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A stress-constrained truss-topology and material-selection problem that can be solved by linear programming

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Abstract We consider the problem of simultaneously selecting the material and determining the area of each bar in a truss structure in such a way that the cost of the structure is minimized subject to stress constraints under a single load condition. We show that such problems can be solved by linear programming to give the global optimum, and that two different materials are always sufficient in an optimal structure.

Key words topology optimization, linear programming, material selection

1 Introduction

A well-studied problem is that of minimizing the cost or weight of a truss structure under a single load condition subject to stress constraints. The design variables are the cross-sectional areas, which are required to be non-negative. If some of the areas become zero, the corresponding elements vanish from the structure, and the corresponding stress constraints also disappear. It was demonstrated by Dorn *et al.* (1964) that this nonconvex problem may equivalently be cast as a linear program if the stress limits are the same in tension and compression. It was also indicated by Dorn *et al.* (1964), Hemp (1973), and Rozvany and Birker (1994) that this holds even if the stress limits are different in tension and compression. This result was proved by Achtziger (1996).

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The purpose of this paper is to present a generalization of the above results to the problem of simultaneously selecting the material, from a finite number of available materials, as well as determining the area of each bar in the ground structure. We also prove that the optimal solution, in fact, always contains at most *two* different materials.

For an overview of stress-constrained truss topology optimization, see Rozvany (2001) and references therein.

This paper is organized as follows. In Sect. 2, we present the considered nonconvex minimum weight problem. In Sect. 3, we show that the considered nonconvex problem may equivalently be cast as a linear program. In Sect. 4, we show the sufficiency of two materials in a optimal solution. In Sect. 5, we show that under some natural assumptions even a single material is sufficient. Furthermore, it is shown by a counterexample that this property does not hold if there are multiple load conditions. The Appendix, finally, contains the proofs of the lemmas stated in the paper.

If $f \in \mathbb{R}$ then $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ denote, respectively, the positive and negative part of f, and then $f = f^+ - f^-$. If $f = (f_1, \dots, f_n)^T \in \mathbb{R}^n$ then $f^+ = (f_1^+, \dots, f_n^+)^T$ and $f^- = (f_1^-, \dots, f_n^-)^T$ denote, respectively, the component-wise positive and negative parts of f, and then $f = f^+ - f^-$.

2 Problem formulation

The elastic equilibrium equations of a given truss structure with d degrees of freedom subject to an external load $p \in \mathbb{R}^d$ are assumed to be given by

$$K(x)u = p\,, \qquad \text{with } K(x) = \sum_{j=1}^{n} x_j (E_j/l_j) r_j r_j^T\,,$$

where $K(x) \in \mathbb{R}^{d\times d}$ denotes the global stiffness matrix, $u \in \mathbb{R}^d$ is the nodal displacement vector, n is the number of potential bars, and $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ are the cross-sectional areas. Furthermore, $E_j > 0$ is Young's modulus and l_j is the length of the j-th bar. Moreover, $r_j \in \mathbb{R}^d$ contains the direction cosines such that $r_j^T u$ is the linearized elongation of the j-th bar. Let R denote the $d \times n$ matrix whose j-th column is r_j. It is assumed that the rows of R are linearly independent. This is equivalent to the assumption that the stiffness matrix is positive definite if $x > 0$.

Assume that m different materials with properties $E_{jk} > 0$ and associated costs $c_{jk} > 0$ per unit area are available for use in the j-th bar. Furthermore, assume that at most one of the available materials may be used in each bar and that the stress limits are material dependent, i.e. the stress limits in tension and compression are given by $\sigma_{jk}^{\max} > 0$ and $\sigma_{jk}^{\min} < 0$. Let x_{jk} and f_{jk} denote the cross-sectional area and the normal force in the j-th bar corresponding to the k-th material. We use the following notation: $f_k = (f_{1k}, \dots, f_{nk})^T \in \mathbb{R}^n$ and $f = (f_1^T, \dots, f_m^T)^T \in \mathbb{R}^{nm}$. The vectors $g_k \in \mathbb{R}^n$, $h_k \in \mathbb{R}^n$, $g \in \mathbb{R}^{nm}$, $h \in \mathbb{R}^{nm}$, and $x \in \mathbb{R}^{nm}$ are defined analogously.

