1)

$$
\begin{aligned}
&Min{:}\ f(x) = 6x_1x_2^{-1} + x_2x_1^{-2} \\
&S.T{:}\ \ h(x) = x_1x2 - 2 = \ 0 \\
&\qquad\ \ g(x) = \ x_1 + x_2 - 1 \geq 0 \\
&\vec{x}_0 = [2,1]^T
\end{aligned}
$$

a) Plot and find the optimum point graphically.
b) Solve the problem using direct quadratic programming.
c) Check the KKT conditions of the optimum found in b.
d) Solve the problem using Sequential Quadratic Programming
e) Explain the difference between these two methods.

2) Wright state has to buy some textbooks for the book store. The information for the textbooks is given below:

Book	Cost	Profit	Width
A	85	13	2
B	65	14	3
C	75	15	1
D	100	18	4

The instructor has specified that the combined total of number of books B and C must be at least 10. The shelf at the bookstore for these books is 100 inches wide. The budget to purchase these books is $5,000. Find the optimal number of each book to purchase in order to maximize profits.

3) Approximation of f(x) and g(x) for a two-bar truss structure. See attached page.

4) Response surface method, see attached page.

5) Given the truss and the structure and the FEA system below. Derive the sensitivity of the vertical displacement at the bottom most node with respect to Area1, Area2, and Area3 using the adjoint method.

E = 10^7

A1 = A2 = A3 = 5

L = 2*sqrt(2)

The reduced (boundary conditions applied) global force vector and reduced global stiffness matrix are given below.

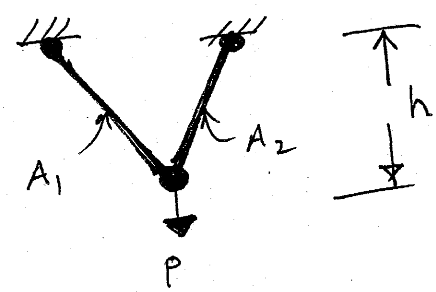
P = $[500, 1000]^T$

$$
\mathrm{K} \ = \frac{\mathrm{E}}{2\sqrt{2}}\begin{bmatrix} A_1 + A_3 & A1 + A_3 \\ A_1 - A_3 & A_1 + 2\sqrt{2}A_2 + A_3 \end{bmatrix}
$$

PROBLEM 1

For the following Truss optimization, structural weight optimized with a stress constraint

3 variables (A_1, A_2, h)



$$f(x) = A_1\sqrt{h^2+36} + A_2\sqrt{h^2+1} \;?$$

$$g(x): \quad 1 - \frac{\sqrt{h^2+1}}{100\,hA_2} \geq 0$$

$$1 \leq h \leq 6, \quad 0 \leq A_1 \leq 0.1, \quad 0 \leq A_2 \leq 0.1$$

Construct a suitable approximatoon and Show its validity upto $\pm 30\%$

$$X_0 = \begin{Bmatrix} 3 \\ 0.05 \\ 0.05 \end{Bmatrix} \quad \text{starting vector}$$

Also, Answer why you chose that approximation.

PROBLEM 2

Optimize the following problem using The Response Surface Approximation Method.

Minimize: $f(x) = x_1 + x_2$

Subject to:

$$g(x): \quad 2 - \frac{1}{x_1} - \frac{1}{x_2} \geq 0$$

Bounds:

$$0.1 \leq x_1 \leq 5$$
$$0.1 \leq x_2 \leq 5$$

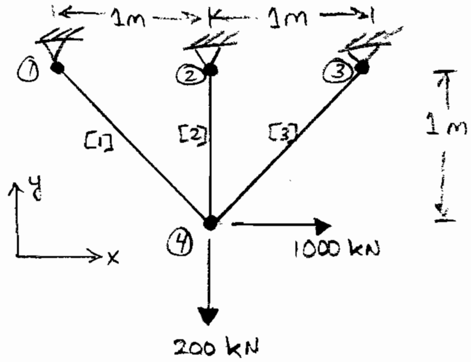
Use the following 3 experimental design points to Start The Optimization

#	x_1	x_2
1	2	2
2	0.5	2
3	1.5	1

Beyond these 3 data points, you are allowed to add only one data point per iteration.

