Computational Methods for Fluid Mechanics – lecture 1 Viscous Incompressible Flow

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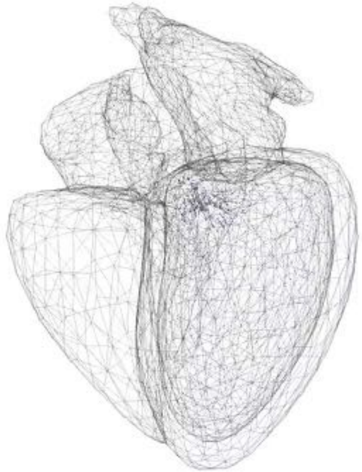
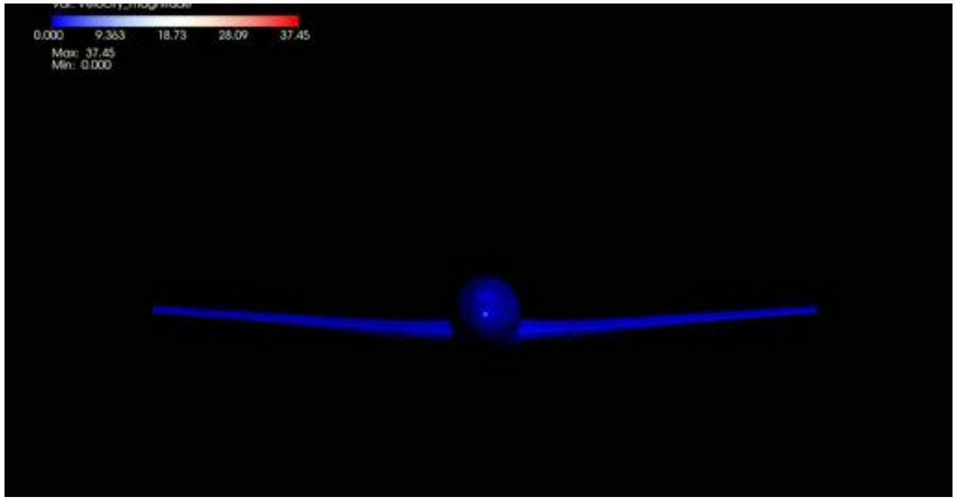
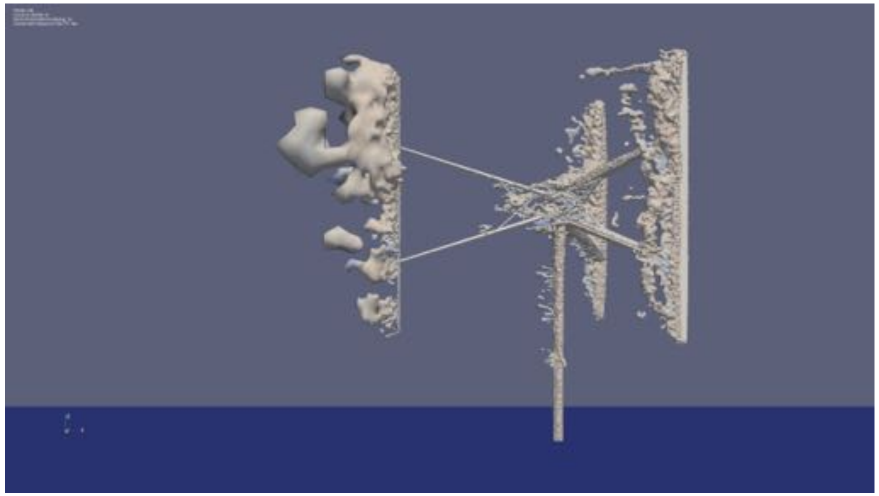
Who am I?



- Professor of numerical analysis at KTH in Stockholm
- Research: fluid dynamics, medicine, renewable energy,…
- https://www.kth.se/profile/jhoffman





Computational Methods for Fluid Mechanics

Lectures

1. Viscous incompressible flow (Monday)
2. The Navier-Stokes equations (Tuesday)
3. Error estimation and adaptive methods (Thursday)
4. Fluid-structure interaction (Friday)

- Lab exercises connected to lectures 1-3
- Lecture notes (covering some of the material)
- Course (open) GitHub repository:

  https://github.com/johanhoffman/Coimbra_2023

Today

- Introduction to Navier-Stokes equations
- Computational fluid mechanics in science and industry.
- Introduction to finite element methods
- Mixed finite element methods for viscous flow

Function spaces: continuous functions

For $\Omega \subset \mathbb{R}^n$, we define the set of functions with k continuous derivatives,

$$C^k(\Omega) = \{\phi : D^\alpha \phi \in C(\Omega),\ |\alpha| \leq k\}, \quad D_j = \partial/\partial x_j$$

with $C(\Omega) = C^0(\Omega)$ and $C^\infty(\Omega) = \cap_{k=0}^{\infty} C^k(\Omega)$. The subset $C_0^k(\Omega)$ consists of the functions $\phi \in C^k(\Omega)$ that have *compact support* in Ω, that is, the support

$$\mathrm{supp}(\phi) = \{x \in \Omega : \phi(x) \neq 0\},$$

is closed and bounded.

