Mathematical Statistics. Exam 1. 2018/10/22

1. (5%) If event A and event B are disjoint, are they independent? Why or why not?

2. (10%) If X is the geometric random variable with probability p, denoted as $X \sim Geo(p)$. Show that X has the memoryless property:

$$P(X > n + k - 1 | X > n - 1) = P(X > k),$$

for some non-negative integers n and k.

3. (10%) Two fair dice are rolled.
   - (a) (5%) What is the probability that the sum of face values is seven?
   - (b) (5%) What is the probability that at least one of the dice came up a three?

4. (15%) Suppose that a random X has a probability density function:

$$f(x) = 1 - \frac{x}{2}, \text{ for } 0 \leq x \leq 2.$$

   - (a) (5%) Let the cumulative distribution function of X be denoted by $F_X(x)$. Find $F_X(x)$.
   - (b) (5%) Let $Z = F_X(X)$. Find the distribution of Z.
   - (c) (5%) Let $U \sim U(0, 1)$. Find a function $g(\cdot)$, so that the random variable $Y = g(U)$ has the same distribution as X.

5. (30%) Suppose that Betty works at the front counter in McDonald, and she serves four customers this morning. For simplicity, a customer is classified as female or male. Assume that the probability that a female customer comes to McDonald with probability $p = 0.6$, and a male customer with probability $q = 0.4$. Denote the gender of a customer by F (female) or M (male). There are two bonus policy. Policy X pays Betty additional 5 dollars for each female customers she serves. Policy Y only focuses on the first and the fourth customers, and pays Betty additional 5 dollars for each male customers. Example: If the sequence of the gender of the customers is FFMF, Betty receives 15 dollars from policy X, and 0 dollar from policy Y. If the sequence of the gender of the customers is MFMF, Betty receives 10 dollars from policy X, and 5 dollars from policy Y.
   - (a) (5%) Write down the sample space of this scenario.
   - (b) (5%) Find the probabilities of each outcome in (a).
   - (c) (5%) Find the joint frequency function of X and Y.
   - (d) (5%) Find the marginal frequency functions of X and Y.
   - (e) (5%) Are X and Y independent? Why or why not?
   - (f) (5%) Find the conditional distribution of X given $Y = 5$.

6. (30%) Assume that X, Y are uniformly distributed on the unit circle, i.e.,

$$f_{X,Y}(x, y) = c, \quad \text{for } x^2 + y^2 \leq 1,$$

for some constant c.

   - (a) (5%) Find c so that $f_{X,Y}(x, y)$ is a legitimate joint probability density function.
   - (b) (5%) Find probability $P(X + Y < 1)$.
   - (c) (5%) Find the marginal probability density function for X, $f_X(x)$.
   - (d) (5%) Let $Z = |X|$. Find the probability density function of Z.
   - (e) (5%) Are X and Y independent? why or why not?
   - (f) (5%) Find the conditional probability density function of $Y|X = 0$.

Mathematical Statistics. Exam 2. 2018/12/3

$\frac{1}{2(z-1)^2}$

$-\frac{1}{2(1+z)^2}$

1. (10%) Let X and Y has a joint probability density function:

$$f(x,y) = 1, \quad y \geq 0, y \leq 1+x, y \leq 1-x.$$

Set $Z = Y/X$. Find the probability density function of Z.

2. (10%) If $X_i \overset{i.i.d.}{\sim} U(0,1)$ for $i = 1, \ldots, 9$. Let $X_{(1)} \leq X_{(2)} \leq X_{(9)}$ be the order statistics. Let $U = X_{(9)}$ be the maximum, $V = X_{(1)}$ be the minimum, and $M = X_{(5)}$ be the median.
   (a) (5%) Find the probability density function of M. $630 \cdot (1-x)^4 x^4$
   (b) (5%) Find the joint probability density function of (U, M, V). $81648 \cdot x^{12} \cdot (1-x)^{12}$

3. (10%) Suppose that $Var(X) = 1$, $Var(Y) = 4$, $Var(Z) = 9$. And the correlations, $\rho_{X,Y} = 0.2$, $\rho_{Y,Z} = -0.3$, and $\rho_{X,Z} = 0.5$. Find the numerical solutions of the following questions.
   (a) (2%) Find $Cov(X,Y)$, $Cov(X,Z)$, and $Cov(Y,Z)$. 0.4, 1.5, -1.8
   (b) (3%) Find $Var(X+Y+Z)$. 14.2
   (c) (5%) Find $Cov(X+2Y+3Z, Y-Z)$. -21.9

4. (20%) Let X and Y be random variables with joint density function:

$$f(x,y) = 1, \quad \text{for } 0 \leq x, 0 \leq y, 2x+y \leq 2.$$

   (a) (5%) Find the conditional expectation of $E[Y|X=1/2]$. $\frac{1}{2}$
   (b) (5%) Find $E[Y|X]$ as a function of X. $1-X$
   (c) (10%) Find the probability density function of $E[Y|X]$. $2z$

5. (20%) Let X be the exponential distribution with rate 1 with density

$$f(x) = \exp^{-x}, \quad \text{for } 0 \leq x.$$

   (a) (5%) Find the moment generating function of X, $M(t)$. (Need to specify the domain of t (2%).)
   (b) (5%) Find $M'(0)$ and $M''(0)$.
   (c) (5%) Find the upper bound of $P(X > 2)$ using Markov inequality. $P(X > t) \leq \frac{E(X)}{t}$
   (d) (5%) Find the upper bound of $P(|X-1| > 2)$ using Chebyshev inequality.