We consider the following stress-constrained minimum-cost truss-topology optimization problem in the variables (x, f, u):

$$
\begin{aligned}
&\underset{x,f,u}{\text{minimize}} \quad \sum_{j=1}^{n} \sum_{k=1}^{m} c_{jk} x_{jk} \\
&\text{subject to} \quad \sum_{k=1}^{m} R f_k = p \,, \\
&f_{jk} = x_{jk} (E_{jk}/l_j) r_j^T u \,, \qquad \forall\,(j,k)\,, \\
&x_{jk} \sigma_{jk}^{\min} \le f_{jk} \le x_{jk} \sigma_{jk}^{\max} \,, \quad \forall\,(j,k)\,, \\
&\text{for each } j, \text{ at most one } x_{jk} > 0 \,, \\
&x \ge 0 \,.
\end{aligned}
\tag{1}
$$

The constraints $\sum_{k=1}^m R f_k = p$ correspond to equilibrium of forces and the constraints $f_{jk} = x_{jk}(E_{jk}/l_j) r_j^T u$ correspond to geometric compatibility and Hooke's law. The constraints $x_{jk}\sigma_{jk}^{\min} \le f_{jk} \le x_{jk}\sigma_{jk}^{\max}$ imply that the stress σ_{jk} in the j-th bar satisfies $\sigma_{jk}^{\min} \le \sigma_{jk} \le \sigma_{jk}^{\max}$ when $x_{jk} > 0$, and that $f_{jk} = 0$ when $x_{jk} = 0$. Hence, the stress constraints are effectively removed when the bar is not present.

Since at most one of the variables $x_{j1}, \dots, x_{jm}$ is non-zero, it follows that at most one of the variables $f_{j1}, \dots, f_{jm}$ is non-zero. Then, the cross-sectional area of the j-th bar is given by this non-zero x_{jk} (if any), while the normal force in the j-th bar is given by the corresponding non-zero f_{jk} (if any).

3 Equivalent LP formulation

To show that (1) can equivalently be cast as a linear program, we proceed as follows. First, we relax the feasible set of problem (1) by removing the complicating compatibility constraints, the displacement variables, and the constraints which state that at most one material can be used in each bar. The resulting problem is a linear program in the variables (x, f). Thereafter, we show that this relaxed problem can equivalently be cast as a linear program in standard form with d equality constraints and $2mn$ nonnegative variables. Finally, we show that, given an optimal basic solution to this linear program in standard form, it is possible to construct a global optimal solution to the original problem (1).

Assume temporarily that the compatibility constraints $f_{jk} = x_{jk}(E_{jk}/l_j) r_j^T u$ and the constraints which state that at most one material can be used in each bar are relaxed. Then, the following linear program in the variables (x, f) is obtained:

$$
\begin{aligned}
&\underset{x,f}{\text{minimize}} \quad \sum_{j=1}^{n} \sum_{k=1}^{m} c_{jk} x_{jk} \\
&\text{subject to} \quad \sum_{k=1}^{m} R f_k = p \,, \\
&x_{jk} \sigma_{jk}^{\min} \le f_{jk} \le x_{jk} \sigma_{jk}^{\max} \,, \qquad \forall\,(j,k)\,, \\
&x \ge 0 \,.
\end{aligned}
\tag{2}
$$

Since $\sigma_{jk}^{\max} > 0$ and $\sigma_{jk}^{\min} < 0$, the inequalities $x_{jk}\sigma_{jk}^{\min} \le f_{jk} \le x_{jk}\sigma_{jk}^{\max}$ are equivalent to the inequality $x_{jk} \ge \max\{f_{jk}/\sigma_{jk}^{\max}, f_{jk}/\sigma_{jk}^{\min}\}$, which in turn is equivalent to $x_{jk} \ge f_{jk}^{+}/\sigma_{jk}^{\max} - f_{jk}^{-}/\sigma_{jk}^{\min}$. It follows that the inequalities $x \ge 0$ are automatically satisfied. Thus, problem (2) is equivalent to the problem

$$
\begin{aligned}
&\underset{x,f}{\text{minimize}} \quad \sum_{j=1}^{n} \sum_{k=1}^{m} c_{jk} x_{jk} \\
&\text{subject to} \quad \sum_{k=1}^{m} R f_k = p \,, \\
&x_{jk} \ge f_{jk}^{+}/\sigma_{jk}^{\max} - f_{jk}^{-}/\sigma_{jk}^{\min} \,, \quad \forall\,(j,k)\,.
\end{aligned}
\tag{3}
$$

