel is used successfully only for the duration of a packet and if it is a successful busy period. Since we know that packet arrivals follow a Poisson distribution

$$\overline{U} = \delta P_S = \delta e^{-\lambda\delta}. \tag{13}$$

The average length of a busy period $\overline{B}$ is either a success with probability $P_S$ or a CI with a probability $1 - P_S$. From our description of the protocol, a successful busy period is one that transmits, processes, and responds with an ACK. Thus the expected length of a successful busy period is $\delta + \omega + \alpha + 2\tau$ ($\tau$ is counted twice for the transmission period of both the large packet and the ACK). Therefore the value of $\overline{B}$ is

$$\overline{B} = P_S(\delta + \omega + \alpha + 2\tau) + (1 - P_S)\overline{C} \tag{14}$$

The average length of a CI can be calculated as

$$\overline{C} = \overline{N} \cdot \overline{X} + \tau, \tag{15}$$

Where $\overline{N}$ is the average number of packets in a CI and $\overline{X}$ is the average length of the inter-arrival times for consecutive packets.

Since packet arrivals follow a Poisson distribution, ~~then~~ $\overline{N}$ is the mean of the geometric random variable with probability of success (which represents the end of a CI) equal to $p = e^{-\delta\lambda}$. Therefore

$$\overline{N} = \frac{1}{p} = e^{\delta\lambda}. \tag{16}$$

The average length of inter-arrival times for consecutive packets in a busy period $\overline{X}$ follows from the assumption of a Poisson distribution, which is $F_X(t) = 1 - e^{-\lambda t}$. It follows that

$$\overline{X} = \int_0^\infty 1 - F_X(t)dt = \int_0^\delta e^{-\lambda t}dt + 0 = [-\frac{e^{-\lambda t}}{\lambda}]_0^\delta = \frac{1}{\lambda}(1 - e^{-\lambda\delta}). \tag{17}$$

Replacing **Equation 15** and **Equation 16** in **Equation 17** results in

$$\overline{C} = \frac{1}{\lambda}(1 - e^{-\lambda\delta})\ e^{\lambda\delta} + \tau = \frac{e^{\lambda\delta} - 1}{\lambda} + \tau. \tag{18}$$

With the previous result, we can solve and simplify **Equation 14** to get

$$\overline{B} = \frac{e^{\lambda\delta} - 1}{\lambda} + \tau + e^{-\lambda\delta}(\alpha + \omega + \tau) \tag{19}$$

Having calculated our values for $\overline{U}$, $\overline{I}$, and $\overline{B}$, we solve for **Equation 11**

$$S = \frac{\lambda\delta e^{-2\lambda\delta}}{1 + \lambda e^{-\lambda\delta}(\tau + \lambda e^{-\lambda\delta}(\alpha + \omega + \tau))}. \tag{20}$$

Equally important, we can obtain the equation for the average packet delay in system $M$. The average delay is the duration between the instant the first bit of the packet starts to be transmitted and the last bit is received correctly at the destination. Suppose a given node has a packet ready to transmit. Assume once again that packet arrivals in the channel have a Poisson distribution. Additionally, assume a fully connected network. Since the protocol is non-persistent, let $\overline{R}$ be the random back-off time a node waits before re-transmitting the packet.

Transmission attempts are events that are independent from each other. The expected number of failed transmissions needed for a successful packet is $\frac{1}{p} - 1$. Since packet arrivals follow a Poisson distribution, the probability of a successful packet is given by $p = e^{-2\lambda\delta}$ [9].

The delay incurred in a failed transmission is $\delta + \tau + \overline{R}$ and in a successful transmission is $\delta + \alpha + \omega + 2\tau$. Therefore the average delay incurred is

$$\overline{D} = (e^{2\lambda\delta} - 1)(\delta + \tau + \overline{R}) + \delta + \alpha + \omega + 2\tau. \tag{21}$$

## 4.2 CSMA

The capacity of ALOHA is limited by the large vulnerability period of a packet. The protocol Carrier Sense Multiple Access (CSMA) tackles this issue by having the sender listen to the channel before transmitting, and only transmitting if the channel is Idle. If the sender senses that the channel is busy, it backs-off for a ranomd duration $k$ before sensing again. For this reason, the vulnerability period is reduced to the propagation delay $\tau$. The same assumptions made for ALOHA also apply for the analysis of CSMA.

