First mid-term exam, Operations Research 1, College of Management, NCTU
Time: 10:10 ~ 12:00

This is a closed book exam.

Calculator, (electronic) dictionary, phone, and computer are NOT allowed.

1. (10%) The NCTU Company desires to blend a new alloy of 40 percent tin, 35 percent zinc, and 25 percent lead from several available alloys having the following properties:

	Alloy				
Property	1	2	3	4	5
Percentage of tin	60	25	45	20	50
Percentage of zinc	10	15	45	50	40
Percentage of lead	30	60	10	30	10
Cost ($/lb)	22	20	25	24	27

The objective is to determine the proportions of these alloys that should be blended to produce the new alloy at a minimum cost. Formulate a linear programming model for this problem.

<Solution>

Let x_i be the proportion of Alloy i used, $i = 1, 2, 3, 4, 5$ (-2%)

Min $22x_1 + 20x_2 + 25x_3 + 24x_4 + 27x_5$ (-2%)

S.T. $60x_1 + 25x_2 + 45x_3 + 20x_4 + 50x_5 = 40$ (-2%)

$10x_1 + 15x_2 + 45x_3 + 50x_4 + 40x_5 = 35$ (-2%)

$30x_1 + 60x_2 + 10x_3 + 30x_4 + 10x_5 = 25$ (-2%)

$x_1 + x_2 + x_3 + x_4 + x_5 = 1$ (-2%)

$x_1, x_2, x_3, x_4, x_5 \geq 0$ (-2%)

2. (10%) Consider the following problem.

Max $Z = 5x_1 + 3x_2 + 4x_3$

S.T. $2x_1 + x_2 + x_3 \leq 20$

$3x_1 + x_2 + 2x_3 \leq 30$

$x_1, x_2 \geq 0$

You are given the information that the basic variables in the optimal solution are x_2 and x_3. Use this information to obtain the optimal solution. Do **not** actually perform any simplex iteration.

<Solution>

Given that x_2 and x_3 are basic variables.

Assume x_4 and x_5 are the slack variables of the first and the second constraint, respectively.

Thus, x_1, x_4, and x_5 are nonbasic variables (value = 0). (-2%)

$x_2 + x_3 = 20$ (-2%)

$x_2 + 2x_3 = 30$ (-2%)

We obtain $x_2 =10$ and $x_3 = 10$ by solving the above equations. (-2%)

The optimal solution is $x_1 =0$, $x_2 =10$ and $x_3 = 10$ (-2%)

3. (30%) Consider the following problem.

Max $Z = 2x_1 + 5x_2 + 3x_3$

S.T. $x_1 - 2x_2 + x_3 \geq 20$

$2x_1 + 4x_2 + x_3 = 50$

$x_1, x_2, x_3 \geq 0$

(a) Using the Big M method, construct the complete first simplex tableau (or in convenient algebraic form). (12%)

(b) Work through the simplex method step by step for **one** simplex iteration. At the end of the first iteration, do we find the feasible solution? (18%)

<Solution>

(a)

$Z - 2x_1 - 5x_2 - 3x_3 + Mx_5 + Mx_6 = 0$ (-2%)

$x_1 - 2x_2 + x_3 - x_4 + x_5 = 20$ (-2%)

$2x_1 + 4x_2 + x_3 + x_6 = 50$ (-2%)

x_4 : surplus variable

x_5 and x_6 : artificial variables

$x_5 = 20 - x_1 + 2x_2 - x_3 + x_4$ (-2%)

$x_6 = 50 - 2x_1 - 4x_2 - x_3$ (-2%)

Substitute into the objection row

$Z + (-3M - 2)x_1 + (-2M - 5)x_2 + (-2M - 3)x_3 + Mx_4 = -70M$ (-2%)

$x_1 - 2x_2 + x_3 - x_4 + x_5 = 20$

$2x_1 + 4x_2 + x_3 + x_6 = 50$

(b)

The basic feasible solution is (0, 0, 0, 0, 20, 50)

This is not an optimal solution since the coefficients of x_1, x_2, and x_3 are negative.

(increase the value of x_1, x_2, or x_3 could increase the objective value) (-2%)

We pick x_1 as the entering basic variable since it has the most negative coefficient. (-2%)

From the first constraint: x_1 could be increased to 20 at most. (-2%)

From the second constraint: x_1 could be increased to 25 at most. (-2%)

According to the minimum ratio test, we pick x_5 as the leaving basic variable. (-2%)

Obtain the next convenient form (tableau form).

