Mathematical Statistics, Exam 2. November 19, 2019

1. (10%) Let U be the uniform distribution on [0,1]. Let $V = 1/U$. Find the density of V.

2. (10%) If $X \sim N(0,1)$. Let $\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ denote the cumulative probability function of the standard normal distribution. Find the distribution of $\Phi(X)$.

3. (10%) Suppose that X and Y are independently discrete random variables and each assumes the values 0, 1, and 2 with probability 1/3 each. Find the frequency function of $X \times Y$.

4. (10%) Let U_1 and U_2 be independent and uniform on [0,1]. Find and sketch the density function of $S = U_1 + U_2$.

5. (10%) Let X be a continuous random variable with probability density function $f(x) = 2x$, $0 \leq x \leq 1$. Moreover, let $Y = X^2$. Find $E(Y)$ and $Var(Y)$.

6. (10%) Suppose that $E(X) = \mu$ and $Var(X) = \sigma^2$. Let $Z = (X - \mu)/\sigma$. Show that $E(Z) = 0$ and $Var(Z) = 1$.

7. (10%) If X_1 and X_2 are independent random variables following a gamma distribution with parameters α and λ. Find $E(R^2)$, where $R^2 = X_1^2 + X_2^2$. (Hint: If X has the gamma distribution with parameters α and λ. Then, X has mean α/λ and variance α/λ^2.)

8. (10%) Some useful inequalities.

   (a) (5%) Use Markov inequality to find an upper bound of $P(X \geq 1)$, where X has the exponential distribution with rate λ.

   (b) (5%) Use Chebyshev's inequality to find an upper bound of $P(|X - 3| > 2)$, where X has the normal distribution with mean 3 and variance 1.

9. (20%) If T_1 and T_2 are independent exponential random variables with rate λ.

   (a) (10%) Find the joint density function of T_1 and T_2.

   (b) (10%) Find the density function $R = T_{(2)} - T_{(1)}$.

Exam 2

1. (10%) Let U be the uniform distribution on [0,1]. Let $V = 1/U$. Find the density of V.

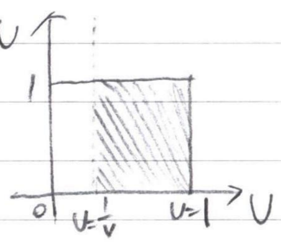
<sol>. consider the cdf of V

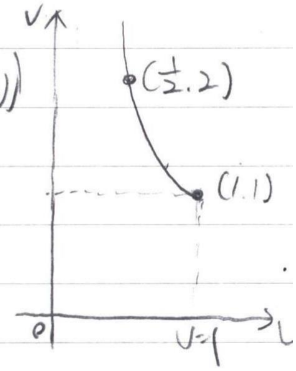
$F_V(v) = P(V \le v) = P(\frac{1}{U} \le v) = P(U \ge \frac{1}{v}) \; (= 1 - P(U \le \frac{1}{v}))$

since $V = \frac{1}{U} \Rightarrow V \in [1, \infty)$

$\Rightarrow P(U \ge \frac{1}{v}) = (1 - \frac{1}{v}) \times 1 = 1 - \frac{1}{v}$

$\Rightarrow f_V(v) = \begin{cases} \frac{1}{v^2}, & v \ge 1 \\ 0, & \text{otherwise} \end{cases}$ #





2. (10%) If $X \sim N(0,1)$. Let $\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ denote the cumulative probability function of the standard normal distribution. Find the distribution of $\Phi(X)$.

<sol> Φ is the cdf of the standard normal distribution.

consider $Y = \Phi(X)$, observe the cdf of Y

$F_Y(y) = P(Y \le y) = P(\Phi(X) \le y) = P(X \le \Phi^{-1}(y))$

$= F_X(\Phi^{-1}(y))$

since $X \sim N(0,1) \Rightarrow F_X(\Phi^{-1}(y)) = \Phi(\Phi^{-1}(y)) = y$

$\Rightarrow F_Y(y) = y$

$\Rightarrow \frac{d}{dy} F_Y(y) = 1$

$\Rightarrow f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$; $\Phi(X) \sim U(0,1)$ #

3. (10%) Suppose that X and Y are independently discrete random variables and each assumes the values 0, 1, and 2 with probability 1/3 each. Find the frequency function of $X \times Y$.

