AN ELEMENTARY INTRODUCTION TO ACOUSTICS

Finn Jacobsen
Acoustic Technology, Department of Electrical Engineering,
DTU Building 352
fja@elektro.dtu.dk

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1. INTRODUCTION

Acoustics is the science of sound, that is, wave motion in gases, liquids and solids, and the effects of such wave motion. Thus the scope of acoustics ranges from fundamental physical acoustics to, say, bioacoustics, psychoacoustics and music, and includes technical fields such as transducer technology, sound recording and reproduction, design of theatres and concert halls, and noise control.

The purpose of this note is to give an introduction to fundamental acoustic concepts, to the physical principles of acoustic wave motion, and to acoustic measurements.

2. FUNDAMENTAL ACOUSTIC CONCEPTS

One of the characteristics of fluids, that is, gases and liquids, is the lack of constraints to deformation. Fluids are unable to transmit shearing forces, and therefore they react against a change of *shape* only because of inertia. On the other hand a fluid reacts against a change in its *volume* with a change of the pressure. Sound waves are compressional oscillatory disturbances that propagate in a fluid. The waves involve molecules of the fluid moving back and forth in the direction of propagation (with no net flow), accompanied by changes in the pressure, density and temperature; see figure 2.1. The *sound pressure*, that is, the difference between the instantaneous value of the total pressure and the static pressure, is the quantity we hear. It is also much easier to measure the sound pressure than, say, the density or temperature fluctuations. Note that sound waves are *longitudinal waves*, unlike bending waves on a beam or waves on a stretched string, which are *transversal waves* in which the particles move back and forth in a direction perpendicular to the direction of propagation.

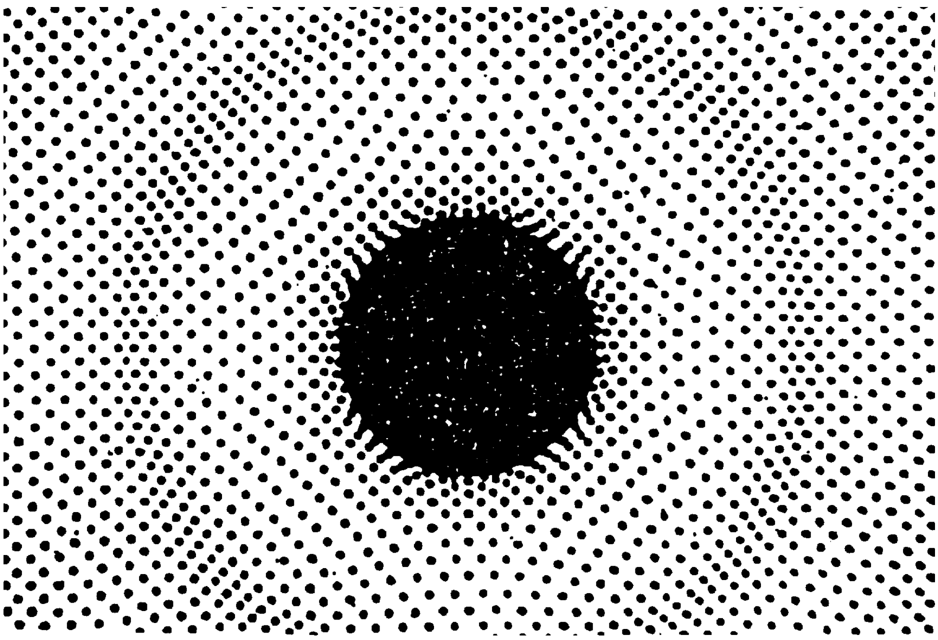

Figure 2.1 Fluid particles and compression and rarefaction in a propagating spherical sound field generated by a pulsating sphere. (From ref. [1].)

In most cases the oscillatory changes undergone by the fluid are extremely small. One can get an idea about the orders of magnitude of these changes by considering the variations in air corresponding to a sound pressure level[1] of 120 dB, which is a very high sound pressure level, close to the threshold of pain. At this level the fractional pressure variations (the sound pressure relative to the static pressure) are about 2×10^{-4}, the fractional changes of the density are about 1.4×10^{-4}, the oscillatory changes of the temperature are less than 0.02 °C, and the particle velocity[2] is about 50 mm/s, which at 1000 Hz corresponds to a particle displacement of less than $8\ \mu\mathrm{m}$. In fact at 1000 Hz the particle displacement at the threshold of hearing is less than the diameter of a hydrogen atom![3]

Sound waves exhibit a number of phenomena that are characteristics of waves; see figure 2.2. Waves propagating in different directions *interfere*; waves will be *reflected* by a rigid surface and more or less *absorbed* by a soft one; they will be *scattered* by small obstacles; because of *diffraction* there will only partly be shadow behind a screen; and if the medium is inhomogeneous for instance because of temperature gradients the waves will be *refracted*, which means that they change direction as they propagate. The speed with which sound waves propagate in fluids is independent of the frequency, but other waves of interest in acoustics, bending waves on plates and beams, for example, are *dispersive*, which means that the speed of such waves depends on the frequency content of the waveform.

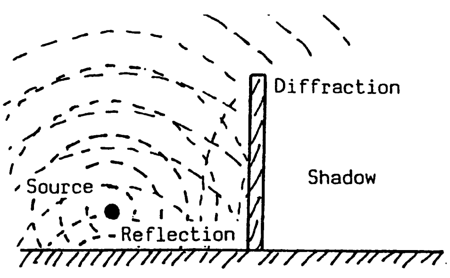

Figure 2.2 Various wave phenomena.

A mathematical description of the wave motion in a fluid can be obtained by combining equations that express the facts that i) mass is conserved, ii) the local longitudinal force caused by a difference in the local pressure is balanced by the inertia of the medium, and iii) sound is very nearly an adiabatic phenomenon, that is, there is no flow of heat. The observation that most acoustic phenomena involve perturbations that are several orders of magnitude smaller than the equilibrium values of the medium makes it possible to simplify the mathematical description by neglecting higher-order terms. The result is the *linearised wave equation.* This is a second-order partial differential equation that, expressed in terms of the sound

[1] See section 1.3.2 for a definition of the sound pressure level.

[2] The concept of fluid particles refers to a macroscopic average, not to individual molecules; therefore the particle velocity can be much less than the velocity of the molecules.

[3] At these conditions the fractional pressure variations amount to about 2.5×10^{-10}. By comparison, a change in altitude of *one metre* gives rise to a fractional change in the static pressure that is about 400000 times larger, about 10^{-4}. Moreover, inside an aircraft at cruising height the static pressure is typically only 80% of the static pressure at sea level. In short, the acoustic pressure fluctuations are *extremely* small compared with commonly occurring static pressure variations.

pressure p, takes the form

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{2.1}$$

in a Cartesian (x, y, z) coordinate system.[4] Here t is the time and, as we shall see later, the quantity

$$c = \sqrt{K_S/\rho} \tag{2.2a}$$

is the *speed of sound*. The physical unit of the sound pressure is pascal (1 Pa = 1 Nm^{-2}). The quantity K_s is the adiabatic bulk modulus, and ρ is the equilibrium density of the medium. For gases, $K_s = \gamma p_0$, where γ is the ratio of the specific heat at constant pressure to that at constant volume ($\simeq 1.401$ for air) and p_0 is the static pressure ($\simeq$ 101.3 kPa for air under normal ambient conditions). The adiabatic bulk modulus can also be expressed in terms of the gas constant R ($\simeq 287\ \mathrm{J{\cdot}kg^{-1}K^{-1}}$ for air), the absolute temperature T, and the equilibrium density of the medium,

$$c = \sqrt{\gamma p_0/\rho} = \sqrt{\gamma RT}\ , \tag{2.2b}$$

which shows that the equilibrium density of a gas can be written as

$$\rho = p_0/RT. \tag{2.3}$$

At 293.15 K = 20°C the speed of sound in air is 343 m/s. Under normal ambient conditions (20°C, 101.3 kPa) the density of air is 1.204 kgm^{-3}. Note that the speed of sound of a gas depends only on the temperature, not on the static pressure, whereas the adiabatic bulk modulus depends only on the static pressure; the equilibrium density depends on both quantities.

Adiabatic compression

Because the process of sound is adiabatic, the fractional pressure variations in a small cavity driven by a vibrating piston, say, a pistonphone for calibrating microphones, equal the fractional density variations multiplied by the ratio of specific heats γ. The physical explanation for the 'additional' pressure is that the pressure increase/decrease caused by the reduced/expanded volume of the cavity is accompanied by an increase/decrease of the temperature, which increases/reduces the pressure even further. The fractional variations in the density are of course identical with the fractional change of the volume (except for the sign); therefore,

$$\frac{p}{p_0} = \gamma\frac{\Delta\rho}{\rho} = -\gamma\frac{\Delta V}{V}.$$

In chapter 4 we shall derive a relation between the volume velocity (= the volume displacement ΔV per unit of time) and the resulting sound pressure.

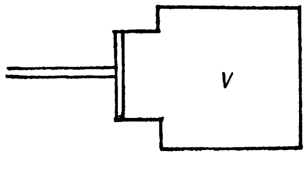

Figure 2.3 A small cavity driven by a vibrating piston.

[4] The left-hand side of eq. (2.1) is the *Laplacian* of the sound pressure, that is, the divergence of the gradient. A negative value of this quantity at a certain point implies that the gradient converges towards the point, indicating a high local value. The wave equation states that this high local pressure tends to decrease.

The linearity of eq. (2.1) is due to the absence of higher-order terms in p in combination with the fact that $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$ are linear operators.[5] This is an extremely important property. It implies that a sinusoidal source will generate a sound field in which the pressure at all positions varies sinusoidally. It also implies linear superposition: sound waves do not interact, they simply pass through each other (see figure 2.5).[6]

The diversity of possible sound fields is of course enormous, which leads to the conclusion that we must supplement eq. (2.1) with some additional information about the sources that generate the sound field, surfaces that reflect or absorb sound, objects that scatter sound, etc. This information is known as *the boundary conditions*. The boundary conditions are often expressed in terms of the particle velocity. For example, the normal component of the particle velocity **u** is zero on a rigid surface. Therefore we need an additional equation that relates the particle velocity to the sound pressure. This relation is known as Euler's equation of motion,

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{0}, \tag{2.4}$$

which is simply Newton's second law of motion for a fluid. The operator ∇ is the gradient (the spatial derivative $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$). Note that the particle velocity is a vector, unlike the sound pressure, which is a scalar.

Sound in liquids

The speed of sound is much higher in liquids than in gases. For example, the speed of sound in water is about 1500 ms^{-1}. The density of liquids is also much higher; the density of water is about 1000 kgm^{-3}. Both the density and the speed of sound depend on the static pressure and the temperature, and there are no simple general relations corresponding to eqs. (2.2b) and (2.3).

2.1 Plane sound waves

The *plane wave* is a central concept in acoustics. Plane waves are waves in which any acoustic variable at a given time is a constant on any plane perpendicular to the direction of propagation. Such waves can propagate in a duct. In a limited area at a distance far from a source of sound in free space the curvature of the spherical wavefronts is negligible and the waves can be regarded as locally plane.

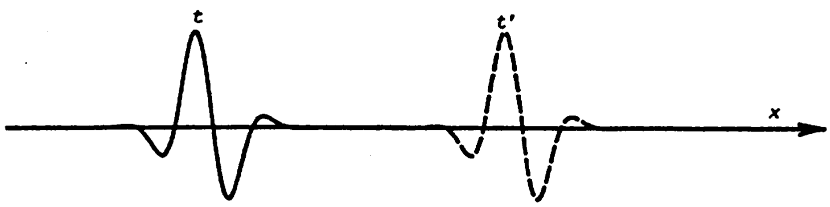

Figure 2.4 The sound pressure in a plane wave of arbitrary waveform at two different instants of time.

[5] This follows from the fact that $\partial^2(p_1 + p_2)/\partial t^2 = \partial^2 p_1/\partial t^2 + \partial^2 p_2/\partial t^2$.

[6] At very high sound pressure levels, say at levels in excess of 140 dB, the linear approximation is no longer adequate. This complicates the analysis enormously. Fortunately, we can safely assume linearity under practically all circumstances encountered in daily life.

The plane wave is a solution to the one-dimensional wave equation,

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{2.5}$$

cf. eq. (2.1). It is easy to show that the expression

$$p = f_1(ct - x) + f_2(ct + x), \tag{2.6}$$

where f_1 and f_2 are arbitrary functions, is a solution to eq. (2.5), and it can be shown this is the general solution. Since the argument of f_1 is constant if x increases as ct it follows that the first term of this expression represents a wave that propagates undistorted and unattenuated in the positive x-direction with constant speed, c, whereas the second term represents a similar wave travelling in the opposite direction. See figures 2.4 and 2.5.

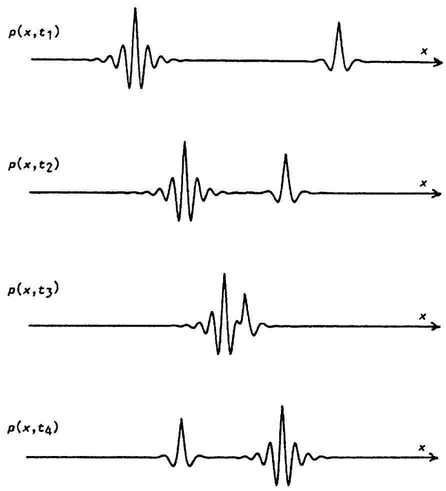

Figure 2.5 Two plane waves travelling in opposite directions are passing through each other.

The special case of a *harmonic* plane progressive wave is of great importance. Harmonic waves are generated by sinusoidal sources, for example a loudspeaker driven with a pure tone. A harmonic plane wave propagating in the x-direction can be written

$$p = p_1 \sin\left(\frac{\omega}{c}(ct - x) + \varphi\right) = p_1 \sin(\omega t - kx + \varphi), \tag{2.7}$$

where $\omega = 2\pi f$ is the angular (or radian) *frequency* and $k = \omega/c$ is the (angular) *wavenumber*. The quantity p_1 is known as the amplitude of the wave, and φ is a phase angle (the arbitrary value of the phase angle of the wave at the origin of the coordinate system at $t = 0$). At any position in this sound field the sound pressure varies sinusoidally with the angular frequency ω, and at any fixed time the sound pressure varies sinusoidally with x with the spatial period

$$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} = \frac{2\pi}{k}. \tag{2.8}$$

The quantity λ is the *wavelength*, which is defined as the distance travelled by the wave in one cycle. Note that the wavelength is inversely proportional to the frequency. At 1000 Hz the wavelength in air is about 34 cm. In rough numbers the audible frequency range goes from 20 Hz to 20 kHz, which leads to the conclusion that acousticians are faced with wavelengths (in air) in the range from 17 m at the lowest audible frequency to 17 mm at the highest audible frequency. Since the efficiency of a radiator of sound or the effect of an obstacle on the sound field depends very much on its size expressed in terms of the acoustic wavelength, it can be realised that the wide frequency range is one of the challenges in acoustics. It simplifies the analysis enormously if the wavelength is very long or very short compared with typical dimensions.

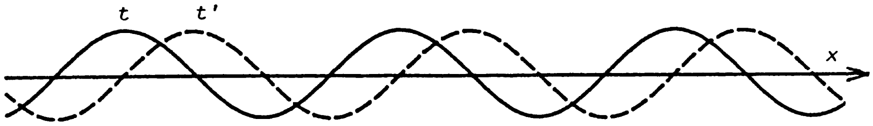

Figure 2.6 The sound pressure in a plane harmonic wave at two different instants of time.

Sound fields are often studied frequency by frequency. As already mentioned, linearity implies that a sinusoidal source with the frequency ω will generate a sound field that varies harmonically with this frequency at all positions.[7] Since the frequency is given, all that remains to be determined is the amplitude and phase at all positions. This leads to the introduction of the complex exponential representation, where the sound pressure is written as a complex function of the position multiplied with a complex exponential. The former function takes account of the amplitude and phase, and the latter describes the time dependence. Thus at any given position the sound pressure can be written as a complex function of the form[8]

$$\hat{p} = A\,\mathrm{e}^{\mathrm{j}\omega t} = |A|\,\mathrm{e}^{\mathrm{j}\varphi}\,\mathrm{e}^{\mathrm{j}\omega t} = |A|\,\mathrm{e}^{\mathrm{j}(\omega t+\varphi)} \tag{2.9}$$

(where φ is the phase of the complex amplitude A), and the real, physical sound pressure is the real part of the complex pressure,

$$p = \mathrm{Re}\{\hat{p}\} = \mathrm{Re}\left\{|A|\,\mathrm{e}^{\mathrm{j}(\omega t+\varphi)}\right\} = |A|\cos(\omega t+\varphi). \tag{2.10}$$

Since the entire sound field varies as $\mathrm{e}^{\mathrm{j}\omega t}$, the operator $\partial/\partial t$ can be replaced by $\mathrm{j}\omega$ (because the derivative of $\mathrm{e}^{\mathrm{j}\omega t}$ with respect to time is $\mathrm{j}\omega\mathrm{e}^{\mathrm{j}\omega t}$),[9] and the operator $\partial^2/\partial t^2$ can be replaced by $-\omega^2$. It follows that Euler's equation of motion can now be written

$$\mathrm{j}\omega\rho\hat{\mathbf{u}} + \nabla\hat{p} = \mathbf{0}, \tag{2.11}$$

and the wave equation can be simplified to

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[7] If the source emitted any other signal than a sinusoidal the waveform would in the general case change with the position in the sound field, because the various frequency components would change amplitude and phase relative to each other. This explains the usefulness of harmonic analysis.

[8] Throughout this note complex variables representing harmonic signals are indicated by carets.

[9] The sign of the argument of the exponential is just a convention. The $\mathrm{e}^{\mathrm{j}\omega t}$ convention is common in electrical engineering, in audio and in related areas of acoustics. The alternative convention $\mathrm{e}^{-\mathrm{j}\omega t}$ is favoured by mathematicians, physicists and acousticians concerned with outdoor sound propagation. With the alternative sign convention $\partial/\partial t$ should obviously be replaced by $-\mathrm{j}\omega$. Mathematicians and physicists also tend to prefer the symbol 'i' rather than 'j' for the imaginary unit.

$$\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial y^2} + \frac{\partial^2 \hat{p}}{\partial z^2} + k^2 \hat{p} = 0, \tag{2.12}$$

which is known as the Helmholtz equation. See the Appendix for further details about complex representation of harmonic signals. We note that the use of complex notation is mathematically very convenient, which will become apparent later.

Written with complex notation the equation for a plane wave that propagates in the x-direction becomes

$$\hat{p} = p_{\mathrm{i}}\, \mathrm{e}^{\mathrm{j}(\omega t - kx)}. \tag{2.13}$$

Equation (2.11) shows that the particle velocity is proportional to the gradient of the pressure. It follows that the particle velocity in the plane propagating wave given by eq. (2.13) is

$$\hat{u}_x = -\frac{1}{\mathrm{j}\omega\rho}\frac{\partial \hat{p}}{\partial x} = \frac{k}{\omega\rho} p_{\mathrm{i}}\, \mathrm{e}^{\mathrm{j}(\omega t - kx)} = \frac{p_{\mathrm{i}}}{\rho c} \mathrm{e}^{\mathrm{j}(\omega t - kx)} = \frac{\hat{p}}{\rho c}. \tag{2.14}$$

Thus the sound pressure and the particle velocity are in phase in a plane propagating wave (see also figure 2.10), and the ratio of the sound pressure to the particle velocity is ρc, the characteristic impedance of the medium. As the name implies, this quantity describes an important acoustic property of the fluid, as will become apparent later. The characteristic impedance of air at 20°C and 101.3 kPa is about 413 kg·m^{-2}s^{-1}.

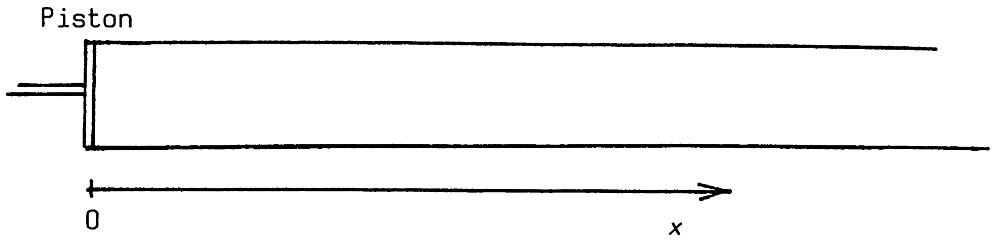

Figure 2.7 A semi-infinite tube driven by a piston.

