DOM1045 Operation Research (I)

Midterm Exam 02

2019.11.26

Name:______________________ Student ID: ___________________

This is a closed book exam. Any electronic device is NOT allowed during the exam. The solution must write on the answer booklet. Both question and answer booklets must be signed and returned to TAs when you leave the classroom. Good luck!

Problem sets	**#1**	**#2**	**#3**	**#4**	**#5**	**#6**	**Total**
Max scores	30	5	20	25	10	10	100
Your scores							

Problem 1. (30 points) Consider the following linear program. Let $x_4, x_5,$ and x_6 are slack variables for the first, second, and third constraints, respectively. Work through the simplex method in matrix form ONLY to solve the problem by either completing the following vectors and matrixes directly or writing down the procedure you learned in the class on page 4. You may utilize information in vectors and matrixes to double check the final solution.

(需使用矩陣方式表達，使用題目提供之表格或自由發揮方式皆可，配分方式原則上依答對之矩陣、向量、空格占 1 分。)

$$\text{Max } Z = x_1 - 7x_2 + 3x_3$$

$$s.t.$$

$$2x_1 + x_2 - x_3 \le 4,$$

$$4x_1 - 3x_2 \le 2,$$

$$-3x_1 + 2x_2 + x_3 \le 3,$$

$$x_1, x_2, x_3 \ge 0.$$

Iteration0

$B = B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $x_B = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$

$c_B = [\ 0\ 0\ 0\ ]$, $Row\ 0 = [\ -1\ 7\ -3\ 0\ 0\ 0\ ]$

x_3 entering

$$\text{Revised } x_3 \text{ coefficients} = \begin{array}{c} B^{-1} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \cdot \begin{array}{c} a_{(k)} \\ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{array} = \begin{array}{c} B^{-1}a_{(k)} \\ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

$$\text{Ratio} = \frac{bi}{a_{ik}} = \left[\frac{b1}{a_{1k}}, \frac{b2}{a_{2k}}, \frac{b3}{a_{3k}} \right] = [\ 4/-1,\ \infty,\ 3\ ]$$

x_6 leaving

Iteration1

> ***Note***
> **$a_{(k)}$:the A matrix column corresponding to the entering**

$$B^{-1}_{(1)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_B = \begin{pmatrix} x_4 \\ x_5 \\ x_3 \end{pmatrix} = B^{-1}_{new} b = \overset{B^{-1}_{new}}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \overset{b}{\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}} = \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}$$

$$c_B = \begin{pmatrix} 0 & 0 & 3 \end{pmatrix}$$

x_1 entering

$$\text{Revised entering coefficients} = B^{-1}_{(1)} a_k = \overset{B^{-1}_{new}}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \cdot \overset{a_{(k)}}{\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}} = \overset{a_{new}}{\begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}}$$

$$\text{Ratio} = \frac{\bar{b}i}{\bar{a}_{ik}} = \frac{\bar{b}1}{\bar{a}_{1k}}, \frac{\bar{b}2}{\bar{a}_{2k}}, \frac{\bar{b}3}{\bar{a}_{3k}} = \begin{pmatrix} 7/-1, & 2/4, & 3/-3 \end{pmatrix} \quad \boxed{x_5} \text{ leaving}$$

Iteration2

$$B^{-1}_{(2)} = \begin{pmatrix} 1 & 1/4 & 1 \\ 0 & 1/4 & 0 \\ 0 & 3/4 & 1 \end{pmatrix}$$

$$x_B = \begin{pmatrix} x_4 \\ x_1 \\ x_3 \end{pmatrix} = B^{-1}_{(2)} b = \overset{B^{-1}_{(2)}}{\begin{pmatrix} 1 & 1/4 & 1 \\ 0 & 1/4 & 0 \\ 0 & 3/4 & 1 \end{pmatrix}} \cdot \overset{b}{\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}} = \begin{pmatrix} 7.5 \\ 0.5 \\ 4.5 \end{pmatrix}$$

$$c_B = \begin{pmatrix} 0 & 1 & 3 \end{pmatrix}_{1x3}$$

optimal solution= Z = $\boxed{14}$

Problem 2. (5 points) Consider the following linear program:

$$\text{Max } Z = 2x_1 - 2x_2 + 3x_3$$
$$s.t.$$
$$\begin{aligned} -x_1 + x_2 + x_3 &\leq 4, \text{ (resource 1)} \\ 2x_1 - x_2 + x_3 &\leq 2, \text{ (resource 2)} \\ x_1 + x_2 + 3x_3 &\leq 12, \text{ (resource 3)} \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Let x_4, x_5, and x_6 are the slack variables of the constraints of resources 1, 2, and 3 respectively. Given the optimal basic variables are $x_2, x_3,$ and x_6, and the corresponding basic inverse is:

$$\begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix}.$$

I. (3 points) Compute shadow prices of all three resources via the LP and the matrix given above only.

II. (2 points) Using the result obtained in (I) to indicate which resource to increase capacity (the right-hand-side values of constraints) such that the objective value increased most.

(依提供解答評分，沒有部份給分。)

I.

$$c_B B^{-1} = \begin{bmatrix} -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 5/2 & 0 \end{bmatrix}$$

II. Resource2

Problem 3 (20 points) Consider the following linear program:

$$\text{Min } Z = -2x_1 + x_2 - 4x_3 + 3x_4$$

$$s.t.$$

$$\begin{aligned}
x_1 + x_2 + 3x_3 + 2x_4 &\le 4,\\
x_1 - x_3 + x_4 &\ge -1,\\
2x_1 + x_2 &\le 2,\\
x_1 + 2x_2 + x_3 + 2x_4 &= 2,\\
x_1 \in R;\ x_2, x_3, x_4 &\ge 0.
\end{aligned}$$

I. (5 points) Convert the problem to standard form (Note that the standard form of LP must have a maximized objective function. All constraints are less than or equal to, and all decision variables are non-negative.)

II. (5 points) Write down the augmented form of the standard form obtained in (I)

III. (5 points) Write down the dual problem of the linear program

IV. (5 points) Convert dual problem in (III) to standard form

(standard form:依三大原則給分:
1.MAX(共兩分，寫 MAX 得一分，目標式寫對得一分)
2.限制式≤(共兩分，≤寫對得一分，其他寫對得一分 ex 若少寫一條限制式等於錯≤及其他，因此扣兩分)
3.變數≥0(共一分，全寫對得一分)
/其他格式依作答完整性與思路，給分方式與 standard form 相同。)

I

$$x_1 = x_1^+ - x_1^-, x_1^+, x_1^- \ge 0$$

$$\text{Max } -Z = 2x_1^+ - 2x_1^- - x_2 + 4x_3 - 3x_4$$

$$s.t.$$

$$\begin{aligned}
x_1^+ - x_1^- + x_2 + 3x_3 + 2x_4 &\le 4,\\
-x_1^+ + x_1^- + x_3 - x_4 &\le 1,\\
2x_1^+ - 2x_1^- + x_2 &\le 2,\\
x_1^+ - x_1^- + 2x_2 + x_3 + 2x_4 &\le 2,\\
-x_1^+ + x_1^- - 2x_2 - x_3 - 2x_4 &\le -2,\\
x_1^+, x_1^-, x_2, x_3, x_4 &\ge 0.
\end{aligned}$$

Ⅱ

$$\text{Max } -Z = 2x_1^+ - 2x_1^- - x_2 + 4x_3 - 3x_4$$

$s.t.$

$$\begin{aligned}
x_1^+ - x_1^- + x_2 + 3x_3 + 2x_4 + x_5 &= 4,\\
-x_1^+ + x_1^- + x_3 - x_4 + x_6 &= 1,\\
2x_1^+ - 2x_1^- + x_2 + x_7 &= 2,\\
x_1^+ - x_1^- + 2x_2 + x_3 + 2x_4 + x_8 &= 2,\\
-x_1^+ + x_1^- - 2x_2 - x_3 - 2x_4 + x_9 &= -2,\\
x_1^+, x_1^-, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 &\geq 0.
\end{aligned}$$