Since $c_{jk} > 0$ for all j and k, it must trivially hold that $x_{jk} = f_{jk}^{+}/\sigma_{jk}^{\max} - f_{jk}^{-}/\sigma_{jk}^{\min}$ for all j and k in any optimal solution (x, f) to (3). Thus, (3) is equivalent to the following problem, in which the variables x have been eliminated:

$$
\begin{aligned}
&\underset{f}{\text{minimize}} \quad \sum_{j=1}^{n} \sum_{k=1}^{m} c_{jk} \left(f_{jk}^{+}/\sigma_{jk}^{\max} - f_{jk}^{-}/\sigma_{jk}^{\min} \right) \\
&\text{subject to} \quad \sum_{k=1}^{m} R f_k = p \,.
\end{aligned}
\tag{4}
$$

The objective function in (4) is not a linear function of f, since f_{jk}^{+} and f_{jk}^{-} are not linear functions of f_{jk}. As will be shown below, however, problem (4) is equivalent to the following linear program in the variables $g \in \mathbb{R}^{nm}$ and $h \in \mathbb{R}^{nm}$:

$$\begin{aligned}
&\underset{g,h}{\text{minimize}} \quad \sum_{j=1}^{n}\sum_{k=1}^{m} c_{jk}\left(g_{jk}/\sigma_{jk}^{\max} - h_{jk}/\sigma_{jk}^{\min}\right) \\
&\text{subject to} \quad \sum_{k=1}^{m} R(g_k - h_k) = p\,, \\
&g, h \geq 0\,.
\end{aligned} \tag{5}$$

Lemma 1. *There always exists an optimal solution* $(\hat{g}, \hat{h})$ *to* (5).

Lemma 2. *If* $(\hat{g}, \hat{h})$ *is an optimal solution to* (5) *and* $\hat{f} = \hat{g} - \hat{h}$ *then* $(\hat{g}, \hat{h}) = (\hat{f}^{+}, \hat{f}^{-})$.

Lemma 3. *If* $(\hat{g}, \hat{h})$ *is an optimal solution to* (5) *then* $\hat{f} = \hat{g} - \hat{h}$ *is an optimal solution to* (4)*, while* $(\hat{x}, \hat{f})$*, with* $\hat{x}_{jk} = \hat{f}_{jk}^{+}/\sigma_{jk}^{\max} - \hat{f}_{jk}^{-}/\sigma_{jk}^{\min}$*, is an optimal solution to both* (2) *and* (3).

The proofs of these lemmas are left to the Appendix.

Let $B \in \mathbb{R}^{d \times d}$ denote the basis matrix consisting of the d linearly independent columns of the $d \times 2nm$ matrix $(R\ -R\ \cdots\ R\ -R)$ corresponding to the basic variables in an optimal basic solution $(\hat{g}, \hat{h})$ to (5). Let the vector $b_k \in \mathbb{R}^{2n}$ be defined by $b_{ik} = l_i \sigma_{ik}^{\max}/E_{ik}$ if $1 \leq i \leq n$ and $b_{ik} = -l_{i-n}\sigma_{(i-n)k}^{\min}/E_{(i-n)k}$ if $n+1 \leq i \leq 2n$ and let $b = (b_1^T, \ldots, b_m^T)^T \in \mathbb{R}^{2nm}$. Furthermore, let b_B denote the vector consisting of the d elements of b corresponding to the basic variables. Let $\hat{u} \in \mathbb{R}^d$ be given as the unique solution to

$$B^T \hat{u} = b_B\,, \tag{6}$$

so that $r_j^T \hat{u} = l_j \sigma_{jk}^{\max}/E_{jk}$ for j and k such that $\hat{g}_{jk}$ is a basic variable and $-r_j^T \hat{u} = -l_j \sigma_{jk}^{\min}/E_{jk}$ for j and k such that $\hat{h}_{jk}$ is a basic variable.