Example 2.1

An semi-infinite tube is driven by a piston with the vibrational velocity $U\mathrm{e}^{\mathrm{j}\omega t}$ as shown in figure 2.7. Because the tube is infinite there is no reflected wave, so the sound field can be written

$$\hat{p}(x) = p_{\mathrm{i}}\mathrm{e}^{\mathrm{j}(\omega t - kx)}, \qquad \hat{u}_x(x) = \frac{p_{\mathrm{i}}}{\rho c}\mathrm{e}^{\mathrm{j}(\omega t - kx)}.$$

The boundary condition at the piston implies that the particle velocity equals the vibrational velocity of the piston:

$$\hat{u}_x(0) = \frac{p_{\mathrm{i}}}{\rho c}\mathrm{e}^{\mathrm{j}\omega t} = U\mathrm{e}^{\mathrm{j}\omega t}.$$

It follows that the sound pressure generated by the piston is

$$\hat{p}(x) = U\rho c\mathrm{e}^{\mathrm{j}(\omega t - kx)}.$$

The general solution to the one-dimensional Helmholtz equation is

$$\hat{p} = p_{\mathrm{i}}\, \mathrm{e}^{\mathrm{j}(\omega t - kx)} + p_{\mathrm{r}}\, \mathrm{e}^{\mathrm{j}(\omega t + kx)}, \tag{2.15}$$

which can be identified as the sum of a wave that travels in the positive x-direction and a wave that travels in the opposite direction (cf. eq. (2.6)). The corresponding expression for the particle velocity becomes, from eq. (2.11),

$$\begin{aligned}\hat{u}_x &= -\frac{1}{\mathrm{j}\omega\rho}\frac{\partial \hat{p}}{\partial x} = \frac{k}{\omega\rho}p_\mathrm{i}\,\mathrm{e}^{\mathrm{j}(\omega t-kx)} - \frac{k}{\omega\rho}p_\mathrm{r}\,\mathrm{e}^{\mathrm{j}(\omega t+kx)} \\ &= \frac{p_\mathrm{i}}{\rho c}\mathrm{e}^{\mathrm{j}(\omega t-kx)} - \frac{p_\mathrm{r}}{\rho c}\mathrm{e}^{\mathrm{j}(\omega t+kx)}.\end{aligned} \tag{2.16}$$

It can be seen that whereas $\hat{p} = \hat{u}_x\rho c$ in a plane wave that propagates in the positive x-direction, the sign is the opposite, that is, $\hat{p} = -\hat{u}_x\rho c$, in a plane wave that propagates in the negative x-direction. The reason for the change in the sign is that the particle velocity is a vector, unlike the sound pressure, so $\hat{u}_x$ is a vector component. It is also worth noting that the general relation between the sound pressure and the particle velocity in this interference field is far more complicated than in a plane propagating wave.

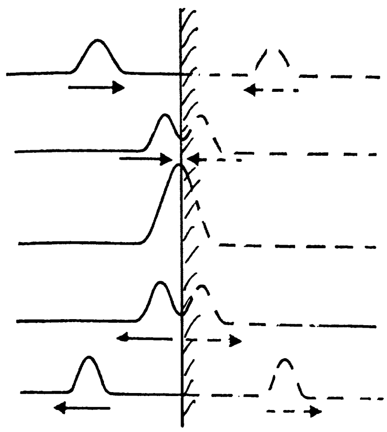

Figure 2.8 Instantaneous sound pressure in a wave that is reflected from a rigid surface at different instants of time. (Adapted from ref. [2].)

A plane wave that impinges on a plane rigid surface perpendicular to the direction of propagation will be reflected. This phenomenon is illustrated in figure 2.8, which shows how an incident transient disturbance is reflected. Note that the normal component of the gradient of the pressure is identically zero on the surface for all values of t. This is a consequence of the fact that the boundary condition at the surface implies that the particle velocity must equal zero here, cf. eq. (2.4).

However, it is easier to analyse the phenomenon assuming harmonic waves. In this case the sound field is given by the general expressions (2.15) and (2.16), and our task is to determine the relation between p_i and p_r from the boundary condition at the surface, say at $x = 0$. As mentioned, the rigid surface implies that the particle velocity must be zero here, which with eq. (2.16) leads to the conclusion that $p_\mathrm{i} = p_\mathrm{r}$, so the reflected wave has the same amplitude as the incident wave. Equation (2.15) now becomes

$$\hat{p} = p_\mathrm{i}\left(\mathrm{e}^{\mathrm{j}(\omega t-kx)} + \mathrm{e}^{\mathrm{j}(\omega t+kx)}\right) = p_\mathrm{i}\left(\mathrm{e}^{-\mathrm{j}kx} + \mathrm{e}^{\mathrm{j}kx}\right)\mathrm{e}^{\mathrm{j}\omega t} = 2p_\mathrm{i}\cos kx\,\mathrm{e}^{\mathrm{j}\omega t}, \tag{2.17}$$

and eq. (2.16) becomes

$$\hat{u}_x = -\mathrm{j}\frac{2p_\mathrm{i}}{\rho c}\sin kx\,\mathrm{e}^{\mathrm{j}\omega t}. \tag{2.18}$$

Note that the amplitude of the sound pressure is doubled on the surface (cf. figure 2.8). Note also the nodal[10] planes where the sound pressure is zero at $x = -\lambda/4$, $x = -3\lambda/4$, etc., and the planes where the particle velocity is zero at $x = -\lambda/2$, $x = -\lambda$, etc. The interference of the two plane waves travelling in opposite directions has produced a *standing wave pattern*, shown in figure 2.9.

The physical explanation of the fact that the sound pressure is identically zero at a distance of a quarter of a wavelength from the reflecting plane is that the incident wave must travel a distance of half a wavelength before it returns to the same point; accordingly the incident and reflected waves are in *antiphase* (that is, 180° out of phase), and since they have the same amplitude they cancel each other. This phenomenon is called *destructive interference*. At a distance of half a wavelength from the reflecting plane the incident wave must travel one wavelength before it returns to the same point. Accordingly, the two waves are in phase and therefore the sound pressure is doubled here (*constructive interference*). The corresponding pattern for the particle velocity is different because the particle velocity is a vector.

Another interesting observation from eqs. (2.17) and (2.18) is that the resulting sound pressure and particle velocity signals as functions of time at any position are 90° out of phase (since $\mathrm{j}\mathrm{e}^{\mathrm{j}\omega t} = \mathrm{e}^{\mathrm{j}(\omega t + \pi/2)}$). Otherwise expressed, if the sound pressure as a function of time is a cosine then the particle velocity is a sine. As we shall see later this indicates that there is no net flow of sound energy towards the rigid surface. See also figure 2.10.

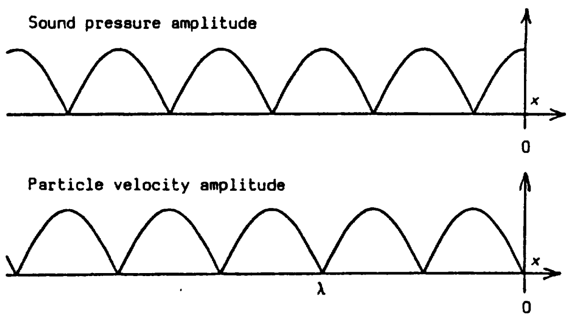

Figure 2.9 Standing wave pattern caused by reflection from a rigid surface at x = 0; amplitudes of the sound pressure and the particle velocity

Example 2.2

The standing wave phenomenon can be observed in a tube terminated by a rigid cap. When the length of the tube, l, equals an odd-numbered multiple of a quarter of a wavelength the sound pressure is zero at the input, which means that it would take very little force to drive a piston here. This is an example of an *acoustic resonance*. In this case it occurs at the frequency

$$f_0 = \frac{c}{4l},$$

[10] A *node* on, say, a vibrating string is a point that does not move, and an *antinode* is a point with maximum displacement. By analogy, points in a standing wave at which the sound pressure is identically zero are called pressure nodes. In this case the pressure nodes coincide with velocity antinodes.

and at odd-numbered multiples of this frequency, $3f_0$, $5f_0$, $7f_0$, etc. Note that the resonances are harmonically related. This means that if some mechanism excites the tube the result will be a musical sound with the fundamental frequency f_0 and overtones corresponding to odd-numbered harmonics.[11]

Brass and woodwind instruments are based on standing waves in tubes. For example, closed organ pipes are tubes closed at one end and driven at the other, open end, and such pipes have only odd-numbered harmonics. See also example 4.4.

The ratio of p_r to p_i is the (complex) *reflection factor R*. The amplitude of this quantity describes how well the reflecting surface reflects sound. In the case of a rigid plane $R = 1$, as we have seen, which implies perfect reflection with no phase shift, but in the general case of a more or less absorbing surface R will be complex and less than unity ($|R| \leq 1$), indicating partial reflection with a phase shift at the reflection plane.

If we introduce the reflection factor in eq. (2.15) it becomes

$$\hat{p} = p_i\left(e^{j(\omega t - kx)} + R\, e^{j(\omega t + kx)}\right), \tag{2.19}$$

from which it can be seen that the amplitude of the sound pressure varies with the position in the sound field. When the two terms in the parenthesis are in phase the sound pressure amplitude assumes its maximum value,

$$p_{max} = p_i\left(1 + |R|\right), \tag{2.20a}$$

and when they are in antiphase the sound pressure amplitude assumes the minimum value

$$p_{min} = p_i\left(1 - |R|\right). \tag{2.20b}$$

The ratio of p_{max} to p_{min} is called the *standing wave ratio*,

$$s = \frac{p_{max}}{p_{min}} = \frac{1+|R|}{1-|R|}. \tag{2.21}$$

From eq. (2.21) it follows that

$$|R| = \frac{s-1}{s+1}, \tag{2.22}$$

which leads to the conclusion that it is possible to determine the acoustic properties of a material by exposing it to normal sound incidence and measuring the standing wave ratio in the resulting interference field. See also chapter 5.

Figure 2.10 shows the instantaneous sound pressure and particle velocity at two different instants of time in a tube that is terminated by a material that does not reflect sound at all (case (a)), by a soft material that partly absorbs the incident sound wave (case (b)), and by a rigid material that gives perfect reflection (case (c)).

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[11] A musical (or complex) tone is not a pure (sinusoidal) tone but a periodic signal, usually consisting of the fundamental and a number of its harmonics, also called partials. These pure tones occur at multiples of the fundamental frequency. The n'th harmonic (or partial) is also called the (n-1)'th overtone, and the fundamental is the first harmonic. The relative position of a tone on a musical scale is called the *pitch* [2]. The pitch of a musical tone essentially corresponds to its fundamental frequency, which is also the distance between two adjacent harmonic components. However, pitch is a subjective phenomenon and not completely equivalent to frequency. We tend to determine the pitch on the basis of the spacing between the harmonic components, and thus we can detect the pitch of a musical tone even if the fundamental is missing.

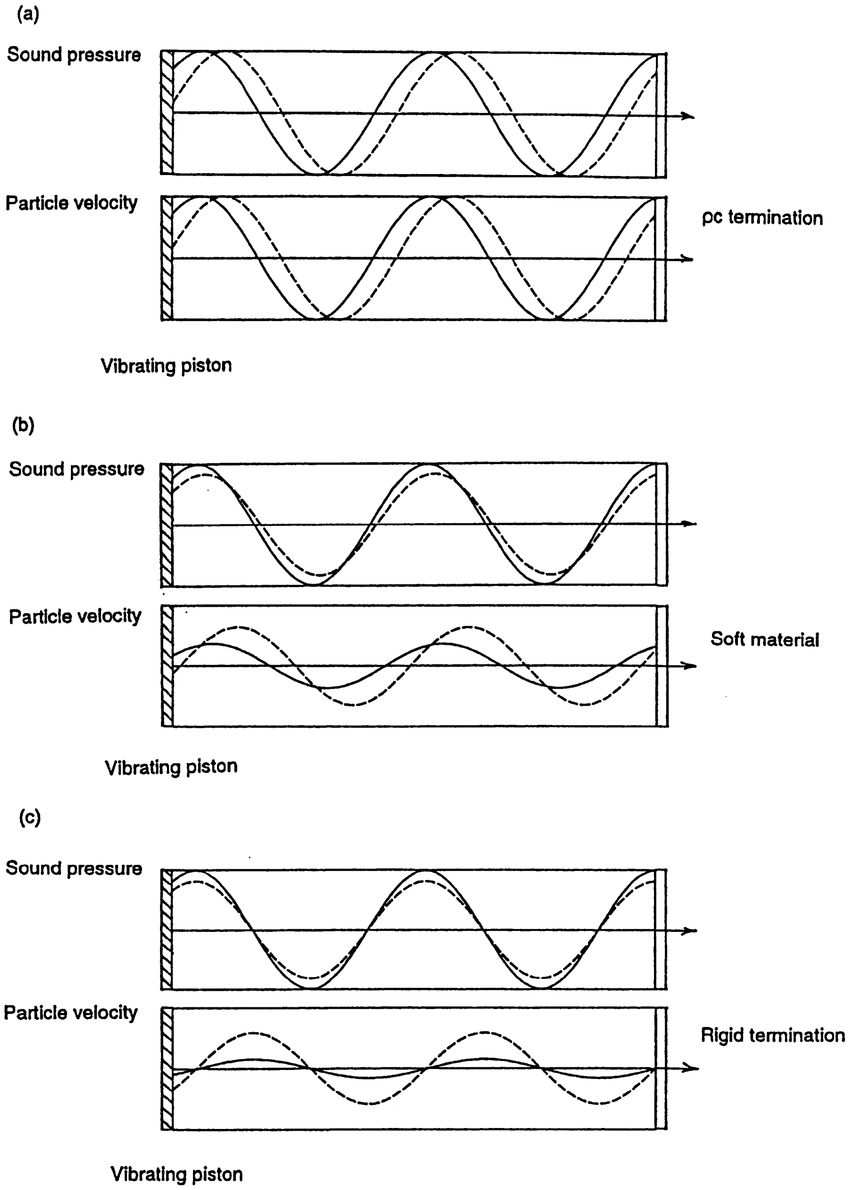

Figure 2.10 Spatial distributions of instantaneous sound pressure and particle velocity at two different instants of time. (a) Case with no reflection (R = 0); (b) case with partial reflection from a soft surface; (c) case with perfect reflection from a rigid surface (R = 1). (From ref. [3].)

Sound transmission between fluids

When a sound wave in one fluid is incident on the boundary of another fluid, say, a sound wave in air is incident on the surface of water, it will be partly reflected and partly transmitted. For simplicity let us assume that a plane wave in fluid 1 strikes the surface of fluid 2 at normal incidence as shown in figure 2.11. Anticipating a reflected wave we can write

$$\hat{p}_1 = p_\mathrm{i}\mathrm{e}^{\mathrm{j}(\omega t - kx)} + p_\mathrm{r}\mathrm{e}^{\mathrm{j}(\omega t + kx)}$$

for fluid 1, and

$$\hat{p}_2 = p_\mathrm{t}\mathrm{e}^{\mathrm{j}(\omega t - kx)}$$

for fluid 2. There are two boundary conditions at the interface: the sound pressure must be the same in fluid 1 and in fluid 2 (otherwise there would be a net force), and the particle velocity must be the same in fluid 1 and in fluid 2 (otherwise the fluids would not remain in contact). It follows that

$$p_i + p_r = p_t \qquad \text{and} \qquad \frac{p_i - p_r}{\rho_1 c_1} = \frac{p_t}{\rho_2 c_2}.$$

Combining these equations gives

$$\frac{p_r}{p_i} = R = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1},$$

which shows that the wave is almost fully reflected in phase ($R \simeq 1$) if $\rho_2 c_2 >> \rho_1 c_1$, almost fully reflected in antiphase ($R \simeq -1$) if $\rho_2 c_2 << \rho_1 c_1$, and not reflected at all if $\rho_2 c_2 = \rho_1 c_1$, irrespective of the individual properties of c_1, c_2, ρ_1 and ρ_2.

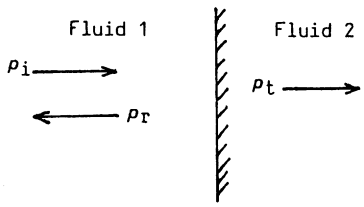

Figure 2.11 Reflection and transmission of a plane wave incident on the interface between two fluids.

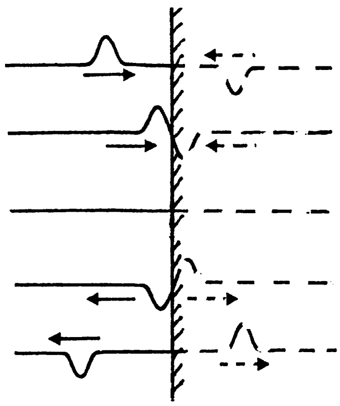

Figure 2.12 Reflection of a pressure wave at the interface between a medium of high characteristic impedance and a medium of low characteristic impedance. (Adapted from ref. [2].)

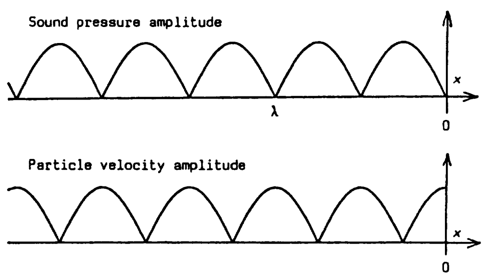

Figure 2.13 Standing wave pattern in a medium of high characteristic impedance caused by caused by reflection from a medium of low characteristic impedance; amplitudes of the sound pressure and the particle velocity.

Because of the significant difference between the characteristic impedances of air and water (the ratio is about 1: 3600) a sound wave in air that strikes a surface of water at normal incidence is almost completely reflected, and so is a sound wave that strikes the air-water interface from the water, but in the latter case the phase of the reflected wave is reversed, as shown in figure 2.12. Compare figures 2.8 and 2.12, and figures 2.9 and 2.13.

2.2 Spherical sound waves

The wave equation can be expressed in other coordinate systems than the Cartesian. If sound is generated by a source in an environment without reflections (which is usually referred to as a free field) it will generally be more useful to express the wave equation in a spherical coordinate system (r, θ, φ). The resulting equation is more complicated than eq. (2.1). However, if the source under study is spherically symmetric there can be no angular dependence, and the equation becomes quite simple,[12]

$$\frac{\partial^2 p}{\partial r^2}+\frac{2}{r}\frac{\partial p}{\partial r}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \tag{2.23a}$$

If we rewrite in the form

$$\frac{\partial^2 (rp)}{\partial r^2}=\frac{1}{c^2}\frac{\partial^2 (rp)}{\partial t^2}, \tag{2.23b}$$

it becomes apparent that this equation is identical in form with the one-dimensional wave equation, eq. (2.5), although p has been replaced by rp. (It is easy to get from eq. (2.23b) to eq. (2.23a); it is more difficult the other way.) It follows that the general solution to eq. (2.23) can be written

$$rp = f_1(ct-r)+f_2(ct+r), \tag{2.24a}$$

---

[12] This can be seen as follows. Since the sound pressure depends only on r we have

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial r}\frac{\partial r}{\partial x},$$

which, with

$$r=\sqrt{x^2+y^2+z^2},$$

becomes

$$\frac{\partial p}{\partial x}=\frac{x}{r}\frac{\partial p}{\partial r}.$$

Similar considerations leads to the following expression for the second-order derivative,

$$\frac{\partial^2 p}{\partial x^2}=\frac{1}{r}\frac{\partial p}{\partial r}+x\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial p}{\partial r}\right)=\frac{1}{r}\frac{\partial p}{\partial r}+\frac{x^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial p}{\partial r}\right)=\frac{1}{r}\frac{\partial p}{\partial r}+\frac{x^2}{r^2}\frac{\partial^2 p}{\partial r^2}-\frac{x^2}{r^3}\frac{\partial p}{\partial r}.$$

Combining eq. (2.1) with this expression and the corresponding relations for y and z finally yields eq. (2.23a):

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\frac{3}{r}\frac{\partial p}{\partial r}+\frac{x^2+y^2+z^2}{r^2}\frac{\partial^2 p}{\partial r^2}-\frac{x^2+y^2+z^2}{r^3}\frac{\partial p}{\partial r}=\frac{2}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial r^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}.$$

that is

$$p = \frac{1}{r}\left(f_1(ct - r) + f_2(ct + r)\right), \tag{2.24b}$$

where f_1 and f_2 are arbitrary functions. The first term is wave that travels outwards, away from the source (cf. the first term of eq. (2.6)). Note that the shape of the wave is preserved. However, the sound pressure is seen to decrease in inverse proportion to the distance. This is *the inverse distance law*.[13] The second term represents a converging wave, that is, a spherical wave travelling inwards. In principle such a wave could be generated by a reflecting spherical surface centred at the source, but that is a rare phenomenon indeed. Accordingly we will ignore the second term when we study sound radiation in chapter 6.

