一百零五學年度 0311 微積分 (二) 期中考
The 105th academic year course 0311 Calculus(2) midterm examination

date: May 5, 2017

Student ID No. 學號: ______________ Name 姓名: ______________

說明 Description:

(1) 答題之前請先檢查所取得之試卷與答案卷是否正確。
Before answering questions, please check if the test papers and answer sheets which you get are correct.

(2) 測驗時間 110 分鐘。試卷加答案卷、答案卡共計 7 頁。
Testing time is 110 minutes. Test papers, answer sheets, and answer cards are of 7 pages in total.

(3) 試卷包括選擇題與填充題, 總分共計 100 分, 占學期成績之 20%。考卷成績將 $\boxed{\text{不}}$ 做爲微積分獎給獎依據。
The test paper includes choices and fill-in-the-blanks, and there is a total score of 100 points, accounting for 20% of the semester grade. The examination result will $\boxed{\text{not}}$ be considered for awarding the Calculus prize.

(4) 請先確實在答案卡與答案卷填入相關個人資料。答題時請依題號作答, 否則不予計分。
Be sure to fill related personal information in answer sheets and answer cards. When answering questions, please answer the question by its question number, or, no score.

P.S. 難易度提示 Difficulty hint: easy<normal<hard
$\star$ $\star\star$ $\star\star\star$ $\star\star\star\star$ $\bigstar$

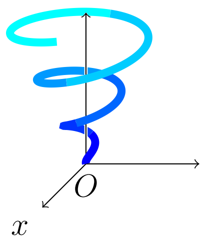
Questions start from the next page

◎ 單選擇題 (單選十題, 每題五分, 共五十分, 答錯不倒扣。)
Single-choice (10 questions, each 5 points, 50 points in total, no penalty for wrong answers.)

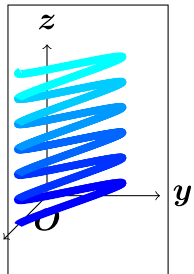
1. Find the **space curve** with the parametric equation:

$$\boldsymbol{x = \cos^2 t, \quad y = \sin^2 t, \quad z = t, \quad t \geq 0.}$$

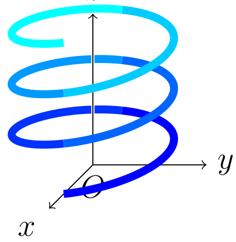
⋆ §13.1 (Ex 13.1.21–26)

(A)


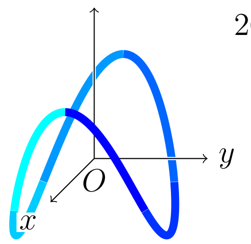
$x = t\cos t,$
$y = t\sin t,$
$z = t.$

(B)


$\boldsymbol{x = \cos^2 t,}$
$\boldsymbol{y = \sin^2 t,}$
$\boldsymbol{z = t.}$

(C)


$x = \cos t,$
$y = \sin t,$
$z = t.$

(D)


$x = \cos t,$
$y = \sin t,$
$z = \cos 2t.$

2. Find the **tangent line** to the curve $\mathbf{r}(t) = \langle e^t, t, \ln t\rangle$ at the point $(e, 1, 0)$.

⋆⋆ §13.2

(A) $x = e,\ y - 1 = z.$ (B) $\dfrac{x-e}{e} = z,\ y = 1.$

(C) $\dfrac{x-e}{e} = y - 1,\ z = 0.$ **(D)** $\boxed{\boldsymbol{\dfrac{x-e}{e} = y - 1 = z.}}$

Solution: $\mathbf{r}(t) = \langle e, 1, 0\rangle$, $t = 1$. $\mathbf{r}'(t) = \langle e^t, 1, \dfrac{1}{t}\rangle$, $\mathbf{r}'(1) = \langle e, 1, 1\rangle$, $x = e + es,\ y = 1 + s,\ z = 0 + s = s.$ $\dfrac{x-e}{e} = y - 1 = z (= s).$