6. (10%) Suppose that $X_i \overset{i.i.d.}{\sim} X$ where X has the probability density function

$$f(x) = 2x, \quad \text{for } 0 \leq x \leq 1.$$

Denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find α, so that $\bar{X}_n$ converges in probability to α. $\frac{2}{3}$. $2\frac{x^3}{3}\Big|_{x=0}^{x=1}$

7. (10%) Show that if $P(Y = aX+b) = 1$ for some given constants a and b. Then, the correlation between X and Y is either 1 or -1.

8. (10%) A point is generated on a unit disk in the following way: The radius, R, is uniform on [0,1], and the angle θ is uniform on $[0, 2\pi]$ and is independent of R. (Hint: $\frac{d}{d\theta}\tan^{-1}(\theta) = \frac{1}{1+\theta^2}$.)
   (a) (5%) Find the joint density of R and θ.
   (b) (5%) Find the joint density of $X = R\cos(\theta)$ and $Y = R\sin(\theta)$.

x_{n-1} -

$a - b + 1$

Mathematical Statistics. Exam 3. 2019/1/7

	R H_0	A $1-\alpha$
H_0	α I	$1-\alpha$
H_0 false	$1-\beta$ power	β II

1. (10%) Describe the Central Limit Theorem.
2. (10%) Explain the type I error and type II error.
3. (10%) Let $X_1, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables; we sometimes refer to them as sample from a normal distribution. The sample mean is $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. The sample variance is $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Find the distribution of $\frac{\bar{X}-\mu}{S/\sqrt{n}}$. $\sim t_{n-1}$
4. (10%) Recall the MLE of μ and σ^2 from an i.i.d. normal sample are $\hat{\mu} = \bar{X}$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Derive the $(1\text{-}\alpha)\times 100\%$ confidence interval for σ^2.
5. (30%) Let $X \sim N(\mu, \sigma^2)$. Then, X has the probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \ -\infty < x < \infty.$$

$\mu_1 = E(X) = \mu$

$\mu_2 = E(X^2) = \sigma^2 + \mu^2$

   (a) (10%) Find the estimators for μ ($\bar{X}$) and σ^2 ($\frac{1}{n}\sum(X_i-\bar{X})^2$) using the method of moments.

   (b) (10%) Recall that a two-parameter member of the exponential family has a density or frequency function of the form

$$f(x|\theta) = \exp\left[\sum_{i=1}^{2} c_i(\theta)T_i(x) + d(\theta) + S(x)\right], \quad x \in A$$

$c_1(\theta) = -\frac{1}{2\sigma^2}$

$T_1(x) = x^2$

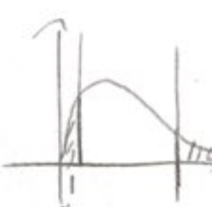
   where the set A does not depend on $\theta = (\theta_1, \theta_2)$.Show that X is a member of two-parameter member of the exponential family, and identify the functions $c_i(\theta)$, $T_i(x)$, $d(\theta)$, $S(x)$, for $i = 1, 2$.

   $d(\theta) = -\log\sigma\sqrt{2\pi} - \frac{\mu^2}{2\sigma^2}$, $S(x) = 0$

   (c) (10%) Find the two sufficient statistics for X. $\sum X_i^2$ $\sum X_i$

6. (30%) Coin 0 has probability of heads equal to 0.5, and coin 1 has probability of heads equal to 0.3. I choose one of the coins, toss it 5 times and tell you the number of heads, do not tell you weather it was coin 0 or coin 1. We set the null hypothesis, (H_0: coin 0 is selected) and the alternative hypothesis, (H_1: coin 1 is selected.) Formally, let X be the number of heads observed. Then, $X \sim Binomial(5, p)$. Then, we have the equivalent statements that $H_0 : p = 0.5$, and $H_1 : p = 0.3$. The following table gives us the probabilities of $X = x$ under H_0 and H_1, and their ratios.

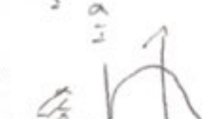
	$X = x$	0	1	2	3	4	5
H_0	coin 0	0.03	0.16	0.31	0.31	0.16	0.03
H_1	coin 1	~~0.17~~	~~0.36~~	0.31	0.13	0.03	0.00
	$\frac{P(x\|H_0)}{P(x\|H_1)}$	0.19	0.43	1.01	2.36	5.51	12.86



Recall that the likelihood ratio test rejects H_0 when $\frac{P(H_0|x)}{P(H_1|x)} < c$.

   (a) (10%) Suppose that $P(H_0) = P(H_1) = \frac{1}{2}$. Show that the likelihood ratio equals to

$$\frac{P(H_0|x)}{P(H_1|x)} = \frac{P(x|H_0)}{P(x|H_1)}.\ \frac{\pi(H_0)}{P(H_1)}$$



α

   (b) (10%) Set $c = 1$. Find the significance level of this test. 0.19.

   (c) (10%) Set $c = 2$. Find the power of this test. 0.84

$1-\beta$