Mid-Term Examination, *Discrete Mathematics*

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1. (4+4 points) (a) What is the coefficient of x^3y^7 in $(2x-3y)^{10}$? (b) What is the coefficient of xy^2z^3 in $(3x+2y+z)^6$?

2. (8 points) Let $f: Z^+ \times Z^+ \to Z^+$ be the closed binary operation defined by $f(a,b) = \gcd(a,b)$. (a) Is f commutative or associative? (b) Does f have an identity element? Repeat questions (a) and (b) for $f(a,b) = \mathrm{lcm}(a,b)$.

3. (10 points) (Pigeon hole principle) Let $p_1, p_2, \ldots, p_n \in Z^+$. Prove that if $p_1 + p_2 + \cdots + p_n - n + 1$ pigeons occupy n pigeonholes, then either the first pigeonhole has p_1 or more pigeons roosting in it, or the second pigeonhole has p_2 or more pigeons roosting in it, ..., or the nth pigeonhole has p_n or more pigeons roosting in it.

4. (3+5 points) (Catalan Numbers) We start at the point $(0,0)$ in the xy-plane and consider two kinds of moves, $R: (x,y) \to (x+1,y)$ and $U: (x,y) \to (x,y+1)$. How many paths are there from point $(0,0)$ to $(10,10)$ without rising above the line $y=x$? Justify your answer.

5. (5+3 points) Let $S(m,n)$ be a Stirling number of the second kind. (a) Give a combinatorial argument for the equality $S(m+1,n) = S(m,n-1) + nS(m,n)$. (b) What is the value of $S(90,100)$?

6. (12 points) (a) Find the number of nonnegative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 38$. (b) Repeat (a) with the condition that $x_1 \geq 3, x_2 \geq 3, x_3 > 3, x_4 > 3$. (c) Repeat (a) with the condition that $x_i \leq 4, 1 \leq i \leq 4$.

7. (5 points) Given a set A with $|A| = n$ and a relation $\mathcal{R}$ on A, let M denote the relation matrix for $\mathcal{R}$. Explain the inequality $M \cap M^{tr} \leq I_n$.

8. (4+4 points) With $\mathcal{U} = \{1, 2, 3\}$ and power set $A = \mathcal{P}(\mathcal{U})$, $\mathcal{R}$ is the subset relation on A such that for $x, y \in A$, $x\mathcal{R}y$ if $x \subseteq y$. (a) Draw the Hasse diagram of the poset $(A, \mathcal{R})$. (b) Give the maximal element(s) and the minimal element(s).

9. (5 points) For positive integer k, let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by $x\mathcal{R}y$ if $x - y$ is a multiple of k. Is $\mathcal{R}$ an equivalence relation? Justify your answer.

10. (10 points) For $\mathcal{I} = \mathcal{O} = \{0, 1\}$, a string $x \in \mathcal{I}^*$ is said to have even parity if it contains an even number of 1s. Construct a $\boxed{\text{state diagram}}$ for a finite state machine that recognizes all nonempty strings of even parity.

11. (8 points) Is it possible to design a $\boxed{\text{finite state machine}}$ to recognize sequences in $A = \{0^i 1^j \mid i, j \in \mathbb{Z}^+, i > j\}$ where the alphabet is $\sigma = \{0, 1\}$? Justify your answer.

12. (10 points) Apply the minimization process to the machine defined by the following table.

	[illegible]		[illegible]	
	0	1	0	1
[illegible]	[illegible]	[illegible]	0	0
[illegible]	[illegible]	[illegible]	0	1
[illegible]	[illegible]	[illegible]	1	1
[illegible]	[illegible]	[illegible]	1	0
[illegible]	[illegible]	[illegible]	0	1
[illegible]	[illegible]	[illegible]	0	0