一百零五學年度 0311 微積分 (二) 期末考
The 105th academic year course 0311 Calculus(2) finalterm examination

date: Jun 9, 2017

Student ID No. 學號: ________________ Name 姓名: ________________

說明 Description:

(1) 答題之前請先檢查所取得之試卷與答案卷是否正確。
Before answering questions, please check if the test papers and answer sheets which you get are correct.

(2) 測驗時間 110 分鐘。試卷加答案卷、答案卡共計 9 頁。
Testing time is 110 minutes. Test papers, answer sheets, and answer cards are of 9 pages in total.

(3) 試卷包括選擇題與填充題, 總分共計 100 分, 占學期成績之 20%。考卷成績將 $\boxed{\text{不}}$ 做爲微積分獎給獎依據。
The test paper includes choices and fill-in-the-blanks, and there is a total score of 100 points, accounting for 20% of the semester grade. The examination result will $\boxed{\text{not}}$ be considered for awarding the Calculus prize.

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答, 否則不予計分。
Be sure to fill related personal information in answer sheets and answer cards. When answering questions, please answer the question by its question number, or, no score.

P.S. 難易度提示 Difficulty hint: easy<normal<hard
★ ★★ ★★★ ★★★★ ★

Questions start from the next page

◎ 單選擇題 (單選十題, 每題五分, 共五十分, 答錯不倒扣。)
Single-choice (10 questions, each 5 points, 50 points in total, no penalty for wrong answers.)

1. Find $\displaystyle\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sin\frac{i\pi}{m}-\sin\frac{j\pi}{n}\right)\frac{\pi^2}{mn}$.
(Hint: $=\iint\limits_R \cdots\, dA$ with $R=[0,\pi]\times[0,\pi]$.) ⋆ §15.1

(A) $\boxed{\mathbf{0.}}$ (B) 4. (C) 4π. (D) π^2.

Solution: $\displaystyle\iint\limits_R(\sin x-\sin y)\,dA=\int_0^\pi\int_0^\pi(\sin x-\sin y)\,dx\,dy$
$\displaystyle=\int_0^\pi\sin x\,dx\int_0^\pi dy-\int_0^\pi dx\int_0^\pi\sin y\,dy$
$\displaystyle=\Big[-\cos x\Big]_0^\pi\Big[y\Big]_0^\pi-\Big[x\Big]_0^\pi\Big[-\cos y\Big]_0^\pi=2\cdot\pi-\pi\cdot2=0.$

2. Find $\displaystyle\int_0^1\int_2^3 ye^{-xy}\,dy\,dx$. ⋆⋆ §15.2

(A) $\boxed{1-e^{-2}+e^{-3}.}$ (B) $-1-e^{-2}+e^{-3}$.
(C) $1+e^{-2}-e^{-3}$. (D) $-1+e^{-2}-e^{-3}$.

Solution: (Change type.) (Ex 15.2.21)
$\displaystyle\int_0^1\int_2^3 ye^{-xy}\,dy\,dx=\int_2^3\int_0^1 ye^{-xy}\,dx\,dy$
$\displaystyle=\int_2^3\Big[-e^{-xy}\Big]_{x=0}^{x=1}dy=\int_2^3(1-e^{-y})\,dy=\Big[y+e^{-y}\Big]_2^3=1-e^{-2}+e^{-3}.$

$\square$ 3. Find $\displaystyle\iint_R \frac{\boldsymbol{xy}}{\boldsymbol{x^2+1}}\,dA$, where $R = \{(x,y) : \mathbf{0 \le \boldsymbol{x} \le 1},\ \mathbf{-3 \le \boldsymbol{y} \le 3}\}$. $\star$ §15.2

(A) $\boxed{\mathbf{0.}}$ (B) 6. (C) $9\ln 2$. (D) $\dfrac{9}{4}\pi$.

Solution: (Ex 15.2.17)

[Sol 1: type II] $\displaystyle\iint_R \frac{xy}{x^2+1}\,dA = \int_{-3}^{3}\int_0^1 \frac{xy}{x^2+1}\,dx\,dy$

$\displaystyle= \int_{-3}^{3}\left[y\ln\sqrt{x^2+1}\right]_{x=0}^{x=1}dy = \int_{-3}^{3}\ln\sqrt{2}y\,dy = \left[\ln\sqrt{2}\frac{y^2}{2}\right]_{-3}^{3} = 0.$

. . . .

[Sol 2: type I] $\displaystyle\iint_R \frac{xy}{x^2+1}\,dA = \int_0^1\int_{-3}^{3} \frac{xy}{x^2+1}\,dy\,dx$

$\displaystyle= \int_0^1\left[\frac{x}{x^2+1}\frac{y^2}{2}\right]_{y=-3}^{y=3}dx = \int_0^1 0\,dx = 0.$

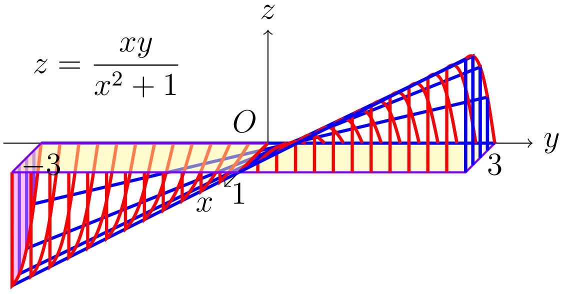
. . . .

[Sol 3: separated] $\displaystyle\iint_R \frac{xy}{x^2+1}\,dA = \int_0^1 \frac{x}{x^2+1}\,dx\int_{-3}^{3} y\,dy$

$\displaystyle= \left[\ln\sqrt{x^2+1}\right]_0^1\left[\frac{y^2}{2}\right]_{-3}^{3} = \ln\sqrt{2}\cdot 0 = 0.$

. . . .

[Sol 4: symmetry] $\dfrac{xy}{x^2+1}$ is odd w.r.t. and R are symmetry to x-axis.



