一百零五學年度 0311 微積分 (二) 期初考
The 105th academic year course 0311 Calculus(2) firstterm examination

date: Mar 31, 2017

Student ID No. 學號 : ________________ Name 姓名 : ________________

說明 Description:

(1) 答題之前請先檢查所取得之試卷與答案卷是否正確。
Before answering questions, please check if the test papers and answer sheets which you get are correct.

(2) 測驗時間 110 分鐘。試卷加答案卷、答案卡共計 7 頁。
Testing time is 110 minutes. Test papers, answer sheets, and answer cards are of 7 pages in total.

(3) 試卷包括選擇題與填充題, 總分共計 100 分, 占學期成績之 20%。考卷成績將 $\boxed{\text{不}}$ 做爲微積分獎給獎依據。
The test paper includes choices and fill-in-the-blanks, and there is a total score of 100 points, accounting for 20% of the semester grade. The examination result will $\boxed{\text{not}}$ be considered for awarding the Calculus prize.

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答, 否則不予計分。
Be sure to fill related personal information in answer sheets and answer cards. When answering questions, please answer the question by its question number, or, no score.

P.S. 難易度提示 Difficulty hint: easy$<$normal$<$hard
$\star$ $\star\star$ $\overset{\star}{\star\star}$ $\overset{\star\star}{\star\star}$ $\bigstar$

———————— Questions start from the next page ————————

◎ 單選擇題 (單選十題, 每題五分, 共五十分, 答錯不倒扣。)
Single-choice (10 questions, each 5 points, 50 points in total, no penalty for wrong answers.)

1. Find the **power series** representation of $\ln \frac{1}{2+x}$. ★★ ★★ §11.9+ 11.10

(A) $\sum_{n=1}^{\infty} \frac{1}{n2^n}x^n$. (B) $\sum_{n=1}^{\infty} \frac{1}{n2^n}x^n - \ln 2$.

(C) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}x^n$. **(D)** $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}x^n - \ln 2.}$

Solution: $\ln \frac{1}{2+x} = \int \frac{-1}{2+x}\,dx = -\frac{1}{2}\int \frac{1}{1-(-\frac{x}{2})}\,dx$

$= -\frac{1}{2}\int \sum_{n=0}^{\infty}(-\frac{x}{2})^n\,dx = -\frac{1}{2}\sum_{n=0}^{\infty}\int (-\frac{x}{2})^n\,dx$

$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n}x^n + C$,

$\ln\frac{1}{2+0} = -\ln 2 = C$, $\ln\frac{1}{2+0} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n}x^n - \ln 2$.

[Quick Sol]

$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$,

$\ln\frac{1}{2+x} = -\ln 2(1+\frac{x}{2}) = -\ln(1+\frac{x}{2}) - \ln 2$

$= -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(\frac{x}{2})^n - \ln 2 = \sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n}x^n - \ln 2$.

[Quicker Sol]

$x=2$, $\sum \frac{1}{n2^n}x^n = \infty$, $\sum \frac{1}{n2^n}x^n - \ln 2 = \infty$, $\sum \frac{(-1)^n}{n2^n}x^n = -\ln 2$,

$\sum \frac{(-1)^n}{n2^n}x^n - \ln 2 = -\ln 4 = \ln\frac{1}{2+x}$.

2. Find the first three **nonzero** terms of the Maclaurin series of $\boldsymbol{\dfrac{e^x}{1+x}}$. ⋆ §11.10

(A) $1+\frac{x^2}{2}+\frac{x^3}{3}$. (B) $1-\frac{x^2}{2}+\frac{x^3}{3}$.

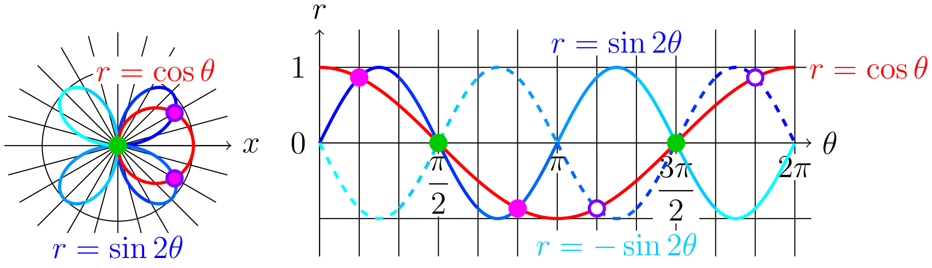
(C) $\boxed{\boldsymbol{1+\frac{x^2}{2}-\frac{x^3}{3}}.}$ (D) $1-\frac{x^2}{2}-\frac{x^3}{3}$.