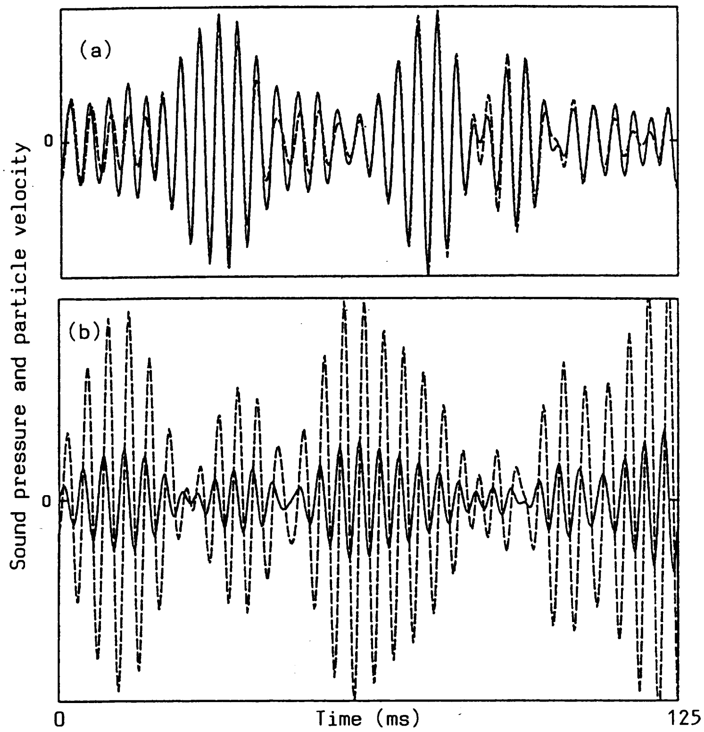

Figure 2.14 (a) Measurement far from a spherical source in free space; (b) measurement close to a spherical source. —, Sound pressure; - - -, particle velocity multiplied by ρc. (From ref. [4].)

A harmonic spherical wave is a solution to the Helmholtz equation

$$\frac{\partial^2(r\hat{p})}{\partial r^2} + k^2 r\hat{p} = 0. \tag{2.25}$$

Expressed in the complex notation the diverging wave can be written

$$\hat{p} = A\frac{e^{j(\omega t - kr)}}{r}. \tag{2.26}$$

[13] The inverse distance law is also known as the inverse square law because the sound intensity is inversely proportional to the square of the distance to the source. See chapters 5 and 6.

The particle velocity component in the radial direction can be calculated from eq. (2.11),

$$\hat{u}_r = -\frac{1}{\mathrm{j}\omega\rho}\frac{\partial \hat{p}}{\partial r} = \frac{A}{\rho c}\frac{\mathrm{e}^{\mathrm{j}(\omega t - kr)}}{r}\left(1+\frac{1}{\mathrm{j}kr}\right) = \frac{\hat{p}}{\rho c}\left(1+\frac{1}{\mathrm{j}kr}\right). \tag{2.27}$$

Because of the spherical symmetry there are no components in the other directions. Note that far[14] from the source the sound pressure and the particle velocity are in phase and their ratio equals the characteristic impedance of the medium, just as in a plane wave. On the other hand, when $kr << 1$ the particle velocity is larger than $|\hat{p}|/\rho c$ and the sound pressure and the particle velocity are almost in *quadrature*, that is, 90° out of phase. These are *near field* characteristics, and such a sound field is also known as a *reactive field*. See figure 2.14.

3. ACOUSTIC MEASUREMENTS

The most important measure of sound is the *rms* sound pressure,[15] defined as

$$p_{\mathrm{rms}} = \sqrt{\overline{p^2(t)}} = \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T p^2(t)\mathrm{d}t\right)^{½}. \tag{3.1}$$

However, as we shall see, a frequency weighting filter[16] is usually applied to the signal before the rms value is determined. Quite often such a single value does not give sufficient information about the nature of the sound, and therefore the rms sound pressure is determined in frequency bands. The resulting sound pressures are practically always compressed logarithmically and presented in decibels.

Example 3.1

The fact that $\sin^2\omega t$ = ½ (1 - cos2ωt) and thus has a time average of ½ leads to the conclusion that the rms value of a sinusoidal signal with the amplitude A is $A/\sqrt{2}$.

3.1 Frequency analysis

Single frequency sound is useful for analysing acoustic phenomena, but most sounds encountered in practice have 'broadband' characteristics, which means that they cover a wide frequency range. If the sound is more or less steady, it will practically always be more useful to analyse it in the frequency domain than to look at the sound pressure as a function of time.

Frequency (or spectral) analysis of a signal involves decomposing the signal into its spectral components. This analysis can be carried out by means of digital analysers that employ the discrete Fourier transform ('FFT analysers'). This topic is outside the scope of this note, but see, e.g., refs. [5, 6]. Alternatively, the signal can be passed through a number of

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[14] In acoustics, dimensions are measured in terms of the wavelength, so that 'far from' means that $r >> \lambda$ (or $kr >> 1$), just as 'near' means that $r << \lambda$ (or $kr << 1$). The dimensionless quantity kr is known as the Helmholtz number.

[15] Root mean square value, usually abbreviated rms. This is the square root of the mean square value, which is the time average of the squared signal.

[16] A filter is a device that modifies a signal by attenuating some of its frequency components.

contiguous analogue or digital bandpass filters[17] with different centre frequencies, a 'filter bank.' The filters can have the same bandwidth or they can have constant relative bandwidth, which means that the bandwidth is a certain percentage of the centre frequency. Constant relative bandwidth corresponds to uniform resolution on a logarithmic frequency scale. Such a scale is in much better agreement with the subjective *pitch* of musical sounds than a linear scale, and therefore frequencies are often represented on a logarithmic scale in acoustics, and frequency analysis is often carried out with constant percentage filters. The most common filters in acoustics are octave band filters and one-third octave band filters.

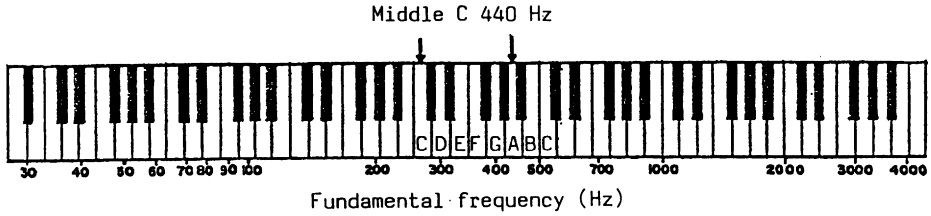

Figure 3.1 The keyboard of a small piano. The white keys from C to B correspond to the seven notes of the C major scale. (Adapted from ref. [7].)

An *octave*[18] is a frequency ratio of 2:1, a fundamental unit in musical scales. Accordingly, the lower limiting frequency of an octave band is half the upper frequency limit, and the centre frequency is the geometric mean, that is,

$$f_\mathrm{l} = f_\mathrm{c}/2^{1/2}, \quad f_\mathrm{u} = 2^{1/2} f_\mathrm{c}, \quad f_\mathrm{c} = \sqrt{f_\mathrm{l} f_\mathrm{u}}, \qquad \text{(3.2a, 3.2b, 3.2c)}$$

where f_c is the centre frequency. In a similar manner a one-third octave[19] band is a band for which $f_\mathrm{u} = 2^{1/3} f_\mathrm{l}$, and

$$f_\mathrm{l} = f_\mathrm{c}/2^{1/6}, \quad f_\mathrm{u} = 2^{1/6} f_\mathrm{c}, \quad f_\mathrm{c} = \sqrt{f_\mathrm{l} f_\mathrm{u}}, \qquad \text{(3.3a, 3.3b, 3.3c)}$$

Since $2^{10} = 1024 \simeq 10^3$ it follows that $2^{10/3} \simeq 10$ and $2^{1/3} \simeq 10^{1/10}$, that is, ten one-third octaves very nearly make a decade, and a one-third octave is almost identical with one tenth of a decade. Decades are practical, and thus in practice 'one-third octave band filters' are actually one-tenth of a decade band filters. Table 3.1 gives the corresponding standardised nominal centre frequencies of octave and one-third octave band filters.[20] As mentioned earlier, the human ear may respond to frequencies in the range from 20 Hz to 20 kHz, that is, a range of three decades, ten octaves or thirty one-third octaves.

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[17] An ideal bandpass filter would allow frequency components in the passband to pass unattenuated and would completely remove frequency components outside the passband. Real filters have, of course, a certain passband ripple and a finite stopband attenuation.

[18] Musical tones an octave apart sound very similar. The diatonic scale contains seven notes per octave corresponding to the white keys on a piano keyboard; see figure 3.1. Thus an octave spans eight notes, say, from C to C'; hence the name octave (from Latin *octo*: eight).

[19] A semitone is one twelfth of an octave on the equally tempered scale (a frequency ratio of $2^{1/12}$:1). Since $2^{1/3} = 2^{4/12}$ it can be seen that a one-third octave is identical with four semitones or a major third (e.g., from C to E, cf. figure 3.1). Accordingly, one-third octave band filters are called Terzfilters in German.

[20] The standardised nominal centre frequencies presented in table 3.1 are based on the fact that the series 1.25, 1.6, 2, 2.5, 3.15, 4, 5, 6.3, 8, 10 is in reasonable agreement with $10^{n/10}$, with n = 1, 2,....., 10.

Table 3.1 Standardised nominal one-third octave and octave (bold characters) centre frequencies (in hertz).

20	25	**31.5**	40	50	**63**	80	100	**125**	160	200	**250**	315	400	**500**	630	800	**1000**
1250	1600	**2000**	2500	3150	**4000**	5000	6300	**8000**	10000	12500	**16000**	20000					

An important property of the mean square value of a signal is that it can be partitioned into frequency bands. This means that if we analyse a signal in, say, one-third octave bands, the sum of the mean square values of the filtered signals equals the mean square value of the unfiltered signal. The reason is that products of different frequency components average to zero, so that all cross terms vanish; the different frequency components are *uncorrelated signals*. This can be illustrated by analysing a sum of two pure tones with different frequencies,

$$\begin{aligned}\overline{(A\sin\omega_1 t + B\sin\omega_2 t)^2} &= A^2\overline{\sin^2\omega_1 t} + B^2\overline{\sin^2\omega_2 t} + 2AB\overline{\sin\omega_1 t\sin\omega_2 t} \\ &= (A^2+B^2)/2.\end{aligned} \tag{3.4}$$

Note that the mean square values of the two signals are added unless $\omega_1 = \omega_2$. The validity of this rule is not restricted to pure tones of different frequency; the mean square value of any stationary signal equals the sum of mean square values of its frequency components, which can be determined with a parallel bank of contiguous filters. Thus

$$p_{\mathrm{rms}}^2 = \sum_i p_{\mathrm{rms},i}^2, \tag{3.5}$$

where $p_{\mathrm{rms},i}$ is the rms value of the output of the i'th filter. Equation (3.5) is known as Parseval's formula.

Random noise

Many generators of sound produce *noise* rather than pure tones. Whereas pure tones and other periodic signals are deterministic, *noise* is a stochastic or random phenomenon. *Stationary* noise is a stochastic signal with statistical properties that do not change with time.

White noise is stationary noise with a flat power spectral density, that is, constant mean square value per hertz. The term white noise is an analogy to white light. When white noise is passed through a bandpass filter, the mean square of the output signal is proportional to the bandwidth of the filter. It follows that when white noise is analysed with constant percentage filters, the mean square of the output is proportional to the centre frequency of the filter. For example, if white noise is analysed with a bank of octave band filters, the mean square values of the output signals of two adjacent filters differ by a factor of two.

Pink noise is stationary noise with constant mean square value in bands with constant relative width, e.g., octave bands. Thus compared with white noise low frequencies are emphasised; hence the name pink noise, which is an analogy to an optical phenomenon. It follows that the mean square value of a given pink noise signal in octave bands is three times larger than the mean square value of the noise in one-third octave bands.

Example 3.2

The fact that noise, unlike periodic signals, has a finite power spectral density (mean square value *per hertz*) implies that one can detect a pure tone in noise irrespective of the signal-to-noise ratio by analysing with sufficiently fine spectral resolution: As the bandwidth is reduced, less and less noise passes through the filter, and the tone will emerge. Compared with filter bank analysers FFT analysers have the advantage that the spectral resolution can be varied over a wide range [6]; therefore FFT analysers are particular suitable for detecting tones in noise.

When several independent sources of noise are present at the same time the mean square sound pressures generated by the individual sources are additive. This is due to the fact that independent sources generate uncorrelated signals, that is, signals whose product average to zero; therefore the cross terms vanish:

$$\overline{(p_1(t)+p_2(t))^2} = \overline{p_1^2(t)} + \overline{p_2^2(t)} + 2\overline{p_1(t)p_2(t)} = \overline{p_1^2(t)} + \overline{p_2^2(t)}. \tag{3.6}$$

It follows that

$$p_{\text{rms,tot}}^2 = \sum_i p_{\text{rms},i}^2. \tag{3.7}$$

Note the similarity between eqs. (3.5) and (3.7). It is of enormous practical importance that the mean square values of uncorrelated signals are additive, because signals generated by different mechanisms are invariably uncorrelated. Almost all signals that occur in real life are mutually uncorrelated.

Example 3.3

Equation (3.7) leads to the conclusion that the mean square pressure generated by a crowd of noisy people in a room is proportional to the number of people (provided that each person makes a given amount of noise, independently of the others). Thus the rms value of the sound pressure in the room is proportional to the *square root* of the number of people.

Example 3.4

Consider the case where the rms sound pressure generated by a source of noise is to be measured in the presence of background noise that cannot be turned off. It follows from eq. (3.7) that it in principle is possible to correct the measurement for the influence of the stationary background noise; one simply subtracts the mean square value of the background noise from the total mean square pressure. For this to work in practice the background noise must not be too strong, though, and it is absolutely necessary that it is completely stationary.

3.2 Levels and decibels

The human auditory system can cope with sound pressure variations over a range of more than a million times. Because of this wide range, the sound pressure and other acoustic quantities are usually measured on a logarithmic scale. An additional reason is that the subjective impression of how loud noise sounds correlates much better with a logarithmic measure of the sound pressure than with the sound pressure itself. The unit is the *decibel*,[21] abbreviated dB, which is a relative measure, requiring a reference quantity. The results are called *levels*. The sound pressure level (sometimes abbreviated SPL) is defined as

$$L_p = 10\log_{10}\frac{p_{\text{rms}}^2}{p_{\text{ref}}^2} = 20\log_{10}\frac{p_{\text{rms}}}{p_{\text{ref}}}, \tag{3.8}$$

where p_{ref} is the reference sound pressure, and $\log_{10}$ is the base 10 logarithm, henceforth written log. The reference sound pressure is 20 μPa for sound in air, corresponding roughly to the lowest audible sound at 1 kHz.[22] Some typical sound pressure levels are given in figure 3.2.

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[21] As the name implies, the decibel is one tenth of a bel. However, the bel is rarely used today. The use of decibels rather than bels is probably due to the fact that most sound pressure levels encountered in practice take values between 0 and 120 when measured in decibels, as can be seen in figure 3.2. Another reason might be that to be audible, the change of the level of a given (broadband) sound must be of the order of one decibel.

[22] For sound in other fluids than atmospheric air (water, for example) the reference sound pressure is 1 μPa. To avoid possible confusion it may be advisable to state the reference sound pressure explicitly, e.g., 'the sound pressure level is 77 dB re 20 μPa.'

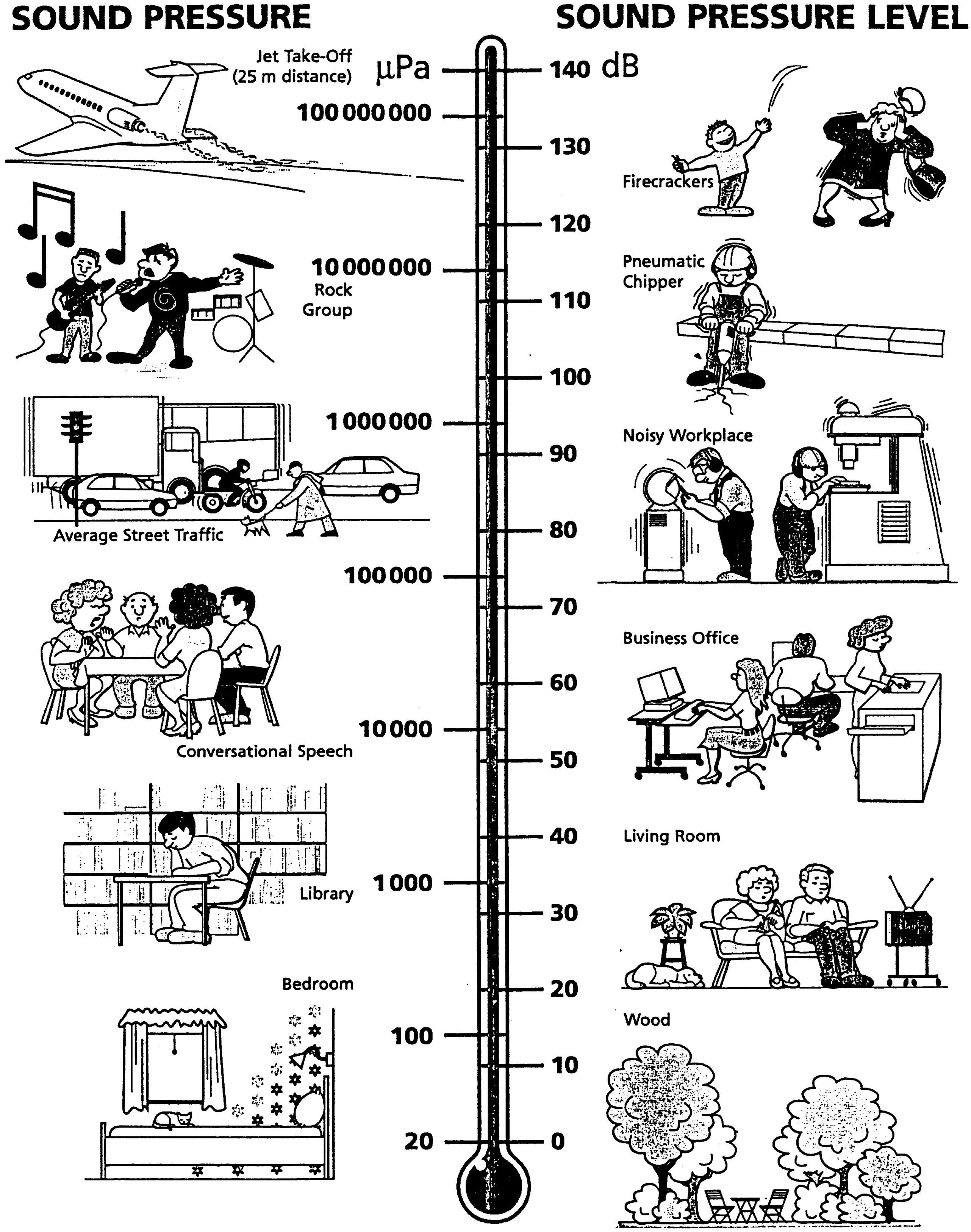

Figure 3.2 Typical sound pressure levels. (Source: Brüel & Kjær.)