3. Find the **arc length** of the curve $\langle 3\sin t, 4t, 3\cos t\rangle$, $0 \le t \le 1$. ⋆ §13.3

(A) $\sqrt{7}$. (B) 3. (C) 4. **(D)** $\boxed{\mathbf{5.}}$

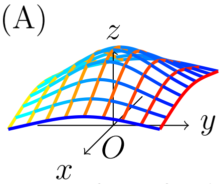
Solution: $\mathbf{r}'(t) = \langle 3\cos t, 4, -3\sin t\rangle$, (Ex 13.3.15)

$|\mathbf{r}'(t)| = \sqrt{(3\cos t)^2 + 4^2 + (-3\sin t)^2} = 5$,

$L = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 5\ dt = 5t\Big|_0^1 = 5.$

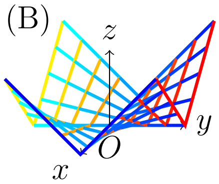
4. Find the **graph** of $z = (x-y)^2$. (Ex 14.1.32) ⋆ §14.1

(A)



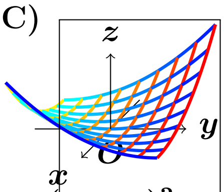
$(1+x^2+y^2)^{-1}$

(B)



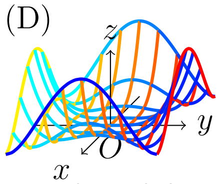
$|xy|$

(C)



$\boldsymbol{(x-y)^2}$

(D)



$(x^2-y^2)^2$

5. Find the **limit** $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz + y^2z + x^2z}{x^2+y^2+z^4}$. ⋆ §14.2

(A) $\boxed{\mathbf{0.}}$ (B) 1. (C) $\frac{1}{3}$. (D) does not exist.

Solution: $x = r\cos\theta$, $y = r\sin\theta$, (EX 14.2.21+39)

$\dfrac{xyz + y^2z + x^2z}{x^2+y^2+z^4} = \dfrac{r^2z(\sin\theta\cos\theta + 1)}{r^2+z^4}$,

$(x,y,z) \to (0,0,0) \iff (r,z) \to (0,0)$.

$\because 0 \le \dfrac{r^2}{r^2+z^4} \le 1$ when $(r,z) \ne (0,0)$,

and $-\sqrt{2} \le \sin\theta\cos\theta \le \sqrt{2}$ for all $\theta \in \mathbb{R}$,

$(1-\sqrt{2})|z| \le \dfrac{r^2z(\sin\theta\cos\theta+1)}{r^2+z^4} \le (1+\sqrt{2})|z|$,

and $\displaystyle\lim_{(r,z)\to(0,0)} (1-\sqrt{2})|z| = \lim_{(r,z)\to(0,0)} (1+\sqrt{2})|z| = 0$.

$\displaystyle\therefore \lim_{(x,y,z)\to(0,0,0)} \frac{xyz + y^2z + x^2z}{x^2+y^2+z^4} \overset{\text{S.T.}}{=} \lim_{(r,z)\to(0,0)} \frac{r^2z(\sin\theta\cos\theta+1)}{r^2+z^4} = 0.$

[Sol 2] $xy \le \max\{x^2, y^2\} \le x^2+y^2 \le x^2+y^2+z^4$,

$-3|z| \le \dfrac{xyz + y^2z + x^2z}{x^2+y^2+z^4} \le 3|z|$.

6. Let $\boldsymbol{F(x,y) = \int_y^x \sqrt[3]{t^2-1}\, dt}$. Find $\boldsymbol{F_x(3,0) - F_y(3,0)}$. $\star$ §14.3

(A) 0. **(B)** $\boxed{\mathbf{1.}}$ (C) 2. (D) 3.

Solution: Let $G(u) = \displaystyle\int_0^u \sqrt[3]{t^2-1}\, dt$. (Ex 14.3.30)

$F = \displaystyle\int_y^x \sqrt[3]{t^2-1}\, dt = \int_0^x \sqrt[3]{t^2-1}\, dt - \int_0^y \sqrt[3]{t^2-1}\, dt = G(x) - G(y)$,

$F_x = \sqrt[3]{x^2-1}$, $F_y = -\sqrt[3]{y^2-1}$,

$F_x(3,0) - F_y(3,0) = \sqrt[3]{3^2-1} - (-\sqrt[3]{0^2-1}) = 2 - 1 = 1.$

7. Find the **total differential** $\boldsymbol{dz}$ of $\boldsymbol{z = xe^{xy}}$. $\star$ §14.4

(A) $x^2e^{xy}\, dx + (1+xy)e^{xy}\, dy$. (B) $(x^3+xy^2+y)e^{xy}$.
(C) $\boxed{\boldsymbol{(xy+1)e^{xy}\, dx + x^2e^{xy}\, dy.}}$ (D) $(2x^2y+x)e^{xy}$.