Ⅲ

$$\text{Max } W = 4y_1 - y_2 + 2y_3 + 2y_4$$

$s.t.$

$$\begin{aligned}
y_1 + y_2 + 2y_3 + y_4 &= -2,\\
y_1 + y_3 + 2y_4 &\leq 1,\\
3y_1 - y_2 + y_4 &\leq -4,\\
2y_1 + y_2 + 2y_4 &\leq 3,
\end{aligned}$$

$y_1, y_3 \leq 0,\quad y_2 \geq 0,\quad y_4$ unconstrained

(黃底為更正部分)

Ⅳ

(全部更正)

Let $y_1 = -y_1'$, $y_3 = -y_3'$ and $y_4 = y_4^+ - y_4^-$

$$\text{Max } W = -4y_1' - y_2 - 2y_3' + 2y_4^+ - 2y_4^-$$

$s.t.$

$$\begin{aligned}
y_1' - y_2 + 2y_3' - y_4^+ + y_4^- &\leq 2,\\
-y_1' + y_2 - 2y_3' + y_4^+ - y_4^- &\leq -2,\\
-y_1' - y_3' + 2y_4^+ - 2y_4^- &\leq 1,\\
-3y_1' - y_2 + y_4^+ - y_4^- &\leq -4,\\
-2y_1' + y_2 + 2y_4^+ - 2y_4^- &\leq 3,
\end{aligned}$$

$y_1, y_2', y_3, y_4^+, y_4^- \geq 0$

Problem 4. (25 points) Consider the following problem:

$$
\begin{aligned}
&\text{Max } Z = 2x_1 - x_2 - x_3 \\
&s.t. \\
&\qquad -x_1 + x_2 + x_3 \le 3, \\
&\qquad 2x_1 + x_2 + x_3 \le 4, \\
&\qquad x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

The simplex method yields the following set of equations in the optimal tableau.

$$
\begin{aligned}
Z + 2x_2 + 2x_3 \qquad\quad + \ x_5 &= 4 \\
\tfrac{3}{2}x_2 + \tfrac{3}{2}x_3 + x_4 + \tfrac{1}{2}x_5 &= 5 \\
x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 \qquad + \tfrac{1}{2}x_5 &= 2
\end{aligned}
$$

(原則上依答對之矩陣上 elements 比例給分。)

I. (5 points) Now, we change the coefficients of x_1 and x_2 as:

$$
\begin{pmatrix} c_1 \\ a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_2 \\ a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.
$$

Write down the revised final tableau in proper form in the following table. (Note that your solution should include the step-by-step procedure in the answer booklet, and the final answer should be filled in the following table.)

		Original variable			Slack variable		
BV	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	1	$\frac{5}{2}$	0	$\frac{3}{2}$	6
x_4	0	0	3	$\frac{3}{2}$	1	$\frac{1}{2}$	5
x_1	0	1	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-2

II. (10 points) Perform the dual simplex method to obtain the optimal solution for the revised problem. (Note that your solution should include the step-by-step procedure in the answer booklet, and the final answer should be filled in the following table.)

		Original variable			Slack variable		
BV	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	4	0	2	1	0	3
x_5	0	-3	0	0	-1	1	1
x_2	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$

III. (4 points) If we introduce a new constraint $x_1 - x_2 + 2x_3 \leq 1$ to the *original* LP (DO NOT consider changes in parts I and II), write down the proper form of the revised final tableau in the following table. (Note that your solution should include the step-by-step procedure in the answer booklet, and the final answer should be filled in the following table.)

		Original variable			Slack variable			
BV	Z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
Z	1	0	2	2	0	1	0	4
x_4	0	0	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	0	5
x_1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	2
x_6	0	0	$-\frac{3}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	-1

IV. (1 point) Continue part III, is the current basic solution feasible or not after adding the new constraint?

☐ Feasible

☑ Infeasible

V. (4 points) A new variable x_6 with coefficients $\begin{bmatrix} c_6 \\ a_{16} \\ a_{26} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is introduced to the original LP (DO NOT consider changes in parts I, II, and III), write down the revised tableau in proper form in the following table.