In the following proposition it is shown how an optimal basic solution $(\hat{g}, \hat{h})$ to the linear program (5) can be used to construct an optimal solution $(\hat{x}, \hat{f}, \hat{u})$ to the original problem (1).

Proposition 1. *Let* $(\hat{g}, \hat{h})$ *be an optimal basic solution to* (5)*. Further, let* $\hat{u}$ *be the corresponding solution to* (6)*,* $\hat{f} = \hat{g} - \hat{h}$*, and* $\hat{x}_{jk} = \hat{g}_{jk}/\sigma_{jk}^{\max} - \hat{h}_{jk}/\sigma_{jk}^{\min}$*. Then* $(\hat{x}, \hat{f}, \hat{u})$ *is an optimal solution to* (1).

Proof. Since, by Lemma 3, $(\hat{x}, \hat{f})$ is optimal to (2), it is sufficient to show that $(\hat{x}, \hat{f}, \hat{u})$ satisfies the relaxed constraints $f_{jk} = x_{jk}(E_{jk}/l_j) r_j^T u$ and that, for each j, at most one $\hat{x}_{jk} > 0$. Since $(\hat{g}, \hat{h})$ is an optimal basic solution, at most one of $\hat{g}_{j1}, \ldots, \hat{g}_{jm}, \hat{h}_{j1}, \ldots, \hat{h}_{jm}$ is nonzero for each j, otherwise the basis matrix would contain linearly dependent columns. It then follows from the definition of $\hat{x}_{jk}$ that, for each j, at most one $\hat{x}_{jk} > 0$. Now, consider elements with $\hat{g}_{jk} > 0$ for some k. For such elements, $\hat{f}_{jk} = \hat{g}_{jk} = \hat{x}_{jk}\sigma_{jk}^{\max}$ and $r_j^T \hat{u} = l_j \sigma_{jk}^{\max}/E_{jk}$. Thus $\hat{f}_{jk} = \hat{x}_{jk}(E_{jk}/l_j) r_j^T \hat{u}$. Next, consider elements with $\hat{h}_{jk} > 0$ for some k. For such elements $\hat{f}_{jk} = -\hat{h}_{jk} = \hat{x}_{jk}\sigma_{jk}^{\min}$ and $r_j^T \hat{u} = l_j \sigma_{jk}^{\min}/E_{jk}$. Thus, $\hat{f}_{jk} = \hat{x}_{jk}(E_{jk}/l_j) r_j^T \hat{u}$. Finally, consider elements with $\hat{g}_{jk} = \hat{h}_{jk} = 0$ for all k. For such elements, $\hat{f}_{jk} = 0$ and $\hat{x}_{jk} = 0$ for all k. Again, it follows that $\hat{f}_{jk} = \hat{x}_{jk}(E_{jk}/l_j) r_j^T \hat{u}$.

4
On the sufficiency of two materials

From problem (5) it can be seen that if the j-th bar is present in the optimal structure, then the material k_1 that minimizes $c_{jk}/\sigma_{jk}^{\max}$ will be chosen if the bar is in tension $(g_{jk} > 0)$ and the material k_2 that minimizes $-c_{jk}/\sigma_{jk}^{\min}$ will be chosen if the bar is in compression $(h_{jk} > 0)$. Hence, it is sufficient to consider at most two possible materials in each bar in the ground structure. Moreover, under natural assumptions on the cost coefficients in the objective function and on the stress limits, it is in fact sufficient to consider at most two materials in the whole structure. What is required is that the cost coefficients can be decomposed as $c_{jk} = l_j q_k$, where l_j is the length of the j-th bar and q_k is a material cost per unit volume, and that the stress limits only depend on the material.