Solution: $f_x = 1\cdot e^{xy} + xe^{xy}\cdot y = (xy+1)e^{xy}$, $f_y = xe^{xy}\cdot x = x^2e^{xy}$,

$dz = f_x(x,y)\, dx + f_y(x,y)\, dy = (xy+1)e^{xy}\, dx + x^2ye^{xy}\, dy$.

8. Let $\boldsymbol{yz + x\ln y = z^2}$. Find $\boldsymbol{\left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle}$ when $\boldsymbol{x = y = z = 1}$. $\star\star\star$ §14.5

(A) $\langle 0,0\rangle$. **(B)** $\boxed{\boldsymbol{\langle 0,2\rangle.}}$ (C) $\langle 2,0\rangle$. (D) $\langle 1,1\rangle$.

Solution: $\dfrac{\partial}{\partial x}$: $yz_x + \ln y = 2zz_x$, (Ex 14.5.34)

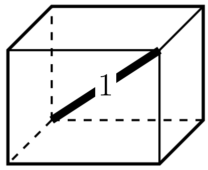
$z_x = \dfrac{\ln y}{2z-y} = \dfrac{\ln 1}{2\cdot 1 - 1} = 0$ or $1\cdot z_x + \ln 1 = 2\cdot 1\cdot z_x$, $z_x = 0$.

$\dfrac{\partial}{\partial y}$: $z + yz_y + \dfrac{x}{y} = 2zz_y$.

$z_y = \dfrac{z + x/y}{2z-y} = \dfrac{1+1/1}{2\cdot 1 - 1} = 2$ or $1 + 1\cdot z_y + 1/1 = 2\cdot 1\cdot z_y$, $z_y = 2$.

$\langle z_x, z_y\rangle = \langle 0, 2\rangle$.

9. Find the **largest possible volume** of a rectangular box whose diagonal(對角線) of length **1**.



$\star$
$\star\star$
§14.7+
14.8

(A) $\frac{1}{2}$. (B) $\frac{1}{6}$. (C) $\frac{\sqrt{2}}{8}$. **(D)** $\boxed{\mathbf{\frac{\sqrt{3}}{9}.}}$

Solution: Let $V = xyz$. (Ex 14.7.53)

Find $\max V$ with $x^2 + y^2 + z^2 = 1$ and $x, y, z > 0$.

[Sol 1] Let $W = V^2 = x^2y^2(1 - x^2 - y^2)$, $\max W \iff \max V$.

$W_x = 2xy^2(1 - 2x^2 - y^2) = 0$ when $x = 0$ or $y = 0$ or $2x^2 + y^2 = 1$.

Similarly, $W_y = 0$ when $x = 0$ or $y = 0$ or $x^2 + 2y^2 = 1$.

So $x^2 = y^2$, $x^2 + 2x^2 = 3x^2 = 1$, $x = \dfrac{1}{\sqrt{3}}$. (negative fails.)

critical points: $(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}})$, $z = \dfrac{1}{\sqrt{3}}$, $V = \left(\dfrac{1}{\sqrt{3}}\right)^3 = \dfrac{1}{3\sqrt{3}} = \dfrac{\sqrt{3}}{9}$.

[Sol 2] Let $V = xyz$ and let $D = x^2 + y^2 + z^2$.

$$\begin{cases} \nabla V = \lambda \nabla D \\ D = 1 \end{cases} \implies \begin{cases} yz = \lambda 2x & \cdots (1) \\ xz = \lambda 2y & \cdots (2) \\ xy = \lambda 2z & \cdots (3) \\ 1 = x^2 + y^2 + z^2 & \cdots (4) \end{cases}$$

$(1) \times x = (2) \times y = (3) \times z$: $\lambda x^2 = \lambda y^2 = \lambda z^2$. $\because \lambda \neq 0$, $x^2 = y^2 = z^2$.

take (4): $x^2 + y^2 + z^2 = 3x^2 = 1$, $x = y = z = \dfrac{1}{\sqrt{3}}$. (negative fails.)