Solution: $\frac{e^x}{1+x}=e^x\frac{1}{1+x}=\sum_{n=0}^{\infty}\frac{1}{n!}x^n\sum_{n=0}^{\infty}(-x)^n$
$=\Big(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots\Big)\Big(1-x+x^2-x^3+\cdots\Big)$
$=1+\Big(1-1\Big)x+\Big(\frac{1}{2}-1+1\Big)x^2+\Big(\frac{1}{6}-\frac{1}{2}+1-1\Big)x^3+\cdots$
$=1+\frac{1}{2}x^2-\frac{1}{3}x^3+\cdots$.

3. Find the **number** of points of **intersection** of the curves $\boldsymbol{r=\cos\theta}$ and $\boldsymbol{r=\sin 2\theta}$. ⋆⋆⋆ §10.4

(A) $\boxed{\textbf{3.}}$ (B) 4. (C) 5. (D) 6.

Solution: $r=\cos\theta=\sin 2\theta=2\sin\theta\cos\theta$ (collision),
& $r=\cos\theta=-\sin 2\theta=-2\sin\theta\cos\theta$.
$\cos\theta=0$ or $\sin\theta=\frac{1}{2}$ or $\sin\theta=-\frac{1}{2}$, $\theta=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{7\pi}{6},\frac{3\pi}{2},\frac{11\pi}{6}$.
$(\frac{\sqrt{3}}{2},\frac{\pi}{6})=(-\frac{\sqrt{3}}{2},\frac{7\pi}{6})$, $(0,\frac{\pi}{2})=(0,\frac{3\pi}{2})$, $(\frac{\sqrt{3}}{2},\frac{11\pi}{6})=(-\frac{\sqrt{3}}{2},\frac{5\pi}{6})$,

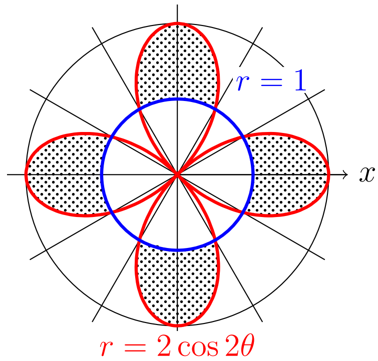


4. Find the **limit** of the sequence $\left\{\dfrac{\boldsymbol{2^n}}{\boldsymbol{n!}}\right\}$. ⋆ §11.1

(A) $\boxed{\mathbf{0.}}$ (B) 1. (C) ∞. (D) does not exist.

Solution: $\dfrac{2^n}{n!} = \dfrac{2}{1}\left(\dfrac{2}{2}\cdots\dfrac{2}{n-1}\right)\dfrac{2}{n} < \dfrac{2}{1}\cdot 1\cdot\dfrac{2}{n} = \dfrac{4}{n} \to 0$ as $n\to\infty$.
(Ex 11.1.55)

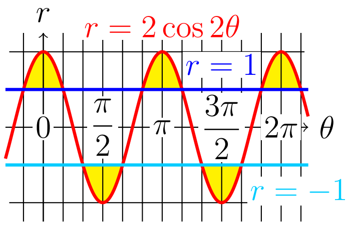
5. Find the **area** of the region **inside** the four-leaved rose $\boldsymbol{r = 2\cos 2\theta}$ and **outside** the circle $\boldsymbol{r = 1}$. ⋆⋆⋆ §10.4



(A) $\boxed{\dfrac{\mathbf{2\pi}}{\mathbf{3}} + \sqrt{\mathbf{3}}}$ (B) $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$

(C) $\dfrac{2\pi}{3} - \sqrt{3}$ (D) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}$

Solution: (102-Calculus1)



$$r = 2\cos 2\theta = 1, \qquad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$
$$r = 2\cos 2\theta = -1, \qquad \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

$$A = 4\int_{-\pi/6}^{\pi/6} \frac{1}{2}\left[(2\cos 2\theta)^2 - 1^2\right]\,d\theta = 4\int_0^{\pi/6} \left[(2\cos 2\theta)^2 - 1^2\right]\,d\theta$$
$$= 4\int_0^{\pi/6} (4\cos^2 2\theta - 1)\,d\theta = 4\int_0^{\pi/6} (1 + 2\cos 4\theta)\,d\theta$$
$$= \Big[4\theta + 2\sin 4\theta\Big]_0^{\pi/6} = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + \sqrt{3}.$$

6. Find the **tangent** line of the polar curve $\boldsymbol{r = \sec\theta\tan\theta}$ at $\boldsymbol{\theta = \frac{\pi}{4}}$. ⋆ §10.3

(A) $x = 0$. (B) $y = 0$. **(C)** $\boxed{\boldsymbol{y = 2x - 1.}}$ (D) $y = -2x - 1$.