The function f(x) is Hölder continuous if

$$|f(x) - f(y)| \leq C|x-y|^\alpha, \quad \forall x, y \in I,$$

where C and α are real numbers such that $C > 0$ and $0 < \alpha \leq 1$. In the special case of $\alpha = 1$, we say that the function is *Lipschitz continuous*, with *Lipschitz constant* $L_f = C$.

Differential operators in R^n

The *gradient* of a scalar function $f \in C^1(\Omega)$ is denoted by

$$\operatorname{grad} f = \nabla f = (D_1 f, ..., D_n f)^T,$$

or in index notation $D_i f$, with the *nabla* operator

$$\nabla = (D_1, ..., D_n)^T.$$

Further, the *directional derivative* $\nabla_v f$, of f in the direction of the vector field $v : \mathbb{R}^n \to \mathbb{R}^n$, is defined as

$$\nabla_v f = (v \cdot \nabla) f = v_j D_j f.$$

Differential operators in R^n

For the $C^1(\Omega)$ vector field $F : \mathbb{R}^n \to \mathbb{R}^m$, we define the *Jacobian* J,

$$J = F' = \nabla F = \begin{bmatrix} D_1F_1 & \cdots & D_nF_1 \\ \vdots & \ddots & \vdots \\ D_1F_m & \cdots & D_nF_m \end{bmatrix} = \begin{bmatrix} (\nabla F_1)^T \\ \vdots \\ (\nabla F_m)^T \end{bmatrix} = D_jF_i,$$

with directional derivative

$$\nabla_v F = (v \cdot \nabla)F = Jv = v_j D_j F_i.$$

Differential operators in R^n

For a scalar function $f \in C^2(\mathbb{R}^n)$, we define the *Laplacian*

$$\Delta f = \nabla^2 f = \nabla^T \nabla f = \nabla \cdot \nabla f = D_1^2 f + ... + D_n^2 f = D_i^2 f,$$

The *vector Laplacian* of a $C^2(\Omega)$ vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, is defined as

$$\Delta F = \nabla^2 F = (\Delta F_1, ..., \Delta F_n)^T,$$

Gauss theorem

For a $C^1(\Omega)$ vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, we define the *divergence*

$$\operatorname{div} F = \nabla \cdot F = D_1F_1 + ... + D_nF_n = \frac{\partial F_i}{\partial x_i},$$

The divergence can be interpreted in terms of *Gauss theorem*, which states that the volume integral of the divergence of F in $\Omega \subset \mathbb{R}^n$, is equal to a surface integral over $\partial\Omega$ of F projected in the direction of the unit outward normal n of $\partial\Omega$,

$$\int_\Omega \nabla \cdot F \, dx = \int_{\partial\Omega} F \cdot n \, ds,$$

with the surface integral defined by a suitable parameterization of $\partial\Omega$.

Function spaces: integrable functions

For $\Omega \subset \mathbb{R}^n$ an open set and p a positive real number, we denote by $L^p(\Omega)$ the class of all Lebesgue measurable functions u defined on Ω, such that

$$\int_\Omega |u(x)|^p \, dx < \infty,$$

where we identify functions that are equal almost everywhere in Ω.

$L^p(\Omega)$ is a Banach space for $1 \leq p < \infty$, with the norm

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}.$$

In the case of a vector valued function $u : \mathbb{R}^n \to \mathbb{R}^m$, we replace the integrand in the definitions by the l^p norm, and for a matrix function $u : \mathbb{R}^n \to \mathbb{R}^{m \times k}$, a generalized Frobenius norm

$$\sum_i^m \sum_j^k |u_{ij}(x)|^p.$$

Function spaces: integrable functions

Theorem 3 (Hölder's inequality for $L^p(\Omega)$). *Let* $1 \leq p, q \leq \infty$ *and* $1/p + 1/q = 1$. *If* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *then* $fg \in L^1(\Omega)$, *and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

$L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = (u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx, \tag{2.1}$$

which induces the $L^2(\Omega)$ norm. For vector valued functions the integrand is replaced by the l_2 inner product, and for matrix functions by the Frobenius inner product. In what follows, we let $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$.

Function spaces: Sobolev spaces

To construct appropriate vector spaces for partial differential equations, we extend the L^p spaces with derivatives. We first define the *Sobolev norms*,

$$\|u\|_{k,p} = \left(\sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p},$$

for $1 \leq p < \infty$, and

$$\|u\|_{k,\infty} = \max_{0\leq|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)},$$

where $D^\alpha u$ refers to weak derivatives. Equipped with the Sobolev norms, we then define the *Sobolev spaces*,

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ 0 \leq |\alpha| \leq k\},$$

for each positive integer k and $1 \leq p \leq \infty$, with $W^{0,p}(\Omega) = L^p(\Omega)$.