$V = \left(\dfrac{1}{\sqrt{3}}\right)^3 = \dfrac{1}{3\sqrt{3}} = \dfrac{\sqrt{3}}{9}$.

$\square$ 10. Find the **maximum value** of $\boldsymbol{z = xy}$ subject to $\boldsymbol{x^2 + 2y^2 = 1}$. ⋆⋆ §14.8

(A) $\frac{1}{4}$. **(B)** $\boxed{\boldsymbol{\frac{\sqrt{2}}{4}}.}$ (C) $\frac{1}{2}$. (D) $\frac{\sqrt{2}}{2}$.

Solution: Let $F = xy$ and let $G = x^2 + 2y^2$.

$$\begin{cases} \nabla F = \lambda \nabla G \\ G = 1 \end{cases} \implies \begin{cases} y = \lambda 2x & \cdots (1) \\ x = \lambda 4y & \cdots (2) \\ 1 = x^2 + 2y^2 & \cdots (3) \end{cases}$$

When $x = 0$ or $y = 0$ or $\lambda = 0$, no solution.

When $x \neq 0$ and $y \neq 0$, (2) $\div$ (1): $\frac{x}{y} = 2\frac{y}{x} \implies x^2 = 2y^2$,

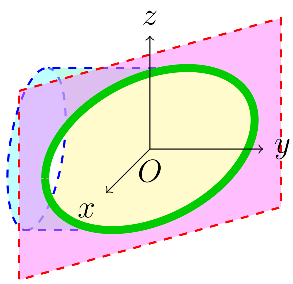
take (3): $2y^2 + 2y^2 = 1 \implies y^2 = \frac{1}{4}, \implies y = \pm\frac{1}{2}, x = \pm\frac{1}{\sqrt{2}}$.

$xy = \pm\frac{1}{2\sqrt{2}} = \pm\frac{\sqrt{2}}{4}$, max/min.

◎ 多選擇題 (多選五題, 每題五分, 共二十五分。錯一個選項扣兩分, 錯兩個選項以上不給分, 分數不倒扣。)

Multiple-choice (5 questions, each 5 points, 25 points in total. One wrong option deducts 2 points, more than one wrong option gets no points, and no penalty for wrong answers.)

11. Find the **vector function** representing the curve of the intersection(交集) of the circular cylinder $\boldsymbol{x^2+z^2=1}$ and the plane $\boldsymbol{x+y=0}$.



⋆ §13.1 Ex 13.1.14

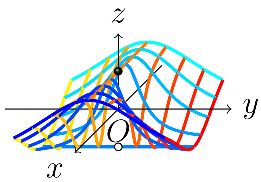
(A) $\boxed{\langle \boldsymbol{\sin t, -\sin t, \cos t} \rangle.}$

(B) $\langle \sin t, -\cos t, \sin t \rangle$.

(C) $\boxed{\langle \boldsymbol{\cos t, -\cos t, \sin t} \rangle.}$

(D) $\langle \cos t, -\sin t, \cos t \rangle$.

12. Let $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$



⋆⋆⋆⋆ §14.2+14.3

Which of the following statements is **correct**.

(A) $\boxed{\boldsymbol{f(x,y) \to 0 \text{ as } (x,y) \to (0,0) \text{ along } x = y.}}$

(B) $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. **(C)** $\boxed{\boldsymbol{f_x(0,0) = 0.}}$ (D) $f_y(0,0) = 0$.

Solution: (A) $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \overset{x=y}{=} \lim_{y\to 0} \frac{y^2-y^2}{y^2+y^2} = \lim_{y\to 0} \frac{0}{2y^2} = 0.$

....

(B) $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) \overset{y=0}{=} \lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$ ($\neq 0$ along $x=y$)

$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

....

(C) $\displaystyle f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^2-0^2}{h^2+0^2}-1}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$

....

(D) $\displaystyle f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{0^2-k^2}{k^2+0^2}-1}{k} = \lim_{k\to 0} \frac{-2}{k}$ does not exist.