Proposition 2. *If the stress limits only depend on the material, i.e.* $\sigma_{jk}^{\max} = \sigma_k^{\max} > 0$ *and* $\sigma_{jk}^{\min} = \sigma_k^{\min} < 0$ *for all* j*, and the cost coefficients can be written as* $c_{jk} = l_j q_k$*, then there exists an optimal solution to* (1) *that consists of at most two materials. Furthermore, the material for which* $q_k/\sigma_k^{\max}$ *is minimized will be used for bars in tension and the material for which* $-q_k/\sigma_k^{\min}$ *is minimized will be used for bars in compression.*

Proof. Let k_1 and k_2 be given by $k_1 = \arg\min_k\{q_k/\sigma_k^{\max}\}$ and $k_2 = \arg\min_k\{-q_k/\sigma_k^{\min}\}$ with ties broken arbitrarily. Further, let $(\hat{g}, \hat{h})$, where $\hat{g} = (\hat{g}_{k_1}^T, \hat{g}_{k_2}^T)^T \in \mathbb{R}^{2n}$ and $\hat{h} = (\hat{h}_{k_1}^T, \hat{h}_{k_2}^T)^T \in \mathbb{R}^{2n}$, be an optimal basic solution to (5) *when only the two materials* k_1 *and* k_2 *are considered* and let $\lambda \in \mathbb{R}^d$ be the corresponding simplex multipliers. A feasible basic solution (g, h) to (5) *when all materials are considered* can then be constructed by letting $g_{k_1} = \hat{g}_{k_1}$, $g_{k_2} = \hat{g}_{k_2}$, $h_{k_1} = \hat{h}_{k_1}$, $h_{k_2} = \hat{h}_{k_2}$, and $g_k = h_k = 0$ for all $k \neq k_1$ and $k \neq k_2$. The set of basic variables is the same as when only two materials were considered. The reduced costs for the variables g_{jk_1}, g_{jk_2}, h_{jk_1}, and h_{jk_2} are the same as before, i.e. they are still ≥ 0. The reduced costs $\bar{c}_{jk}$ for the variables g_{jk} for $k \neq k_1$ and $k \neq k_2$ are given by $\bar{c}_{jk} = l_j q_k/\sigma_k^{\max} - \lambda^T r_j \geq l_j q_{k_1}/\sigma_{k_1}^{\max} - \lambda^T r_j = \bar{c}_{jk_1} \geq 0$, where we have used $q_k/\sigma_k^{\max} \geq q_{k_1}/\sigma_{k_1}^{\max}$ by the definition of k_1. Similarly, the reduced costs $\bar{c}_{jk}$ for the variables h_{jk} for $k \neq k_1$ and $k \neq k_2$ are given by $\bar{c}_{jk} = -l_j q_k/\sigma_k^{\min} + \lambda^T r_j \geq -l_j q_{k_2}/\sigma_{k_2}^{\min} + \lambda^T r_j = \bar{c}_{jk_2} \geq 0$, where we have used $-q_k/\sigma_k^{\min} \geq -q_{k_2}/\sigma_{k_2}^{\min}$ by the definition of k_2. This shows that (g, h) is not only a feasible basic solution to (5) but also an *optimal* basic solution. Hence, it is sufficient to consider only the two materials k_1 and k_2 in problem (5). The second claim follows from above.

5
On the sufficiency of one material

It follows from the above proof that if $-\sigma_k^{\min} = \sigma_k^{\max}$ for all k and the assumptions of Proposition 2 hold, then it is always sufficient with *one* material, namely the one that minimizes $q_k/\sigma_k^{\max}$ over k. This result may seem "obvious", but, in fact, it does not hold if there are several load conditions, as will be demonstrated on the three-bar truss with two load conditions shown in Fig. 1.

The lengths of all the elements are equal to one, the horizontal load condition has a magnitude equal to $\sqrt{2}$ and the vertical load condition has a magnitude equal to 4. In this example, there are two different materials to choose between. They both have the same material cost per volume, $q_k = 1$ for $k = 1, 2$, and the same stress limits, $\sigma_k^{\max} = -\sigma_k^{\min} = 1$ for $k = 1, 2$. However, they have different Young's moduli, $E_1 = 7$ and $E_2 = 5$, respectively, which means that the first material is stiffer than the second.