The fact that the mean square sound pressures of independent sources are additive (cf. eq. (3.7)) leads to the conclusion that the levels of such sources are combined as follows:

$$L_{p,\text{tot}} = 10\log\left(\sum_i 10^{0.1L_{p,i}}\right). \tag{3.9}$$

Another consequence of eq. (3.7) is that one can correct a measurement of the sound pressure level generated by a source for the influence of steady background noise as follows:

$$L_{p,\text{source}} = 10\log\left(10^{0.1L_{p,\text{tot}}} - 10^{0.1L_{p,\text{background}}}\right). \tag{3.10}$$

This corresponds to subtracting the mean square sound pressure of the background noise from the total mean square sound pressure as described in example 3.5. However, since all measurements are subject to random errors, the result of the correction will be reliable only if the background level is at least, say, 3 dB below the total sound pressure level. If the background noise is more than 10 dB below the total level the correction is less than 0.5 dB.

Example 3.5

Expressed in terms of sound pressure levels the inverse distance law states that the level decreases by 6 dB when the distance to the source is doubled.

Example 3.6

When each of two independent sources in the absence of the other generates a sound pressure level of 70 dB at a certain point, the resulting sound pressure level is 73 dB (**not** 140 dB!), because $10\log 2 \simeq 3$. If one source creates a sound pressure level of 65 dB and the other a sound pressure level of 59 dB, the total level is $10\log(10^{6.5} + 10^{5.9}) \simeq 66$ dB.

Example 3.7

Say the task is to determine the sound pressure level generated by a source in background noise with a level of 59 dB. If the total sound pressure level is 66 dB, it follows from eq. (3.10) that the source would have produced a sound pressure level of $10\log(10^{6.6} - 10^{5.9}) \simeq 65$ dB in the absence of the background noise.

Example 3.8

When two sinusoidal sources emit pure tones of the same frequency they create an interference field, and depending on the phase difference the total sound pressure amplitude at a given position will assume a value between the sum of the two amplitudes and the difference:

$$\left||A| - |B|\right| \le \left|A\mathrm{e}^{\mathrm{j}\omega t} + B\mathrm{e}^{\mathrm{j}\omega t}\right| = |A + B| = \left||A|\mathrm{e}^{\mathrm{j}\varphi_A} + |B|\mathrm{e}^{\mathrm{j}\varphi_B}\right| \le |A| + |B|.$$

For example, if two pure tone sources of the same frequency each generates a sound pressure level of 70 dB in the absence of the other source then the total sound pressure level can be anywhere between 76 dB (constructive interference) and - ∞ dB (destructive interference). Note that eqs. (3.7) and (3.9) do **not** apply in this case because the signals are not uncorrelated. See also figure A2 in the Appendix.

Other first-order acoustic quantities, for example the particle velocity, are also often measured on a logarithmic scale. The reference velocity is 1 nm/s = 10^{-9} m/s.[23] This reference is also used in measurements of the vibratory velocities of vibrating structures.

The acoustic second-order quantities sound intensity and sound power, defined in chapter 5, are also measured on a logarithmic scale. The sound intensity level is

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[23] The prefix n (for 'nano') represents a factor of 10^{-9}.

$$L_I = 10\log\frac{|I|}{I_{\text{ref}}}, \tag{3.11}$$

where I is the intensity and $I_{\text{ref}} = 1\ \text{pWm}^{-2} = 10^{-12}\ \text{Wm}^{-2}$,[24] and the sound power level is

$$L_W = 10\log\frac{P_{\text{a}}}{P_{\text{ref}}}, \tag{3.12}$$

where P_{a} is the sound power and $P_{\text{ref}} = 1$ pW. Note than levels of linear quantities (pressure, particle velocity) are defined as twenty times the logarithm of the ratio of the rms value to a reference value, whereas levels of second-order (quadratic) quantities are defined as *ten* times the logarithm, in agreement with the fact that if the linear quantities are doubled then quantities of second order are quadrupled.

Example 3.9

It follows from the constant spectral density of white noise that when such a signal is analysed in one-third octave bands, the level increases 1 dB from one band to the next $(10\log(2^{1/3}) \simeq 1\,\text{dB})$.

3.3 Noise measurement techniques and instrumentation

A sound level meter is an instrument designed to measure sound pressure levels. Today such instruments can be anything from simple devices with analogue filters and detectors and a moving coil meter to advanced digital analysers. Figure 3.3 shows a block diagram of a simple sound level meter. The microphone converts the sound pressure to an electrical signal, which is amplified and passes through various filters. After this the signal is squared and averaged with a detector, and the result is finally converted to decibels and shown on a display. In the following a very brief description of such an instrument will be given; see e.g. refs. [8, 9] for further details.

The most commonly used microphones for this purpose are condenser microphones, which are more stable and accurate than other types. The diaphragm of a condenser microphone is a very thin, highly tensioned foil. Inside the housing of the microphone cartridge is the other part of the capacitor, the back plate, placed very close to the diaphragm (see figure 3.4). The capacitor is electrically charged, either by an external voltage on the back plate or (in case of prepolarised *electret* microphones) by properties of the diaphragm or the back plate. When the diaphragm moves in response to the sound pressure, the capacitance changes, and this produces an electrical voltage proportional to the instantaneous sound pressure.

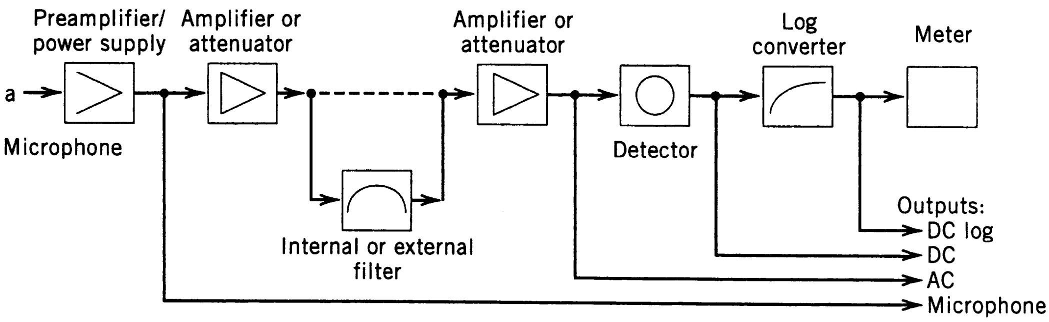

Figure 3.3 A sound level meter. (From ref. [10].)

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[24] The prefix p (for 'pico') represents a factor of 10^{-12}.

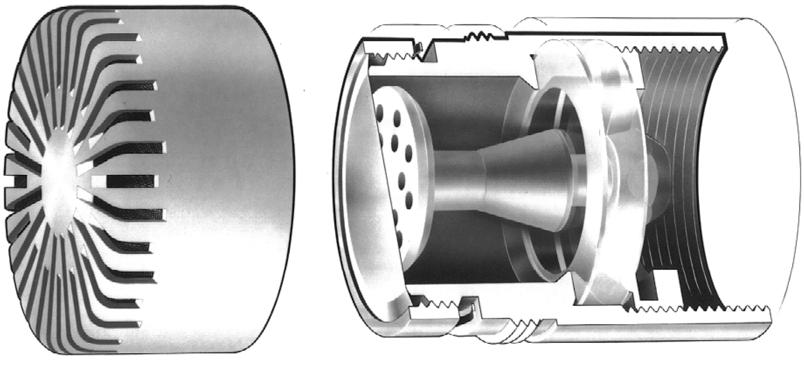

Figure 3.4 A condenser microphone. (From ref. [11].)

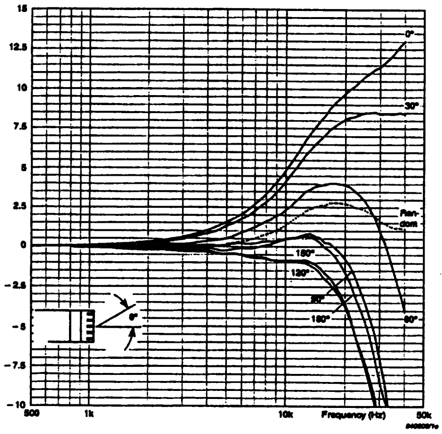

Figure 3.5 The 'free-field correction' of a typical measurement microphone for sound coming from various directions. The free-field correction is the fractional increase of the sound pressure (usually expressed in dB) caused by the presence of the microphone in the sound field. (From ref. [11].)

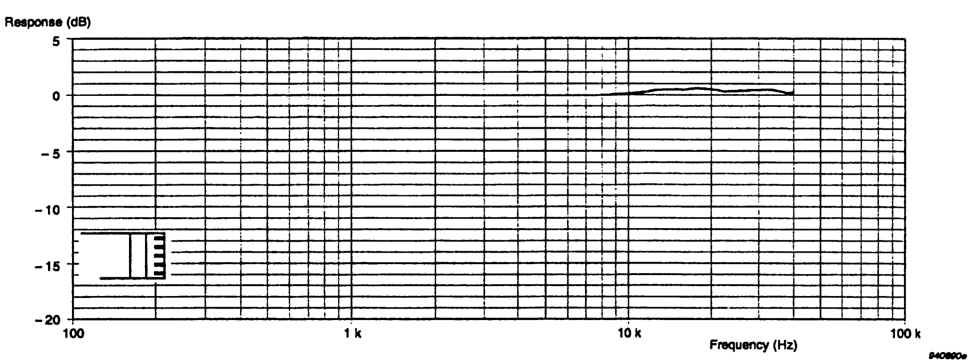

Figure 3.6 Free-field response of a microphone of the 'free-field' type at axial incidence. (From ref. [11].)

The microphone should be as small as possible so as not to disturb the sound field. However, this is in conflict with the requirement of a high sensitivity and a low inherent noise level, and typical measurement microphones are '½-inch' microphones with a diameter of about 13 mm. At low frequencies, say below 1 kHz, such a microphone is much smaller than the wavelength and does not disturb the sound field appreciably. In this frequency range the microphone is *omnidirectional* as of course it should be since the sound pressure is a scalar and has no direction. However, from a few kilohertz and upwards the size of the microphone is no longer negligible compared with the wavelength, and therefore it is no longer omnidirectional, which means that its response varies with the nature of the sound field; see figure 3.5.

One can design condenser microphones to have a flat response in as wide a frequency range as possible under specified sound field conditions. For example, 'free-field' microphones are designed to have a flat response for axial incidence (see figure 3.6), and such microphones should therefore be pointed towards the source. 'Random-incidence' microphones are designed for measurements in a diffuse sound field where sound is arriving from all directions, and 'pressure' microphones are intended for measurements in small cavities.

The sensitivity of the human auditory system varies significantly with the frequency in a way that changes with the level. In particular the human ear is, at low levels, much less sensitive to low frequencies than to medium frequencies. This is the background for the standardised frequency weighting filters shown in figure 3.7. The original intention was to simulate a human ear at various levels, but it has long ago been realised that the human auditory system is far more complicated than implied by such simple weighting curves, and B- and D-weighting filters are little used today. On the other hand the A-weighted sound pressure level is the most widely used single-value measure of sound, because the A-weighted sound pressure level correlates in general much better with the subjective effect of noise than measurements of the sound pressure level with a flat frequency response. C-weighting, which is essentially flat in the audible frequency range, is sometimes used in combination with A-weighting, because a large difference between the A-weighted level and the C-weighted level is a clear indication of a prominent content of low frequency noise. The results of measurements of the A- and C-weighted sound pressure level are denoted L_A and L_C respectively, and the unit is dB.[25] If no weighting filter is applied, the level is sometimes denoted L_Z.

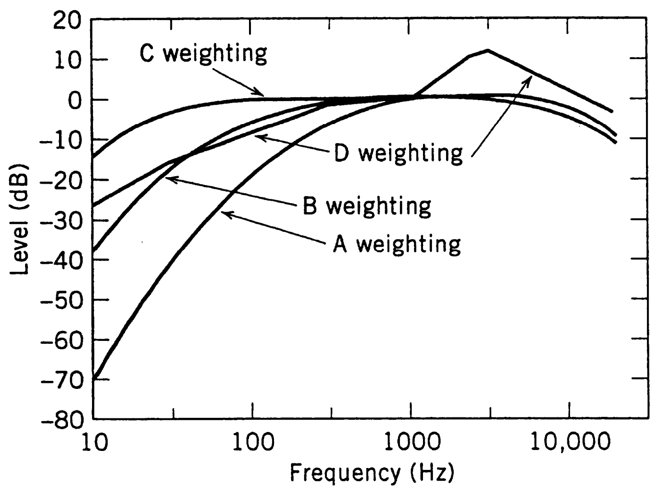

Figure 3.7 Standardised frequency weighting curves. (From ref. [8].)

[25] In practice the unit is often written dB (A) and dB (C), respectively.

Table 3.2 The response of standard A- and C-weighting filters in one-third octave bands.

Centre frequency (Hz)	A-weighting (dB)	C-weighting (dB)
8	-77.8	-20.0
10	-70.4	-14.3
12.5	-63.4	-11.2
16	-56.7	-8.5
20	-50.5	-6.2
25	-44.7	-4.4
31.5	-39.4	-3.0
40	-34.6	-2.0
50	-30.2	-1.3
63	-26.2	-0.8
80	-22.5	-0.5
100	-19.1	-0.3
125	-16.1	-0.2
160	-13.4	-0.1
200	-10.9	0.0
250	-8.6	0.0
315	-6.6	0.0
400	-4.8	0.0
500	-3.2	0.0
630	-1.9	0.0
800	-0.8	0.0
1000	0.0	0.0
1250	0.6	0.0
1600	1.0	-0.1
2000	1.2	-0.2
2500	1.3	-0.3
3150	1.2	-0.5
4000	1.0	-0.8
5000	0.5	-1.3
6300	-0.1	-2.0
8000	-1.1	-3.0
10000	-2.5	-4.4
12500	-4.3	-6.2
16000	-6.6	-8.5
20000	-9.3	-11.2

In the measurement instrument the frequency weighting filter is followed by a squaring device, a lowpass filter that smooths out the instantaneous fluctuations, and a logarithmic converter. The lowpass filter corresponds to applying a time weighting function. The most common time weighting in sound level meters is exponential, which implies that the squared signal is smoothed with a decaying exponential so that recent data are given more weight than older data:

$$L_p(t) = 10\log\left(\left(\frac{1}{\tau}\int_{-\infty}^{t} p^2(u)\,\mathrm{e}^{-(t-u)/\tau}\,\mathrm{d}u\right)\bigg/ p_{\mathrm{ref}}^2\right). \tag{3.13}$$

Two values of the time constant τ are standardised: S (for 'slow') corresponds to a time constant of 1 s, and F (for 'fast') is exponential averaging with a time constant of 125 ms.

The alternative to exponential averaging is linear (or integrating) averaging, in which all the sound is weighted uniformly during the integration. The equivalent sound pressure level is defined as

$$L_{\text{eq}} = 10\log\left(\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p^2(t)\text{d}t\right)\bigg/ p_{\text{ref}}^2\right). \tag{3.14}$$

Measurements of random noise with a finite integration time are subject to random errors that depend on the bandwidth of the signal and on the integration time. It can be shown that the variance of the measurement result is inversely proportional to the product of the bandwidth and the integration time [6].[26]

As can be seen by comparing with eqs. (3.1) and (3.8), the equivalent sound pressure level is just the sound pressure level corresponding to the rms sound pressure determined with a specified integration period. The A-weighted equivalent sound pressure level L_{Aeq} is the level corresponding to a similar time integral of the A-weighted instantaneous sound pressure. Sometimes the quantity is written $L_{\text{Aeq},T}$ where T is the integration time.

Whereas exponential averaging corresponds to a running average and thus gives a (smoothed) measure of the sound at any instant of time, the equivalent sound pressure level (with or without A-weighting) can be used for characterising the total effect of fluctuating noise, for example noise from road traffic. Typical values of T are 30 s for measurement of noise from technical installations, 8 h for noise in a working environment and 24 h for traffic noise.

Sometimes it is useful to analyse noise signals in one-third octave bands, cf. section 3.1. From eq. (3.5) it can be seen that the total sound pressure level can be calculated from the levels in the individual one-third octave bands, L_i, as follows:

$$L_{\text{Z}} = 10\log\left(\sum_i 10^{0.1L_i}\right). \tag{3.15}$$

In a similar manner one can calculate the A-weighted sound pressure level from the one-third octave band values and the attenuation data given in table 3.2,

$$L_{\text{A}} = 10\log\left(\sum_i 10^{0.1(L_i + K_i)}\right), \tag{3.16}$$

where K_i is the relative response of the A-weighting filter (in dB) in the i'th band.

Example 3.10

A source gives rise to the following one-third octave band values of the sound pressure level at a certain point,

Centre frequency (Hz)	Sound pressure level (dB)
315	52
400	68
500	76
630	71
800	54

[26] In the literature reference is sometimes made to the equivalent integration time of exponential detectors. This is two times the time constant (e.g. 250 ms for ‘F’), because a measurement of random noise with an exponential detector with a time constant of τ has the same statistical uncertainty as a measurement with linear averaging over a period of 2τ [9].

and less than 50 dB in all the other bands. It follows that

$$L_Z \simeq 10\log\left(10^{5.2}+10^{6.8}+10^{7.6}+10^{7.1}+10^{5.4}\right) \simeq 77.7 \text{ dB},$$

and

$$L_A \simeq 10\log\left(10^{(5.2-0.66)}+10^{(6.8-0.48)}+10^{(7.6-0.32)}+10^{(7.1-0.19)}+10^{(5.4-0.08)}\right) \simeq 74.7 \text{ dB}.$$

Noise that changes its level in a regular manner is called *intermittent noise*. Such noise could for example be generated by machinery that operates in cycles. If the noise occurs at several steady levels, the equivalent sound pressure level can be calculated from the formula

$$L_{\text{eq},T} = 10\log\left(\sum_i \frac{t_i}{T}10^{0.1L_i}\right). \tag{3.17}$$

This corresponds to adding the mean square values with a weighting that reflects the relative duration of each level.

Example 3.11

The A-weighted sound pressure level at a given position in an industrial hall changes periodically between 84 dB in intervals of 15 minutes, 95 dB in intervals of 5 minutes and 71 dB in intervals of 20 minutes. From eq. (3.17) it follows that the equivalent sound pressure level over a working day is

$$L_{\text{Aeq}} = 10\log\left(\frac{15}{40}10^{8.4}+\frac{5}{40}10^{9.5}+\frac{20}{40}10^{7.1}\right) \simeq 87.0 \text{ dB}.$$

Most sound level meters have also a peak detector for determining the highest absolute value of the instantaneous sound pressure (without filters and without time weighting), p_{peak}. The *peak level* is calculated from this value and eq. (3.8) in the usual manner, that is,

$$L_p = 20\log\frac{p_{\text{peak}}}{p_{\text{ref}}}. \tag{3.18}$$

Example 3.12

The *crest factor* of a signal is the ratio of its peak value to the rms value (sometimes expressed in dB). From example 3.1 it follows that the crest factor of a pure tone signal is $\sqrt{2}$ or 3 dB.

The sound exposure level (sometimes abbreviated SEL) is closely related to L_{Aeq}, but instead of dividing the time integral of the squared A-weighted instantaneous sound pressure by the actual integration time one divides by t_0 = 1 s. Thus the sound exposure level is a measure of the total energy[27] of the noise, normalised to 1 s:

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[27] In signal analysis it is customary to use the term ‘energy’ in the sense of the integral of the square of a signal, without regard to its units. This should not be confused with the potential energy density of the sound field introduced in chapter 5.

$$L_{\mathrm{AE}} = 10\log\left(\left(\frac{1}{t_0}\int_{t_1}^{t_2} p_{\mathrm{A}}^2(t)dt\right)\Big/ p_{\mathrm{ref}}^2\right) \tag{3.19}$$

This quantity is used for measuring the total energy of a 'noise event' (say, a hammer blow or the take off of an aircraft), independently of its duration. Evidently the measurement interval should encompass the entire event.