4. Evaluate the double integral $\iint\limits_T \boldsymbol{2xy}\ dA$, where T is the triangular region with vertices(頂點) $\mathbf{(0,0)}$, $\mathbf{(1,0)}$ and $\mathbf{(0,1)}$. ⋆ §15.3

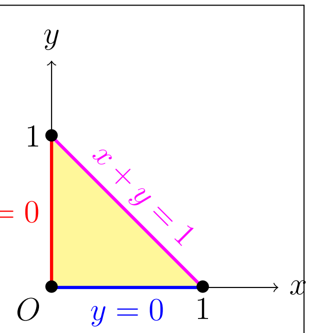
(A) $\boxed{\mathbf{\frac{1}{12}.}}$ (B) $\frac{1}{4}$. (C) $\frac{5}{12}$. (D) $\frac{1}{2}$.

Solution: (Ex 15.3.22)

$$\iint\limits_T 2xy\ dA = \int_0^1 \int_0^{1-x} 2xy\ dy\ dx$$

$$= \int_0^1 \left[xy^2\right]_{y=0}^{y=1-x} dx = \int_0^1 (x - 2x^2 + x^3)\ dx$$

$$= \left[\frac{x^2}{2} - 2\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$



5. Evaluate the iterated integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \boldsymbol{e^{x^3}\ dx\ dy}$. ⋆ §15.3

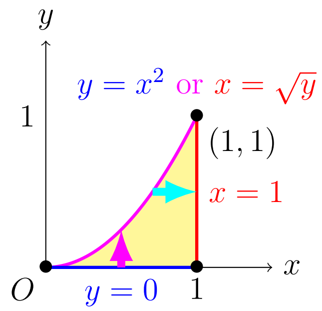
(A) $\boxed{\mathbf{\frac{e-1}{3}.}}$ (B) $\frac{e}{3}$. (C) $e-1$. (D) e.

Solution: (Change type)

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\ dx\ dy = \int_0^1 \int_0^{x^2} e^{x^3}\ dy\ dx$$

$$= \int_0^1 \left[ye^{x^3}\right]_{y=0}^{y=x^2} dx = \int_0^1 x^2 e^{x^3}\ dx$$

$$= \left[\frac{1}{3}e^{x^3}\right]_0^1 = \frac{e-1}{3}.$$



6. Find the **area** of the region enclosed by cycles $\boldsymbol{x^2+y^2=2y}$ and $\boldsymbol{x^2+y^2=2\sqrt{3}x}$.
(Hint: Polar.)

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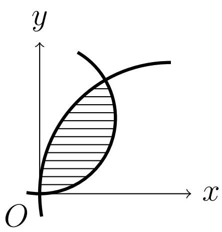
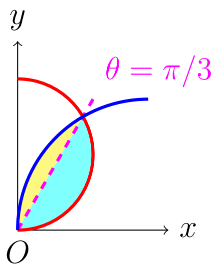
§15.4



(A) $\frac{1}{3}\pi-\sqrt{3}$. **(B)** $\boxed{\boldsymbol{\frac{5}{6}\pi-\sqrt{3}.}}$ (C) $\frac{7}{6}\pi-\sqrt{3}$. (D) $\frac{2}{3}\pi-\sqrt{3}$.

Solution:

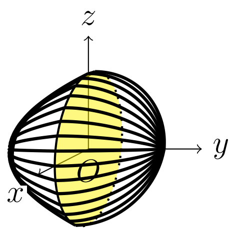


$r^2=x^2+y^2=2y=2r\sin\theta$, $r=2\sin\theta$;
$r^2=x^2+y^2=2\sqrt{3}x=2\sqrt{3}r\cos\theta$, $r=2\sqrt{3}\cos\theta$.
$r=0$ or $2\sin\theta=r=2\sqrt{3}\cos\theta$, $\tan\theta=\sqrt{3}$, $\theta=\pi/3$.

$D=\{(r,\theta):0\le\theta\le\pi/3,\ 0\le r\le 2\sin\theta\}$
$\quad\cup\{(r,\theta):\pi/3\le\theta\le\pi/2,\ 0\le r\le 2\sqrt{3}\cos\theta\}$.

$$A(D)=\iint_D dA=\int_0^{\pi/3}\int_0^{2\sin\theta} r\,dr\,d\theta+\int_{\pi/3}^{\pi/2}\int_0^{2\sqrt{3}\cos\theta} r\,dr\,d\theta$$
$$=\int_0^{\pi/3} 2\sin^2\theta\,d\theta+\int_{\pi/3}^{\pi/2} 6\cos^2\theta\,d\theta$$
$$=\int_0^{\pi/3}(1-\cos 2\theta)\,d\theta+\int_{\pi/3}^{\pi/2}3(1+\cos 2\theta)\,d\theta$$
$$=\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi/3}+\left[3\theta+\frac{3\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$
$$=\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)+\left(\frac{\pi}{2}-\frac{3\sqrt{3}}{4}\right)=\frac{5}{6}\pi-\sqrt{3}.$$

$\square$ 7. Find the **volume** of the solid enclosed by the paraboloid $\boldsymbol{y = x^2 + z^2 - 1}$ and the half-sphere $\boldsymbol{y = \sqrt{1 - x^2 - z^2}}$.



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§15.4+ 15.7 + 15.8

(A) $\frac{2}{3}\pi$. (B) $\frac{5}{6}\pi$. (C) π. **(D)** $\boxed{\boldsymbol{\frac{7}{6}\pi.}}$

Solution: $x^2 + z^2 - 1 = y = \sqrt{1 - x^2 - z^2}$, $x^2 + z^2 = 1$.

$E = \{(x, y, z) : (x, z) \in D,\ x^2 + z^2 - 1 \le y \le \sqrt{1 - x^2 - z^2}\}$.

$D = \{(r, \theta) : 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

$$V(E) = \iiint_E dV = \iint_D \int_{x^2+z^2-1}^{\sqrt{1-x^2-z^2}} dy\, dA$$

$$= \iint_D (\sqrt{1 - x^2 - z^2} + 1 - x^2 - z^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (\sqrt{1 - r^2} + 1 - r^2) \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r\sqrt{1 - r^2} + r - r^3)\, dr$$

$$= \Big[\theta\Big]_0^{2\pi} \Big[-\frac{1}{3}\sqrt{1 - r^2}^3 + \frac{r^2}{2} - \frac{r^4}{4}\Big]_0^1 = 2\pi \cdot (\frac{1}{3} + \frac{1}{2} - \frac{1}{4}) = \frac{7}{6}\pi.$$

8. Assume that a **black-hole**(黑洞) is a **spherical object**(球體) whose **density**(密度) function is **inversely proportional**(成反比) to **the square of the distance**(距離平方) from its center. The **mass** of a black-hole of **radius** $\boldsymbol{R}$ is **proportional**(成正比) to? (Hint: $m = \iiint_E \rho \, dV$.) ⋆ §15.7+15.9

(A) $\boxed{\boldsymbol{R.}}$ (B) R^2. (C) R^3. (D) R^4.