Maximum of 4 iterations is allowed

3-Bar Truss Structure:

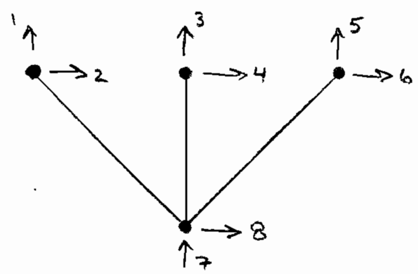


$A_1 = 25\ cm^2$
$A_2 = 25\ cm^2$
$A_3 = 25\ cm^2$
$E = 72.4\ GPa$
$\rho = 2780\ \frac{kg}{m^3}$

Find:

Sensitivity of axial stress in third element and displacement in horizontal direction at node 4 with respect to areas A_1, A_2, A_3

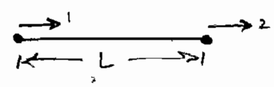
Finite Element Solution:



Global Degrees of Freedom

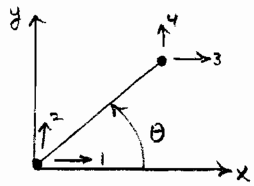
Structure modeled with truss elements

Truss element in local coordinates,



$$\underline{k}_e = \frac{A_e E_e}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Truss element rotated to global coordinates according to



$$\underline{K}_e = \underline{T}_e^T\, \underline{k}_e\, \underline{T}_e$$

$$\underline{T}_e = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \end{bmatrix}$$

For the 3-Bar Truss Structure,

$A_1 = 25 cm^2 = 0.0025\ m^2$	$A_2 = 0.0025\ m^2$	$A_3 = 0.0025\ m^2$
$E_1 = 72.4\times10^9\ Pa$	$E_2 = 72.4\times10^9\ Pa$	$E_3 = 72.4\times10^9\ Pa$
$L_1 = \sqrt{1^2+1^2} = 1.414\ m$	$L_2 = 1.0\ m$	$L_3 = 1.414\ m$
$\theta_1 = -45^\circ$	$\theta_2 = -90^\circ$	$\theta_3 = -135^\circ$

$$\underline{T}_1 = \begin{bmatrix} \cos(-45^\circ) & \sin(-45^\circ) & 0 & 0 \\ 0 & 0 & \cos(-45^\circ) & \sin(-45^\circ) \end{bmatrix} = \begin{bmatrix} 0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & 0.7071 & -0.7071 \end{bmatrix}$$

$$\underline{T}_2 = \begin{bmatrix} \cos(-90^\circ) & \sin(-90) & 0 & 0 \\ 0 & 0 & \cos(-90^\circ) & \sin(-90^\circ) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\underline{T}_3 = \begin{bmatrix} \cos(-135^\circ) & \sin(-135^\circ) & 0 & 0 \\ 0 & 0 & \cos(-135^\circ) & \sin(-135^\circ) \end{bmatrix} = \begin{bmatrix} -0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & -0.7071 & -0.7071 \end{bmatrix}$$

$$\underline{K}_1 = \begin{bmatrix} 0.7071 & 0 \\ -0.7071 & 0 \\ 0 & 0.7071 \\ 0 & -0.7071 \end{bmatrix} \frac{(0.0025)(72.4\times10^9)}{1.414} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & 0.7071 & -0.7071 \end{bmatrix}$$

$$\underline{K}_1 = 6.3993\times10^7 \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\underline{K}_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & - \end{bmatrix} \frac{(0.0025)(72.4\times10^9)}{1.0} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\underline{K}_2 = 1.81\times10^8 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\underline{K}_3 = \begin{bmatrix} -0.7071 & 0 \\ -0.7071 & 0 \\ 0 & -0.7071 \\ 0 & -0.7071 \end{bmatrix} \frac{(0.0025)(72.4\times10^9)}{1.0} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -0.7071 & -0.7071 & 0 & 0 \\ 0 & 0 & -0.7071 & -0.7071 \end{bmatrix}$$

$$\underline{K}_3 = 6.3993\times10^7 \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