<sol>

$Y \backslash X$ (P)	0	1	2
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
2	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

$Y \backslash X$ (XY)	0	1	2
0	0	0	0
1	0	1	2
2	0	2	4

$\Rightarrow f_{X \times Y}(X.Y) = P(X \times Y) = \begin{cases} \frac{1}{9}, & X \times Y = 4 \\ \frac{2}{9}, & X \times Y = 2 \\ \frac{1}{9}, & X \times Y = 1 \\ \frac{5}{9}, & X \times Y = 0 \\ 0, & \text{otherwise} \end{cases}$ #

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4. (10%) Let U_1 and U_2 be independent and uniform on [0.1]. Find and sketch the density function of $S = U_1 + U_2$.

<sol> see CH3 Problem 43

5. (10%) Let X be a continuous random variable with probability density function $f(x) = 2x$, $0 \le x \le 1$. Moreover, let $Y = X^2$. Find $E(Y)$ and $Var(Y)$.

<sol> $E(Y) = E(X^2) = \int_0^1 x^2 f(x) = \int_0^1 2x^3\,dx = \frac{2}{4}x^4\Big|_0^1 = \frac{1}{2}$

$$\left(E(X^2) = Var(X) + (E(X))^2 = \int (x-\mu)^2 f(x)\,dx + \mu^2 = \int x^2 f(x)\,dx - 2\mu \overbrace{\boxed{\int x f(x)\,dx}}^{E(X)=\mu} + \mu^2 \underbrace{\boxed{\int f(x)\,dx}}_{1} + \mu^2 = \int x^2 f(x)\,dx \right)$$

$Var(Y) = E(Y^2) - (E(Y))^2$

$= E(X^4) - (\frac{1}{2})^2 = \int_0^1 x^4\, 2x\,dx - \frac{1}{4} = \frac{2}{6}x^6\Big|_0^1 - \frac{1}{4} = \frac{1}{12}$ #

6. (10%) Suppose that $E(X) = \mu$ and $Var(X) = \sigma^2$. Let $Z = (X - \mu)/\sigma$. Show that $E(Z) = \cancel{1}\,0$ and $Var(Z) = 1$.

<sol> $E(Z) = E(\frac{X-\mu}{\sigma}) = \frac{1}{\sigma}(E(X) - \mu) = \frac{1}{\sigma}(\mu - \mu) = 0$

$Var(Z) = E(Z^2) - E(Z)^2 = E((\frac{X-\mu}{\sigma})^2) - 0$

$= \frac{1}{\sigma^2}E(X^2 - 2\mu X + \mu^2) = \frac{1}{\sigma^2}(E(X^2) - 2E(X)\cdot E(X) + E(X)^2)$

$= \frac{1}{\sigma^2}(E(X^2) - E(X)^2) = \frac{1}{\sigma^2} Var(X) = 1$ #

7. (10%) If X_1 and X_2 are independent random variables following a gamma distribution with parameters α and λ. Find $E(R^2)$, where $R^2 = X_1^2 + X_2^2$. (Hint: If X has the gamma distribution with parameters α and λ. Then, X has mean α/λ and variance α/λ^2.)

<sol> $E(R^2) = E(X_1^2 + X_2^2) = E(X_1^2) + E(X_2^2)$

$= Var(X_1) + E(X_1)^2 + Var(X_2) + E(X_2)^2$

$= \frac{\alpha}{\lambda^2} + (\frac{\alpha}{\lambda})^2 + \frac{\alpha}{\lambda^2} + (\frac{\alpha}{\lambda})^2 = \frac{2(\alpha+\alpha^2)}{\lambda^2}$ #

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8. (10%) Some useful inequalities.

(a) (5%) Use Markov inequality to find an upper bound of $P(X \geq 1)$, where X has the exponential distribution with rate λ.