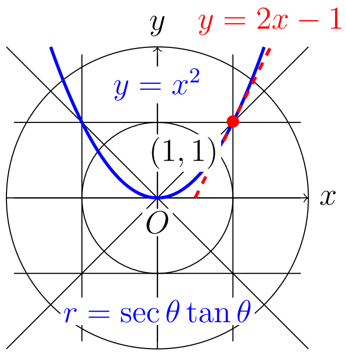
Solution:

$x = r\cos\theta = \tan\theta$,

$y = r\sin\theta = \tan^2\theta$,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\tan\theta\sec^2\theta}{\sec^2\theta(\neq 0)} = 2\tan\theta.$$

$$y = \frac{dy}{dx}\Big|_{\theta=\pi/4}(x - \tan\frac{\pi}{4}) + \tan^2\frac{\pi}{4}$$
$$= 2\cdot 1(x-1) + 1^2 = 2x - 1.$$



[Quick Sol]

$y = x^2$, $\frac{dy}{dx} = 2x$. When $\theta = \frac{\pi}{4}$, $(x, y) = (1, 1)$ and $\frac{dy}{dx} = 2$.

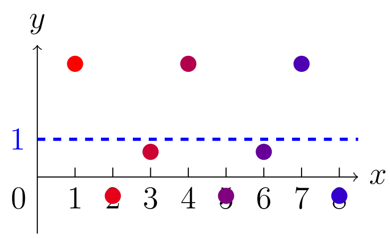
Tangent line: $y = 2(x-1) + 1 = 2x - 1$.

7. Let $\boldsymbol{a_1 = 3}$, $\boldsymbol{a_{n+1} = \frac{1}{1 - a_n}}$ for $\boldsymbol{n \geq 2}$. Find the **limit** $\lim_{n\to\infty} a_n$. ⋆⋆ §11.1

(A) 0. (B) 1. (C) ∞. **(D)** $\boxed{\textbf{does not exist.}}$

Solution: Solve $x = \frac{1}{1-x}$, no real solution for x. (Ex 11.1.82)

$\{a_n\} = 3, -\frac{1}{2}, \frac{2}{3}, 3, -\frac{1}{2}, \frac{2}{3}, \ldots$, $\lim_{n\to\infty} a_n$ does not exist.

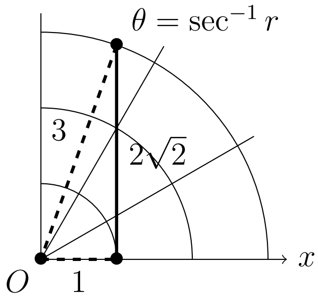


8. Find the arc **length** of the polar curve $\boldsymbol{\theta = \sec^{-1} r}$ for $\mathbf{1 \le r \le 3}$. ★★ §10.4

(A) $2\sqrt{2}-1$ (B) 2 **(C)** $\boxed{\mathbf{2\sqrt{2}}}$ (D) 3

Solution: $r = \sec\theta$, $\sqrt{r^2 + (r')^2} = \sqrt{\sec^2\theta + \sec^2\theta\tan^2\theta} = \sec^2\theta$,

$$L = \int_0^{\sec^{-1} 3} \sqrt{r^2 + (r')^2}\, d\theta$$
$$= \int_0^{\sec^{-1} 3} \sec^2\theta\, d\theta$$
$$= \tan\theta\Big|_0^{\sec^{-1} 3} = \tan(\sec^{-1} 3) = 2\sqrt{2}.$$



[Quick Sol] $x = r\cos\theta = 1$, $\sqrt{3^2 - 1^2} = 2\sqrt{2}$.

9. Find the **value** of $\boldsymbol{p}$ for which the series $\sum \dfrac{\boldsymbol{\ln n}}{\boldsymbol{n^p}}$ is **divergent**. ★★ §11.3+ 11.4

(A) $p > 1$. **(B)** $\boxed{\boldsymbol{p \le 1.}}$ (C) $p \in \mathbb{R}$. (D) does not exist.

Solution: $1 < \ln n < \dfrac{n^\varepsilon}{\varepsilon}$ for $\varepsilon > 0$ and $n \ge 3$. (Ex 11.3.32)

$\dfrac{1}{n^p} < \dfrac{\ln n}{n^p} < \dfrac{n^\varepsilon}{\varepsilon n^p} = \dfrac{1}{\varepsilon n^{p-\varepsilon}}$,

$\dfrac{1}{n^p}$ and hence $\dfrac{\ln n}{n^p}$ diverges for $p \le 1$,

$\dfrac{1}{n^{p-\varepsilon}}$ and hence $\dfrac{\ln n}{n^p}$ converges for $p - \varepsilon > 1 \iff p > 1$.