Example 3.13

It is clear from eqs. (3.14) and (3.19) that $L_{\mathrm{Aeq},T}$ of a noise event of finite duration decreases with the logarithm of T if the T exceeds its duration:

$$L_{\mathrm{Aeq},T} = 10\log\left(\left(\frac{1}{T}\int_{-\infty}^{\infty} p_{\mathrm{A}}^2(t)\mathrm{d}t\right)\Big/ p_{\mathrm{ref}}^2\right) = L_{\mathrm{AE}} - 10\log\frac{T}{t_0}.$$

Example 3.14

If n identical noise events each with a sound exposure level of L_{AE} occur within a period of T (e.g., one working day) then the A-weighted equivalent sound level is

$$L_{\mathrm{Aeq},T} = L_{\mathrm{AE}} + 10\log n - 10\log\frac{T}{t_0},$$

because the integrals of the squared signals are additive; cf. eq. (3.7).[28]

4. THE CONCEPT OF IMPEDANCE

By definition an *impedance* is the ratio of the complex amplitudes of two signals representing cause and effect, for example the ratio of an AC voltage across a part of an electric circuit to the corresponding current, the ratio of a mechanical force to the resulting vibrational velocity, or the ratio of the sound pressure to the particle velocity. The term has been coined from the verb 'impede' (obstruct, hinder), indicating that it is a measure of the opposition to the flow of current etc. The reciprocal of the impedance is the *admittance*, coined from the verb 'admit' and indicating lack of such opposition. Note that these concepts require complex representation of harmonic signals; it makes no sense to divide, say, the instantaneous sound pressure with the instantaneous particle velocity. There is no simple way of describing properties corresponding to a complex value of the impedance without the use of complex notation.

The mechanical impedance is perhaps simpler to understand than the other impedance concepts, since it is intuitively clear that it takes a certain vibratory force to generate mechanical vibrations. The mechanical impedance of a structure at a given point is the ratio of the complex amplitude of a harmonic point force acting on the structure to the complex amplitude of the resulting vibratory velocity at the same point,[29]

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[28] Strictly speaking this requires that the instantaneous product of the 'event' and any of its time shifted versions time average to zero. In practice this will always be the case.

[29] Note that the sign of the imaginary part of the impedance changes if the $e^{-i\omega t}$ convention is used instead of the $e^{j\omega t}$ convention. Cf. footnote no 9 on p. 10.

$$Z_{\mathrm{m}} = \frac{\hat{F}}{\hat{v}}. \tag{4.1a}$$

The unit is kg/s. The mechanical admittance is the reciprocal of the mechanical impedance,

$$Y_{\mathrm{m}} = \frac{\hat{v}}{\hat{F}}. \tag{4.1b}$$

This quantity is also known as the mobility. The unit is s/kg.

Example 4.1

It takes a force of $F = a \cdot M$ to set a mass M into the acceleration a (Newton's second law of motion); therefore the mechanical impedance of the mass is

$$Z_{\mathrm{m}} = \frac{\hat{F}}{\hat{v}} = \frac{\hat{F}}{\hat{a}/\mathrm{j}\omega} = \mathrm{j}\omega M.$$

Example 4.2

It takes a force of $F = \xi K$ to stretch a spring with the stiffness K a length of ξ (Hooke's law); therefore the mechanical impedance of the spring is

$$Z_{\mathrm{m}} = \frac{\hat{F}}{\hat{v}} = \frac{\hat{F}}{\mathrm{j}\omega\hat{\xi}} = \frac{K}{\mathrm{j}\omega}.$$

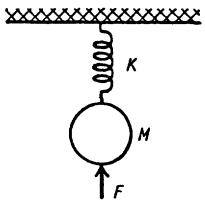

Figure 4.1 A mass hanging from a spring.

Example 4.3

A simple mechanical oscillator consists of a mass M suspended from a spring with a stiffness constant of K, as sketched in figure 4.1. In order to set the mass into vibrations one will have to move the mass *and* displace the spring from its equilibrium value. It follows that the mechanical impedance of this system is the sum of the impedance of the mass and the impedance of the spring,

$$Z_{\mathrm{m}} = \mathrm{j}\omega M + \frac{K}{\mathrm{j}\omega} = \mathrm{j}\left(\omega M - \frac{K}{\omega}\right) = \mathrm{j}\omega M\left(1 - \left(\omega_0/\omega\right)^2\right),$$

where

$$\omega_0 = \sqrt{K/M}$$

is the angular resonance frequency. Note that the impedance is zero at the resonance, indicating that even a very small harmonic force at this frequency will generate an infinite velocity. In practice there will always be some losses, of course, so the impedance is very small but not zero at the resonance frequency.

The *acoustic impedance* is associated with average properties on a surface. This quantity is mainly used under conditions where the sound pressure is more or less constant on the surface. It is defined as the complex ratio of the average sound pressure to the *volume velocity*, which is the surface integral of the normal component of the particle velocity,

$$\hat{q} = \int_S \hat{\mathbf{u}} \cdot \mathrm{d}\mathbf{S}, \tag{4.2}$$

where S is the surface area. Thus the acoustic impedance is

$$Z_\mathrm{a} = \hat{p}_\mathrm{av} / \hat{q}\ . \tag{4.3}$$

The unit is $\mathrm{kgm^{-4}s^{-1}}$. Since the total force acting on the surface equals the product of the average sound pressure and the area, and since $\hat{q} = S\hat{u}_n$ if the velocity is uniform, it can be seen that there is a simple relation between the two impedance concepts under such conditions:

$$Z_\mathrm{m} = Z_\mathrm{a} S^2. \tag{4.4}$$

This equation makes it possible to calculate the force it would take to drive a massless piston with the velocity $\hat{u}_n$. In other words, the acoustic impedance describes the load on a (real or fictive) piston caused by the medium. If the piston is real, the impedance is called the *radiation impedance*. This quantity is used for describing the load on, for example, a loudspeaker membrane caused by the motion of the medium.[30]

The concept of acoustic impedance is essentially associated with approximate low-frequency models. For example, it is a very good approximation to assume that the sound field in a tube is one-dimensional when the wavelength is long compared with the cross-sectional dimensions of the tube. Under such conditions the sound field can be described by eqs. (2.15) and (2.16), and a tube of a given length behaves as an acoustic two-port.[31] It is possible to calculate the transmission of sound through complicated systems of pipes using fairly simple considerations based the assumption of continuity of the sound pressure and the volume velocity at each junction [12].[32] The acoustic impedance is also useful in studying the properties of acoustic transducers. Such transducers are usually much smaller than the wavelength in a significant part of the frequency range. This makes it possible to employ so-called lumped parameter models where the system is described by an analogous electrical circuit composed of simple lumped element, inductors, resistors and capacitors, representing masses, losses and springs [13, 14]. Finally it should be mentioned that the acoustic impedance can be used for describing the acoustic properties of materials exposed to normal sound incidence.[33]

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[30] The load of the medium on a vibrating piston can be described either in terms of the acoustic radiation impedance (the ratio of the sound pressure to the volume velocity) or the mechanical radiation impedance (the ratio of the force to the velocity).

[31] 'Two-port' is a term from electric circuit theory denoting a network with two terminals. Such a network is completely described by the relations between four quantities, the voltage and current at the input terminal and the voltage and current at the output terminal. By analogy, an acoustic two-port is completely described by the relations between the sound pressures and the volume velocities at the two terminals. In case of a cylindrical tube such relations can easily be derived from eqs. (2.15) and (2.16) [12].

[32] Such systems act as acoustic filters. Silencers (or mufflers) are composed of coupled tubes.

[33] In the general case we need to describe the properties of acoustic materials with the local ratio of the sound pressure on the surface to the resulting vibrational velocity. In most literature this quantity, which is used mainly in theoretical work, is called the specific acoustic impedance. In many practical applications the properties of acoustic materials are described in terms of absorption coefficients (or absorption factors), assuming either normal or diffuse sound incidence (see chapter 5). It is possible to calculate the absorption coefficient of a material from its specific acoustic impedance, but not the impedance from the absorption coefficient.

Example 4.4

The acoustic input impedance of a tube terminated by a rigid cap can be deduced from eqs. (2.17) and (2.18) (with $x = -l$),

$$Z_{\mathrm{a}} = -\mathrm{j}\frac{\rho c}{S}\cot kl,$$

where l is the length of the tube and S is its cross-sectional area. Note that the impedance goes to infinity when l equals a multiple of half a wavelength, indicating that it would take an infinitely large force to drive a piston at the inlet of the tube at these frequencies (see figure 4.2). Conversely, the impedance is zero when l equals an odd-numbered multiple of a quarter of a wavelength; at these frequencies the sound pressure on a vibrating piston at the inlet of the tube would vanish. Cf. example 2.2.

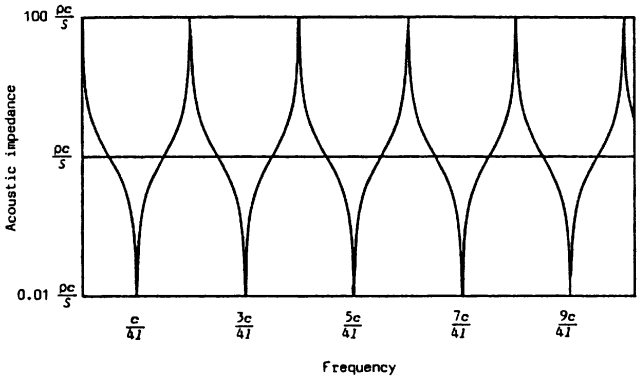

Figure 4.2 The acoustic input impedance of a tube terminated rigidly.

At low frequencies the acoustic impedance of the rigidly terminated tube analysed in example 4.4 can be simplified. The factor cotkl approaches $1/kl$, and the acoustic impedance becomes

$$Z_{\mathrm{a}} \simeq -\mathrm{j}\frac{\rho c}{Slk} = \frac{\rho c^2}{\mathrm{j}\omega V}, \tag{4.5}$$

where $V = Sl$ is the volume of the tube, indicating that the air in the tube acts as a spring. Thus the acoustic impedance of a cavity much smaller than the wavelength is spring-like, with a stiffness that is inversely proportional to the volume and independent of the shape of the cavity. Since, from eq. (2.2b),

$$\rho c^2 = \gamma p_0, \tag{4.6}$$

it can be seen that the acoustic impedance of a cavity at low frequencies also can be written

$$Z_{\mathrm{a}} = \frac{\gamma p_0}{\mathrm{j}\omega V}, \tag{4.7}$$

in agreement with the considerations on p. 7.

Example 4.5

A Helmholtz resonator is the acoustic analogue to the simple mechanical oscillator described in example 4.3; see figure 4.3. The dimensions of the cavity are much smaller that the wavelength; therefore it behaves as a spring with the acoustic impedance

$$Z_a = \frac{\rho c^2}{j\omega V},$$

where V is the volume; cf. eq. (4.5). The air in the neck moves back and forth uniformly as if it were incompressible; therefore the air in the neck behaves as a lumped mass with the mechanical impedance

$$Z_m = j\omega\rho S l_{eff},$$

where l_{eff} is the effective length and S is the cross-sectional area of the neck. (The effective length of the neck is somewhat longer than the physical length, because some of the air just outside the neck is moving along with the air in the neck.) The corresponding acoustic impedance follows from eq. (4.4):

$$Z_a = \frac{j\omega\rho l_{eff}}{S}.$$

By analogy with example 4.3 we conclude that the angular resonance frequency is

$$\omega_0 = c\sqrt{\frac{S}{V l_{eff}}}.$$

Note that the resonance frequency is independent of the density of the medium.

It is intuitively clear that a larger volume or a longer neck would correspond to a lower frequency, but it is perhaps less obvious that a smaller neck area gives a lower frequency.

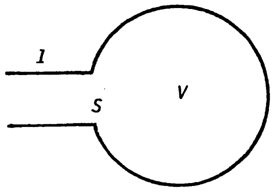

Figure 4.3 A Helmholtz resonator.

Yet another impedance concept, the characteristic impedance, has already been introduced. As we have seen in section 2.1, the complex ratio of the sound pressure to the particle velocity in a plane propagating wave equals the characteristic impedance of the medium (cf. eq. (2.14)), and it approximates this value in a free field far from the source (cf. eq. (2.27)). Thus, the characteristic impedance describes a property of the medium, as we have seen on p. 16. The unit is $\mathrm{kgm^{-2}s^{-1}}$.

5. SOUND ENERGY, SOUND INTENSITY, SOUND POWER AND SOUND ABSORPTION

The most important quantity for describing a sound field is the sound pressure. However, sources of sound emit sound power, and sound fields are also energy fields in which potential and kinetic energies are generated, transmitted and dissipated. Some typical sound power levels are given in table 5.1.

It is apparent that the radiated sound power is a negligible part of the energy conversion of almost any source. However, energy considerations are nevertheless of great practical importance in acoustics. The usefulness is due to the fact that a statistical approach where the energy of the sound field is considered turns out to give very useful approximations in room acoustics and in noise control. In fact determining the sound power of sources is a central point in noise control engineering. The value and relevance of knowing the sound power radiated by a source is due to the fact that this quantity is largely independent of the surroundings of the source in the audible frequency range.

Table 5.1 Typical sound power levels.

Aircraft turbojet engine	10 kW	160 dB
Gas turbine (1 MW)	32 W	135 dB
Small airplane	5 W	127 dB
Tractor (150 hp)	100 mW	110 dB
Large electric motor (0.5 MW)	10 mW	100 dB
Vacuum cleaner	100 µW	80 dB
Office machine	32 µW	75 dB
Speech	10 µW	70 dB
Whisper	10 nW	40 dB

5.1 The energy in a sound field

It can be shown that the instantaneous potential energy density at a given position in a sound field (the potential sound energy per unit volume) is given by the expression

$$w_{\mathrm{pot}}(t) = \frac{p^2(t)}{2\rho c^2}. \tag{5.1}$$

This quantity describes the local energy stored per unit volume of the medium because of the compression or rarefaction; the phenomenon is analogous to the potential energy stored in a compressed or elongated spring, and the derivation is similar.

The instantaneous kinetic energy density at a given position in a sound field (the kinetic energy per unit volume) is

$$w_{\mathrm{kin}}(t) = \frac{1}{2}\rho u^2(t). \tag{5.2}$$

This quantity describes the energy per unit volume at the given position represented by the mass of the particles of the medium moving with the velocity u. This corresponds to the kinetic energy of a moving mass, and the derivation is similar.

The instantaneous sound intensity at a given position is the product of the instantaneous sound pressure and the instantaneous particle velocity,

$$\mathbf{I}(t) = p(t)\mathbf{u}(t). \tag{5.3}$$

This quantity, which is a vector, expresses the magnitude and direction of the instantaneous flow of sound energy per unit area at the given position, or the work done by the sound wave per unit area of an imaginary surface perpendicular to the vector.

Energy conservation

By combining the fundamental equations that govern a sound field (the conservation of mass, the relation between the sound pressure and density changes, and Euler's equation of motion), one can derive the equation

$$\nabla \cdot \mathbf{I}(t) = -\frac{\partial w(t)}{\partial t},$$

where $\nabla \cdot \mathbf{I}(t)$ is the divergence of the instantaneous sound intensity and $w(t)$ is the sum of the potential and kinetic energy densities. This is the equation of conservation of sound energy, which expresses the simple fact that the rate of change of the total sound energy at a given point in a sound field is equal to the flow of converging sound energy; if the sound energy density at the point increases there must be a net flow of energy towards the point, and if it decreases there must be net flow of energy diverging away from the point.

The global version of this equation is obtained using Gauss's theorem,[34]

$$\int_V \nabla \cdot \mathbf{I}(t)\mathrm{d}V = \int_S \mathbf{I}(t) \cdot \mathrm{d}\mathbf{S} = -\frac{\partial}{\partial t}\left(\int_V w(\mathrm{t})\mathrm{d}V\right) = -\frac{\partial E(t)}{\partial t},$$

where S is the area of an arbitrary, closed surface, V is the volume inside the surface, and $E(t)$ is the total instantaneous sound energy within the surface. This equation shows that the rate of change of the total sound energy within a closed surface is identical with the surface integral of the normal component of the instantaneous sound intensity.

In practice the time-averaged energy densities,

$$w_{\text{pot}} = \frac{p_{\text{rms}}^2}{2\rho c^2}, \qquad w_{\text{kin}} = \frac{1}{2}\rho u_{\text{rms}}^2, \tag{5.4a, 5.4b}$$

are more important than the instantaneous quantities, and the time-averaged sound intensity (which is usually referred to just as the 'sound intensity'),

$$\mathbf{I} = \overline{\mathbf{I}(t)} = \overline{p(t)\mathbf{u}(t)}, \tag{5.5}$$

is more important than the instantaneous intensity $\mathbf{I}(t)$. Energy conservation considerations lead to the conclusion that the integral of the normal component of the sound intensity over a closed surface is zero,

$$\int_S \mathbf{I} \cdot \mathrm{d}\,\mathbf{S} = 0 \tag{5.6}$$

in any sound field unless there is generation or dissipation of sound power within the surface S. If, on the other hand, the surface encloses a source the integral equals the radiated sound power of the source, irrespective of the presence of other sources of noise outside the surface:

$$\int_S \mathbf{I} \cdot \mathrm{d}\,\mathbf{S} = P_{\text{a}} \tag{5.7}$$

Often we will be concerned with harmonic signals and make use of complex notation, as in chapters 2 and 4. Expressed in the complex notation eqs. (5.4) and (5.5) become

$$w_{\text{pot}} = \frac{|\hat{p}|^2}{4\rho c^2}, \qquad w_{\text{kin}} = \frac{1}{4}\rho|\hat{u}|^2, \tag{5.8a, 5.8b}$$

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[34] According to Gauss's theorem the volume integral of the divergence of a vector equals the corresponding surface integral of the (outward pointing) normal component of the vector.

$$\mathbf{I} = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{\mathbf{u}}^*\right\}. \tag{5.9}$$

(Note that the two complex exponentials describing the time dependence of the sound pressure and the particle velocity cancel each other because one of them is conjugated; see the Appendix.) The component of the sound intensity in the x-direction is

$$I_x = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{u}_x^*\right\}. \tag{5.10}$$

Inserting the expressions for the sound pressure (eq. (2.13)) and the particle velocity (eqs. (2.14)) in a plane propagating wave into eq. (5.10) shows that

$$I_x = \frac{|\hat{p}|^2}{2\rho c} = \frac{p_{\mathrm{rms}}^2}{\rho c} \tag{5.11}$$

in this particular sound field. Moreover, inserting expressions for the sound pressure and the particle velocity in a simple spherical wave, eqs. (1.2.26) and (1.2.27), into eq. (1.5.10) gives the same relation for the radial sound intensity:

$$I_r = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{u}_r^*\right\} = \mathrm{Re}\left\{\frac{A\,\mathrm{e}^{\mathrm{j}(\omega t-kr)}}{r}\frac{A^*\,\mathrm{e}^{-\mathrm{j}(\omega t-kr)}}{\rho c r}\left(1-\frac{1}{\mathrm{j}kr}\right)\right\} = \frac{|A|^2}{2\rho c r^2} = \frac{|\hat{p}|^2}{2\rho c} = \frac{p_{\mathrm{rms}}^2}{\rho c}. \tag{5.12}$$

It is apparent that there is a simple relation between the sound intensity and the mean square sound pressure in these two extremely important cases.[35] However, it should be emphasised that in the general case eq. (5.11) is **not** valid, and one will have to measure both the sound pressure and the particle velocity simultaneously and average the instantaneous product over time in order to measure the sound intensity. Equipment for such measurements has been commercially available since the early 1980s [3].

Example 5.1

It follows from eq. (5.11) that the sound intensity in a plane propagating wave with an rms sound pressure of 1 Pa is $(1\ \mathrm{Pa})^2/(1.2\,\mathrm{kgm}^{-3}\cdot 343\,\mathrm{ms}^{-1}) \simeq 2.4\ \mathrm{mW/m}^2$.

Example 5.2

The sound intensity in the interference field generated by a plane sound wave reflected from a rigid surface at normal incidence can be determined by inserting eqs. (2.17) and (2.18) into eq. (5.10):

$$I_x = \frac{1}{2}\mathrm{Re}\left\{2p_{\mathrm{i}}\cos kx\frac{\mathrm{j}2p_{\mathrm{i}}^*}{\rho c}\sin kx\right\} = \mathrm{Re}\left\{\frac{2\mathrm{j}|p_{\mathrm{i}}|^2}{\rho c}\sin 2kx\right\} = 0.$$

This result shows that there is no net flow of sound energy towards the rigid surface.