Global Stiffness Matrix and Global Force Vector,

$$\underline{K} = 10^8 \begin{bmatrix} 0.6399 & -0.6399 & 0 & 0 & 0 & 0 & -0.6399 & 0.6399 \\ -0.6399 & 0.6399 & 0 & 0 & 0 & 0 & 0.6399 & -0.6399 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.81 & 0 & 0 & 0 & -1.81 \\ 0 & 0 & 0 & 0 & 0.6399 & 0.6399 & -0.6399 & -0.6399 \\ 0 & 0 & 0 & 0 & 0.6399 & 0.6399 & -0.6399 & -0.6399 \\ -0.6399 & 0.6399 & 0 & 0 & -0.6399 & -0.6399 & 1.2799 & 0 \\ 0.6399 & -0.6399 & 0 & -1.81 & -0.6399 & -0.6399 & 0 & 3.0899 \end{bmatrix}$$

$$\underline{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -200{,}000 \\ 1{,}000{,}000 \end{bmatrix}$$

Reduced System (eliminate DoFs 1-6),

$$10^8 \begin{bmatrix} 1.2799 & 0 \\ 0 & 3.0899 \end{bmatrix} \begin{Bmatrix} u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} -200{,}000 \\ 1{,}000{,}000 \end{Bmatrix}$$

Displacement Solution : $\underline{U} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -0.0016 \ 0.0032]^T$ m

<u>Sensitivity Analysis:</u> <u>Horizontal Displacement</u>, u_8

$$\underset{(1\times1)}{\frac{du_8}{dA_i}} = \underset{(1\times1)}{\cancel{\frac{\partial u_8}{\partial A_i}}^{0}} + \underset{(1\times8)}{\left(\frac{\partial u_8}{\partial \underline{U}}\right)^T} \underset{(8\times1)}{\frac{d\underline{U}}{dA_i}}$$

From FEA equations $\frac{d\underline{U}}{dA_i} = \underline{K}^{-1}\left[\cancel{\frac{d\underline{F}}{dA_i}}^{0} - \frac{d\underline{K}}{dA_i}\underline{U}\right]$ (□)

Need to determine $\frac{dK}{dA_i}$ for A_1, A_2, A_3

Solving $\frac{dK}{dA_1}$ for example, go back to formulation of global stiffness

- only element that contributes terms with A_1 to $\underline{K}$ if $\underline{k}_1$ so we really just need dk_1/dA_1

$$\frac{d\underline{k}_1}{dA_1} = T_1^T \frac{d}{dA_1}\left(\frac{A_1E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)T_1 = T_1^T \frac{E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} T_1$$

(in this case just $\underline{k}_1/A_1$)

$$\frac{dk_1}{dA_1} = 2.5597\times10^{10}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

- now placing in global coordinates

$$\frac{dK}{dA_1} = 2.5597\times10^{10}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

• similarly, we obtain $\frac{dK}{dA_2}$ & $\frac{dK}{dA_3}$

$$\frac{dK}{dA_2} = 7.2\times10^{10}\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&-1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&1 \end{bmatrix}, \quad \frac{dK}{dA_3} = 2.5597\times10^{10}\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&-1&-1\\ 0&0&0&0&1&1&-1&-1\\ 0&0&0&0&-1&-1&1&1\\ 0&0&0&0&-1&-1&1&1 \end{bmatrix}$$

For displacement in global degree of freedom 8,

$$\frac{\partial u_8}{\partial \underline{u}} = \left[\frac{\partial u_8}{\partial u_1}\ \frac{\partial u_8}{\partial u_2}\ \frac{\partial u_8}{\partial u_3}\ \frac{\partial u_8}{\partial u_4}\ \frac{\partial u_8}{\partial u_5}\ \frac{\partial u_8}{\partial u_6}\ \frac{\partial u_8}{\partial u_7}\ \frac{\partial u_8}{\partial u_8}\right]^T$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