$\square$ 10. Find the **interval** of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}(x-2)^n$. ⋆ §11.8

(A) $\boxed{\mathbf{(0,4].}}$ (B) $(-4,0]$. (C) $[0,4)$. (D) $[-4,0)$.

Solution: $\displaystyle\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n(x-2)^n}{n2^n}}\right| = \lim_{n\to\infty}\frac{n}{n+1}\frac{|x-2|}{2} = \frac{|x-2|}{2}.$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n}(x-2)^n$ converges if $\dfrac{|x-2|}{2} < 1 \iff |x-2| < 2$ and diverges if $\dfrac{|x-2|}{2} > 1 \iff |x-2| > 2$ by Ratio T.

$R = 2$, $0 < x < 4$.

If $x = 4$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n}(x-2)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}(= -\ln 2)$ converges.

If $x = 0$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n2^n}(x-2)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

$I = (0,4]$.

◎ 多選擇題 (多選五題, 每題五分, 共二十五分。錯一個選項扣兩分, 錯兩個選項以上不給分, 分數不倒扣。)

Multiple-choice (5 questions, each 5 points, 25 points in total. One wrong option deducts 2 points, more than one wrong option gets no points, and no penalty for wrong answers.)

11. Which of the following series is **absolutely convergent**? ★ §11.4+ 11.5 + 11.6

(A) $\boxed{\boldsymbol{\sum \frac{1}{n^{2+1/n}}}}$. (B) $\sum (-1)^n \sin\left(\frac{\pi}{n}\right)$.

(C) $\boxed{\boldsymbol{\sum \frac{\sin^3 n}{n^3}}}$. (D) $\sum (-1)^n \frac{n^n}{2^n n!}$.

Solution: $\left|\frac{1}{n^{2+1/n}}\right| = \frac{1}{n^{2+1/n}}$ and $\lim_{n\to\infty} \frac{1/n^{2+1/n}}{1/n^2} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$,

then $\sum \frac{1}{n^2}$ and hence $\sum \frac{1}{n^{2+1/n}}$ absolutely converges by LCT.

....

$\left|(-1)^n \sin\left(\frac{\pi}{n}\right)\right| = \sin\left(\frac{\pi}{n}\right)$ and $\lim_{n\to\infty} \frac{\sin(\pi/n)}{1/n} = \pi$,

then $\sum \frac{1}{n}$ and hence $\sum \sin\left(\frac{\pi}{n}\right)$ diverges by LCT.

$\sin\left(\frac{\pi}{n}\right) \searrow 0$, then $\sum (-1)^n \sin\left(\frac{\pi}{n}\right)$ is convergent by AST.

$\therefore \sum (-1)^n \sin\left(\frac{\pi}{n}\right)$ is conditionally convergent.

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$\left|\frac{\sin^3 n}{n^3}\right| \le \frac{1}{n^3}$, then $\sum \frac{1}{n^3}$ and hence $\sum \left|\frac{\sin^3 n}{n^3}\right|$ converges by CT.

$\therefore \sum \frac{\sin^3 n}{n^3}$ is absolutely convergent.

....

$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)^{n+1}/2^{n+1}(n+1)!}{(-1)^n n^n/2^n n!}\right| = \lim_{n\to\infty} \frac{1}{2}(1+\frac{1}{n})^n = \frac{e}{2} > 1$,

$\sum (-1)^n \frac{n^n}{2^n n!}$ diverges by Ratio T.

12. Let $\sum a_n$ be a **convergent** series and let f be a **continuous** function on $\mathbb{R}$. Which of the following statements is **always true**?

(A) $\boxed{\{|a_n|\} \text{ is convergent.}}$ (B) $\boxed{\{f(a_n)\} \text{ is convergent.}}$

(C) $\sum |a_n|$ is convergent. (D) $\sum f(a_n)$ is convergent.

⋆⋆ §11.1+ 11.5 + 11.6

Solution: $\sum a_n$ converges $\implies \lim a_n = 0 \iff \lim |a_n| = 0$.

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$\sum \dfrac{(-1)^{n-1}}{n}$ converges but $\sum \left|\dfrac{(-1)^{n-1}}{n}\right| = \sum \dfrac{1}{n}$ diverges.
(Conditionally convergent series)

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$\sum a_n$ converges $\implies \lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n) = f(0)$.

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$\sum \dfrac{1}{n^3}$ converges and $f(x) = \sqrt[3]{x}$, but $\sum f(\dfrac{1}{n^3}) = \sum \dfrac{1}{n}$ diverges.
(Or $f(0) \neq 0$, by T4D.)