Solution: Let the center of the black-hole B be the origin O.
Then $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq R^2\}$ and $\rho = \dfrac{K}{x^2 + y^2 + z^2}$.

$$m = \iiint\limits_B \rho(x, y, z) \, dV = \iiint\limits_B \frac{K}{x^2 + y^2 + z^2} \, dV$$

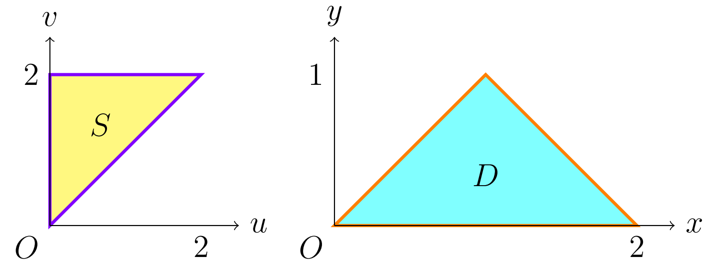
$$= K \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \frac{1}{\rho^2} \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= K \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^{R} d\rho = K \cdot 2 \cdot 2\pi \cdot R = 4K\pi R \propto R.$$

9. Evaluate $\displaystyle\iint_D \boldsymbol{\cos\left(\frac{x-y}{x+y}\right)}\, dA$, where D is the triangular region bounded by lines $\boldsymbol{x = y}$, $\boldsymbol{x + y = 2}$ and $\boldsymbol{y = 0}$. (Hint: Jacobian.) $\star$ §15.10

(A) $1 - \cos 1$. (B) $\frac{1}{2}$. (C) $\cos 1$. **(D)** $\boxed{\textbf{sin 1.}}$

Solution:



Let $u = x - y$ and $v = x + y$, then $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(v - u)$.

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}(\neq 0).$$

$x = y \to u = 0$, $x + y = 2 \to v = 2$, $y = 0 \to u = v$.

$S = T^{-1}(D) = \{(u, v) : 0 \leq v \leq 2,\ 0 \leq u \leq v\}$. (Type II !)

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dA = \iint_S \cos\frac{u}{v} \cdot \left|\frac{1}{2}\right| dA = \int_0^2 \int_0^v \frac{1}{2} \cos\frac{u}{v}\, du\, dv$$

$$= \int_0^2 \left[\frac{v}{2} \sin\frac{u}{v}\right]_{u=0}^{u=v} dv = \int_0^2 \frac{v}{2} \sin 1\, dv = \left[\frac{v^2}{4} \sin 1\right]_0^2 = \sin 1.$$

$\square$ 10. Let $\boldsymbol{u = \sqrt{x+y}}$, $\boldsymbol{v = \sqrt{y+z}}$, $\boldsymbol{w = \sqrt{z+x}}$. Find the **Jacobian** of the transformation from uvw-space to xyz-space. (Hint: $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.) $\star$ §15.10

(A) $\boxed{\boldsymbol{4uvw.}}$ (B) $4\sqrt{(x+y)(y+z)(z+x)}$.

(C) $\dfrac{1}{4uvw}$. (D) $\dfrac{1}{4\sqrt{(x+y)(y+z)(z+x)}}$.

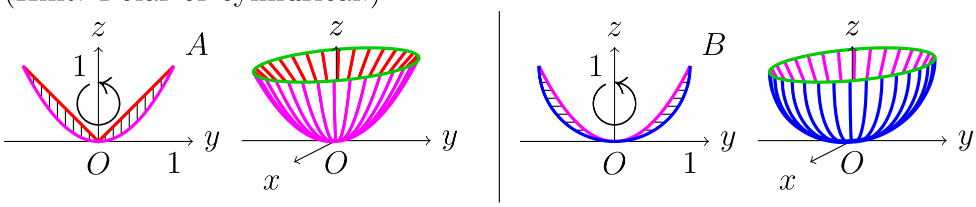
Solution: $u^2 = x + y$, $v^2 = y + z$, $w^2 = z + x$.
$x = \frac{1}{2}(u^2 - v^2 + w^2)$, $y = \frac{1}{2}(u^2 + v^2 - w^2)$, $z = \frac{1}{2}(-u^2 + v^2 + w^2)$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} u & -v & w \\ u & v & -w \\ -u & v & w \end{vmatrix} = 4uvw.$$

◎ 多選擇題 (多選五題, 每題五分, 共二十五分。錯一個選項扣兩分, 錯兩個選項以上不給分, 分數不倒扣。)

Multiple-choice (5 questions, each 5 points, 25 points in total. One wrong option deducts 2 points, more than one wrong option gets no points, and no penalty for wrong answers.)

11. Let $\boldsymbol{A}$ be the **volume** of the region **below** the cone $\boldsymbol{z = \sqrt{x^2 + y^2}}$ and **above** the paraboloid $\boldsymbol{z = x^2 + y^2}$, and let $\boldsymbol{B}$ be the **volume** of the region **below** the paraboloid $\boldsymbol{z = x^2 + y^2}$ and **above** the half-sphere $\boldsymbol{z = 1 - \sqrt{1 - x^2 - y^2}}$. Which of the following statement is **correct**? (Hint: Polar or cylindrical.) ⋆⋆⋆ §15.4+15.8



(A) $A = \pi/3$. **(B)** $\boxed{\boldsymbol{B = \pi/6.}}$ (C) $A > B$. (D) $A + B = \pi/2$.

Solution: $D = \{(r, \theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le 1\}$.