(b) (5%) Use Chebyshev's inequality to find an upper bound of $P(|X - 3| > 2)$, where X has the normal distribution with mean 3 and variance 1.

there exist

<sol> (a) Markov Inequality:

If X is a random variable with $P(X \geq 0) = 1$ and $\exists E(X)$,

$$P(X \geq t) \leq E(X)/t$$

$$\Rightarrow P(X \geq 1) \leq E(X)/1$$
$$= E(X) = \int x \cdot f(x)\, dx$$
$$= \int_0^\infty x \cdot \lambda \cdot e^{-\lambda x}\, dx$$
$$= \lambda \int_0^\infty x \cdot e^{-\lambda x}\, dx$$

Intergration by parts

$$= \lambda \left(-\frac{1}{\lambda} x \cdot e^{-\lambda x} \Big|_0^\infty - \int_0^\infty -\frac{1}{\lambda} e^{-\lambda x}\, dx \right)$$
$$= \lambda \left(0 - 0 - \left(\frac{1}{\lambda^2} e^{-\lambda x} \Big|_0^\infty \right) \right)$$
$$= \lambda \cdot \left(-\left(0 - \frac{1}{\lambda^2} \cdot 1\right) \right) = \frac{1}{\lambda}$$

$\Rightarrow$ the upper baund of $P(X \geq 1) = \frac{1}{\lambda}$

(b) Chebyshev's Inequality:

Let X be a random variable with mean μ and variance σ^2,

$$P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}, \ \forall t > 0$$ for any

$$\Rightarrow P(|X - 3| \geq 2) \leq \frac{1}{2^2} = \frac{1}{4}$$

$\Rightarrow$ the upper baund of $P(|X-3| \geq 2) = \frac{1}{4}$ ※

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9. (20%) If T_1 and T_2 are independent exponential random variables with rate λ.

   (a) (10%) Find the joint density function of T_1 and T_2.

   (b) (10%) Find the density function $R = T_{(2)} - T_{(1)}$.

<sol> (a) since T_1 and T_2 are independent

$\Rightarrow f_{T_1T_2}(t_1, t_2) = f_{T_1}(t_1) \cdot f_{T_2}(t_2) = \lambda \cdot e^{-\lambda t_1} \cdot \lambda \cdot e^{-\lambda t_2}$

$\Rightarrow f_{T_1T_2}(t_1 t_2) = \begin{cases} \lambda^2 e^{-\lambda(t_1+t_2)} & , t_1 \geq 0, \ t_2 \geq 0 \\ 0 & , \text{otherwise} \end{cases}$

(b) consider $U = T_{(1)}$ and $V = T_{(2)}$

(just for a easier symbal to avoid confusing)

since U. V is ordered,

$\Rightarrow f_{UV}(u.v) = 2 \cdot f_{T_1T_2}(u.v) = 2 \cdot \lambda^2 \cdot e^{-\lambda(u+v)}$ (the joint density function of $T_{(1)}$ and $T_{(2)}$)

(see CH3 Problem 73)

since $R = V - U$

$\Rightarrow F_R(r) = P(R \leq r) = P(V - U \leq r)$

$= \iint_{V-U\leq r} f_{UV}(u.v) \, du dr$

here Jacobian $= \left| \det \begin{pmatrix} \frac{du}{du} & \frac{du+r}{dr} \\ \frac{du}{dr} & \frac{du+r}{dr} \end{pmatrix} \right| = \left| \det \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \right| = 1$

$= \int_0^\infty \int_0^\infty f_{UV}(u.u+r) \cdot \boxed{J} \, du dr$

$\Rightarrow f_R(r) = \int_0^\infty f_{UV}(u.u+r) \, du$

$= \int_0^\infty 2\lambda^2 \cdot e^{-\lambda(2u+r)} \, du = -\lambda \cdot e^{-\lambda(2u+r)} \Big|_0^\infty = \lambda e^{-\lambda r}$

$\Rightarrow f_R(r) = \begin{cases} \lambda \cdot e^{\lambda r} & , r \geq 0 \\ 0 & , \text{otherwise} \end{cases}$ #