13. Find **all** $\boldsymbol{n}$ for which applies the method of the **Remainder Estimate for the Integral Test** to approximate(逼近) $\frac{\pi^2}{6}$ by the series $\sum \frac{\mathbf{1}}{\boldsymbol{n}^{\mathbf{2}}}$ with accuracy(準確) to within **0.01** ($|R_n| \le 0.01$). ⋆ §11.5

(A) 10. **(B)** $\boxed{\mathbf{100.}}$ **(C)** $\boxed{\mathbf{101.}}$ **(D)** $\boxed{\mathbf{200.}}$

Solution: $R_n \le \displaystyle\int_n^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_n^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{x}\right]_n^t$
$= \displaystyle\lim_{t\to\infty}\left(\frac{1}{n}-\frac{1}{t}\right) = \frac{1}{n} \le 0.01$, $n \ge 100$.
♦ $R_{99} \approx 0.01005$, $R_{100} \approx 0.00995$, $R_{101} \approx 0.00985$.

14. Suppose that both $\sum \boldsymbol{c_n 2^n}$ and $\sum \boldsymbol{c_n(-1)^n}$ are **convergent**. Find **all possible interval** of convergence of $\sum \boldsymbol{c_n(x-3)^n}$. ⋆ §11.8

(A) $(2,4)$. (B) $[2,4]$. (C) $(1,5)$. **(D)** $\boxed{\mathbf{[1,5].}}$

Solution: $\sum c_n(x-3)^n$ converges for $x-3=2 \iff x=5$ and $x-3=-1 \iff x=2$. The radius of convergence $R \ge 2$, and the interval of convergence $(1,5] \subseteq I$.

15. Determining the convergence of $\displaystyle\sum_{\boldsymbol{n=2}}^{\boldsymbol{n}} \frac{\mathbf{1}}{\boldsymbol{n(\ln n)^2}}$ by the following steps:

Step 1. $\dfrac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing for $x > 1$.

Step 2. $\displaystyle\int_1^\infty \frac{1}{x(\ln x)^2}\,dx$ is convergent.

Step 3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent by the Integral Test.

Step 4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} = \int_2^\infty \frac{1}{x(\ln x)^2}\,dx.$

Which of the above steps is **incorrect**?

$\star$
$\star\star$
§11.3

(A) Step 1. **(B)** $\boxed{\textbf{Step 2.}}$ (C) Step 3. **(D)** $\boxed{\textbf{Step 4.}}$

Solution: $\displaystyle\frac{d}{dx}\frac{1}{x(\ln x)^2} = -\frac{\ln x + 2x}{x^2(\ln x)^4} < 0$ for $x > 1$. (Ex 11.3.22)

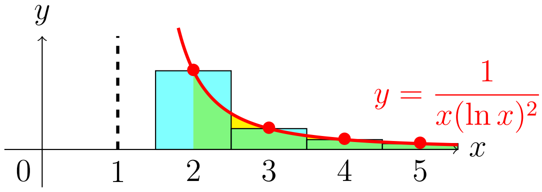
$$\int \frac{dx}{x(\ln x)^2} = \frac{-1}{\ln x} + C,$$

$$\int_1^\infty \frac{1}{x(\ln x)^2}\,dx = \int_1^2 \frac{1}{x(\ln x)^2}\,dx + \int_2^\infty \frac{1}{x(\ln x)^2}\,dx$$

$$= \lim_{t\to 1^+} \int_t^2 \frac{1}{x(\ln x)^2}\,dx + \lim_{s\to\infty} \int_2^s \frac{1}{x(\ln x)^2}\,dx$$

$$= \lim_{t\to 1^+} \left(\frac{1}{\ln t} - \cancel{\frac{1}{\ln 2}}\right) + \lim_{s\to\infty} \left(\cancel{\frac{1}{\ln 2}} - \frac{1}{\ln s}\right) = \infty + 0 = \infty.$$

$$\sum_{n=2}^{n} \frac{1}{n(\ln n)^2} \approx 2.10974, \quad \int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln 2} \approx 1.44270.$$



◎ 填空題 (五題, 每題五分, 共二十五分, 答錯不倒扣。)
Fill-in-the-blank (5 questions, each worth 5 points, 25 points in total, no penalty for wrong answers.)

16. Find the **sum** of the series $\sum \frac{1}{n^2+3n}$. ⋆ §11.2

Solution: $\frac{11}{18}$. (Ex 11.2.45)

....

