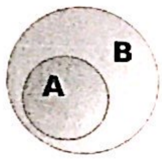
Mathematical Statistics, Exam 1. October 8, 2019

Note: Just give a formula for these questions. No need to calculate the final values.

1. (10%) In the expansion of $(x+y)^{10}$?
   (a) (5%) What is the coefficient for x^3y^7?
   (b) (5%) What is the coefficient for x^3y^5?

2. (10%) If event A is in event B, it means that for every element in A, it will also in B. The following Venn diagram gives an example of event A included in event B.



Suppose that event A is in event B. are they independent? Why or why not? (Need to give your reason by formal mathematical definition.)

3. (10%) If a four-letter word is formed at random (meaning that all sequences of ~~five~~ four letters are equally likely), what is the probability that no letter occurs more than once? (Note that there are a total of 26 alphabets: a, b, ..., z.)

4. (20%) Mickey Mouse is trapped in a cave. There are three doors: A, B, and C. Mickey Mouse selects these doors with equal probabilities. The probabilities for Mickey Mouse to survive are 1/3, 1/2, 1/6, when he selects doors A, B, C, respectively. Otherwise, Mickey mouse will starve to death.
   (a) (10%) What is the probability that Mickey Mouse escapes?
   (b) (10%) If Mickey Mouse survives, what is the probability that he chooses door C?

5. (10%) Driving to work, a computer passes through a sequence of three intersection with traffic lights. At each light, she either stops, s, or continues, c. Assume that each traffic light stops with probability p for some $0 < p < 1$ and continues with probability $(1-p)$. And these traffic lights stop mutually independently. Let X denote the number of stops that this worker encounters through his trip to work, and Y denote the number of stop at the first traffic light. Find X's and Y's probability mass functions.

6. (10%) If X is the geometric random variable with probability p, denoted as $X \sim Geo(p)$. Show that X has the memoryless property:

$$P(X > n+k-1 | X > n-1) = P(X > k),$$

for some non-negative integers n and k.

7. (15%) Suppose that X is a continuous random varialbe with pdf

$$f(x) = 2x, \text{ for } 0 \le x \le 1.$$

   (a) (5%) Sketch the pdf $f(x)$, and show that $\int_{-\infty}^{\infty} f(x)dx = 1$.
   (b) (10%) Find the cdf of X?

8. (15%) Suppose X is a continuous random variable with probability density function

$$f(x) = \begin{cases} (x+1) & \text{for } -1 \le x \le 0; \\ -(x-1); & \text{for } 0 < x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = X^2$.

   (a) (5%) Sketch $y = x^2$ and the density plot of X, $f(x)$.
   (b) (5%) Find the cdf of Y.
   (c) (5%) Find the pdf of Y.

Exam 1

■ NO /

■ DATE /

1. (10%) In the expansion of $(x+y)^{10}$?

   (a) (5%) What is the coefficient for x^3y^7?

   (b) (5%) What is the coefficient for x^3y^5?

<sol> $(x+y)^{10} = \underbrace{(x+y)(x+y)\cdots(x+y)}_{10 \text{ brackets}}$

(a) pick 3 brackets as x, the rest as y

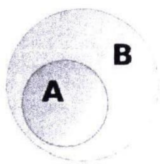
$\binom{10}{3}\binom{7}{7} = \frac{10\times9\times8}{3\times2\times1} = 120.$

(b) pick 3 brackets as x, 5 brackets as y,

the rest brackets has either x or y

$\Rightarrow$ the coefficient for $x^3y^5 = 0$. #

2. (10%) If event A is in event B, it means that for every element in A, it will also in B. The following Venn diagram gives an example of event A included in event B.



Suppose that event A is in event B, are they independent? Why or why not? (Need to give your reason by formal mathematical definition.)

<sol> event A and event B are independent $\Leftrightarrow P(A\cap B) = P(A)\cdot P(B)$

since $A \subset B \Rightarrow A \rightarrow B \Rightarrow P(A\cap B) = P(A) \geq P(A)\cdot P(B)$ ✗

$\Rightarrow$ event A and event B are not independent #

3. (10%) If a four-letter word is formed at random (meaning that all sequences of ~~five~~ letters are equally likely), what is the probability that no letter occurs more than once? (Note that there are a total of 26 alphabets: a, b, ..., z.)

four

<sol> $P = \dfrac{\text{ways that no letter occured more than once}}{\text{all ways to form a four-letter word}}$

$= \dfrac{26\times25\times24\times23}{26^4} \left(= \dfrac{25\times3\times23}{13^3} \doteqdot 78.52\%\right)$

KING EAGLE

■ NO /

■ DATE /

Exam 1

4. (20%) Mickey Mouse is trapped in a cave. There are three doors: A, B, and C. Mickey Mouse selects these doors with equal probabilities. The probabilities for Mickey Mouse to survive are 1/3, 1/2. 1/6, when he selects doors A, B, C, respectively. Otherwise, Mickey mouse will starve to death.

   (a) (10%) What is the probability that Mickey Mouse escapes?

   (b) (10%) If Mickey Mouse survives, what is the probability that he chooses door C?

