introduction to proposition logic

- Basic elements of proposition logic
- Fundamental operations on propositions logic
- Truth table
- Logical equivalences
- Algebraic operations on propositions
- Predicates and quantifiers

Discrete Mathematics - Propositional Logic

The rules of mathematical logic specify methods of reasoning mathematical statements. Greek philosopher, Aristotle, was the pioneer of logical reasoning. Logical reasoning provides the theoretical base for many areas of mathematics and consequently computer science. It has many practical applications in computer science like design of computing machines, artificial intelligence, definition of data structures for programming languages etc.

Propositional Logic is concerned with statements to which the truth values, "true" and "false", can be assigned. The purpose is to analyze these statements either individually or in a composite manner.

Prepositional Logic – Definition

A proposition is a collection of declarative statements that has either a truth value "true" or a truth value "false". A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, etc). The connectives connect the propositional variables.

Some examples of Propositions are given below −

- "Man is Mortal", it returns truth value "TRUE"
- "12 + 9 = 3 – 2", it returns truth value "FALSE"

The following is not a Proposition −

- "A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true or false.

Connectives

In propositional logic generally we use five connectives which are −

- OR ($\vee$)

- AND ($\wedge$)

- Negation/ NOT ($\neg$)

- Implication / if-then ($\rightarrow$)

- If and only if ($\Leftrightarrow$).

OR (∨) – The OR operation of two propositions A and B (written as $A \lor B$) is true if at least

any of the propositional variable A or B is true.

The truth table is as follows –

A	B	A ∨ B
True	True	True
True	False	True
False	True	True
False	False	False

AND ($\wedge$) – The AND operation of two propositions A and B (written as $A \wedge B$) is true if both

the propositional variable A and B is true.

The truth table is as follows –

A	B	A ∧ B
True	True	True
True	False	False
False	True	False
False	False	False

Negation (¬) − The negation of a proposition A (written as $\neg A$) is false when A is true and is true when A is false.

The truth table is as follows −

A	¬ A
True	False
False	True

Implication / if-then ($\rightarrow$) – An implication $A \rightarrow B$ is the proposition "if A, then B". It is false if A is true and B is false. The rest cases are true.

The truth table is as follows –

A	B	A → B
True	True	True
True	False	False
False	True	True
False	False	True

If and only if ($\Leftrightarrow$ **) −** $A \Leftrightarrow B$ is bi-conditional logical connective which is true when p and q are same, i.e. both are false or both are true.

The truth table is as follows −

A	B	A ⇔ B
True	True	True
True	False	False
False	True	False
False	False	True

Tautologies

A Tautology is a formula which is always true for every value of its propositional variables.

Example – Prove $[(A \rightarrow B) \wedge A] \rightarrow B$ is a tautology

The truth table is as follows −

A	B	A → B	(A → B) ∧ A	[(A → B) ∧ A] → B
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

As we can see every value of $[(A \rightarrow B) \wedge A] \rightarrow B$ is "True", it is a tautology.

Contradictions

A Contradiction is a formula which is always false for every value of its propositional variables.

Example − Prove $(A \lor B) \land [(\neg A) \land (\neg B)]$ is a contradiction

The truth table is as follows −

A	B	A ∨ B	¬ A	¬ B	(¬ A) ∧ (¬ B)	(A ∨ B) ∧ [(¬ A) ∧ (¬ B)]
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

As we can see every value of $(A \lor B) \land [(\neg A) \land (\neg B)]$ is "False", it is a contradiction.

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

Example − Prove $(A \lor B) \land (\neg A)$ a contingency

The truth table is as follows −

A	B	A ∨ B	¬ A	(A ∨ B) ∧ (¬ A)
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

As we can see every value of $(A \lor B) \land (\neg A)$ has both "True" and "False", it is a contingency.

Propositional Equivalences

Two statements X and Y are logically equivalent if any of the following two conditions hold −

- The truth tables of each statement have the same truth values.

- The bi-conditional statement $X \Leftrightarrow Y$ is a tautology.

Example − Prove $\neg(A \vee B) and [(\neg A) \wedge (\neg B)]$ are equivalent

Testing by 1st method (Matching truth table)

A	B	A ∨ B	¬ (A ∨ B)	¬ A	¬ B	[(¬ A) ∧ (¬ B)]
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Here, we can see the truth values of $\neg(A \vee B) and [(\neg A) \wedge (\neg B)]$ are same, hence the statements are equivalent.