$z = \sqrt{x^2 + y^2} = r$, $z = x^2 + y^2 = r^2$,

$z = 1 - \sqrt{1 - x^2 - y^2} = 1 - \sqrt{1 - r^2}$.

$$I = \int_0^{2\pi} \int_0^1 r \cdot r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r^2\, dr = 2\pi \cdot \frac{1}{3} = \frac{2}{3}\pi,$$

$$II = \int_0^{2\pi} \int_0^1 r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r^3\, dr = 2\pi \cdot \frac{1}{4} = \frac{1}{2}\pi,$$

$$III = \int_0^{2\pi} \int_0^1 (1 - \sqrt{1 - r^2}) \cdot r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 (r - r\sqrt{1 - r^2})\, dr$$

$$= \Big[\theta\Big]_0^{2\pi} \Big[\frac{r^2}{2} + \frac{1}{3}(1 - r^2)^{3/2}\Big]_0^1 = 2\pi \cdot \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}\pi.$$

$A = I - II = \frac{\pi}{6}$, $B = II - III = \frac{\pi}{6}$, $A = B$.

12. When evaluating $I = \iiint\limits_E \sqrt{\boldsymbol{x^2+2xy+5y^2+9z^2}}\,dV$, where E is the region inside the ellipsoid $\boldsymbol{x^2+2xy+5y^2+9z^2=1}$, let $\boldsymbol{u=x+y}$, $\boldsymbol{v=2y}$ and $\boldsymbol{w=3z}$. Which of the following statement is **correct**? ⋆⋆ §15.9+ 15.10

(A) $I = \iiint\limits_B \sqrt{u^2+v^2+w^2}\,dV$, where B is a unit ball.

(B) $(x,y,z) = \left(u+\frac{v}{2}, \frac{v}{2}, \frac{w}{3}\right)$. **(C)** $\boxed{\boldsymbol{\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{6}}.}$ **(D)** $\boxed{\boldsymbol{I = \frac{\pi}{6}}.}$

Solution: $T(u,v,w) = (x,y,z) = (u-\frac{v}{2}, \frac{v}{2}, \frac{w}{3})$. .... (B)

$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{vmatrix} = \frac{1}{6}$. .... (C)

$B = T^{-1}(E) = \{(u,v,w) : u^2+v^2+w^2 \le 1\}$, a unit ball.

$I = \iiint\limits_E \sqrt{x^2+2xy+5y^2+9z^2}\,dV$

$= \iiint\limits_B \sqrt{u^2+v^2+w^2} \cdot \left|\frac{1}{6}\right|\,dV$ .... (A)

$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 \frac{1}{6}\rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ (Spherical)

$= \frac{1}{6}\int_0^{\pi}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^1\rho^3\,d\rho$

$= \frac{1}{6}\cdot 2\cdot 2\pi\cdot\frac{1}{4} = \frac{\pi}{6}$. .... (D)

13. Which changing of **types** of iterated integrals is **correct**? §15.3 ⋆⋆

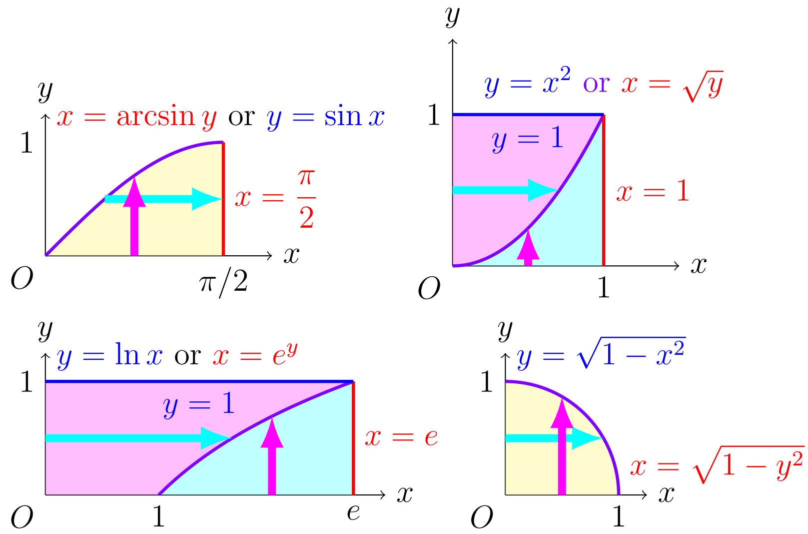
(A) $\boxed{\displaystyle \int_0^{\pi/2}\int_0^{\sin x} f(x,y)\,dy\,dx = \int_0^1\int_{\arcsin y}^{\pi/2} f(x,y)\,dx\,dy.}$

(B) $\displaystyle \int_0^1\int_0^{x^2} f(x,y)\,dy\,dx = \int_0^1\int_0^{\sqrt{y}} f(x,y)\,dx\,dy.$

(C) $\displaystyle \int_1^e\int_0^{\ln x} f(x,y)\,dy\,dx = \int_0^1\int_0^{e^y} f(x,y)\,dx\,dy.$

(D) $\boxed{\displaystyle \int_0^1\int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx = \int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.}$

Solution: (Ex 15.3.53,17,47,46)



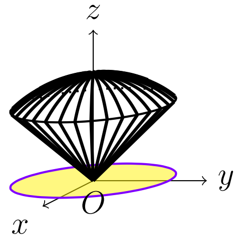
The correct should be:

$$\int_0^1\int_0^{x^2} f(x,y)\,dy\,dx = \int_0^1\int_{\sqrt{y}}^{1} f(x,y)\,dx\,dy,$$

$$\int_1^e\int_0^{\ln x} f(x,y)\,dy\,dx = \int_0^1\int_{e^y}^{e} f(x,y)\,dx\,dy.$$

14. When evaluating the **volume** of the solid enclosed by the sphere $\boldsymbol{x^2+y^2+z^2=2}$ and the half cone $\boldsymbol{z=\sqrt{x^2+y^2}}$, which **iterated integral** is **correct**?