13. Which statement for a function f of two variables is **always true**? ⋆⋆ §14.4

(A) If f has partial derivatives, then f is differentiable.

(B) $\boxed{\textbf{If } \boldsymbol{f} \textbf{ is differentiable, then } \boldsymbol{f} \textbf{ is continuous.}}$

(C) If f is continuous, then f has partial derivatives.

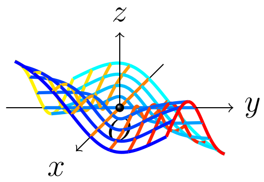
(D) If f is continuous, then f is differentiable.

Solution: (A) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$
has $f_x(0,0) = f_y(0,0) = 0$, but no limit at $(0,0)$ $\implies$ discontinuous $\implies$ not differentiable.

(B) differentiable $\implies$ continuous.

(C,D) $f(x,y) = |x+y|$ at $(0,0)$ is continuous but not differentiable, and $f_x(0,0)$ and $f_y(0,0)$ do not exist.

14. Let $f(x,y) = \begin{cases} \dfrac{xy^2 - x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.



★ §14.2+ 14.3 + 14.4

What about f at $\mathbf{(0,0)}$ is **correct**?

(A) $\boxed{\textbf{\textit{f} has a limit.}}$ **(B)** $\boxed{\textbf{\textit{f} has partial derivatives.}}$

(C) $\boxed{\textbf{\textit{f} is continuous.}}$ (D) f is differentiable.

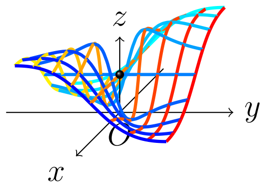
Solution: $-|x| - |y| \leq f(x,y) \leq |x| + |y|$, by Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$,
$\Longrightarrow$ f has limit 0 and is continuous at $(0,0)$.

$f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-0}{h} = 0$, $f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k)-0}{k} = 0$.

If f is differentiable, then $f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \varepsilon_1 x + \varepsilon_2 y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(x,y) \to (0,0)$.

When $y = x \neq 0$, $0 = \varepsilon_1 x + \varepsilon_2 x$, $\varepsilon_1 + \varepsilon_2 = 0$; when $y = -x \neq 0$, $x = \varepsilon_1 x - \varepsilon_2 x$, $\varepsilon_1 - \varepsilon_2 = 1$; $\Longrightarrow$ $\varepsilon_1 = \dfrac{1}{2} \not\to 0$, $\varepsilon_2 = -\dfrac{1}{2} \not\to 0$.

15. Let $f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$.



★★ §14.2+ 14.3 + 14.4

What about f at $\mathbf{(0,0)}$ is **correct**?

(A) f has a limit. **(B)** $\boxed{\textbf{\textit{f} has partial derivatives.}}$

(C) f is continuous. (D) f is differentiable.

Solution: $f(x,y) \to 1$ along $x = 0$ and $\to 0$ along $x = y$.

$f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{(h-0)^2}{h^2+0^2} - 1}{h} = \lim\limits_{h\to 0} 0 = 0$,

$f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{\frac{(0-k)^2}{0^2+k^2} - 1}{k} = \lim\limits_{k\to 0} 0 = 0$.

differentiable $\overset{\not\Leftarrow}{\Longrightarrow}$ continuous $\overset{\not\Leftarrow}{\Longrightarrow}$ limits.

◎ 填空題 (五題, 每題五分, 共二十五分, 答錯不倒扣。)
Fill-in-the-blank (5 questions, each worth 5 points, 25 points in total, no penalty for wrong answers.)

16. Find the **directional derivative** of $\boldsymbol{f(x,y) = \frac{y^2}{x}}$ at $\mathbf{(1,1)}$ in the direction of $\mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$. (Hint: $\boldsymbol{D}_{\mathbf{u}}f = \nabla f \bullet \mathbf{u}$.) $\star\star\star$ §14.6

Solution: $\frac{3}{\sqrt{5}}$ or $\frac{3\sqrt{5}}{5}$.

....