$Z + (-8M - 9)x_2 + (M-1)x_3 + (-2M - 2)x_4 + (3M + 2)x_5 = -10M + 40$ (-2%)

$x_1 - 2x_2 + x_3 - x_4 + x_5 = 20$ (-2%)

$8x_2 - x_3 + 2x_4 - 2x_5 + x_6 = 10$ (-2%)

The basic feasible solution is (20, 0, 0, 0, 0, 10)

The value of x_6 (artificial variable) is not equal to zero ($x_6 = 10$). (-1%)

Thus, we haven't found the initial feasible solution. (-1%)

4. (10%) Consider the following linear programming model.

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to

$$x_1 + 2x_2 + x_3 \le 40 \text{ (resource 1)}$$
$$3x_1 + 2x_3 \le 60 \text{ (resource 2)}$$
$$x_1 + 4x_2 \le 30 \text{ (resource 3)}$$
$$x_1, x_2, x_3 \ge 0$$

Let x_4, x_5 and x_6 be the slack variables of constraints 1, 2 and 3 respectively. The optimum tableau is given as follows. Suppose the unit prices of each resource are all equal to 1.5, identify which resource should be given an increase in level?

Basic Variable	Coefficients of:							RHS
	Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	1	4	0	0	1	2	0	160
x_2	0	-1/4	1	0	1/2	-1/4	0	5
x_3	0	3/2	0	1	0	1/2	0	30
x_6	0	2	0	0	-2	1	1	10

<Solution>

The shadow price for resource 1, 2, 3 are 1, 2, and 0, respectively. (-6%)
($y_1 = 1, y_2 = 2, y_3 = 0$).

The unit price of each resource is 1.5
Only the shadow price of resource 2 is greater than 1.5. (-2%)
Thus, resource 2 should be given an increase in level. (-2%)

5. (10%) Consider a problem with two decision variable x_1 and x_2, which represent the level of activities 1 and 2, respectively. For each variable, the permissible values are 0, 1, 2, and feasible combinations of these values (i.e., pairs (x_1, x_2)) are determined from a variety of constraints. The objective is to maximize profit denoted by Z, whose dependence on x_1 and x_2 is summarized in the following table:

Z		x_2 = 0	x_2 = 1	x_2 = 2
x_1	0	0	4	8
	1	3	8	13
	2	6	12	18

Based on this information, indicate which assumptions of LP is/are violated. Justify your answers. The four LP assumptions are: proportionality, additivity, divisibility, and certainty.

<Solution>

Additivity: This assumption is violated. For example, the objective function value with (x_1, x_2)=(1, 1) is not equal to the objective function value with (x_1, x_2)=(1, 0) plus the objective function value with (x_1 , x_2)=(0, 1). (-5%)

Divisibility: This assumption is violated. Because the decision variables can only take on integer values of 0, 1,or 2. (-5%)

6. (30%) Label the following statement as being **true** or **false**. If false, **briefly state** your comments.

   (a) When an artificial problem is created by introducing artificial variables and using the Big M method, if all artificial variables in current solution for the artificial problem are equal to zero, then the current solution is an infeasible solution in the real (original) problem.

   False. When all the artificial variables are equal to zero, we have a feasible solution of the original problem.

   (b) The simplex method's rule for choosing the entering basic variable always leads to the best adjacent BF solution (Largest Z).

   False. The simplex method picks the entering basic variable with the largest incremental rate of the objective value.

   (c) If there is no leaving basic variable at some iteration, then the problem has no feasible solution.

   False. It implies the case of unbounded.

   (d) In a particular iteration of the simplex method, if there is a tie for which variable should be the leaving basic variable, then the next basic feasible solution must have at least one basic variable equal to zero.

   True. We will have at least one degenerate basic variable.

   (e) Only CPF solutions can be optimal, so the number of optimal solutions cannot exceed the number of CPF solutions.

   False. When there are multiple optimal solutions, the optimal solution is not necessary to be an CPF.

   (f) Assume that the feasible region exists and is bounded. If multiple optimal solutions exist, then an optimal CPF solution may have an adjacent CPF solution that also is optimal (the same value of Z).

   True. There are at least two adjacent optimal CPF when multiple optimal solutions occur.

   (g) To choose the new CPF solution to move to from the current CPF solution, the simplex method identifies all the adjacent CPF solutions and determines which one gives the largest rate of improvement in the value of the objective function.

   False. The simplex method does not identify all the adjacent CPF.

   (h) Given that a standard form (maximization and all constraints inequalities are less than and equal to) LP formulation with 5 decision variables and 4 functional constraints, there are 5 basic variables and 4 nonbasic variables.

False. There are 4 basic variables and 5 nonbasic variables.

(i) In terms of linear programming, infeasible solution implies that the objective value can increase without limit.

False. Infeasible implies the feasible region is empty.

(j) If a linear programming problem has an unbounded feasible region, then the problem must have multiple optimal solutions.

False. It could be unbounded or has one optimal solution.