Under conditions where the sound pressure and the particle velocity are constant over a surface in phase as well as in amplitude we can write

---

[35] Eq. (5.11) implies that the sound intensity level is almost identical with the sound pressure level in air at 20°C and 101.3 kPa:

$$L_I = 10\log\left(I/I_{\mathrm{ref}}\right) = 10\log\left(p_{\mathrm{rms}}^2/(\rho c)/I_{\mathrm{ref}}\right) = 10\log\left(p_{\mathrm{rms}}^2/p_{\mathrm{ref}}^2\right) - 10\log\left(\rho c I_{\mathrm{ref}}/p_{\mathrm{ref}}^2\right) \simeq L_p - 0.14\ \mathrm{dB} \simeq L_p.$$

$$\hat{p} = \hat{q} Z_{\mathrm{a}} \tag{5.13}$$

(cf. eq. (4.4)), and the sound power passing through the surface can be expressed in terms of the acoustic impedance:

$$P_{\mathrm{a}} = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{q}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{|\hat{q}|^2 Z_{\mathrm{a}}\right\} = \frac{|\hat{q}|^2}{2}\mathrm{Re}\left\{Z_{\mathrm{a}}\right\}. \tag{5.14}$$

This expression demonstrates that the radiated sound power of a vibrating surface is closely related to the volume velocity and to the real part of the radiation impedance.

Equation (5.7) implies that one can determine the sound power radiated by a source by integrating the normal component of the sound intensity over a surface that encloses the source. This is the *sound intensity method* of measuring sound power. Note that special equipment for such measurements is required.

In an environment without reflecting surfaces the sound field generated by a source of finite extent is locally plane far from the source, as mentioned in section 2.2, and therefore the local sound intensity is to a good approximation given by eq. (5.11). With eq. (5.7) we now conclude that one can estimate the radiated sound power of a source by integrating the mean square pressure generated by the source over a spherical surface centred at the source:

$$P_{\mathrm{a}} = \int_S \left(p_{\mathrm{rms}}^2/(\rho c)\right)\mathrm{d}S. \tag{5.15}$$

However, whereas eq. (5.7) is valid even in the presence of sources outside the measurement surface eq. (5.11) is not; therefore only the source under test must be present. In practice one measures the sound pressure at a finite number of discrete points. This is the *free-field method* of measuring sound power. Note that an anechoic room (a room without any reflecting surfaces) is required.

Yet another method of measuring sound power requires a diffuse sound field in a reverberation room.

5.2 Sound absorption

Most materials absorb sound. As we have seen in chapter 2 we need a precise description of the boundary conditions for solving the wave equation, which leads to a description of material properties in terms of the specific acoustic impedance, as mentioned in chapter 4. However, in many practical applications, for example in architectural acoustics, a simpler measure of the acoustic properties of materials, the absorption coefficient (or absorption factor), is more useful. By definition the absorption coefficient of a given material is the absorbed fraction of the incident sound power. From this definition it follows that the absorption coefficient takes values between naught and unity. A value of unity implies that all the incident sound power is absorbed.

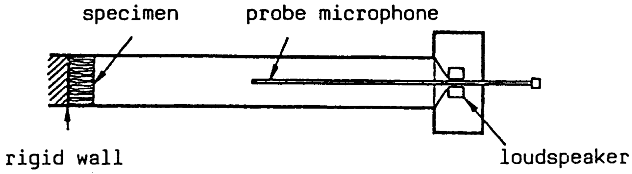

Figure 5.1 A standing wave tube for measuring the normal incidence absorption coefficient. (From ref. [15].)

In general the absorption coefficient of a given material depends on the structure of the sound field (plane wave incidence of a given angle of incidence, for example, or random or diffuse incidence in a room). Here we will study only the absorption for plane waves of normal incidence.

Consider the sound field in a tube driven by a loudspeaker at one end and terminated by the material under test at the other end, as sketched in figure 5.1. This is a one-dimensional field, which means that it has the general form given by eqs. (2.15) and (2.16). The amplitudes p_i and p_r depend on the boundary conditions, that is, the vibrational velocity of the loudspeaker and the properties of the material at the end of the tube. The sound intensity is obtained by inserting eqs. (2.15) and (2.16) into eq. (5.10):

$$\begin{aligned} I_x &= \mathrm{Re}\left\{\left(p_\mathrm{i}\,\mathrm{e}^{-\mathrm{j}kx} + p_\mathrm{r}\,\mathrm{e}^{\mathrm{j}kx}\right)\frac{\left(p_\mathrm{i}^*\,\mathrm{e}^{\mathrm{j}kx} - p_\mathrm{r}^*\,\mathrm{e}^{-\mathrm{j}kx}\right)}{2\rho c}\right\} = \frac{|p_\mathrm{i}|^2 - |p_\mathrm{r}|^2}{2\rho c} \\ &= \frac{\left(|p_\mathrm{i}|+|p_\mathrm{r}|\right)\left(|p_\mathrm{i}|-|p_\mathrm{r}|\right)}{2\rho c} = \frac{p_\mathrm{max}\,p_\mathrm{min}}{2\rho c}, \end{aligned} \tag{5.16}$$

where the last equation sign follows from eq. (2.20). (Note that p_max and p_min are amplitudes.) The *incident* sound intensity is the value associated with the incident wave, that is,

$$I_\mathrm{inc} = \frac{|p_\mathrm{i}|^2}{2\rho c}. \tag{5.17}$$

The absorption coefficient is the ratio of I_x to I_inc,

$$\alpha = \frac{I_x}{I_\mathrm{inc}} = \frac{|p_\mathrm{i}|^2 - |p_\mathrm{r}|^2}{|p_\mathrm{i}|^2} = 1 - |R|^2 = 1 - \left(\frac{s-1}{s+1}\right)^2 = \frac{4s}{(1+s)^2}, \tag{5.18}$$

where we have introduced the reflection factor and the standing wave ratio (cf. eqs. (2.19) and (2.22)). Note that the absorption coefficient is independent of the phase angle of R, which shows that there is more information in the complex reflection factor than in the absorption coefficient. Equation (5.18) demonstrates that one can determine the normal incidence absorption coefficient of a material by exposing it to normal sound incidence in a tube and measuring the standing wave ratio of the resulting interference field.

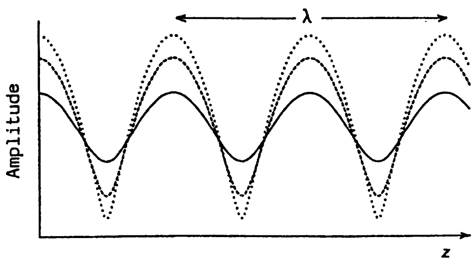

Figure 5.2 Standing wave patterns for various values of the absorption coefficient: 0.9 (——); 0.6 (– –); 0.3 (···).

Example 5.3

If the material under test is completely reflecting then $|R| = 1$, corresponding to an absorption coefficient of zero. In this case the standing wave ratio is infinitely large. If the material is completely absorbing, $R = 0$, corresponding to an absorption coefficient of unity. In the latter case there is no reflected wave, so the sound pressure amplitude is constant in the tube, corresponding to a standing wave ratio of one.

6. RADIATION OF SOUND

Sound can be generated by many different mechanisms. In this note we will study only the simplest one, which is also the most important: that of a solid vibrating surface. As we shall see, the most efficient mechanism for radiation of sound involves a net volume displacement.

6.1 Point sources

The simplest source to describe mathematically is a pulsating sphere, that is, a sphere that expands and contracts harmonically with spherical symmetry as the one shown in figure 2.1. In free space such a source generates the simple spherical sound field we studied in section 2.2. Say the source has a radius of a. From eq. (2.27) we know that the particle velocity on the surface of the source is

$$\hat{u}_r(a) = \frac{A}{\rho c}\frac{\mathrm{e}^{\mathrm{j}(\omega t - ka)}}{a}\left(1 + \frac{1}{\mathrm{j}ka}\right). \tag{6.1}$$

The boundary condition on the surface implies that the vibrational velocity $U\mathrm{e}^{\mathrm{j}\omega t}$ must equal the normal component of the particle velocity; therefore

$$A = \frac{\mathrm{j}\rho cka^2U\mathrm{e}^{\mathrm{j}ka}}{1+\mathrm{j}ka} = \frac{\mathrm{j}\rho\omega Q\mathrm{e}^{\mathrm{j}ka}}{4\pi\left(1+\mathrm{j}ka\right)}, \tag{6.2}$$

where we have introduced the volume velocity of the pulsating sphere,

$$Q = 4\pi a^2 U, \tag{6.3}$$

by multiplying with the surface area of the sphere. Inserting into eq. (2.26) gives an expression for the sound pressure generated by the source,

$$\hat{p} = \frac{\mathrm{j}\rho\omega Q\mathrm{e}^{\mathrm{j}(\omega t - k(r-a))}}{4\pi r\left(1+\mathrm{j}ka\right)}. \tag{6.4}$$

We can now calculate the radiation impedance of the pulsating sphere. This is the ratio of the sound pressure on the surface of the sphere to the volume velocity (cf. eq. (4.3)):

$$Z_{\mathrm{a,r}} = \frac{\hat{p}(a)}{Q\mathrm{e}^{\mathrm{j}\omega t}} = \frac{\mathrm{j}\rho\omega}{4\pi a\left(1+\mathrm{j}ka\right)} \simeq \frac{\rho ck^2}{4\pi} + \frac{\mathrm{j}\omega\rho}{4\pi a}, \tag{6.5}$$

where the approximation to the right is based on the assumption that $ka \ll 1$. Note that the imaginary part of the radiation impedance is much larger than the real part at low frequencies, indicating that most of the force it takes to expand and contract the sphere goes to moving the mass of the air in a region near the sphere (cf. example 4.1). This air moves back and forth almost as if it were incompressible.

In the limit of a vanishingly small sphere the source becomes a *monopole*, also known as a *point source* or a *simple source*. With $ka \ll 1$, the expression for the sound pressure generated by a point source with the volume velocity $Q\mathrm{e}^{\mathrm{j}\omega t}$ becomes

$$\hat{p} = \frac{\mathrm{j}\rho\omega Q\mathrm{e}^{\mathrm{j}(\omega t - kr)}}{4\pi r}. \tag{6.6}$$

A vanishingly small sphere with a finite volume velocity[36] may seem to be a rather academic source. However, the monopole is a central concept in theoretical acoustics. At low frequencies it is a good approximation to any source that produces a net displacement of volume, that is, any source that is small compared with the wavelength and changes its volume as a function of time, irrespective of its shape and the way it vibrates. An enclosed loudspeaker is to a good approximation a monopole at low frequencies. A source that injects fluid, the outlet of an engine exhaust system, for example, is also in effect a monopole.

The sound intensity generated by the monopole can be determined from eq. (5.10):

$$I_r = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{u}_r^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\frac{\mathrm{j}\rho\omega Q}{4\pi r}\frac{-\mathrm{j}\rho\omega Q^*}{4\pi r\rho c}\left(1-\frac{1}{\mathrm{j}kr}\right)\right\} = \frac{\left(\rho\omega|Q|\right)^2}{32\pi^2 r^2\rho c}. \tag{6.7}$$

By multiplying with the surface of the area of a sphere with the radius r we get the sound power radiated by the monopole,

$$P_\mathrm{a} = \frac{\left(\rho\omega|Q|\right)^2}{32\pi^2 r^2\rho c}4\pi r^2 = \frac{\rho c k^2|Q|^2}{8\pi}. \tag{6.8}$$

We could also obtain this result from eqs. (5.14) and (6.5), of course. Note that the sound power is proportional to the square of the frequency, indicating that a small pulsating sphere is not a very efficient radiator of sound at low frequencies.

Reciprocity

The *reciprocity principle* states that if a monopole source at a given point generates a certain sound pressure at a another point then the monopole would generate the same sound pressure if we interchange listener and source position, irrespective of the presence of reflecting or absorbing surfaces. This is a strong statement with many practical implications.

It is easy to take account of a large reflecting plane surface, say, at z = 0, if one makes use of the concept of *image sources*. If the surface is rigid the boundary condition implies that $u_z = 0$ at $z = 0$, and simple symmetry considerations show that this is satisfied if we replace the rigid plane with an image source; see figure 6.1. The resulting sound pressure is simply the sum of the sound pressures generated by the source and the image source,

$$\hat{p} = \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t-kR_1)}}{4\pi R_1} + \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t-kR_2)}}{4\pi R_2} = \frac{\mathrm{j}\rho\omega Q}{4\pi R_1}\mathrm{e}^{\mathrm{j}(\omega t-kR_1)}\left(1+\frac{R_1}{R_2}\mathrm{e}^{\mathrm{j}k(R_1-R_2)}\right). \tag{6.9}$$

The parenthesis shows the effect of the reflecting plane, that is, it represents the sound pressure normalised by the free field value. The normalised equation can be used for studying outdoor sound propagation over a hard surface, and it is common practice to present the 'ground effect', that is, the effect of reflections from the ground on outdoor sound propagation, in this form.

At very low frequencies $k(R_1 - R_2) \ll 1$, and the rigid surface can be seen to have the effect of increasing the sound pressure by a factor of $1+ R_1/R_2$. Destructive interference occurs when the second term in the parenthesis is real and negative, and the first interference

[36] The volume velocity of the monopole is sometimes referred to as the source strength. However, some authors use other definitions of the source strength. The term 'volume velocity' is unambiguous.

dip occurs when $k(R_1 - R_2) = \pi$, corresponding to $(R_1 - R_2)$ being half a wavelength. Figure 6.2 shows the sound pressure relative to free field for sound propagation over a rigid plane surface.

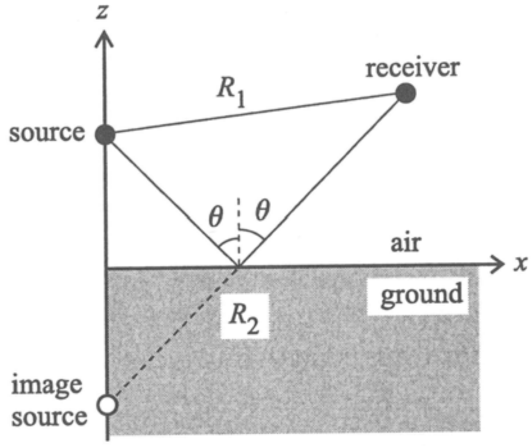

Figure 6.1 The sound pressure generated by a monopole above a rigid plane is the sum of two terms: direct sound and the contribution from the image source.(From ref. [16].)

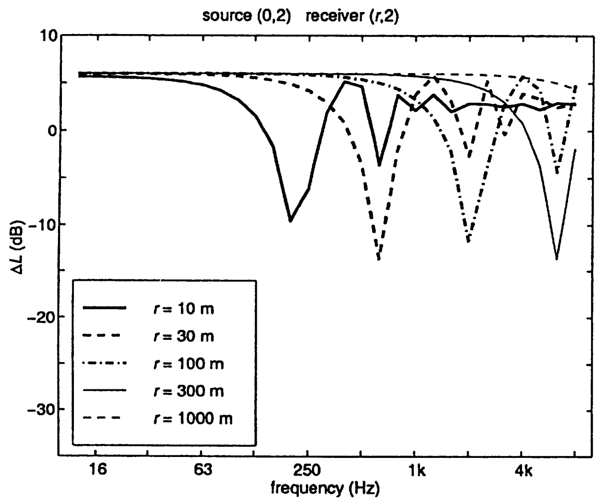

Figure 6.2 The sound pressure in one-third octave bands generated by a monopole above a rigid plane and shown relative to free field for five different source-receiver distances.(From ref. [16].)

If the distance between the source and the observation point is much longer than the distance between the source and the reflecting plane (see figure 1.6.3) we can make use of the *far-field approximation* and let $r_1 \simeq r_2 \simeq r$ in the denominator of eq, (1.6.6). However, the two contributions will arrive with a different phase no matter how far from the source we are. If the observation point is sufficiently far we can approximate the two distances by $r_1 \simeq r - h\cos\theta$ and $r_2 \simeq r + h\cos\theta$ in the complex exponentials. The resulting sound pressure now becomes

$$
\begin{aligned}
\hat{p} &= \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t - kr_1)}}{4\pi r_1} + \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t - kr_2)}}{4\pi r_2} \\
&\simeq \frac{\mathrm{j}\rho\omega Q}{4\pi r}\left(\mathrm{e}^{\mathrm{j}(\omega t - k(r - h\cos\theta))} + \mathrm{e}^{\mathrm{j}(\omega t - k(r + h\cos\theta))}\right) = \frac{\mathrm{j}\rho\omega Q}{2\pi r}\cos(kh\cos\theta)\,\mathrm{e}^{\mathrm{j}(\omega t - kr)} .
\end{aligned}
\qquad (6.10)
$$

Inspection of eq. (6.10) leads to the conclusion that the sound pressure in the far field depends on kh and on θ unless $kh << 1$, in which case the sound pressure is simply doubled.

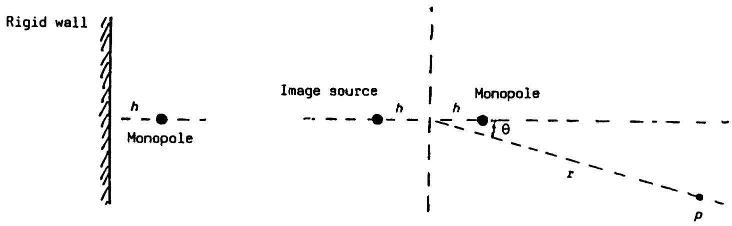

Figure 6.3 Far field sound pressure generated by a monopole near a rigid plane surface.

The sound power of the monopole is affected by the presence of the reflecting surface unless it is far away, $kh >> 1$. We can calculate the sound power by integrating the sound intensity over a hemisphere, cf. eq. (5.7). (Since the normal component of the particle velocity is zero at all points on the plane between the source and the image source, the normal component of the intensity is also zero, so this surface does not contribute to the integral.) Moreover, the considerations that lead to eq. (5.15) are also valid for combinations of sources. It follows that

$$\begin{aligned} P_{\mathrm{a}} &= \int_0^{\pi/2}\int_0^{2\pi} \frac{|\hat{p}|^2}{2\rho c} r^2 \sin\theta \mathrm{d}\varphi \mathrm{d}\theta = \frac{\rho c k^2 |Q|^2}{4\pi} \int_0^{\pi/2} \cos^2(kh\cos\theta)\sin\theta \mathrm{d}\theta \\ &= \frac{\rho c k^2 |Q|^2}{4\pi kh} \int_0^{kh} \cos^2 x \mathrm{d}x = \frac{\rho c k^2 |Q|^2}{8\pi}\left(1+\frac{\sin(2kh)}{2kh}\right). \end{aligned} \tag{6.11}$$

Figure 6.4 shows the factor in parentheses. It is apparent that the sound power is doubled if the source is very close to the surface, and that the rigid surface has an insignificant influence on the sound power output of the source when h exceeds a quarter of a wavelength, corresponding to $kh = \pi/2$.

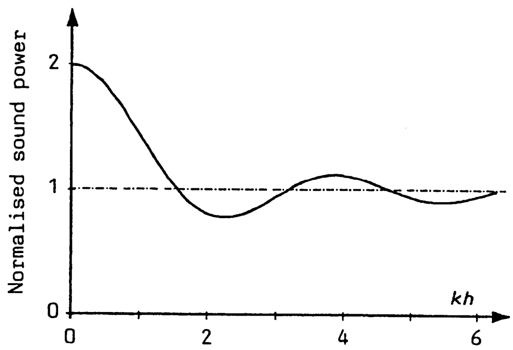

Figure 6.4 The influence of a rigid surface on the sound power of a monopole.

Example 6.1

It can be deduced from eq. (6.11) that two identical monopoles (two enclosed loudspeakers driven with the same signal, for example) when placed in close proximity at very low frequencies will radiate twice as much sound power as they do when they are far from each other. The physical explanation is that the radiation load on each source is doubled; the sound pressure on each source is not only generated by the source itself but also by the neighbouring source. Alternatively one might regard the two loudspeakers as one compound source with twice the volume velocity of each loudspeaker. Because of the quadratic relation between volume velocity and power (cf. eq. (6.8)) this compound source will radiate four times more sound power than one single loudspeaker in isolation.

Two monopoles of the same volume velocity but vibrating in antiphase constitute a *point dipole* if the distance between them is much less than the wavelength; see figure 6.5. It is clear that the combined source has no net volume velocity. A point dipole is a good approximation to a small vibrating body that does not change its volume as a function of time. Such a source exerts a force on the fluid. The oscillating sphere shown in figure 6.6, for example, is in effect a dipole, and so is an unenclosed loudspeaker unit. Other examples include vibrating beams and wires.

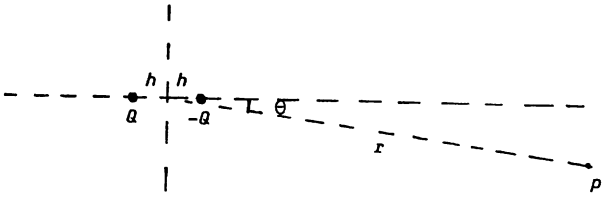

Figure 6.5 A point dipole.