		Original variable				Slack variable		
BV	Z	x_1	x_2	x_3	x_6	x_4	x_5	RHS
Z	1	0	2	2	-2	0	1	4
x_4	0	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	1	$\frac{1}{2}$	5
x_1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	2

VI. (1 point) Continue part V, is the current basic solution feasible or not after adding the new variable?

☑ Feasible

☐ Infeasible

Problem 5. (10 points) Given the following problem:

$$\text{Max } Z = c_1x_1 + c_2x_2$$

$s.t.$

$$2x_1 - x_2 \le b_1,$$
$$x_1 - x_2 \le b_2,$$
$$x_1 \ge 0, x_2 \ge 0.$$

Let x_3 and x_4 be the slack variables for the first and second functional constraints, respectively. If $c_1 = 3,\ c_2 = -2,\ b_1 = 30,$ and $b_2 = 10,$ the simplex method yields the following final tableau.

BV	Z	x_1	x_2	x_3	x_4	RHS
	1	0	0	1	1	40
x_2	0	0	1	1	-2	10
x_1	0	1	0	1	-1	20

(依提供解答給分。

1.第一小題若出現使用圖解法觀念非 algebraic analysis、只計算到△c_1範圍，發生以上狀況只給兩分。

2.第二小題若出現未依題目要求之答案，例如:寫b_2<12.5，以上狀況只給兩分)

I. (5 points) Use algebraic analysis to determine the allowable range for c_1 without changing the optimal basis.

Increasing c_1 by $\triangle c_1$ $(c_1 = 3 + \triangle c_1)$ causes the coefficient of x_1 in row 0 of the final tableau to become $-\triangle c_1$. To make it 0, add $\triangle c_1$ times row 2 to row 0:

$(-\triangle c_1 \quad 0 \quad 1 \quad 1) + \triangle c_1(1 \quad 0 \quad 1 \quad -1) = (0 \quad 0 \quad 1+\triangle c_1 \quad 1 - \triangle c_1).$

For optimality, we need $1 + \triangle c_1 \ge 0$ and $1 - \triangle c_1 \ge 0$, so $-1 \le \triangle c_1 \le 1.$

$$2 \le c_1 \le 4$$

II. (5 points) If b_1 is decreased by 5, how much can b_2 increase at most without changing the optimal basis?

The allowable range for b_1 is $b_1 \ge 20$, and the allowable range for b_2 is $b_2 \le 15$.

b_1 decreased by 50% $(\frac{30-25}{30-20} = 50\%)$.

By the 100% Rule, b_2 can increase at most 50%.
Thus, $(15 - 10) \times 50\% = 2.5$, b_2 can increase by 2.5 at most.

Problem 6. (10 points**)** Give the linear program:

$$\text{Max } c^T x$$

$$s.t.$$

$$\mathrm{A}x \leq b,$$

$$x \geq 0.$$

c is the column vector of the objective function coefficients, x is the column vector of the decision variables, A is the matrix of the left-hand-side coefficients, and b is the column vector of the right-hand-side values. Let B be the basic matrix and N be the non-basic matrix. Also, x_B is the vector of the basic variables, x_N is the vector of non-basic variables, c_B is the vector of the objective function coefficients corresponding to basic variables and c_N is the vector of the objective function coefficients corresponding to non-basic variables. Derive I and II using matrix operations only.

I. (5 points) $x_B = \mathrm{B}^{-1}b - \mathrm{B}^{-1}\mathrm{N}x_N$

II. (5 points) $\mathrm{Z} = c_B\mathrm{B}^{-1}b - (c_B\mathrm{B}^{-1}\mathrm{N} - c_N)x_N$

(可部份給分，依同學推導過程合理性給分。

I.II. 分別由 $\mathrm{Bx_B + Nx_N = b}$及$\mathrm{Z = c_Bx_B + c_Nx_N}$ 開始推導，才有分)

I.

$\mathrm{B}x_B + Nx_N = b$

$\mathrm{B}x_B = b - Nx_N$

$x_B = B^{-1}b - B^{-1}Nx_N$

II.

$\mathrm{Z} = c_Bx_B + c_Nx_N$

$\mathrm{Z} = c_B(B^{-1}b - B^{-1}Nx_N) + c_Nx_N$

$\quad = c_BB^{-1}b - c_BB^{-1}Nx_N + c_Nx_N$

$\quad = c_BB^{-1}b - (c_BB^{-1}N - c_N)\, x_N$