$\mathbf{u} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$, $\nabla f(x,y) = \langle -\frac{y^2}{x^2}, \frac{2y}{x} \rangle$, $\nabla f(1,1) = \langle -1, 2 \rangle$,

$\boldsymbol{D}_{\mathbf{u}}f(1,1) = \nabla f(1,1) \bullet \mathbf{u} = (-1)\frac{1}{\sqrt{5}} + 2\frac{2}{\sqrt{5}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$.

17. Find **all** the **saddle points** of $\boldsymbol{z = y \sin \frac{1}{x}}$. (Hint: $(?, ?), \cdots$.) $\star$ §14.7

Solution: $\left(\frac{1}{n\pi}, 0\right)$, $n \in \mathbb{Z} \setminus \{0\}$.

....

$f_x = -\frac{y}{x^2}\cos\frac{1}{x} = 0$ when $y = 0$ or $x = \frac{2}{(2n-1)\pi}$, $n \in \mathbb{Z}$.

$f_y = \sin\frac{1}{x} = 0$ when $x = \frac{1}{n\pi}$, $n \in \mathbb{Z} \setminus \{0\}$.

critical points: $(x,y) = (\frac{1}{n\pi}, 0)$, $n \in \mathbb{Z} \setminus \{0\}$.

$f_{xy} = f_{yx} = -\frac{1}{x^2}\cos\frac{1}{x}$, $f_{yy} = 0$, $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = -\frac{1}{x^4}\cos^2\frac{1}{x}$,

$D(\frac{1}{n\pi}, 0) = -n^4\pi^2 < 0$, all are saddle points.

18. Let $\mathbf{r} = \displaystyle\int_0^{\pi/2} (3\sin^2 u\cos u\ \mathbf{i} + 3\sin u\cos^2 u\ \mathbf{j} + 2\sin u\cos u\ \mathbf{k})\,du$. Find the **unit vector** in the direction of $\mathbf{r}$. (Hint: $\langle ?, ?, ?\rangle$ or $?\mathbf{i}+?\mathbf{j}+?\mathbf{k}$.) ⋆⋆⋆ §13.2

Solution: $\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ or $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$. (Ex 13.2.37)

....

$\mathbf{r} = \langle \sin^3 t, 1-\cos^3 t, \sin^2 t\rangle\Big|_0^{\pi/2} = \langle 1,1,1\rangle = \mathbf{i}+\mathbf{j}+\mathbf{k}$,

$\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\langle 1,1,1\rangle}{\sqrt{1^2+1^2+1^2}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$.

19. Find the **tangent plane** to the surface $z = f(x,y) = x^2 + 3xy - y^2$ at the point $(1,2)$. ⋆⋆⋆ §14.4

Solution: $z - 3 = 8(x-1) - (y-2)$ or $8x - y - z = 3$.

....

$f(1,2) = (1)^2 + 3(1)(2) - (2)^2 = 1 + 6 - 4 = 3$,

$f_x = 2x + 3y,\ f_x(1,2) = 2(1) + 3(2) = 8$,

$f_y = 3x - 2y,\ f_y(1,2) = 3(1) - 2(2) = -1$,

tangent plane: $z - 3 = 8(x-1) - (y-2)$ or $8x - y - z = 3$.

20. Let $f(x,y) = e^{xy}\cos y$. Find $f_{xy}(0,0)$. ⋆⋆ §14.3

Solution: 1. (Ex 14.3.60)

....

$f_x = ye^{xy}\cos y$,

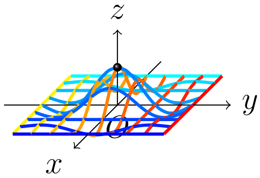
$f_{xy} = e^{xy}\cos y + xye^{xy}\cos y + ye^{xy}(-\sin y)$

$= e^{xy}(\cos y + xy\cos y - y\sin y)$,

$f_{xy}(0,0) = e^{0\cdot 0}(\cos 0 + 0\cdot 0\cdot\cos 0 - 0\cdot\sin 0) = 1(1+0-0) = 1$.

⊕ 加分題 (共十五分。總分超過100分以100分計。)
Bonus (15 points in total. The total score more than 100 points will only get 100 points.)

Let $g(t) = \begin{cases} \dfrac{\sin t}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$, and $f(x,y) = g(x)g(y)$.