The sound pressure generated by the two monopoles is

$$\hat{p} = \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t - kr_1)}}{4\pi r_1} - \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t - kr_2)}}{4\pi r_2}. \tag{6.12}$$

The near field of this combination of sources is fairly complicated. However, the far field is relatively simple. We can calculate the sound pressure in the far field in the same way we used in deriving eq. (6.10),

$$\begin{aligned}\hat{p} &\simeq \frac{\mathrm{j}\rho\omega Q}{4\pi r}\left(\mathrm{e}^{\mathrm{j}(\omega t - k(r + h\cos\theta))} - \mathrm{e}^{\mathrm{j}(\omega t - k(r - h\cos\theta))}\right) = \frac{\rho\omega Q}{2\pi r}\sin(kh\cos\theta)\,\mathrm{e}^{\mathrm{j}(\omega t - kr)} \\ &\simeq \frac{\rho c h k^2 Q}{2\pi r}\cos\theta\,\mathrm{e}^{\mathrm{j}(\omega t - kr)}.\end{aligned} \tag{6.13}$$

Note that the sound pressure is proportional to $h|Q|$, varies as $\cos\theta$ and is identically zero in the plane between the two monopoles.[37]

The sound power of the dipole is calculated by integrating the mean square sound pressure over a spherical surface centred midway between the two monopoles:

$$\begin{aligned}P_\mathrm{a} &= \int_0^{\pi}\int_0^{2\pi} \frac{|\hat{p}|^2}{2\rho c} r^2 \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{\rho c h^2 k^4 |Q|^2}{4\pi}\int_0^{\pi}\cos^2\theta\sin\theta\,\mathrm{d}\theta \\ &= \frac{\rho c h^2 k^4 |Q|^2}{4\pi}\int_{-1}^{1} x^2\,\mathrm{d}x = \frac{\rho c h^2 k^4 |Q|^2}{6\pi}.\end{aligned} \tag{6.14}$$

Note that the sound power of the dipole is proportional to the fourth power of the frequency, indicating very poor sound radiation at low frequencies. The physical explanation of the poor radiation efficiency of the dipole is of course that the two monopoles almost cancel each other.

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[37] The quantity *2hQ* is referred to by some authors as the dipole strength. However, other authors use other definitions.

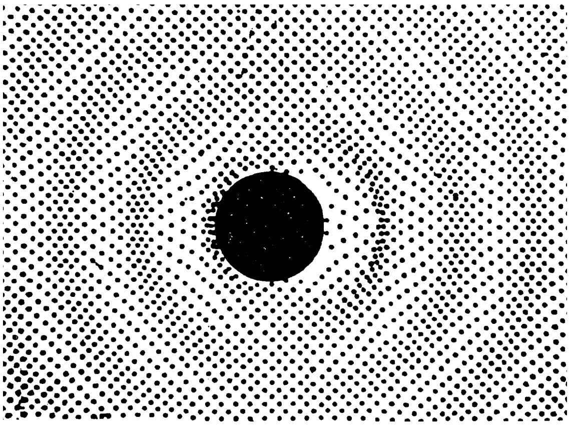

Figure 6.6 Fluid particles in the sound field generated by an oscillating sphere. (From ref. [1].)

6.2 Sound radiation from a circular piston in an infinite baffle

Apart from the pulsating sphere, a vibrating circular piston in an infinite, rigid baffle is one of the simplest cases of a spatially extended sound source that can be dealt with analytically. It is often used in connection with loudspeaker modelling.

The basic approach to extended sound sources is to consider them as composed of many simple sources, just as a dipole is made up of two monopoles. Thus, the piston is the sum of many monopoles that all radiate in phase. Because of the infinite baffle, each monopole gives rise to an image source that coincides with the monopole, cf. eqs. (6.9) and (6.10)); in other words, the baffle has the effect of doubling the volume velocity of each monopole. Let the piston vibrate with the velocity $U\mathrm{e}^{\mathrm{j}\omega t}$. It follows that the volume velocity of each elementary monopole is $U\mathrm{d}S$. By linear superposition we conclude that the sound pressure radiated by the piston can be evaluated at any position in front of the baffle simply by integrating over the surface of the piston,

$$\hat{p} = \mathrm{j}\omega\rho\int_S \frac{\mathrm{e}^{\mathrm{j}(\omega t - kh)}}{2\pi h} U\mathrm{d}S\,, \tag{6.15}$$

where h is the distance between the observation point and the running position on the piston, and S is the surface of the piston of radius a (see figure 6.7).This is a special case of what is known as Rayleigh's integral, which can be used for computing the sound radiation into half space of any plane infinite surface with a given vibrational velocity [17]. Note the factor of two in the denominator instead of four for the monopole (cf. Eq. (6.6)), which is due to the contribution of the image sources.

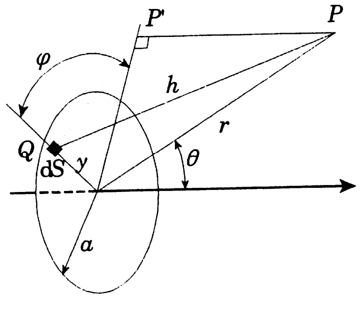

Figure 6.7 Definition of the variables. (From ref. [18].)

The far field sound pressure, that is, the sound pressure at long distances from the centre of the piston compared with the radius and the wavelength, can be calculated by expanding h in the complex exponential,

$$h = \sqrt{r^2 + y^2 - 2ry\sin\theta\cos\varphi} \simeq r\sqrt{1 - 2\frac{y}{r}\sin\theta\cos\varphi} \simeq r - y\sin\theta\cos\varphi, \tag{6.16}$$

while retaining only the first term of eq. (6.16) in the denominator (cf. eq. (6.10)). Thus the expression for the sound pressure becomes

$$\hat{p}(r,\theta) \simeq \frac{\mathrm{j}\omega\rho U}{2\pi r} \mathrm{e}^{\mathrm{j}(\omega t - kr)} \int_0^{2\pi} \int_0^{a} \mathrm{e}^{\mathrm{j}ky\sin\theta\cos\varphi} y\,\mathrm{d}y\,\mathrm{d}\varphi. \tag{6.17}$$

The calculation makes use of the Bessel functions $\mathrm{J}_0(z)$ and $\mathrm{J}_1(z)$, defined by

$$\mathrm{J}_0(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{j}z\cos\beta} \mathrm{d}\beta \tag{6.18}$$

and

$$\mathrm{J}_1(z) = \frac{1}{z} \int_0^{z} \beta\, \mathrm{J}_0(\beta) \mathrm{d}\beta \tag{6.19}$$

(see figure 6.8), and leads to the following expression for the far field sound pressure,

$$\hat{p}(r,\theta) = \frac{\mathrm{j}\omega\rho U}{r} \frac{a\,\mathrm{J}_1(ka\sin\theta)}{k\sin\theta} \mathrm{e}^{\mathrm{j}(\omega t - kr)} = \frac{\mathrm{j}\omega\rho Q}{2\pi r} \left[\frac{2\,\mathrm{J}_1(ka\sin\theta)}{ka\sin\theta} \right] \mathrm{e}^{\mathrm{j}(\omega t - kr)}, \tag{6.20}$$

where we have introduced the volume velocity of the piston, $Q = \pi a^2 U$. The factor in brackets is called the directivity of the piston, which is a frequency dependent function that describes the directional characteristics of the source in the far field,

$$D(f,\theta) = \left[\frac{2\,\mathrm{J}_1(ka\sin\theta)}{ka\sin\theta} \right]. \tag{6.21}$$

This function has its maximum value, unity, when $\theta = 0$, indicating maximum radiation in the axial direction all frequencies. Figure 6.9 shows the directivity for different values of the normalised frequency ka. Note that the piston is an omnidirectional source (a monopole placed on a rigid surface) at low frequencies, just as one would expect. At high frequencies the radiation of the piston is concentrated in a beam near the axial direction.

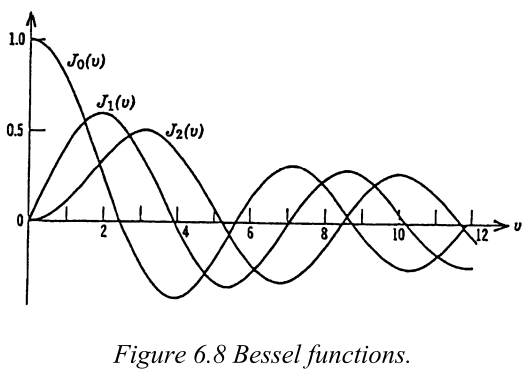

Figure 6.8 Bessel functions.

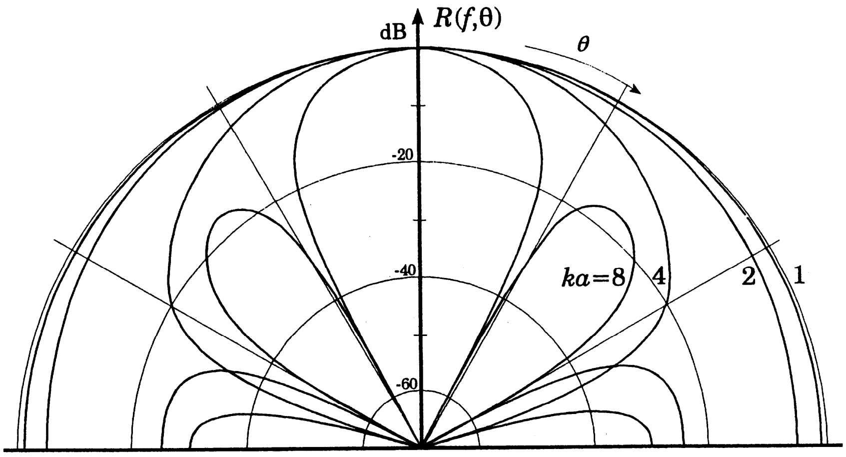

Figure 6.9 Directivity of the piston as a function of the normalised frequency ka. (From ref. [18].)

The sound pressure on the axis of the piston can be evaluated fairly easily. Since sin θ = 0 on the axis, the expression for the distance h reduces to

$$h = \sqrt{r^2 + y^2}, \tag{6.22}$$

from which,

$$\mathrm{d}h = \frac{y\mathrm{d}y}{\sqrt{r^2 + y^2}} = \frac{y}{h}\mathrm{d}y. \tag{6.23}$$

Thus the sound pressure on the axis is given by

$$\hat{p} = \frac{\mathrm{j}\omega\rho U}{2\pi}\mathrm{e}^{\mathrm{j}\omega t}\int_0^{2\pi}\int_r^{\sqrt{r^2+a^2}}\mathrm{e}^{-\mathrm{j}kh}\mathrm{d}h\mathrm{d}\varphi = \rho c U\mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{-\mathrm{j}kr} - \mathrm{e}^{-\mathrm{j}k\sqrt{r^2+a^2}}\right]. \tag{6.24}$$

If we introduce the quantity

$$\Delta = \left(\sqrt{r^2 + a^2} - r\right)/2, \tag{6.25}$$

the sound pressure can be written

$$\hat{p} = 2\mathrm{j}\rho c U\,\mathrm{e}^{\mathrm{j}(\omega t - k[r+\Delta])}\sin(k\Delta). \tag{6.26}$$

It can be seen that the sound pressure is zero when $k\Delta$ is a multiple of π, that is, when Δ is a multiple of half a wavelength, corresponding to the positions

$$r = a\left[\frac{1}{2n}\frac{a}{\lambda} - \frac{n}{2}\frac{\lambda}{a}\right] \tag{6.27}$$

on the axis, where n is a positive integer. In a similar way, the sound pressure assumes a maximum value for

$$2\Delta = \sqrt{r^2 + a^2} - r = (2m+1)\lambda/2 \tag{6.28}$$

(where m is a positive integer), that is, for

$$r = a\left[\frac{1}{2m+1}\frac{a}{\lambda} - \frac{2m+1}{4}\frac{\lambda}{a}\right]. \tag{6.29}$$

Figure 6.10 shows the normalised sound pressure on the axis of the piston as a function of the distance, which for a given frequency is defined by the corresponding *ka*-factor.

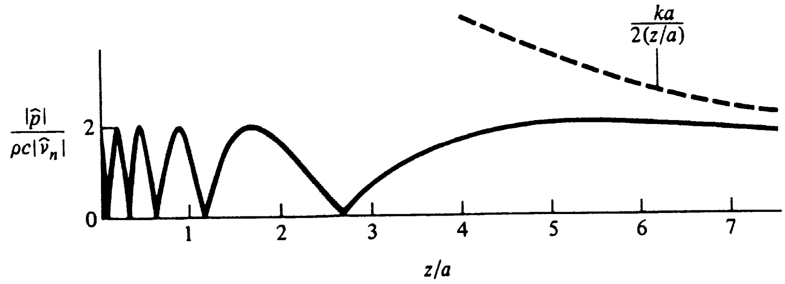

Figure 6.10 Sound pressure on the axis of a baffled piston for ka/2π = 5.5. (From ref. [19].)

It may seem surprising that the sound pressure is zero at some positions right in front of the vibrating piston. The phenomenon is due to destructive interference, caused by the fact that the distance from such a position to the various parts of the piston varies in such a manner that the contributions cancel out.

Example 6.2

In the far field, when $r >> a$ and $r >> a^2/\lambda$, one obtains

$$\Delta \simeq \frac{1}{2}\left[r\left(1+\frac{a^2}{2r^2}\right)-r\right] = \frac{a^2}{4r},$$

and the sound pressure reduces to

$$\hat{p} = \mathrm{j}\rho c U\left(\frac{ka^2}{2r}\right)\mathrm{e}^{\mathrm{j}(\omega t - kr)} = \frac{\mathrm{j}\rho c k Q}{2\pi r}\mathrm{e}^{\mathrm{j}(\omega t - kr)}.$$

This expression agrees with eq. (6.20) for $\theta = 0$ ($D(f) = 1$), as of course it should. This asymptotic expression is plotted as a dashed line in figure 6.10.

Example 6.3

The distances at which the minima occur, normalised by the radius of the piston, are given in terms of normalised frequencies by

$$\frac{r}{a} = \left[\frac{ka}{4\pi n} - \frac{\pi n}{ka}\right].$$

Minima of order n only occur for $ka \geq 2\pi n > 6$. Thus for a loudspeaker with a radius of 50 mm, minima only occur at frequencies higher than 6900 Hz, that is, far above the frequencies at which the piston approximation is valid. It follows that the minima are never observed in front of loudspeakers in real life.

In the near field there is no possible approximation except on the axis. However, by developing the spherical monopole field in cylindrical coordinates, the *force* exerted on the piston can be calculated analytically. The calculations are rather complicated (see ref. [19] or

[20] for a complete treatment), and lead to an expression in terms of special functions such as Bessel and Struve functions. The result is,

$$\hat{F} = \int_S \hat{p}\mathrm{d}S = \rho c \pi a^2 U \mathrm{e}^{\mathrm{j}\omega t}\left[1 - \frac{\mathrm{J}_1(2ka)}{ka} + \mathrm{j}\frac{\mathrm{H}_1(2ka)}{ka}\right], \tag{6.30}$$

where H_1 is the first Struve function.

The radiation impedance is the impedance seen by the piston, that is, the ratio of the average sound pressure to the volume velocity,

$$Z_{\mathrm{a,r}} = \frac{\hat{p}_{\mathrm{av}}}{Q\mathrm{e}^{\mathrm{j}\omega t}} = \frac{\hat{F}}{SQ\mathrm{e}^{\mathrm{j}\omega t}} \tag{6.31}$$

(cf. eq. (4.3)). Combining eqs.(6.30) and (6.31) gives

$$Z_{\mathrm{a,r}} = \frac{\rho c}{\pi a^2}\left[1 - \frac{\mathrm{J}_1(2ka)}{ka} + \mathrm{j}\frac{\mathrm{H}_1(2ka)}{ka}\right]. \tag{6.32}$$

Figure 6.11 shows the normalised, dimensionless radiation impedance (the bracket in eq. (6.32)),

$$\frac{Z_{\mathrm{a,r}}\pi a^2}{\rho c} = R_1 + \mathrm{j}X_1. \tag{6.33}$$

At low frequencies and at high frequencies the radiation impedance takes simple expressions:

$$ka << 1 \qquad Z_{\mathrm{a,r}} = r_{\mathrm{a,r}} + \mathrm{j}\omega m_{\mathrm{a,r}} = \frac{1}{2\pi}\rho c k^2 + \mathrm{j}\omega\rho\frac{8}{3a\pi^2}, \tag{6.34a}$$

$$ka >> 1 \qquad Z_{\mathrm{a,r}} = \frac{\rho c}{\pi a^2} + \mathrm{j}\omega\rho\frac{2}{\pi^2 k^2 a^3} = \frac{\rho c}{\pi a^2}\left(1 + \mathrm{j}\frac{4/\pi}{2ka}\right). \tag{6.34b}$$

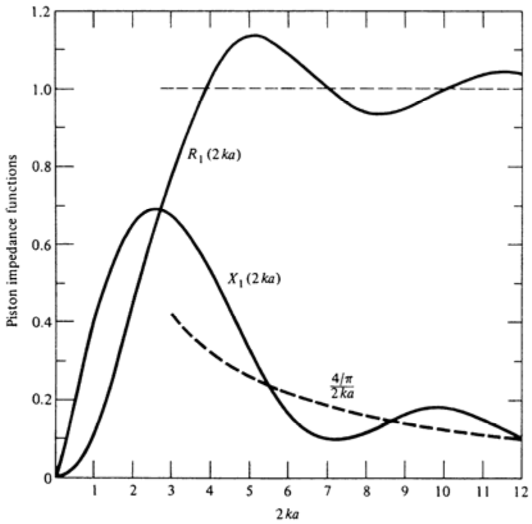

Figure 6.11 Radiation impedance of a piston as a function of the normalised frequency. (From ref. [19].)

The first expression is fundamental for designing loudspeakers. Note that the real part of the radiation impedance equals that of a small pulsating sphere, eq. (6.5), multiplied by a factor of two because of the rigid plane. The quantity $m_{a,r}$ can be interpreted as the acoustic mass of the air driven along by the piston. Interference effects in the near field make it different from the imaginary part of impedance of the pulsating sphere. However, as in the case of the pulsating sphere (eq. (6.5)), the imaginary part of the acoustic radiation impedance diverges when the radius a goes to zero.

Example 6.4

The mechanical radiation impedance is given by eqs. (4.4) and (6.33) as $Z_{m,r} = \rho c\pi a^2(R_1+jX_1)$. Its low frequency approximation is therefore:

$$Z_{\mathrm{m,r}} = \frac{\pi a^4 \rho c k^2}{2} + \mathrm{j}\omega\rho\frac{8a^3}{3}.$$

The imaginary part of this impedance is the impedance of the mass of a layer of air in front of the piston. This layer of air is moving back and forth as if it were incompressible.

The radiated sound power is defined in chapter 5 as the integral of the normal component of the sound intensity over a surface than encloses the source. This method can also be used for computing the sound power of a piston in an infinite baffle. However, by far the simplest approach is to use eq. (5.14), which expresses the sound power in terms of the mean square volume velocity and the real part of the acoustic radiation impedance:

$$P_{\mathrm{a}} = \tfrac{1}{2}|Q|^2 \,\mathrm{Re}\{Z_{\mathrm{a,r}}\} = \tfrac{1}{2}|Q|^2 \frac{\rho c}{\pi a^2} R_1 = \tfrac{1}{2}|Q|^2 \frac{\rho c}{\pi a^2}\left[1 - \frac{\mathrm{J}_1(2ka)}{ka}\right]. \tag{6.35}$$

At low frequencies this becomes, with eq. (6.34a),

$$P_{\mathrm{a}} = \frac{\rho c k^2 |Q|^2}{4\pi}, \tag{6.36}$$

which is just what we would expect since the piston acts as a monopole on a rigid plane in this frequency range (cf. eq. (6.11)).