First, assume that the same material is used in all three elements. Then, it can be shown (by enumerating all possible topologies) that the globally optimal cross-sectional areas are given by $(x_1, x_2, x_3) = (1.0, 3.0, 1.0)$. The optimal objective value is 5.0. (The same result is obtained regardless of whether we use the first material in all elements or the second material in all elements.)

Next, assume that the stiffest material is used in elements 1 and 3 while the more flexible material is used in element 2. Then, it can be shown (again by enumerating all possible topologies) that the globally optimal cross-sectional areas are given by $(x_1, x_2, x_3) = (1.0, 2.6, 1.0)$.

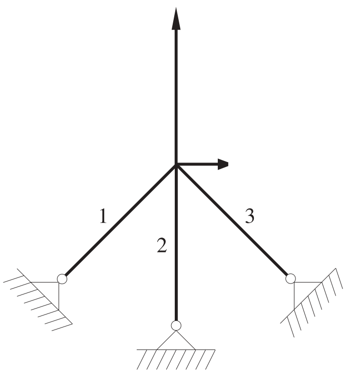

Fig. 1 Three-bar truss

The optimal objective value is now 4.6, so it is clearly not optimal to use a single material in this example.

Appendix

Proof of Lemma 1. Since R has d linearly independent rows, there is at least one solution to $\sum_{k=1}^{m} Rf_k = p$. Let f denote such a feasible solution. Then, (f^+, f^-) is a feasible solution to (5). Thus, the feasible set of (5) is nonempty. Further, the objective function in (5) is bounded from below (by zero). It is well known, see e.g. Luenberger (1989), that these observations guarantee the existence of an optimal solution to the linear program (5).

Proof of Lemma 2. If $(\hat{g}, \hat{h})$ is optimal to (5), then for each j and k it must hold that $\hat{g}_{jk}\hat{h}_{jk} = 0$, since if both $\hat{g}_{jk} > 0$ and $\hat{h}_{jk} > 0$ for some j and k, then a strictly better solution would be obtained by subtracting $\min\{\hat{g}_{jk}, \hat{h}_{jk}\}$ from both $\hat{g}_{jk}$ and $\hat{h}_{jk}$. Therefore, if $\hat{g}_{jk} > 0$ then $\hat{h}_{jk} = 0$ and $\hat{f}_{jk} = \hat{g}_{jk}$, while if $\hat{h}_{jk} > 0$ then $\hat{g}_{jk} = 0$ and $\hat{f}_{jk} = -\hat{h}_{jk}$. In both cases, $\hat{f}_{jk}^+ = \hat{g}_{jk}$ and $\hat{f}_{jk}^- = \hat{h}_{jk}$.

Proof of Lemma 3. $\hat{f}$ is feasible to (4) since $\sum_{k=1}^{m} R\hat{f}_k = \sum_{k=1}^{m} R(\hat{g}_k - \hat{h}_k) = p$. Let f be an arbitrary feasible solution to (4). Then, (f^+, f^-) is a feasible solution to (5), and thus $\sum_{j=1}^{n}\sum_{k=1}^{m} c_{jk}(f_{jk}^+/\sigma_{jk}^{\max} - f_{jk}^-/\sigma_{jk}^{\min}) \geq \sum_{j=1}^{n}\sum_{k=1}^{m} c_{jk}(\hat{g}_{jk}/\sigma_{jk}^{\max} - \hat{h}_{jk}/\sigma_{jk}^{\min})$ (since $(\hat{g}, \hat{h})$ is optimal to (5)). According to Lemma 2, $(\hat{g}, \hat{h}) = (\hat{f}^+, \hat{f}^-)$, so we get that $\sum_{j=1}^{n}\sum_{k=1}^{m} c_{jk}(f_{jk}^+/\sigma_{jk}^{\max} - f_{jk}^-/\sigma_{jk}^{\min}) \geq \sum_{j=1}^{n}\sum_{k=1}^{m} c_{jk}(\hat{f}_{jk}^+/\sigma_{jk}^{\max} - \hat{f}_{jk}^-/\sigma_{jk}^{\min})$, which shows that $\hat{f}$ is optimal to (4).

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