⋆ ∼ ⋆⋆
§14

(Hint: $\cos t \leq \dfrac{\sin t}{t} \leq 1$.)

(Command: When the answer does not exist, answer "does not exist".)

(a). [1 pts] $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) =$ .... $\boxed{1}$.

(b). [1 pts] $f_x(0,0) =$ .... $\boxed{0}$.

(c). [1 pts] $f_y(0,0) =$ .... $\boxed{0}$.

(d). [1 pts] Is f **continuous** at $(0,0)$? (Yes/No) .... $\boxed{\text{Yes}}$.

(e). [1 pts] Is f **differentiable** at $(0,0)$? (Yes/No) .... $\boxed{\text{Yes}}$.

(f). [2 pts] Find the **tangent plane** to $z = f(x,y)$ at $(0,0)$.

Solution: $z = 1$.

(g). [2 pts] Find the **linearization** $L(x,y)$ of $f(x,y)$ at $(0,0)$.

Solution: $L(x,y) = 1$.

(h). [2 pts] When $\boldsymbol{x = 0 \neq y}$, $f_x(x,y) = \ \ldots$ $\boxed{0}$.

(i). [2 pts] When $\boldsymbol{x \neq 0 = y}$, $f_x(x,y) = \ \ldots.$ $\boxed{\dfrac{x\cos x - \sin x}{x^2}}$.

(j). [2 pts] When $\boldsymbol{xy \neq 0}$, $f_x(x,y) = \ \ldots\ldots$ $\boxed{\dfrac{x\cos x - \sin x}{x^2}\dfrac{\sin y}{y}}$.

Solution: $\because \sin t \le t \le \tan t \iff \cos t \le \dfrac{\sin t}{t} \le 1$,
$\cos x \cos y \le f(x,y) \le 1$, and $\lim\limits_{(x,y)\to(0,0)} \cos x \cos y = \lim\limits_{(x,y)\to(0,0)} 1 = 1$,
$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) \overset{S.T.}{=} 1 = f(0,0)$, f is continuous at $(0,0)$.

$f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{\sin h}{h}\cdot 1 - 1}{h} = \lim\limits_{h\to 0} \dfrac{\sin h - h}{h^2}$
$\overset{l'H}{=} \lim\limits_{h\to 0} \dfrac{\cos h - 1}{2h} \overset{l'H}{=} \lim\limits_{h\to 0} \dfrac{-\sin h}{2} = 0$; similarly, $f_y(0,0) = 0$.
When $x = 0$ and $y \neq 0$, $f(x,y) = g(y)$ and $f_x(x,y) = 0 = f_x(0,0)$;
when $x \neq 0$, $\lim\limits_{(x,y)\to(0,0)} f_x(x,y) = \lim\limits_{(x,y)\to(0,0)} \left[\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right)\cdot g(y)\right]$
$= \lim\limits_{(x,y)\to(0,0)} \left(\dfrac{x\cos x - \sin x}{x^2} g(y)\right) = \lim\limits_{x\to 0} \dfrac{x\cos x - \sin x}{x^2} \lim\limits_{y\to 0} \dfrac{\sin y}{y}$
$\overset{l'H}{=} \lim\limits_{x\to 0} \dfrac{-x\sin x}{2x}\cdot 1 = \lim\limits_{x\to 0} \dfrac{-\sin x}{2} = 0 = f_x(0,0)$.
So f_x and similarly f_y are continuous $\implies$ f is differentiable at $(0,0)$.

$z = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 := L(x,y)$.

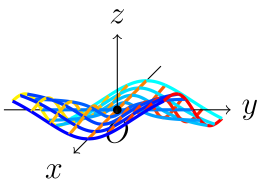
When $x = 0 \neq y$, $f(x,y) = \dfrac{\sin y}{y}$, $f_x = 0$.

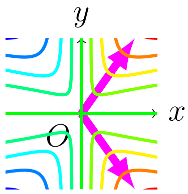
When $x \neq 0 = y$, $f(x,y) = \dfrac{\sin x}{x}$, $f_x = \dfrac{x\cos x - \sin x}{x^2}$.