<u>Direct Method</u>: Solve (□) for $i=1,2,3$ and substitute into (☆) below

$$\frac{du_8}{dA_i} = -\underbrace{\left(\frac{\partial u_8}{\partial \underline{u}}\right)^T}_{(1\times8)} \underbrace{\frac{d\underline{u}}{dA_i}}_{(8\times1)} \qquad (☆)$$

Here since we must invert $\underline{K}$, we use reduced relations consistent with boundary conditions in FEA problem ($\underline{K}$ 8×8 is singular)

For example for A_1, $\frac{d\underline{u}}{dA_1} = \underline{K}^{-1}\left(\cancel{\frac{dF}{dA_1}}^{0} - \frac{dK}{dA_1}\underline{u}\right)$

$$\underset{(\text{reduced})}{\frac{d\underline{u}}{dA_1}} = \underbrace{\begin{bmatrix} 1.2799\times10^{8} & 0 \\ 0 & 3.0899\times10^{8}\end{bmatrix}^{-1}}_{\underline{K}^{-1}} \underbrace{2.5597\times10^{10}\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}}_{\frac{dK}{dA_1}} \underbrace{\begin{Bmatrix} -0.0016 \\ 0.0032 \end{Bmatrix}}_{\underline{u}}$$

(see page E6 for $\frac{d\underline{u}}{dA_1}, \frac{d\underline{u}}{dA_2}, \frac{d\underline{u}}{dA_3}$)

Solving for each sensitivity yields,

$$\frac{du_8}{dA_1} = -0.3976, \quad \frac{du_8}{dA_2} = -0.7583, \quad \frac{du_8}{dA_3} = -0.1387$$

<u>Adjoint Method</u>: introduce adjoint variable $\underline{\lambda}$ to (☆) & (□)

$$\frac{du_8}{dA_i} = \underline{\lambda}^T \frac{d\underline{K}}{dA_i}\underline{u} \qquad (*)$$

where $\underline{\lambda}^T = \left(\frac{\partial u_8}{\partial \underline{u}}\right)^T \underline{K}^{-1} \Rightarrow \underline{K}\,\underline{\lambda} = \frac{\partial u_8}{\partial \underline{u}}$

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Adjoint problem (reduced with same boundary conditions as original FEA)

$$\underline{K}\underline{\lambda} = \frac{\partial U_8}{\partial \underline{U}} \Rightarrow 10^8 \begin{bmatrix} 1.2799 & 0 \\ 0 & 3.0899 \end{bmatrix} \begin{Bmatrix} \lambda_7 \\ \lambda_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Solving yields, $\begin{Bmatrix} \lambda_7 \\ \lambda_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.3236 \times 10^{-8} \end{Bmatrix}$

Substituting back into (*)

$$\frac{dU_8}{dA_i} = \begin{bmatrix} 0 & 0.3236 \times 10^8 \end{bmatrix} \frac{d\underline{K}}{dA_i} \underline{U}$$

Solving for $i = 1, 2, 3$ (again on reduced indices coorresponding to BCs yields)

$$\frac{dU_8}{dA_1} = -0.3976, \quad \frac{dU_8}{dA_2} = -0.7583, \quad \frac{dU_8}{dA_3} = -0.1387$$ (same as direct method)

Sensitivity Analysis: Axial Stress, σ_3

Recall one way (there are others) to compute axial stress in element is from axial load in local coordinates.

$$\underset{(2\times1)}{\underline{\sigma}_3} = \frac{\overset{(2\times1)}{\underline{f}_3}}{A_3}$$

$\underset{(2\times1)}{\underline{f}_3}$ ... axial force in element 3 in local (2x2) coordinates

$$\underset{(2\times1)}{\underline{f}_3} = \underset{(2\times2)}{\underline{K}_3} \underset{(2\times1)}{\underline{U}_3} \quad \underline{\underline{OR}} \quad \underset{(2\times1)}{\underline{f}_3} = \underset{(2\times4)}{\underline{T}_3} \underset{(4\times4)}{\underline{K}_3} \underset{(4\times1)}{\underline{U}_3}$$