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i^2+3i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{3}\Big(\frac{1}{i} - \frac{1}{i+3}\Big)$
$= \lim_{n\to\infty} \frac{1}{3}\Big[\Big(\frac{1}{1} - \cancel{\frac{1}{4}}\Big) + \Big(\frac{1}{2} - \cancel{\frac{1}{5}}\Big) + \Big(\frac{1}{3} - \cancel{\frac{1}{6}}\Big) + \cdots$
$+ \Big(\cancel{\frac{1}{n-2}} - \frac{1}{n+1}\Big) + \Big(\cancel{\frac{1}{n-1}} - \frac{1}{n+2}\Big) + \Big(\cancel{\frac{1}{n}} - \frac{1}{n+3}\Big)\Big]$
$= \lim_{n\to\infty} \frac{1}{3}\Big(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\Big)$
$= \frac{1}{3}\Big(1 + \frac{1}{2} + \frac{1}{3}\Big) = \frac{11}{18}$.

17. Find the **interval** of convergence of the Maclaurin series of $\mathbf{\ln \frac{1}{1-x}}$. ⋆ §11.9

Solution: $[-1, 1)$.

....

$\ln \frac{1}{1-x} = \int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx = \sum_{n=0}^{\infty} \int x^n\,dx$
$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} \frac{x^n}{n} + C$. Take $x = 0$, $\ln \frac{1}{1-0} = 0 = C$.
$\ln \frac{1}{1-x} = \sum \frac{x^n}{n}$ with $R = 1$, $-1 < x < 1$.
If $x = 1$, $\sum \frac{1}{n}$ diverges. If $x = -1$, $\sum \frac{(-1)^n}{n}$ converges. $I = [-1, 1)$.

18. Find the **radius** of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}(x-3)^{2n}$. ⋆⋆ §11.8

Solution: $\sqrt{e}$.

. . . .

$\displaystyle\lim_{n\to\infty}\left|\frac{(n+1)!(x-3)^{2(n+1)}/(n+1)^{n+1}}{n!(x-3)^{2n}/n^n}\right| = \lim_{n\to\infty}\frac{(x-3)^2}{(1+1/n)^n} = \frac{(x-3)^2}{e}$.

by Ratio T, converges if $\dfrac{(x-3)^2}{e} < 1 \iff |x-3| < \sqrt{e}$ and diverges if $\dfrac{(x-3)^2}{e} > 1 \iff |x-3| > \sqrt{e}$.

♦ When $x = 3 \pm \sqrt{e}$, $\displaystyle\sum \frac{n!e^n}{n^n}$ diverges. $I = (3-\sqrt{e}, 3+\sqrt{e})$.)

19. Let $f(x) = x\cos x$. Evaluate the **value** of $f^{(101)}(0)$. ⋆⋆ §11.10

Solution: 101.

. . . .

$x\cos x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n+1}$. $2n+1 = 101$, $n = 50$, $c_{101} = \dfrac{f^{(101)}(0)}{101!}$,

$f^{(101)}(0) = 101!c_{101} = 101!\dfrac{(-1)^{50}}{100!} = 101$.

[Another sol]

$f' = \cos x - x\sin x$, $f'' = -2\sin x - x\cos x$, $f''' = -3\cos x + x\sin x$,

$f''' = 4\sin x + x\cos x$,...

Guess $f^{(n)}(x) = n(\cos x)^{(n-1)} + x(\cos x)^{(n)}$.

$f^{(n+1)}(x) = (f^{(n)})'(x) = (n(\cos x)^{(n-1)} + x(\cos x)^{(n)})'$

$= n(\cos x)^{(n)} + (\cos x)^{(n)} + x(\cos x)^{(n+1)}$

$= (n+1)(\cos x)^{(n+1-1)} + x(\cos x)^{(n+1)})$, by induction.

$f^{(101)}(0) = 101(\cos x)^{(100)}(0) + 0\cdot(\cos x)^{(101)}(0) = 101\cos 0 = 101$.

20. Find the **third**-degree Taylor polynomial $\boldsymbol{T_3(x)}$ of $\boldsymbol{\sin x}$ at $\boldsymbol{\frac{\pi}{2}}$. $\star\star\star$ §11.10+ 11.11

Solution: $1 - \frac{1}{2}(x - \frac{\pi}{2})^2$.

....

$T_3(x) = f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x - \frac{\pi}{2}) + \frac{1}{2!}f''(\frac{\pi}{2})(x - \frac{\pi}{2})^2 + \frac{1}{3!}f'''(\frac{\pi}{2})(x - \frac{\pi}{2})^3$
$= \sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2})(x - \frac{\pi}{2}) - \frac{1}{2!}\sin(\frac{\pi}{2})(x - \frac{\pi}{2})^2 - \frac{1}{3!}\cos(\frac{\pi}{2})(x - \frac{\pi}{2})^3$
$= 1 + 0 \cdot (x - \frac{\pi}{2}) - \frac{1}{2!} \cdot 1 \cdot (x - \frac{\pi}{2})^2 - \frac{1}{3!} \cdot 0 \cdot (x - \frac{\pi}{2})^3$
$= 1 - \frac{1}{2}(x - \frac{\pi}{2})^2$.