Testing by 2nd method (Bi-conditionality)

Testing by 2nd method (Bi-conditionality)

A	B	¬ (A ∨ B)	[(¬ A) ∧ (¬ B)]	[¬ (A ∨ B)] ⇔ [(¬ A) ∧ (¬ B)]
True	True	False	False	True
True	False	False	False	True
False	True	False	False	True
False	False	True	True	True

As $[\neg(A \vee B)] \Leftrightarrow [(\neg A) \wedge (\neg B)]$ is a tautology, the statements are equivalent.

Inverse, Converse, and Contra-positive

Implication / if-then $(\rightarrow)$ is also called a conditional statement. It has two parts −

- Hypothesis, p
- Conclusion, q

As mentioned earlier, it is denoted as $p \rightarrow q$.

Example of Conditional Statement − "If you do your homework, you will not be punished." Here, "you do your homework" is the hypothesis, p, and "you will not be punished" is the conclusion, q.

Inverse − An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is "If p, then q", the inverse will be "If not p, then not q". Thus the inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

Example − The inverse of "If you do your homework, you will not be punished" is "If you do not do your homework, you will be punished."

Converse − The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is "If p, then q", the converse will be "If q, then p". The converse of $p \rightarrow q$ is $q \rightarrow p$.

Example − The converse of "If you do your homework, you will not be punished" is "If you will not be punished, you do your homework".

Contra-positive − The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is "If p, then q", the contra-positive will be "If not q, then not p". The contra-positive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

Example − The Contra-positive of " If you do your homework, you will not be punished" is "If you are punished, you did not do your homework".

Normal Forms

We can convert any proposition in two normal forms −

- Conjunctive normal form
- Disjunctive normal form

Conjunctive Normal Form

A compound statement is in conjunctive normal form if it is obtained by operating AND among variables (negation of variables included) connected with ORs. In terms of set operations, it is a compound statement obtained by Intersection among variables connected with Unions.

Examples

- $(A \lor B) \land (A \lor C) \land (B \lor C \lor D)$

- $(P \cup Q) \cap (Q \cup R)$

Disjunctive Normal Form

A compound statement is in disjunctive normal form if it is obtained by operating OR among variables (negation of variables included) connected with ANDs. In terms of set operations, it is a compound statement obtained by Union among variables connected with Intersections.

Examples

- $(A \land B) \lor (A \land C) \lor (B \land C \land D)$

- $(P \cap Q) \cup (Q \cap R)$

Discrete Mathematics - Predicate Logic

Predicate Logic deals with predicates, which are propositions containing variables.

Predicate Logic – Definition

A predicate is an expression of one or more variables defined on some specific domain. A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.

The following are some examples of predicates −

- Let E(x, y) denote "x = y"
- Let X(a, b, c) denote "a + b + c = 0"
- Let M(x, y) denote "x is married to y"

Well Formed Formula

Well Formed Formula (wff) is a predicate holding any of the following –

- All propositional constants and propositional variables are wffs

- If x is a variable and Y is a wff, $\forall x Y$ and $\exists x Y$ are also wff

- Truth value and false values are wffs

- Each atomic formula is a wff

- All connectives connecting wffs are wffs

Quantifiers

The variable of predicates is quantified by quantifiers. There are two types of quantifier in predicate logic − Universal Quantifier and Existential Quantifier.

Universal Quantifier

Universal quantifier states that the statements within its scope are true for every value of the specific variable. It is denoted by the symbol $\forall$.

$\forall x P(x)$ is read as for every value of x, P(x) is true.

Example − "Man is mortal" can be transformed into the propositional form $\forall x P(x)$ where P(x) is the predicate which denotes x is mortal and the universe of discourse is all men.

Existential Quantifier

Existential quantifier states that the statements within its scope are true for some values of the specific variable. It is denoted by the symbol $\exists$.

$\exists x P(x)$ is read as for some values of x, P(x) is true.

Example – "Some people are dishonest" can be transformed into the propositional form $\exists x P(x)$ where P(x) is the predicate which denotes x is dishonest and the universe of discourse is some people.

Nested Quantifiers

If we use a quantifier that appears within the scope of another quantifier, it is called nested quantifier.

Example

- $\forall a \, \exists b \, P(x, y)$ where $P(a, b)$ denotes $a + b = 0$

- $\forall a \, \forall b \, \forall c \, P(a, b, c)$ where $P(a, b)$ denotes $a + (b + c) = (a + b) + c$

Note – $\forall a \, \exists b \, P(x, y) \neq \exists a \, \forall b \, P(x, y)$