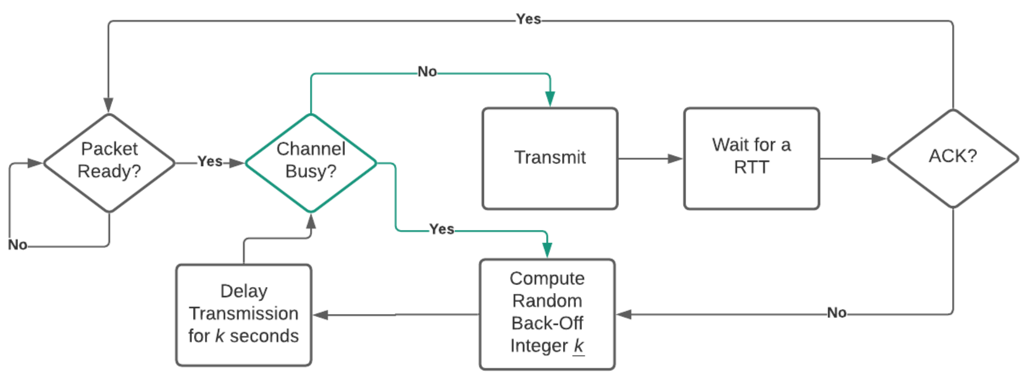


Figure 6: CSMA protocol [8].

Since arrivals follow a Poisson distribution, a packet succeeds with probability $P_S = e^{-\lambda(\tau+\omega)}$. Therefore we get

$$\overline{U} = \delta P_S = \delta e^{-\lambda(\tau+\omega)} \tag{22}$$

Furthermore, collisions in CSMA happen only when a node transmits before it receives a busy signal. Let $Y$ be a random variable defined by the time between packet transmissions in a CI and $0 \leq Y \leq \tau + \omega$. Let $y$ be the time of the last transmission.

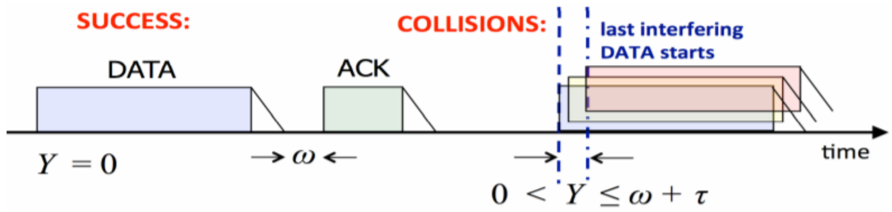


Figure 7: Packet transmissions in CSMA [8].

The probability that there are no packet arrivals in the time $[y, \omega + \tau]$ is expressed by $P[Y \leq y] = e^{-\lambda(\omega+\tau-y)}$. It follows that

$$\overline{Y} = \int_0^\infty 1 - F_Y(t)dt = \int_0^{\omega+\tau} 1 - e^{-\lambda(\omega+\tau-t)}dt = \omega + \tau - \frac{1 - e^{-\lambda(\omega+\tau}}{\lambda} \tag{23}$$

Observe that for a packet to arrive successfully, $Y = 0$ must be true. Similar to **Equation 18**, we can construct the equation for $\overline{B}$ as

$$\begin{aligned}
\overline{B} &= \overline{Y} + \delta + \tau + e^{-\lambda(\omega+\tau)}(\omega + \alpha + \tau) \\
&= \omega + \tau - \frac{1 - e^{-\lambda(\omega+\tau)}}{\lambda} + \delta + \tau + e^{-\lambda(\omega+\tau)}(\omega + \alpha + \tau) \\
&= \delta + \omega + 2\tau - \frac{1}{\lambda} + e^{-\lambda(\omega+\tau)}(\omega + \alpha + \tau + \frac{1}{\lambda})
\end{aligned} \tag{24}$$