local coordinates | global coordinates

Using global coordinate relation for $\underline{f}_3$

$$\frac{d\sigma_3}{dA_i} = \frac{\partial \sigma_3}{\partial A_i} + \frac{\partial \sigma_3}{\partial \underline{f}_3} \frac{d\underline{f}_3}{dA_i}$$

note $\underline{f}_3$ has $\underline{K}_3(A_i)$ & $\underline{U}_3(A_i)$

$$\frac{d\underline{f}_3}{dA_i} = \cancelto{0}{\frac{\partial \underline{f}_3}{\partial A_i}} + \frac{\partial \underline{f}_3}{\partial \underline{K}_3} \frac{d\underline{K}_3}{dA_i} + \frac{\partial \underline{f}_3}{\partial \underline{U}_3} \frac{d\underline{U}_3}{dA_i}$$

$$\frac{d\underline{U}_3}{dA_i} = \cancelto{0}{\frac{\partial \underline{U}_3}{\partial A_i}} + \left(\frac{\partial \underline{U}_3}{\partial \underline{U}}\right)^T \frac{d\underline{U}}{dA_i}$$

back to needing global displacement vector derivative!

$$\frac{d\sigma_3}{dA_i} = \frac{\partial \sigma_3}{\partial A_i} + \underset{①}{\frac{\partial \sigma_3}{\partial \underline{f}_3}} \left(\underset{②}{\frac{\partial \underline{f}_3}{\partial \underline{K}_3}} \frac{d\underline{K}_3}{dA_i} + \underset{③}{\frac{\partial \underline{f}_3}{\partial \underline{U}_3}} \left(\frac{\partial \underline{U}_3}{\partial \underline{U}}\right)^T \frac{d\underline{U}}{dA_i} \right)$$

$$\frac{d\underline{U}}{dA_i} = \underline{K}^{-1}\left(\frac{d\underline{F}}{dA_i} - \frac{d\underline{K}}{dA_i}\underline{U}\right)$$

Using knowledge of $\sigma_3 = \frac{f_3}{A_3}$ can determine terms ①, ②, ③

① $\underset{(1\times1)}{\frac{\partial \sigma_3}{\partial f_3}} = \underset{(1\times1)}{\frac{1}{A_3}}$ ② $\underset{(2\times1)}{\frac{\partial f_3}{\partial \underline{K}_3}} = \underset{(2\times4)(4\times1)}{\underline{T}_3 \underline{U}_3}$ ③ $\underset{(2\times4)}{\frac{\partial f_3}{\partial \underline{U}_3}} = \underset{(2\times4)(4\times4)}{\underline{T}_3 \underline{K}_3}$

$$\underset{(2\times1)}{\frac{d\sigma_3}{dA_i}} = \underset{(2\times1)}{\frac{\partial \sigma_3}{\partial A_i}} + \underset{(1\times1)}{\frac{1}{A_3}}\left(\underset{(2\times4)(4\times4)(4\times1)}{\underline{T}_3 \frac{d\underline{K}_3}{dA_i} \underline{U}_3} + \underset{(2\times4)(4\times4)}{\underline{T}_3 \underline{K}_3} \underset{(4\times8)}{\underbrace{\left(\frac{\partial \underline{U}_3}{\partial \underline{U}}\right)^T}_{②}} \underset{(8\times1)}{\underbrace{\frac{d\underline{U}}{dA_i}}_{①}}\right) \quad (*)$$

Direct Method

① From previous direct method for displacement sensitivity, already have $\frac{d\underline{U}}{dA_i}$

$$\frac{d\underline{U}}{dA_1} = [0\ 0\ 0\ 0\ 0\ 0\ -0.9598\ 0.3976]$$
$$\frac{d\underline{U}}{dA_2} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0.7583]$$
$$\frac{d\underline{U}}{dA_3} = [0\ 0\ 0\ 0\ 0\ 0\ 0.3347\ 0.1367]$$

② Recall that $\underline{U}_3 = [U_5\ U_6\ U_7\ U_8]^T$
→ displacement vector for element 3 with global coordinate indices