⋆ §15.7+ 15.8 + 15.9



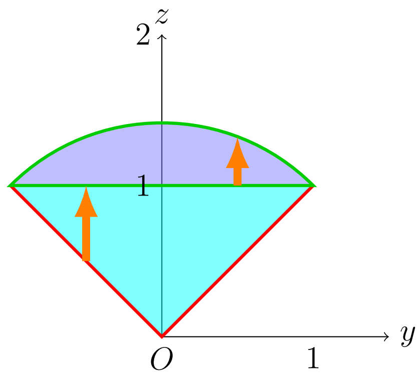
(A) $\boxed{\displaystyle \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz\; dy\; dx.}$

(B) $\displaystyle \int_{-1}^{1}\int_{y}^{\sqrt{2-y^2}}\int_{-\sqrt{z^2-y^2}}^{\sqrt{2-z^2-y^2}} dx\; dz\; dy.$

(C) $\boxed{\displaystyle \int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{\sqrt{2-r^2}} r\; dz\; dr\; d\theta.}$

(D) $\displaystyle \int_{0}^{\pi/4}\int_{0}^{2\pi}\int_{0}^{2} \rho^2\sin\phi\; d\rho\; d\theta\; d\phi.$

Solution: (15.9.Ex)



$$\int_{-1}^{1}\int_{1}^{\sqrt{2-y^2}}\int_{-\sqrt{2-z^2-y^2}}^{\sqrt{2-z^2-y^2}} dx\; dz\; dy$$

$$+$$

$$\int_{-1}^{1}\int_{y}^{1}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} dx\; dz\; dy.$$

$$\int_{0}^{\pi/4}\int_{0}^{2\pi}\int_{0}^{\sqrt{2}} \rho^2\sin\phi\; d\rho\; d\theta\; d\phi.$$

15. Find the **iterated integrals** over the same **region** as the iterated integral $\int_0^1 \int_0^{1-z} \int_0^{y^2} dx\, dy\, dz$. (Hint: Find other types.)

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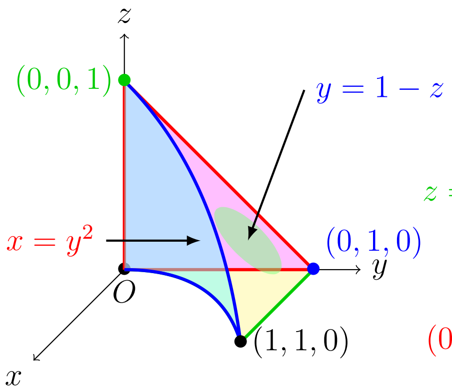
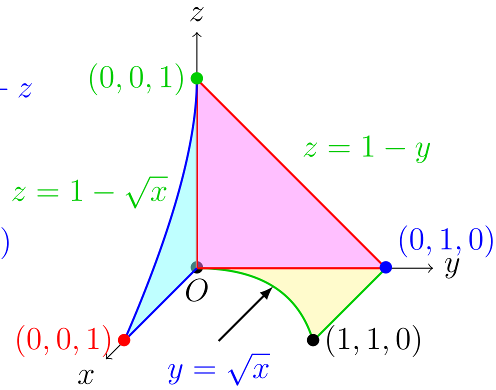
§15.3

(A) $\boxed{\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} dy\, dx\, dz.}$ (B) $\boxed{\int_0^1 \int_0^{y^2} \int_0^{1-y} dz\, dx\, dy.}$

(C) $\boxed{\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} dy\, dz\, dx.}$ (D) $\boxed{\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz\, dy\, dx.}$

Solution: (Ex 15.7.33)





$$\int_0^1 \int_0^{1-y} \int_0^{y^2} dx\, dz\, dy = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{y^2} \int_0^{1-y} dz\, dx\, dy = \int_0^1 \int_0^{1-z} \int_0^{y^2} dx\, dy\, dz$$

$$= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} dy\, dx\, dz = \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} dy\, dz\, dx.$$

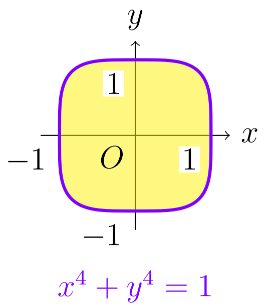
◎ 填空題 (每題五分) Fill-in-the-blank (each 5 points)

16. Evaluate the double integral: $\displaystyle\iint_D \boldsymbol{x^3}\, dA$, $D = \{(x, y) : \boldsymbol{x^4 + y^4 \leq 1}\}$. ⋆⋆ §15.3

Solution: 0.

. . . .

[Sol 1: type I]



$$\iint_D x^3\, dA = \int_{-1}^{1}\int_{-\sqrt[4]{1-x^4}}^{\sqrt[4]{1-x^4}} x^3\, dy\, dx$$

$$= \int_{-1}^{1} \left[yx^3\right]_{y=-\sqrt[4]{1-x^4}}^{y=\sqrt[4]{1-x^4}} dx$$

$$= \int_{-1}^{1} 2x^3\sqrt[4]{1-x^4}\, dx$$

$$= \left[-\frac{2}{5}(1-x^4)^{5/4}\right]_{-1}^{1} = 0.$$

[Sol 2: type II]

$$\int_{-1}^{1}\int_{-\sqrt[4]{1-y^4}}^{\sqrt[4]{1-y^4}} x^3\, dx\, dy = \int_{-1}^{1}\left[\frac{x^4}{4}\right]_{x=-\sqrt[4]{1-y^4}}^{x=\sqrt[4]{1-y^4}} dy = \int_{-1}^{1} 0\, dy = 0.$$

[Sol 3: Symmetry] x^3 is odd w.r.t. and D is symmetry to y-axis.

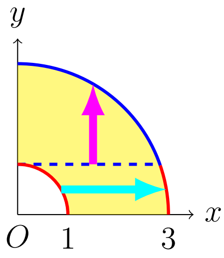
17. Evaluate the integral (Hint: Polar and combine.)

$$\int_0^{2\sqrt{2}} \int_1^{\sqrt{9-x^2}} \tan^{-1}\left(\frac{y}{x}\right) dy\, dx + \int_0^1 \int_{\sqrt{1-y^2}}^{\sqrt{9-y^2}} \tan^{-1}\left(\frac{y}{x}\right) dx\, dy.$$ ⋆⋆ §15.4

Solution: $\dfrac{\pi^2}{2}$.

. . . .