When $xy \neq 0$, $f(x,y) = \dfrac{\sin x}{x} g(y)$, $f_x = \dfrac{x\cos x - \sin x}{x^2}\dfrac{\sin y}{y}$.

⊛ 挑戰題 (共十分。總分超過100分以100分計。)
Challenge (10 points in total. The total score more than 100 points will only get 100 points.)

Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ and $\mathbf{u} = \langle a, b\rangle$ be a unit vector.





(α). [3 pts] Express the directional derivative $\boldsymbol{D}_{\mathbf{u}}f(0,0)$ of f at $(0,0)$ in the direction of $\mathbf{u}$ as a function $F(a,b)$. (Hint: by definition.) ⁂

Solution: ab^2.

....

$\boldsymbol{D}_{\mathbf{u}}f(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+ah, 0+bh) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{ab^2h^3}{a^2h^2+b^2h^2} - 0}{h}$
$= \lim\limits_{h\to 0} ab^2 = ab^2$.

(β). [3 pts] Find the **maximum value** of $F(a,b)$ subject to $a^2+b^2=1$. ⁂⁂

Solution: $\dfrac{2}{3\sqrt{3}}$ or $\dfrac{2\sqrt{3}}{9}$.

....

[Sol 1] $F(a,b) = ab^2 = a(1-a^2) = a - a^3 = h(a)$,
$h'(a) = 1 - 3a^2 = 0$ and $h''(a) = -6a \lesseqgtr 0$ when $a = \pm\dfrac{1}{\sqrt{3}}$,
$h(\dfrac{1}{\sqrt{3}}) = \dfrac{1}{\sqrt{3}}(1 - \dfrac{1}{3}) = \dfrac{2}{3\sqrt{3}} = \dfrac{2\sqrt{3}}{9}$.

[Sol 2] Let $G(a,b) = a^2 + b^2$. By the method of Lagrange multiplier,

$$\begin{cases} \nabla F = \lambda \nabla G \\ G = 1 \end{cases} \implies \begin{cases} b^2 = \lambda 2a & \cdots (1) \\ 2ab = \lambda 2b & \cdots (2) \\ 1 = a^2 + b^2 & \cdots (3) \end{cases}$$

When $a = 0$ or $b = 0$ or $\lambda = 0$, $\implies a = b = \lambda = 0$, no solution.

When $a, b, \lambda \neq 0$, (1) $\div$ (2): $\dfrac{b}{2a} = \dfrac{a}{b} \implies b^2 = 2a^2$,

take (3): $a^2 + 2a^2 = 1 \implies a^2 = \dfrac{1}{3}, \implies a = \pm\dfrac{1}{\sqrt{3}}$ & $b = \pm\dfrac{\sqrt{2}}{\sqrt{3}}$.

$F(\pm\dfrac{1}{\sqrt{3}}, \pm\dfrac{\sqrt{2}}{\sqrt{3}}) = F(\pm\dfrac{1}{\sqrt{3}}, \mp\dfrac{\sqrt{2}}{\sqrt{3}}) = \pm\dfrac{2}{3\sqrt{3}} = \pm\dfrac{2\sqrt{3}}{9}$, max/min.

(γ). [4 pts] Find all $\mathbf{u}$ such that the maximum value of $\boldsymbol{D}_{\mathbf{u}} f(0,0)$ occurs. ★

Solution: $\left\langle \dfrac{1}{\sqrt{3}}, \pm\dfrac{\sqrt{2}}{\sqrt{3}} \right\rangle$, or $\left\langle \dfrac{1}{\sqrt{3}}, \dfrac{\sqrt{2}}{\sqrt{3}} \right\rangle$ and $\left\langle \dfrac{1}{\sqrt{3}}, -\dfrac{\sqrt{2}}{\sqrt{3}} \right\rangle$.

....

[Sol 1] $a = \dfrac{1}{\sqrt{3}}$, $b = \pm\sqrt{1 - a^2} = \pm\dfrac{\sqrt{2}}{\sqrt{3}}$.

[Sol 2] $F(\dfrac{1}{\sqrt{3}}, \pm\dfrac{\sqrt{2}}{\sqrt{3}}) = \dfrac{2}{3\sqrt{3}} = \dfrac{2\sqrt{3}}{9}$.

Questions End