$[1\ 0]\begin{bmatrix}1&0&1&1&0\\0&1&0&1&1\end{bmatrix} = [1\ 0\ 1\ 1\ 0]$

$[1\ 1]\begin{bmatrix}1&0&1&1&0\\0&1&0&1&1\end{bmatrix} = [1\ 1\ 1\ 0\ 1]$

Final Examination of *Discrete Mathematics*, Fall Semester of 2011
Dept. of Information & Finance Management
National Chiao Tung University

1. (6 points) Show that Boolean expression $wx+\overline{x}\overline{z}+(y+\overline{z})$ can be simplify to $x+y+\overline{z}$.

2. (10 points) Find a minimal-sum-of-products representation for

$$f(w,x,y) = \sum m(1,2,5,6).$$

3. (3+3 points) Show that for every G, (a) the identity of G is unique; (b) the inverse of each element of G is unique.

4. (3+3 points) $(\mathbb{Z}_7^*, \bullet)$ is a group defined on integers $\{1,2,3,4,5,6\}$ with operator $\bullet$ as multiplication modulo 7. Determine the inverse of 4 and the inverse of 6.

5. (8+12 points) (a) Give the 6 permutations of the symmetric group S_3. (b) Give the table of binary operations defined in S_3.

6. (10 points) Find all subgroups of $(\mathbb{Z}_{12}, +)$.

7. (10 points) Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Show that A, A^2, A^3, A^4 form a cyclic group under the regular matrix multiplication. Is the group abelian (commutative)?

8. (5 points) Let $x = (0,1,0,0,1) \in \mathbb{Z}_2^5$. Based on Hamming metrics, list the elements $y \in \mathbb{Z}_2^5$ that fall in the sphere of radius 3 centered at x.

9. (5+5 points) Let $E : W \longrightarrow C$ be an encoding function with the set of messages $W \subseteq \mathbb{Z}_2^m$ and the set of code words $E(W) \subseteq \in \mathbb{Z}_2^n$, where $m < n$. Fill in the answer to (?) (a) If our objective is **error detection**, then for $k \in \mathbb{Z}^+$, we can detect all transmission errors of weight k if and only if the minimum distance between code words is at least **(?)**. (b) If our objective is **error correction**, then for $k \in \mathbb{Z}^+$, we can correct all transmission errors of weight k if and only if the minimum distance between code words is at least **(?)**.

10. (6+4+4+6 points) The encoding function $E : \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^5$ is defined by the generator matrix $G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$. (a) Find the associated parity-check matrix H. (b) What is the capability of error detection of this encoding function? (c) What about the capability of error correction? (d) Decode the following two codewords: 10101 and 00110.

$[0\ 0]\begin{bmatrix}1&0&1&1&0\\0&1&0&1&1\end{bmatrix} = [0\ 0\ 0\ 0\ 0]$

$[1\ 1]\begin{bmatrix}1&0&1&1&0\\0&1&0&1&1\end{bmatrix} = [1\ 1\ 1\ 0\ 1]$

$[1\ 0]\begin{bmatrix}1&0&1&1&0\\0&1&0&1&1\end{bmatrix} = [1\ 0\ 1\ 1\ 0]$

$[0\ 1]\begin{bmatrix}1&0&1&1&0\\0&1&0&1&1\end{bmatrix} = [0\ 1\ 0\ 1\ 1]$

$a_n = n! - C_3^n + C_5^n - C_6^n + \cdots + (-1)^n C_n^n 4321$ $\quad$ $4+3+2+1$

$a_4 = 4! - C_3^4 + C_4^4$ $\quad$ 1432 4132

1. (7%+8%)(a) Find the number of integers of {1, 2, …, 1000} that are not divisible by 2, 3,and 7.
(b) Repeat the same question for integers that are not divisible by 2 and 3, but divisible by 7.

2. (15%) Explain (or prove) the correctness of the Principles of Inclusion and Exclusion.

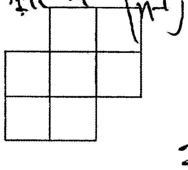
$$\begin{aligned}\overline{N} = N &- [N(c_1) + N(c_2) + N(c_3) + \cdots + N(c_t)] \\ &+ [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)] \\ &- [N(c_1c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_1c_2c_t) + N(c_1c_3c_4) + \cdots \\ &+ N(c_1c_3c_t) + \cdots + N(c_{t-2}c_{t-1}c_t)] + \cdots + (-1)^t N(c_1c_2c_3\cdots c_t),\end{aligned}$$

$A_0 = \frac{2A_2+3A_1}{4}$ $\quad$ 166 71 47 998 333

$n \cdot a_{n-1} - \binom{n}{2}$

3. (10%) Find the rook polynomial of the following board.

$a_n = 2a_{n-1} + \binom{n-1}{2} + \binom{n-1}{3} + \cdots + \binom{n-1}{n-1}$



$C_3^4 \times 4$ $\quad$ $8A_1 n + 8A_0 - 6A_1 = 6n$

$8A_2 n^2 - (2A_2 - 8A_1)n - (4A_2 + 6A_1 - 8A_0)$

$\frac{1}{2} n(n-1)$ $\quad$ $3A_2 = 2A_1 \quad A_1 = \frac{3}{2}A_2$

4. (8%+12%) Consider the non-homogeneous recurrence relation "$a_{n+2} - 8a_{n+1} + 15a_n = f(n)$, *for n* ≥ 0." (a) Find the solution of the homogeneous part. (b) Find particular solutions for (b.1) $f(n) = 9$; (b.2) $f(n)=6n$; (b.3) $f(n)=5n^2$. $\quad a_0 = 1,\ a_1 = 2$

$2a_{n-1}$

5. (15%) Use generating functions to solve the recurrence relation "$a_{n+2} - 5a_{n+1} + 6a_n = 2$, $n \geq 0$, $a_0 = 3$, $a_1 = 7$."

$n! - 1 - C_2^n$ $\quad$ $a_n = C_2^3 + C_3^3$

6. (15%) Let a_n be the number of permutations of {1, ..., n} that avoid the pattern of three-term increasing subsequences. For $n = 3$, these allowed permutations are 132, 213, 231, 312 and 321. For $n = 4$, they are 1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312 and 4321. Give a convolution recurrence of a_n.

$24 - 1 - C_2^4 \quad n-1 \quad a_n = n! - 2a_{n-1}$

7. (15%) For $n \in$ **Z**+, d_n denotes the number of derangements of {1, 2, 3, . . . , n}. If $n > 2$, explain why d_n satisfies the recurrence relation $d_n = (n-1)(d_{n-1} + d_{n-2})$, $d_2=1$, $d_1=0$.

$C_3^4 \times 4 - C_2^4 + C_1^4$ $\quad$ $a_3 = 5 \quad a_4 = 14$

$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

$d_{n-1} + d_{n-2}$

$(n-1)(n-1)!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right]$

$+ (n-1)!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-2}\frac{1}{(n-2)!}\right)$

$n d_{n-1} + n d_{n-2} - d_{n-1} - d_{n-2}$

$n!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right)$

123, 124, 134, 234 $\quad$ $C_3^4 \times 4$

4321 2341 4231