$R = \{(x, y) : 0 \le x \le 2\sqrt{2},\ 1 \le y \le \sqrt{9 - x^2}\}$
$\cup \{(x, y) : 0 \le y \le 1,\ \sqrt{1 - y^2} \le x \le \sqrt{9 - y^2}\}$
$= \{(r, \theta) : 1 \le r \le 3,\ 0 \le \theta \le \pi/2\}$.
$\tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{r\sin\theta}{r\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta.$



$$I = \iint\limits_R \tan^{-1}\left(\frac{y}{x}\right) dA = \int_0^{\pi/2}\int_1^3 \theta \cdot r\ dr\ d\theta = \int_0^{\pi/2} \theta\ d\theta \int_1^3 r\ dr$$
$$= \left[\frac{\theta^2}{2}\right]_0^{\pi/2}\left[\frac{r^2}{2}\right]_1^3 = \frac{\pi^2}{8} \cdot 4 = \frac{\pi^2}{2}.$$

18. Find the **area** of the finite part of the paraboloid $\boldsymbol{2x = y^2 + z^2}$ cut off by the plane $\boldsymbol{x = 4}$. ⋆⋆⋆ §15.6

Solution: $\dfrac{52}{3}\pi$.

. . . .

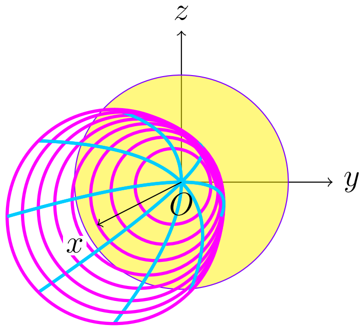
(projective to yz-plane) (Ex 15.6.23)

$x = f(y,z) = \dfrac{y^2}{2} + \dfrac{z^2}{2}$,

$2x = y^2 + z^2 = r^2$ and $x = 4$, $r = \sqrt{8}$.

$D = \{(y,z) : x^2 + z^2 \leq 8\}$

$\quad = \{(r,\theta) : 0 \leq r \leq \sqrt{8}, 0 \leq \theta \leq 2\pi\}$.

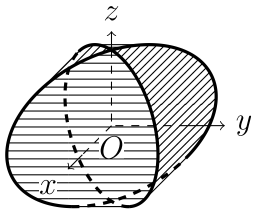


$$A(S) = \iint\limits_D \sqrt{(f_y)^2 + (f_z)^2 + 1}\; dA = \iint\limits_D \sqrt{y^2 + z^2 + 1}\; dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{8}} \sqrt{r^2+1} \cdot r\; dr\; d\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{8}} r\sqrt{r^2+1}\; dr$$

$$= \Big[\theta\Big]_0^{2\pi} \left[\frac{1}{2}\frac{2}{3}(r^2+1)^{3/2}\right]_0^{\sqrt{8}} = 2\pi \cdot \frac{27-1}{3} = \frac{52}{3}\pi.$$

19. Find the **volume** of the **bicylinder**(雙柱) bounded by the cylinders $\boldsymbol{x^2+z^2=R^2}$ and $\boldsymbol{y^2+z^2=R^2}$ of the same radius R. (Hint: Use symmetry in the first octant.) ⋆⋆ §15.7

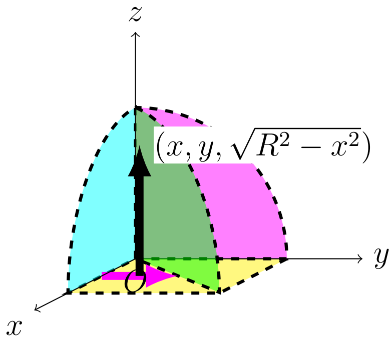


Solution: $\dfrac{16}{3}R^3$.

....

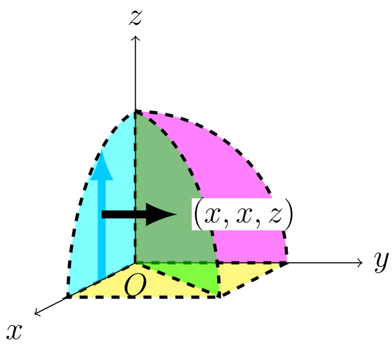
[Sol 1]

$$V = 16\int_0^R \int_0^x \int_0^{\sqrt{R^2-x^2}} dz\ dy\ dx$$
$$= 16\int_0^R \int_0^x \sqrt{R^2-x^2}\ dy\ dx$$
$$= 16\int_0^R x\sqrt{R^2-x^2}\ dy\ dx$$
$$= 16\Big[-\frac{1}{3}\sqrt{R^2-x^2}^3\Big]_0^R = \frac{16}{3}R^3.$$



[Sol 2]

$$V = 16\int_0^R \int_0^{\sqrt{R-x^2}} \int_0^x dy\ dz\ dx$$
$$= 16\int_0^{\pi/2} \int_0^R r\cos\theta \cdot r\ dr\ d\theta$$
$$= 16\int_0^{\pi/2} \cos\theta\ d\theta \int_0^R r^2 r\ dr$$
$$= 16\Big[\sin\theta\Big]_0^{\pi/2}\Big[\frac{r^3}{3}\Big]_0^R = \frac{16}{3}R^3.$$



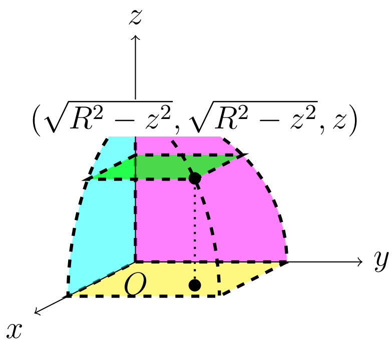
[Sol 3]

$$V = 8\int_0^R (R^2 - z^2)\ dz$$
$$= 8\Big[R^2 z - \frac{z^3}{3}\Big]_0^R = \frac{16}{3}R^3.$$

[Sol 4]

$V(Bicylinder) : V(Ball) = 4 : \pi,$

$$V = \frac{4}{\pi}\cdot\frac{4}{3}\pi R^3 = \frac{16}{3}R^3.$$



20. Evaluate the triple integral $\displaystyle\iiint_E \boldsymbol{\sin(x+y+z)}\ dV$,

where $E = \{(x, y, z) : \boldsymbol{0 \le y \le \pi/2,\ 0 \le x \le y,\ 0 \le z \le x}\}$. ★ §15.7

Solution: $\frac{1}{3}$.