[Quick sol]

$\sin x = \sin(\frac{\pi}{2} + x - \frac{\pi}{2}) = \sin(\frac{\pi}{2})\cos(x - \frac{\pi}{2}) + \cos(\frac{\pi}{2})\sin(x - \frac{\pi}{2})$
$= 1 \cdot \cos(x - \frac{\pi}{2}) + 0 \cdot \sin(x - \frac{\pi}{2}) = \cos(x - \frac{\pi}{2})$
$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x - \frac{\pi}{2})^{2n} = 1 - \frac{1}{2!}(x - \frac{\pi}{2})^2 + \frac{1}{4!}(x - \frac{\pi}{2})^4 + \cdots,$
$T_3(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2$.

⊕ 加分題 (共十五分。總分超過100分以100分計。)
Bonus (15 points in total. The total score more than 100 points will only get 100 points.)

(a). Match functions with power series. (Each blank **1 pt**.) ⋆ ∼ ⁂

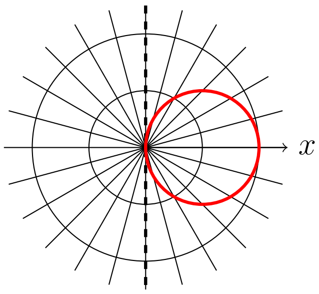
(A) $\sum_{n=0}^{\infty} x^n$	(J) $\sum_{n=0}^{\infty} (-1)^n x^n$	(S) $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n$
(B) $\sum_{n=0}^{\infty} x^{2n}$	(K) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$	(T) $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n x^n$
(C) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$	(L) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$	(U) $\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n$
(D) $\sum_{n=1}^{\infty} -\frac{x^n}{n}$	(M) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$	(V) $\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^n$
(E) $\sum_{n=1}^{\infty} \frac{x^n}{n}$	(N) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	(W) $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{2n}$
(F) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	(O) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	(X) $\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n x^{2n}$
(G) $\sum_{n=1}^{\infty} \frac{x^{2n}}{2n}$	(P) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n}$	(Y) $\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n}$
(H) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	(Q) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	(Z) $\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^{2n}$
(I) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$	(R) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	

(Hint: Fill uppercase alphabet as A, B, $\cdots$.)

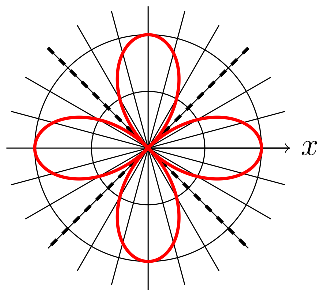
(i) $e^x = \boxed{\text{C}}$.	(vi) $\frac{1}{\sqrt{1+x^2}} = \boxed{\text{Y}}$.
(ii) $\frac{1}{1-x} = \boxed{\text{A}}$.	(vii) $\ln \frac{1}{1-x} = \boxed{\text{E}}$.
(iii) $\tan^{-1} x = \boxed{\text{R}}$.	(viii) $\sqrt{1+x} = \boxed{\text{S}}$.
(iv) $\sin x = \boxed{\text{Q}}$.	(ix) $\ln(1+x) = \boxed{\text{N}}$.
(v) $\frac{1}{1+x^2} = \boxed{\text{K}}$.	(x) $\cos x = \boxed{\text{O}}$.

(b). Sketch the polar curves. (Each graph **1 pts**.) $\star \sim \star\star\star$

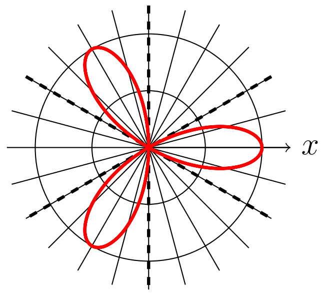
(xi) $r = 2\cos\theta$



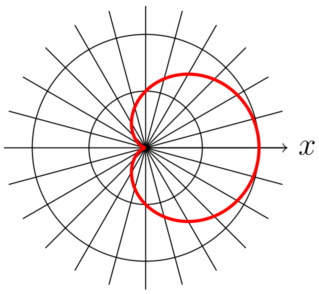
(xii) $r = 2\cos 2\theta$



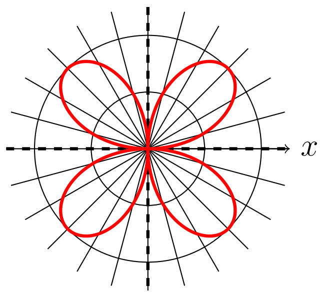
(xiii) $r = 2\cos 3\theta$



(xiv) $r = 1 + \cos\theta$



(xv) $r = 2\sin 2\theta$



⊛ 挑戰題 (共五分。總分超過100分以100分計。)
Challenge (5 points in total. The total score more than 100 points will only get 100 points.)