<sol> (a) $\frac{1}{3}\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{3}$

(b) $P(\text{chooses } C \mid \text{survives}) = \dfrac{P((\text{chooses } C)\cap(\text{survives}))}{P(\text{survives})}$

$= \dfrac{\frac{1}{3}\times\frac{1}{6}}{\frac{1}{3}} = \frac{1}{6}$ #

5. (10%) Driving to work, a computer passes through a sequence of three intersection with traffic lights. At each light, she either stops, s, or continues, c. Assume that each traffic light stops with probability p for some $0 < p < 1$ and continues with probability $(1-p)$. And these traffic lights stop mutually independently. Let X denote the number of stops that this worker encounters through his trip to work, and Y denote the number of stop at the first traffic light. Find X's and Y's probability mass functions.

<sol>

$$P(X=x)=\begin{cases}\binom{3}{0}\cdot p^0(1-p)^3, & x=0\\ \binom{3}{1}\cdot p^1(1-p)^2, & x=1\\ \binom{3}{2}\, p^2\cdot(1-p)^1, & x=2\\ \binom{3}{3}\, p^3\cdot(1-p)^0, & x=3\\ 0, & \text{otherwise}\end{cases}$$

$$P(Y=y)=\begin{cases}\binom{1}{0}\, p^0(1-p)^1, & y=0\\ \binom{1}{1}\, p^1(1-p)^0, & y=1\\ 0, & \text{otherwise}\end{cases}$$ #

6. (10%) If X is the geometric random variable with probability p, denoted as $X \sim Geo(p)$. Show that X has the memoryless property:

$$P(X > n+k-1 | X > n-1) = P(X > k),$$

for some non-negative integers n and k.

<sol> see CH2. Problem 21

Exam 1

■ NO /

■ DATE /

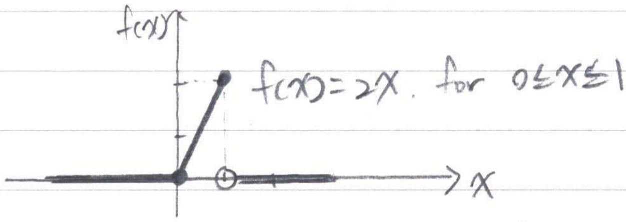
7. (15%) Suppose that X is a continuous random varialbe with pdf

$$f(x) = 2x, \text{ for } 0 \le x \le 1.$$

(a) (5%) Sketch the pdf $f(x)$, and show that $\int_{-\infty}^{\infty} f(x)dx = 1$.

(b) (10%) Find the cdf of X?

<sol> (a)



$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= 0 + x^2\Big|_0^1 + 0 = 1 - 0 = 1$$

(b) $F(k) = P(X \le k) = \int_{-\infty}^{k} f(x)\,dx$

- $k < 0$: $\int_{-\infty}^{k} f(x)\,dx = 0$
- $0 \le k < 1$: $\int_{-\infty}^{k} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{-\infty}^{k} f(x)\,dx$
  $= 0 + x^2\Big|_0^k = k^2 - 0 = k^2.$
- $k \ge 1$: $\int_{-\infty}^{k} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{k} f(x)\,dx$
  $= 0 + 1 + 0 = 1$

$$\Rightarrow F(x) = P(X < x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$ #

Exam 1

8. (15%) Suppose X is a continuous random variable with probability density function

$$f(x) = \begin{cases} (x+1) & \text{for } -1 \le x \le 0; \\ -(x-1); & \text{for } 0 < x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

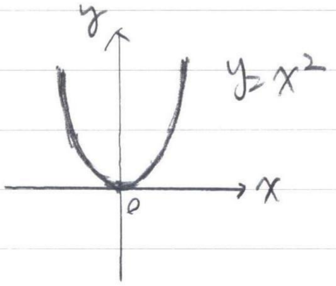
Let $Y = X^2$.

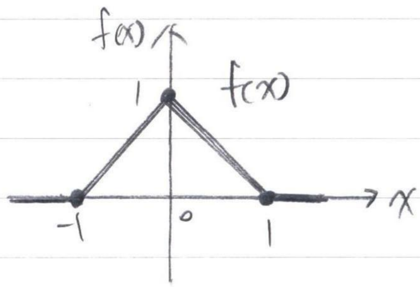
(a) (5%) Sketch $y = x^2$ and the density plot of X, $f(x)$.

(b) (5%) Find the cdf of Y.

(c) (5%) Find the pdf of Y.

<sol> (a)





(b) consider the cdf of $Y = X^2$

$$F_Y(y) = P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\Rightarrow F_Y(y) = \begin{cases} 0, & y < 0 \text{ (notice that } y = x^2 \ge 0) \\ 2\sqrt{y} - y, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

(c) $f_Y(y) = \frac{d}{dy}\left(F_X(\sqrt{y}) - F_X(-\sqrt{y})\right)$

$$= \frac{d}{d\sqrt{y}} F_X(\sqrt{y}) \frac{d\sqrt{y}}{dy} - \frac{d}{d-\sqrt{y}} F_X(-\sqrt{y}) \frac{d-\sqrt{y}}{dy}$$

$$= f_X(\sqrt{y}) \cdot \frac{1}{2}\frac{1}{\sqrt{y}} - \left(f_X(-\sqrt{y}) \cdot \left(-\frac{1}{2}\frac{1}{\sqrt{y}}\right)\right)$$
$$= \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$$

$$= \frac{1}{\sqrt{y}} f_X(\sqrt{y})$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{\sqrt{y}} - 1, & 0 < y \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \#$$