. . . .

$$\int_0^{\pi/2}\int_0^y\int_0^x \sin(x+y+z)\ dz\ dx\ dy$$
$$= \int_0^{\pi/2}\int_0^y \Big[-\cos(x+y+z)\Big]_{z=0}^{z=x}\ dx\ dy$$
$$= \int_0^{\pi/2}\int_0^y \Big[-\cos(2x+y) + \cos(x+y)\Big]\ dx\ dy$$
$$= \int_0^{\pi/2} \Big[-\frac{1}{2}\sin(2x+y) + \sin(x+y)\Big]_{x=0}^{x=y}\ dy$$
$$= \int_0^{\pi/2} \Big[-\frac{1}{2}\sin(3y) + \sin(2y) - \frac{1}{2}\sin y\Big]\ dy$$
$$= \Big[\frac{1}{6}\cos 3y - \frac{1}{2}\cos 2y + \frac{1}{2}\cos y\Big]_0^{\pi/2} = \frac{1}{3}.$$

⊕ 加分題 (共二十分。總分超過100分以100分計。)
Bonus (20 points in total. The total score more than 100 points will only get 100 points.)

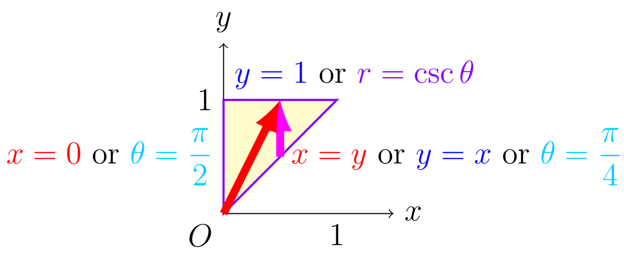
(a). Fill the **upper and lower limits** of the following iterated integrals in given orders or coordinates. (**1 pt** for **both** limits are correct.)

(I) $\displaystyle\int_{\mathbf{0}}^{\mathbf{1}}\int_{\mathbf{0}}^{\boldsymbol{y}} f(x,y)\ \boldsymbol{dx}\ \boldsymbol{dy}$ ⋆

$$= \int_{\boxed{0}}^{\boxed{1}}\int_{\boxed{x}}^{\boxed{1}} f(x,y)\ \boldsymbol{dy}\ \boldsymbol{dx}$$

$$= \int_{\boxed{\pi/4}}^{\boxed{\pi/2}}\int_{\boxed{0}}^{\boxed{\csc\theta}} g(r,\theta)\ \boldsymbol{dr}\ \boldsymbol{d\theta}.$$

Solution:

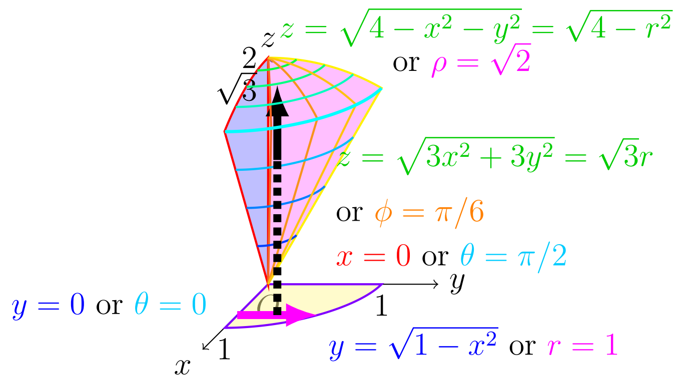


(II) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} f(x,y,z)\, \boldsymbol{dz\ dy\ dx}$ $\star\star$

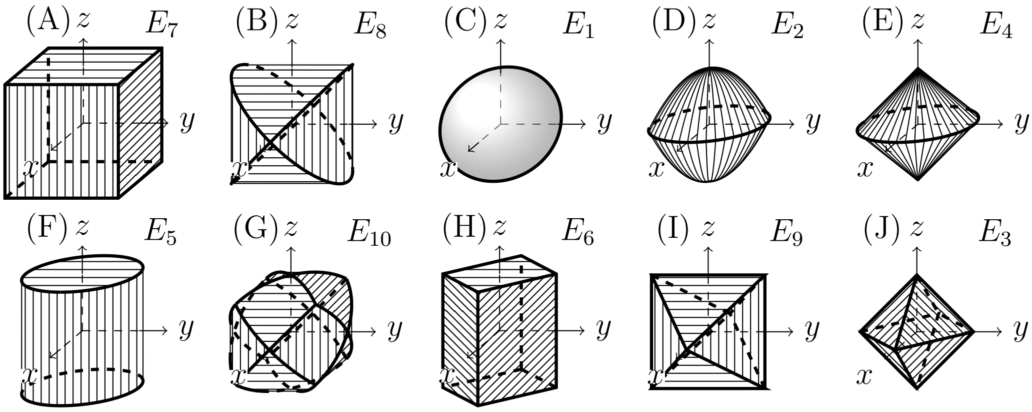
$$= \int_{\boxed{0}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{\sqrt{3}r}}^{\boxed{\sqrt{4-r^2}}} g(r,\theta,z)\, \boldsymbol{dz\ dr\ d\theta}$$

$$= \int_{\boxed{0}}^{\boxed{\pi/6}} \int_{\boxed{0}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{2}} h(\rho,\theta,\phi)\, \boldsymbol{d\rho\ d\theta\ d\phi}.$$

Solution:



(b). Match the **graph** with the regions $E_1 \sim E_{10}$. (Each **1 pt**) $\star$



(Command: Fill uppercase alphabet as A, B, ...)