The value of $\overline{I}$ is the same for ALOHA as it is for CSMA. Therefore, we get the equation for the throughput

$$S = \frac{\delta}{\omega + \alpha + \tau + \frac{1}{\lambda} + e^{\lambda(\omega+\tau)}(\delta + \omega + 2\tau)} \tag{25}$$

## 4.3 Protocol Comparison

In the real case, the propagation delay $\tau$, the processing time $\omega$, and the packet size of an ACK $\alpha$ are very small when compared to average packet sizes. For the sake of comparing both protocols assume $\tau = \alpha = \omega = 0$. Let $G = \lambda\delta$. Thus we can simplify the throughput shown in **Equation 20** for ALOHA as

$$S_{\textbf{ALOHA}} = \delta e^{-2\lambda\delta} = Ge^{-2G} \tag{26}$$

Similarly, we can simplify the throughput equation for CSMA. We can normalize on packet size as $a = \frac{\tau}{\delta}$ to get

$$S_{\textbf{CSMA}} = \frac{aGe^{-aG}}{1 + a - e^{-aG}} \tag{27}$$

As shown in **Figure 8**, plotting the previous equations shows that the throughput of CSMA is much higher than in ALOHA. When studied computationally (by running both

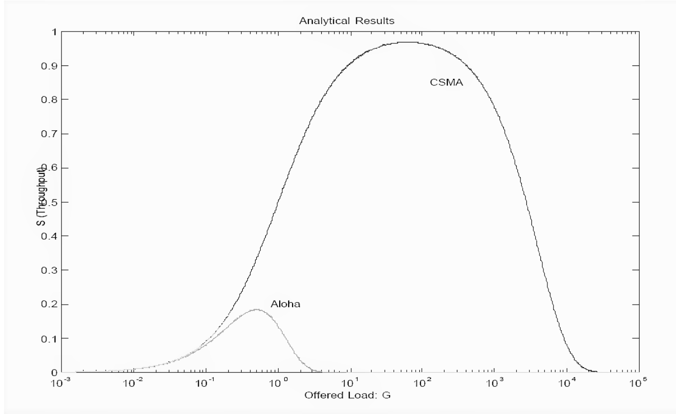


Figure 8: Plotting of the Throughput Equations for ALOHA and CSMA [8].

protocols in a MAC network simulation), the resulting graphs are similar to those in **Figure 8**[15]. Thus, our analysis correctly predicts the behaviour of both protocols and serves as a way to mathematically compare their behavior.

# 5 Conclusion

Using a Markov Chain model when analyzing MAC protocols leads to accurate and complete formulations of the behavior of the system. This method is particularly useful since it follows a similar process for any MAC system and is repeatable and adaptable to most protocols.

Several other methods have been made for analyzing MAC protocol behavior, but this methods explore protocol behavior holistically instead of mathematically. For example, the approach performed by Fouad A. Tobagi and Leonard Kleinrock [11] leads to the same results we derived in **Equation 26** and **Equation 27**. They introduced the construction and assumptions we made for our MAC system, but their analysis relies heavily on the understanding of principles of communication. Without the properties of a Markov Chain, Kleinrock and Tobagi implement renewal theory and other statistical methods. Our approach is much simpler. In like manner, the book *Multiple Access Protocols: Performance and analysis* by Raphael Rom and Moshe Sidi [15] develops an analysis similar to that of Kleinrock and Tobagi. A combination of observations and reasoning specific to each protocol leads to the same equations. Consequently, their approach lacks the flexibility to implement the same method to other protocols, thus it is not reliable for MAC protocols analyisis.

Another key point is that Markov Chain models are useful to other problems in computer networking. The field of computer networks is broad and varied, and for the most part, statistical analysis is at the core of its understanding. Not only does Markov Chains modeling allow for more elevated analysis of MAC protocols (i.e. determining optimal window size [4], analyzing GTS transmissions [12] [2]), but it also allows for the study of other networking problems [14]. All things considered, Markov Chains represent a powerful method for analyzing network systems and problem solving of computer networking problems. The reliability, repeatability, and flexibility of this method is paramount for network analysis, thus making it a deep field of research based on rigorous mathematics and statistics.

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