Then,
$$\underset{(8\times4)}{\frac{d\underline{U}_3}{\partial \underline{U}}} = \begin{bmatrix} \partial U_5/\partial U_1 & \partial U_6/\partial U_1 & \partial U_7/\partial U_1 & \partial U_8/\partial U_1 \\ \partial U_5/\partial U_2 & \partial U_6/\partial U_2 & \partial U_7/\partial U_2 & \partial U_8/\partial U_2 \\ \partial U_5/\partial U_3 & \vdots & \vdots & \vdots \\ \partial U_5/\partial U_4 & & & \\ \vdots & & & \\ \partial U_5/\partial U_8 & \partial U_6/\partial U_8 & \partial U_7/\partial U_8 & \partial U_8/\partial U_8 \end{bmatrix} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}$$

$$\frac{d\sigma_3}{dA_1} = \cancel{\frac{\partial \sigma_3}{\partial A_1}}^{0} + \frac{1}{A_3}\left(\underline{T}_3 \overset{\text{zero matrix}}{\cancel{\frac{d\underline{K}_3}{dA_1}}} \underline{U}_3 + \underline{T}_3 \underline{K}_3 \left(\frac{\partial \underline{U}_3}{\partial \underline{U}}\right)^T \frac{d\underline{U}}{dA_1}\right) = 2.0353\times10^{10}\begin{bmatrix}-1\\1\end{bmatrix}$$

(element 3 does explicitly depend on A_1 or A_2)

$$\frac{d\sigma_3}{dA_2} = \cancel{\frac{\partial \sigma_3}{\partial A_2}}^{0} + \frac{1}{A_3}\left(\underline{T}_3 \cancel{\frac{d\underline{K}_3}{dA_2}}^{0} \underline{U}_3 + \underline{T}_3 \underline{K}_3 \left(\frac{\partial \underline{U}_3}{\partial \underline{U}}\right)^T \frac{d\underline{U}}{dA_2} = 2.7452\times10^{10}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\frac{d\sigma_3}{dA_3} = \underbrace{\frac{\partial \sigma_3}{\partial A_3}}_{-\frac{f_3}{A_3^2}} + \frac{1}{A_3}\left(\underline{T}_3 \frac{d\underline{K}_3}{dA_3} \underline{U}_3 + \underline{T}_3 \underline{K}_3 \left(\frac{\partial \underline{U}_3}{\partial \underline{U}}\right)^T \frac{d\underline{U}}{dA_3} = 1.713\times10^{10}\begin{bmatrix}1\\-1\end{bmatrix}$$

Adjoint Method: Introducing adjoint variable $\underline{\lambda}$ to (*)

$$\frac{d\sigma_3}{dA_i} = \frac{\partial \sigma_3}{\partial A_i} + \frac{1}{A_3}\left(\underline{T}_3 \frac{dK_3}{dA_i}\underline{u}_3 + \underline{\lambda}^T\left(\frac{d\underline{F}}{dA_i} + \frac{d\underline{K}}{dA_i}\underline{u}\right)\right) \quad (\Delta)$$

$$\text{where} \quad \underline{\lambda}^T = \underline{T}_3 \underline{K}_3 \left(\frac{\partial \underline{u}_3}{\partial \underline{u}}\right)^T \underline{K}^{-1}$$

So the adjoint problem becomes,

$$\underline{K}\,\underline{\lambda} = \left(\underline{T}_3 \underline{K}_3 \frac{\partial \underline{u}_3}{\partial \underline{u}}^T\right)^T$$

which takes a reduced form

$$\begin{bmatrix} 1.2799\times10^8 & 0 \\ 0 & 3.0899\times10^8 \end{bmatrix} \underline{\lambda} = \begin{bmatrix} 9.05\times10^7 & -9.05\times10^7 \\ 9.05\times10^7 & -9.05\times10^7 \end{bmatrix}$$

So in reduced form $\underline{\lambda} = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.2929 & -0.2929 \end{bmatrix}$

and full global indices,

$$\underline{\lambda} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.7071 & -0.7071 \\ 0.2929 & -0.2929 \end{bmatrix}$$

Substitution into (Δ) & solving for $i = 1, 2, 3$ yields same answers as direct method for stresses