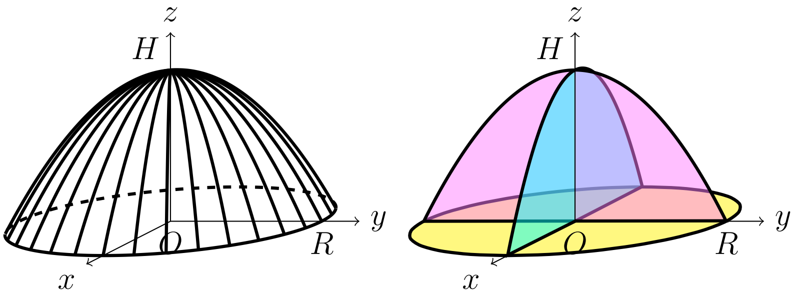
(i) $E_1 = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$	C
(ii) $E_2 = \{(x, y, z) : x^2 + y^2 + \lvert z\rvert \leq 1\}$	D
(iii) $E_3 = \{(x, y, z) : \lvert x\rvert + \lvert y\rvert + \lvert z\rvert \leq 1\}$	J
(iv) $E_4 = \{(x, y, z) : \sqrt{x^2 + y^2} + \lvert z\rvert \leq 1\}$	E
(v) $E_5 = \{(x, y, z) : x^2 + y^2 \leq 1, \lvert z\rvert \leq 1\}$	F
(vi) $E_6 = \{(x, y, z) : \lvert x\rvert + \lvert y\rvert \leq 1, \lvert z\rvert \leq 1\}$	H
(vii) $E_7 = \{(x, y, z) : \lvert x\rvert \leq 1, \lvert y\rvert \leq 1, \lvert z\rvert \leq 1\}$	A
(viii) $E_8 = \{(x, y, z) : x^2 + y^2 \leq 1, x^2 + z^2 \leq 1\}$	B
(ix) $E_9 = \{(x, y, z) : \lvert x\rvert + \lvert y\rvert \leq 1, \lvert x\rvert + \lvert z\rvert \leq 1\}$	I
(x) $E_{10} = \{(x, y, z) : x^2 + y^2 \leq 1, x^2 + z^2 \leq 1, z^2 + y^2 \leq 1\}$	G

⊛ 挑戰題 (共十分。總分超過100分以100分計。)
Challenge (10 points in total. The total score more than 100 points will only get 100 points.)

Let S be a solid with the region E enclosed by the paraboloid

$$\boldsymbol{\frac{x^2}{R^2}+\frac{y^2}{R^2}+\frac{z}{H}=1}$$

and the $\boldsymbol{xy}$**-plane** with the **density** function $\boldsymbol{\rho(x,y,z)=z}$.



(Command: Answer in terms of $\boldsymbol{R}$ and $\boldsymbol{H}$.)

(α). [2 pts] The **volume** of E is .... $\boxed{\frac{\pi}{2}HR^2}$ ⋆⋆

Solution: $E=\{(r,\theta,z): 0\le\theta\le 2\pi, 0\le r\le R, 0\le z\le H-\frac{H}{R^2}r^2\}$.

$$V=\iiint\limits_E dV=\int_0^{2\pi}\int_0^R\int_0^{H-Hr^2/R^2} r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^R\Big[rz\Big]_{z=0}^{z=H-Hr^2/R^2} dr\,d\theta=\int_0^{2\pi}\int_0^R\Big(H-\frac{H}{R^2}r^2\Big)r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^R(Hr-\frac{H}{R^2}r^3)\,dr$$

$$=\Big[\theta\Big]_0^{2\pi}\Big[H\frac{r^2}{2}-\frac{H}{R^2}\frac{r^4}{4}\Big]_0^R=2\pi\cdot\frac{HR^2}{4}=\frac{\pi}{2}HR^2.$$

(β). [2 pts] The **mass** of S is .... $\boxed{\frac{\pi}{6}H^2R^2}$ ⋆⋆⋆

Solution: $M = \iiint\limits_E \rho(x,y,z)dV = \int_0^{2\pi}\int_0^R\int_0^{H-Hr^2/R^2} z\cdot r\ dz\ dr\ d\theta$

$= \int_0^{2\pi}\int_0^R \left[r\frac{z^2}{2}\right]_{z=0}^{z=H-Hr^2/R^2} dr\ d\theta = \int_0^{2\pi}\int_0^R r\frac{(H-Hr^2/R^2)^2}{2}\ dr\ d\theta$

$= \int_0^{2\pi} d\theta \int_0^R \left(\frac{H^2}{2}r - \frac{H^2}{R^2}r^3 + \frac{H^2}{2R^4}r^5\right) dr$

$= \Big[\theta\Big]_0^{2\pi}\left[\frac{H^2}{2}\frac{r^2}{2} - \frac{H^2}{R^2}\frac{r^4}{4} + \frac{H^2}{2R^4}\frac{r^6}{6}\right]_0^R = 2\pi\cdot\frac{H^2R^2}{12} = \frac{\pi}{6}H^2R^2.$

(γ). [3 pts] The **centroid** of E is .... $\boxed{\left(0,0,\frac{H}{3}\right)}$ ★★★★

Solution: By the symmetry, $\iiint_E x\ dV = \iiint_E y\ dV = 0$,

$\iiint\limits_E z\ dV = \iiint\limits_E \rho\ dV = M.$

$\frac{\iiint_E z\ dV}{\iiint_E\ dV} = \frac{M}{V} = \frac{\pi}{6}H^2R^2 \div \frac{\pi}{2}HR^2 = \frac{H}{3}.$

(δ). [3 pts] The **center of mass** of S is ..... $\boxed{\left(0,0,\frac{H}{2}\right)}$ ★

Solution: By the symmetry, $M_{yz} = M_{xz} = 0$.

$M_{xy} = \iiint\limits_E z\rho(x,y,z)dV = \int_0^{2\pi}\int_0^R\int_0^{H-Hr^2/R^2} z\cdot z\cdot r\ dz\ dr\ d\theta$

$= \int_0^{2\pi}\int_0^R \left[r\frac{z^3}{3}\right]_{z=0}^{z=H-Hr^2/R^2} dr\ d\theta = \int_0^{2\pi}\int_0^R r\frac{(H-Hr^2/R^2)^3}{3}\ dr\ d\theta$

$= \int_0^{2\pi} d\theta \int_0^R \left(\frac{H^3}{3}r - \frac{H^3}{R^2}r^3 + \frac{H^3}{R^4}r^5 - \frac{H^3}{3R^6}r^7\right) dr$

$= \Big[\theta\Big]_0^{2\pi}\left[\frac{H^3}{3}\frac{r^2}{2} - \frac{H^3}{R^2}\frac{r^4}{4} + \frac{H^3}{R^4}\frac{r^6}{6} - \frac{H^3}{3R^6}\frac{r^8}{8}\right]_0^R = 2\pi\cdot\frac{H^3R^2}{24} = \frac{\pi}{12}H^3R^2.$

$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi}{12}H^3R^2 \div \frac{\pi}{6}H^2R^2 = \frac{H}{2}.$

Questions End