Example 6.5

Instead of using the volume velocity and the acoustic impedance we could equally well compute the sound power from the mean square velocity and the real part of the mechanical radiation impedance, since, with eq. (4.4),

$$P_{\mathrm{a}} = \tfrac{1}{2}|Q|^2 \,\mathrm{Re}\{Z_{\mathrm{a,r}}\} = \tfrac{1}{2}|U|^2 \,\mathrm{Re}\{Z_{\mathrm{m,r}}\}.$$

Example 6.6

Equation (6.36) shows that the sound power of the piston is proportional to $|\omega Q|^2$ at low frequencies, that is, the sound power is independent of the frequency if the volume *acceleration* is independent of the frequency. This implies that the displacement of the piston should be inversely proportional to the *square* of the frequency if we want the sound power to be independent of the frequency. In other words, it implies very large displacements at low frequencies. Since mechanical systems such as loudspeakers only allow a limited excursion, the low frequency sound power output of a loudspeaker is always limited: the only way to increase the sound power is to increase the size of the membrane. This explains why very large loudspeakers are found in subwoofers.

The *directivity factor* of a source is defined as the sound intensity on the axis in the far field normalised by the sound intensity of an omnidirectional source with the same sound power. From eq. (6.20) the sound intensity on the axis is

$$I_r = \frac{1}{2}\rho c k^2 \left(\frac{|Q|}{2\pi r} \right)^2 \tag{6.37}$$

(see also example 6.2). Normalising with $P_a/4\pi r^2$ (eq. (6.35)) gives the directivity factor *Q(f)*,

$$Q(f) = \frac{(ka)^2}{R_1} = \frac{(ka)^2}{1 - \dfrac{\mathrm{J}_1(2ka)}{ka}}. \tag{6.38}$$

The directivity factor of the piston is plotted in figure 6.12 as a function of the normalised frequency *ka*. Note that the directivity factor approaches two at low frequencies rather than one, reflecting the fact that all the sound power is radiated in only half a sphere. At high frequencies the directivity becomes proportional to the square of the frequency.

In practice, one often uses the directivity index, defined by

$$DI(f) = 10 \log Q(f). \tag{6.40}$$

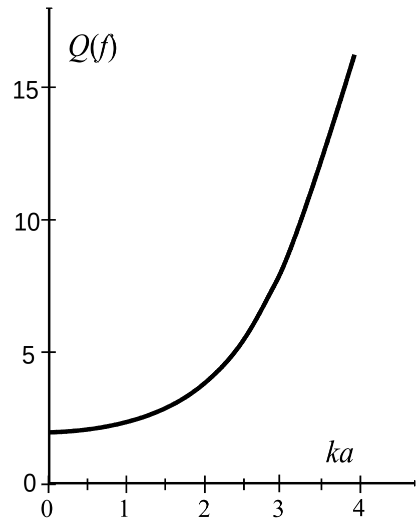

Figure 6.12 Directivity factor of a piston in a baffle. (From ref. [18].)

REFERENCES

1 L. Cremer and M. Hubert: *Vorlesungen über Technische Akustik* (3rd edition). Springer-Verlag, Berlin, 1985.
2 T.D. Rossing, F.R. Moore and P.A. Wheeler: *The Science of Sound* (3nd edition). Addison Wesley, San Francisco, CA, 2002.
3 M.J. Crocker and F. Jacobsen: Sound intensity. Chapter 156 in *Encyclopedia of Acoustics*, ed. M.J. Crocker. John Wiley & Sons, New York, 1997.
4 F. Jacobsen: A note on instantaneous and time-averaged active and reactive sound intensity. *Journal of Sound and Vibration* **147**, 1991, 489-496.
5 F. Jacobsen: An elementary introduction to applied signal analysis. Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark, Note no 7001, 2011.
6 R.B. Randall: *Frequency analysis* (3rd edition). Brüel & Kjær, Nærum, 1987.
7 D.W. Martin and W.D. Ward: Subjective evaluation of musical scale temperament in pianos. *Journal of the Acoustical Society of America* **33**, 1961, 582-585.
8 R.W. Krug: Sound level meters. Chapter 155 in *Encyclopedia of Acoustics*, ed. M.J. Crocker, John Wiley & Sons, New York, 1997.
9 J. Pope: Analyzers. Chapter 107 in *Handbook of Acoustics*, ed. M.J. Crocker. John Wiley & Sons, New York, 1998.
10 P.V. Brüel, J. Pope and H.K. Zaveri: Introduction to acoustical measurement and instrumentation. Chapter 154 in *Encyclopedia of Acoustics*, ed. M.J. Crocker. John Wiley & Sons, New York, 1997.
11 Anon.: *Microphone Handbook*. Brüel & Kjær, Nærum, 1996.
12 F. Jacobsen: Propagation of sound waves in ducts. Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark, Note no 31260, 2011.
13 K. Rasmussen: *Analogier mellem mekaniske, akustiske og elektriske systemer* (4th edition). Polyteknisk Forlag, Lyngby, 1994.
14 W. Marshall Leach, Jr.: *Introduction to Electroacoustics and Audio Amplifier Design* (2nd edition). Kendall/Hunt Publishing Company, Dubuque, IA, 1999.
15 Z. Maekawa and P. Lord: Environmental and Architectural Acoustics. E & FN Spon, London, 1994.
16 E.M. Salomons: *Computational Atmospheric Acoustics*. Kluwer Academic Publishers, Dordrecht, 2001.
17 W.S. Rayleigh: On the passage of waves through apertures in plane screens, and allied theorems. *Philosophical Magazine* **43**, 1897, 259-272.
18 K. Rasmussen: Lydfelter. Acoustic Technology, Technical University of Denmark, Note no 2107, 1996.
19 A.D. Pierce: *Acoustics. An Introduction to Its Physical Principles and Applications*. The Acoustical Society of America, New York, 1989.
20 P.M. Morse and K.U. Ingard: *Theoretical Acoustics*. McGraw-Hill, New York, 1968.
21 L.E. Kinsler, A.R. Frey, A.B. Coppens and J.V. Sanders: *Fundamentals of Acoustics* (4th edition). John Wiley & Sons, New York, 2000.

BIBLIOGRAPHY

Recommended reading includes the following. The book by Everest and Pohlmann manages to deal with many acoustic phenomena practically without mathematics. The books by Fahy, Beranek, and Kinsler *et al*. are also introductory. More advanced treatments can be found in the Nelson's chapter, in the books by Morse, Morse & Ingard, and Filippi *et al*., and in Pierce's chapters and book.

1. F. Alton Everest and K.C. Pohlmann: *Master Handbook of Acoustics* (5th edition). McGraw-Hill, New York, 2009.
2. L.L. Beranek: *Acoustics* (2nd edition). The American Institute of Physics for the Acoustical Society of America, Cambridge, MA, 1986.
3. F. Fahy: *Foundations of Engineering Acoustics*. Academic Press, San Diego, 2000.
4. L.E. Kinsler, A.R. Frey, A.B. Coppens and J.V. Sanders: *Fundamentals of Acoustics* (4th edition). John Wiley & Sons, New York, 2000.
5. P.A. Nelson: An introduction to acoustics. Chapter 1 in *Fundamentals of Noise and Vibration*, ed. F.J. Fahy and J. Walker. E & FN Spon, London, 1998.
6. P.M. Morse: *Vibration and Sound* (2nd edition). The American Institute of Physics, New York, 1983.
7. P.M. Morse and K.U. Ingard: *Theoretical Acoustics*. McGraw-Hill, New York, 1968.
8. A.D. Pierce: *Acoustics. An Introduction to Its Physical Principles and Applications*. The Acoustical Society of America, New York, 1989.
9. A.D. Pierce: Mathematical theory of wave propagation. Chapter 2 in *Encyclopedia of Acoustics*, ed. M.J. Crocker, John Wiley & Sons, New York, 1997.
10. A.D. Pierce: Basic linear acoustics. Chapter 3 in *Springer Handbook of Acoustics*, ed. T.D. Rossing, Springer, New York, 2007.
11. P. Filippi, D. Habault, J.P. Lefebre and A. Bergassoli: *Acoustics: Basic Physics, Theory and Methods.* Academic Press, London, 1999.

APPENDIX: COMPLEX NOTATION

In a harmonic sound field the sound pressure at any point is a function of the type $\cos(\omega t + \varphi)$. It is common practice to use *complex notation* in such cases. This is a symbolic method that makes use of the fact that complex exponentials give a more condensed notation that trigonometric functions because of the complicated multiplication theorems of sines and cosines.

We recall that a complex number A can be written either in terms of its real and imaginary part or in terms of its magnitude (also called absolute value or modulus) and phase angle,

$$A = A_\mathrm{r} + \mathrm{j}A_\mathrm{i} = |A|\mathrm{e}^{\mathrm{j}\varphi_A}, \tag{A.1}$$

where

$$\mathrm{j} = \sqrt{-1} \tag{A.2}$$

is the imaginary unit, and

$$A_\mathrm{r} = \mathrm{Re}\{A\} = |A|\cos\varphi_A, \qquad A_\mathrm{i} = \mathrm{Im}\{A\} = |A|\sin\varphi_A, \tag{A.3, A.4}$$

$$|A| = \sqrt{A_\mathrm{r}^2 + A_\mathrm{i}^2} \tag{A.5}$$

(see figure A1). The complex conjugate of A is

$$A^* = A_\mathrm{r} - \mathrm{j}A_\mathrm{i} = |A|\mathrm{e}^{-\mathrm{j}\varphi_A}; \tag{A.6}$$

therefore the magnitude can also be written

$$|A| = \sqrt{A \cdot A^*}. \tag{A.7}$$

Multiplication and division of two complex numbers are most conveniently carried out if they are given in terms of magnitudes and phase angles,

$$AB = |A||B|\mathrm{e}^{\mathrm{j}(\varphi_A + \varphi_B)}, \qquad A/B = \frac{|A|}{|B|}\mathrm{e}^{\mathrm{j}(\varphi_A - \varphi_B)}. \tag{A.8, A.9}$$

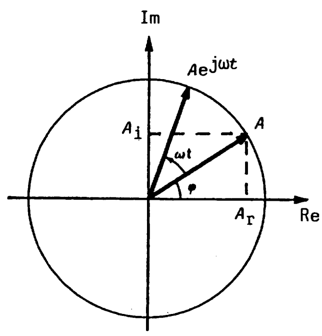

Figure A1. Complex representation of a harmonic signal.

Complex representation of harmonic signals makes use of the fact that

$$e^{jx} = \cos x + j\sin x \tag{A.10}$$

(Euler's equation) or, conversely,

$$\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right), \quad \sin x = -j\frac{1}{2}\left(e^{jx} - e^{-jx}\right). \qquad \text{(A.11a, A.11b)}$$

In a harmonic sound field the sound pressure at a given position can be written

$$\hat{p} = Ae^{j\omega t}, \tag{A.12}$$

where A is the *complex amplitude* of the sound pressure. The real, physical sound pressure is of course a real function of the time,

$$p = \mathrm{Re}\{\hat{p}\} = \mathrm{Re}\left\{|A|e^{j(\omega t+\varphi_A)}\right\} = |A|\cos(\omega t + \varphi_A), \tag{A.13}$$

which is seen to be an expression of the form $\cos(\omega t + \varphi)$. The magnitude of the complex quantity $|A|$ is called the *amplitude* of the pressure, and φ_A is its phase. It can be concluded that complex notation implies the mathematical trick of adding another solution, an expression of the form $\sin(\omega t + \varphi)$, multiplied by a constant, the imaginary unit j. This trick relies on linear superposition.

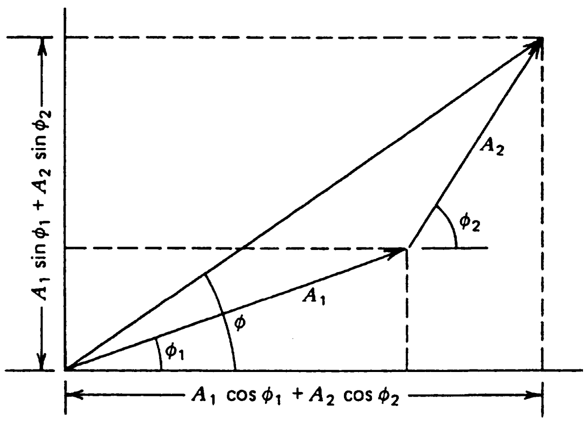

Figure A2. Two simple harmonic signals with identical frequencies. (From ref. [21].)

The mathematical convenience of the complex representation of harmonic signals can be illustrated by an example. A sum of two harmonic signals of the same frequency, $A_1e^{j\omega t}$ and $A_2e^{j\omega t}$, is yet another harmonic signal with an amplitude of $|A_1 + A_2|$ (see figure A2). Evidently, this can also be derived without complex notation,

$$\begin{aligned} p &= |A_1|\cos(\omega t + \varphi_1) + |A_2|\cos(\omega t + \varphi_2) \\ &= \left(|A_1|\cos\varphi_1 + |A_2|\cos\varphi_2\right)\cos\omega t - \left(|A_1|\sin\varphi_1 + |A_2|\sin\varphi_2\right)\sin\omega t \\ &= \left[\left(|A_1|\cos\varphi_1 + |A_2|\cos\varphi_2\right)^2 + \left(|A_1|\sin\varphi_1 + |A_2|\sin\varphi_2\right)^2\right]^{1/2}\cos(\omega t + \varphi), \end{aligned} \tag{A.14}$$

where

$$\varphi = \arctan \frac{|A_1| \sin \varphi_1 + |A_2| \sin \varphi_2}{|A_1| \cos \varphi_1 + |A_2| \cos \varphi_2}, \tag{A.15}$$

but the expedience and convenience of the complex method seems indisputable.

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\mathrm{j}\omega t} = \mathrm{j}\omega \mathrm{e}^{\mathrm{j}\omega t}, \tag{A.16}$$

it follows that differentiation with respect to time corresponds to multiplication by a factor of $\mathrm{j}\omega$. Conversely, integration with respect to time corresponds to division with $\mathrm{j}\omega$. If, for example, the vibrational velocity of a surface is, in complex representation,

$$\hat{v} = B\mathrm{e}^{\mathrm{j}\omega t} = |B| \mathrm{e}^{\mathrm{j}(\omega t + \varphi_B)}, \tag{A.17}$$

which means that the real, physical velocity is

$$v = \mathrm{Re}\{\hat{v}\} = |B| \cos(\omega t + \varphi_B), \tag{A.18}$$

then the acceleration is written

$$\hat{a} = \mathrm{j}\omega\hat{v}, \tag{A.19}$$

which means that the physical acceleration is

$$a = \mathrm{Re}\{\hat{a}\} = \mathrm{Re}\{\mathrm{j}\omega B\mathrm{e}^{\mathrm{j}\omega t}\} = -\omega |B| \sin(\omega t + \varphi_B), \tag{A.20}$$

and this is seen to agree with the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t} \cos(\omega t + \varphi_B) = -\omega \sin(\omega t + \varphi_B). \tag{A.21}$$

In a similar manner we find the displacement,

$$\hat{\xi} = \frac{\hat{v}}{\mathrm{j}\omega}, \tag{A.22}$$

which means that

$$\xi = \mathrm{Re}\{\hat{\xi}\} = \mathrm{Re}\left\{\frac{1}{\mathrm{j}\omega} B\mathrm{e}^{\mathrm{j}\omega t}\right\} = \frac{1}{\omega} |B| \sin(\omega t + \varphi_B), \tag{A.23}$$

in agreement with the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\omega}\sin(\omega t+\varphi_B)\right)=\cos(\omega t+\varphi_B). \tag{A.24}$$

Acoustic second-order quantities involve time averages of squared harmonic signals and, more generally, products of harmonic signals. Such quantities are dealt with in a special way, as follows. Expressed in terms of the complex pressure amplitude $\hat{p}$, the mean square pressure becomes

$$\overline{p^2}=p_{\mathrm{rms}}^2=\left|\hat{p}\right|^2/2, \tag{A.25}$$

in agreement with the fact that the average value of a squared cosine is ½. Note that it is the squared *magnitude* of $\hat{p}$ that enters into the expression, not the square of $\hat{p}$, which in general would be a complex number proportional to $\mathrm{e}^{2\mathrm{j}\omega t}$.

The time average of a product is given by the following expression

$$\overline{xy}=\frac{1}{2}\mathrm{Re}\left\{\hat{x}\hat{y}^*\right\}=\frac{1}{2}\mathrm{Re}\left\{\hat{x}^*\hat{y}\right\}. \tag{A.26}$$

This can be seen as follows,

$$\frac{1}{2}\mathrm{Re}\left\{\hat{x}\hat{y}^*\right\}=\frac{1}{2}\mathrm{Re}\left\{\left|\hat{x}\right|\mathrm{e}^{\mathrm{j}(\omega t+\phi_x)}\left|\hat{y}\right|\mathrm{e}^{-\mathrm{j}(\omega t+\phi_y)}\right\}=\frac{1}{2}\left|\hat{x}\right|\left|\hat{y}\right|\cos(\phi_x-\phi_y), \tag{A.27}$$

which is seen to in agree with

$$\overline{xy}=\overline{\left|\hat{x}\right|\cos(\omega t+\varphi_x)\left|\hat{y}\right|\cos(\omega t+\varphi_y)}=\frac{1}{2}\left|\hat{x}\right|\left|\hat{y}\right|\cos(\varphi_x-\varphi_y). \tag{A.28}$$

Note that either $\hat{x}$ or $\hat{y}$ must be conjugated.

LIST OF SYMBOLS

a	radius of sphere [m]; acceleration [m/s^2]
c	speed of sound [m/s]
D	directivity [dimensionless]
DI	directivity index [dB]
E	total acoustic energy [J]
f	frequency [Hz]
F	force [N]
h	distance [m]
H_1	Struve function
$\mathbf{I}$	sound intensity [W/m^2]
I_{ref}	reference sound intensity [W/m^2]
I_x	component of sound intensity [W/m^2]
J_{m}	Bessel fuction
k	wavenumber [m^{-1}]
K	stiffness constant [N/m]
K_{s}	adiabatic bulk modulus [N/m^2]
l	length [m]
L_{A}	A-weighted sound pressure level [dB re p_{ref}]
L_{Aeq}	equivalent A-weighted sound pressure level [dB re p_{ref}]
L_{AE}	sound exposure level [dB re p_{ref}]
L_{C}	C-weighted sound pressure level [dB re p_{ref}]
L_{eq}	equivalent sound pressure level [dB re p_{ref}]
L_I	sound intensity level [dB re I_{ref}]
L_p	sound pressure level [dB re p_{ref}]
L_W	sound power level [dB re P_{ref}]
L_{Z}	sound pressure level measured without frequency weighting [dB re p_{ref}]
M	mass [kg]
n	natural number [dimensionless]
p	sound pressure [Pa]
$p_{\mathrm{A}}(t)$	instantaneous A-weighted sound pressure [Pa]
p_{ref}	reference sound pressure [Pa]
p_{rms}	rms value of sound pressure [Pa]
p_0	static pressure [Pa]
P	power [W]
P_{a}	sound power [W]
P_{ref}	reference sound power [W]
q	volume velocity associated with a fictive surface [m^3/s]
Q	volume velocity of source [m^3/s]; directivity factor [dimensionless]
r	radial distance in spherical coordinate system [m]
R	gas constant [m^{2}s^{-2}K^{-1}]; reflection factor [dimensionless]
s	standing wave ratio [dimensionless]
S	surface area [m^2]
t	time [s]
T	absolute temperature [K]; averaging time [s]

$\mathbf{u}$	particle velocity [m/s]
u_x	component of the particle velocity [m/s]
U	velocity [m/s]
v	velocity [m/s]
V	volume [m^3]
w_{kin}	kinetic energy density [J/m^3]
w_{pot}	potential energy density [J/m^3]
x, y, z	Cartesian coordinates [m]
Z_{a}	acoustic impedance [kg $m^{-4}s^{-1}$]
$Z_{\mathrm{a,\,r}}$	acoustic radiation impedance [kg $m^{-4}s^{-1}$]
Z_{m}	mechanical impedance [kg/s]
$Z_{\mathrm{m,\,r}}$	mechanical radiation impedance [kg/s]
Y	mobility (mechanical admittance) [s/kg]
α	absorption coefficient [dimensionless]
γ	ratio of specific heats [dimensionless]
ΔV	volume displacement [m^3]
θ	polar angle in spherical coordinate system [dimensionless]
λ	wavelength [m]
ξ	displacement [m]
ρ	density [kgm^{-3}]
τ	time constant [s]
φ	phase angle [radian]; azimuth angle in spherical coordinate system [radian]
ω	angular frequency [radian/s]
^	indicates complex representation of a harmonic variable

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