The sequence $\{a_n\}$ is defined by the *recurrence relation*(遞迴關係)

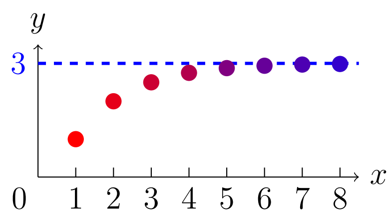
$$a_1 = 1, \qquad a_2 = 2, \qquad a_n = \frac{3a_{n-1} - a_{n-2}}{2} \quad \text{for } n \geq 3.$$

(α). [2 pts] Find the **limit** $\lim\limits_{n\to\infty} a_n = \boxed{3}$. ⋆⋆⋆⋆

(β). [3 pts] Find the **formula** of $a_n = \boxed{3 - \frac{1}{2^{n-2}}}$. ★

Solution:

$$\begin{array}{rlll} 2\,a_3 &= 3\,a_2 & -\,a_1 & \\ 2\,a_4 &= 3\,a_3 & -\,a_2 & \\ & \vdots & & \\ 2\,a_{n-1} &= 3\,a_{n-2} & -\,a_{n-3} & \\ 2\,a_n &= 3\,a_{n-1} & -\,a_{n-2} & \\ \hline 2\,a_n &= 2\,a_2 & -\,a_1 & +\,a_{n-1}. \end{array}$$



If $\{a_n\}$ converges with $\lim\limits_{n\to\infty} a_n = a$, then $2a = 2 \times 2 - 1 + a$, $a = 3$.

$a_n \nearrow$ $(a_{n+1} = \dfrac{3a_n - a_{n-1}}{2} > \dfrac{3a_n - a_n}{2} = a_n)$ is easy,
but $a_n \leq 3$ is hard, $\therefore$ cannot apply MCT.

[Method of Iteration]

$$a_n - a_{n-1} = \frac{1}{2}(a_{n-1} - a_{n-2}) = \cdots = \frac{1}{2^{n-2}}(a_2 - a_1) = \frac{1}{2^{n-2}},$$

$$\begin{aligned} a_n &= (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - a_1) + a_1 \\ &= \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} + \cdots + 1 + a_1 \\ &= \sum_{i=0}^{n-2} \frac{1}{2^i} + 1 \; = \; 3 - \frac{1}{2^{n-2}}, \end{aligned}$$

$$\lim_{n\to\infty} a_n = 3.$$

Solution: (Method of Power Series)

Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$.

$$
\begin{array}{lrcl}
 & \text{Let } f(x) & = & \sum a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
-) & \frac{3x}{2} f(x) & = & \frac{3a_1}{2} x^2 + \frac{3a_2}{2} x^3 + \cdots \\
+) & \frac{x^2}{2} f(x) & = & \frac{a_1}{2} x^3 + \cdots \\
\hline
 & (1 - \frac{3x}{2} + \frac{x^2}{2}) f(x) & = & a_1 x + (a_2 - \frac{3a_1}{2}) x^2 \\
 & & & + \sum_{n\geq 3} (a_n - \frac{3a_{n-1} - a_{n-2}}{2}) x^n \\
 & & = & x + \frac{x^2}{2},
\end{array}
$$

$$
\begin{aligned}
f(x) &= \frac{x + x^2/2}{(1-x)(1-x/2)} = \frac{3x}{1-x} - \frac{2x}{1-x/2} \\
&= 3\sum_{n=1}^{\infty} x^n - 4\sum_{n=1}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty} \left[3 - \frac{4}{2^n}\right] x^n, \\
a_n &= 3 - \frac{1}{2^{n-2}}, \ \lim_{n\to\infty} a_n = 3.
\end{aligned}
$$

Solution: (Method of Characteristic Polynomial)

Assume $a_n = x^n$, $x \neq 0$, $x^n = \frac{3x^{n-1} - x^{n-2}}{2}$, $x^2 = \frac{3x-1}{2}$.

Solve characteristic polynomial $x^2 = \frac{3x-1}{2} \implies x = 1, \frac{1}{2}$.

$\implies$ general solution $a_n = a \times 1^n + b \times (\frac{1}{2})^n$.

Solve boundary condition: $\begin{cases} n = 1, & a + \frac{1}{2} b = a_1 = 1, \\ n = 2, & a + \frac{1}{4} b = a_2 = 2, \end{cases}$

$\implies \begin{cases} a = & 3, \\ b = & -4, \end{cases} \implies a_n = 3 - 4(\frac{1}{2})^n, \ \lim_{n\to\infty} a_n = 3.$

Questions End