Function spaces: Sobolev spaces

The Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$ are Hilbert spaces with the inner product and associated norm

$$(u, v)_k = \sum_{0 \le \alpha \le k} (D^\alpha u, D^\alpha v), \quad \|u\|_k = (u, u)_k^{1/2},$$

for which Cauchy-Schwarz inequality is satisfied,

$$|(u, v)_k| \le \|u\|_k \|v\|_k.$$

We denote by $H^{-k}(\Omega)$ the dual space of $H_0^k(\Omega)$, with the norm

$$\|u\|_{-k} = \sup_{v \in H_0^k(\Omega)} \frac{|(u, v)|}{\|v\|_k} = \sup_{v \in H_0^k(\Omega): \|v\|_k = 1} |(u, v)|,$$

satisfying a generalized Hölder inequality for $u \in H^{-k}(\Omega)$ and $v \in H_0^k(\Omega)$,

$$|(u, v)| \le \|u\|_{-k} \|v\|_k.$$

Partial integration/Green's theorem

For a scalar function $f : \mathbb{R}^n \to \mathbb{R}$, and a vector function $F : \mathbb{R}^n \to \mathbb{R}^n$, we have the following generalization of partial integration over a domain $\Omega \subset \mathbb{R}^n$, referred to as *Green's theorem*,

$$(\nabla f, F) = -(f, \nabla \cdot F) + \langle f, F \cdot n \rangle,$$

with n the unit outward normal vector for the boundary $\partial\Omega$, and where we use the notation

$$\langle v, w \rangle = (v, w)_{L^2(\partial\Omega)} \tag{2.2}$$

for the boundary integral. With $F = \nabla g$ and $g : \mathbb{R}^n \to \mathbb{R}$ a scalar function,

$$(\nabla f, \nabla g) = -(f, \Delta g) + \langle f, \nabla g \cdot n \rangle.$$

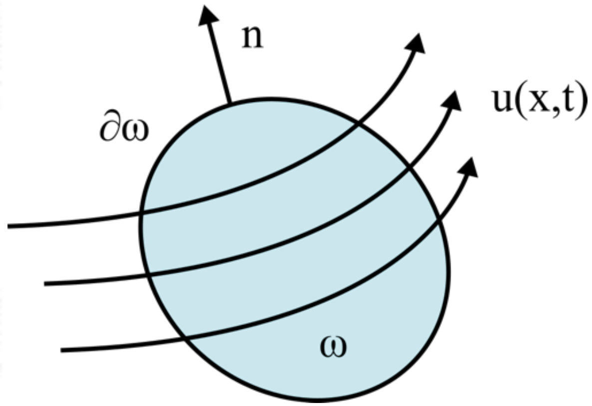
Conservation laws

Consider an arbitrary open subdomain $\omega \subset \mathbb{R}^n$. For a time $t > 0$, the total flow of a quantity with density $\phi(x,t)$ through the boundary $\partial\omega$ is given by

$$\int_{\partial\omega} \phi u \cdot n \, ds,$$

where n is the outward unit normal of $\partial\omega$, and $u = u(x,t)$ is the velocity of the flow. The change of the total quantity ϕ in ω is equal to the volume source or sink $s = s(x,t)$, minus the total flow of the quantity through the boundary $\partial\omega$,

$$\frac{d}{dt}\int_{\omega} \phi(x,t)\, dx = -\int_{\partial\omega} \phi u \cdot n \, ds + \int_{\omega} s(x,t)\, dx$$



Conservation laws

$$\frac{d}{dt}\int_\omega \phi(x,t)\,dx = -\int_{\partial\omega} \phi u \cdot n\,ds + \int_\omega s(x,t)\,dx,$$

Gauss' theorem,

$$\int_\omega \left(\frac{\partial}{\partial t}\phi(x,t) + \nabla\cdot(\phi u) - s\right)\,dx = 0,$$

and assuming the integrand is continuous in ω, we are lead to the general conservation equation

$$\dot{\phi} + \nabla\cdot(\phi u) - s = 0,$$

for $t > 0$ and $x \in \omega$, with $\omega \subset \mathbb{R}^n$ any open domain for which the equation is sufficiently regular.

Conservation of mass

Now consider the flow of a continuum with $\rho = \rho(x,t)$ the mass density of the continuum. The general continuity equation (5.1) with $\phi = \rho$ and zero source $s = 0$, gives the equation for conservation of mass

$$\dot{\rho} + \nabla \cdot (\rho u) = 0.$$

We say that a flow is *incompressible* if

$$\nabla \cdot u = 0,$$

or equivalently, if the *material derivative* is zero,

$$\frac{D\rho}{Dt} = \dot{\rho} + u \cdot \nabla \rho = 0,$$

since

$$0 = \dot{\rho} + \nabla \cdot (\rho u) = \frac{D\rho}{Dt} + \rho \nabla \cdot u.$$

Conservation of momentum

Newton's 2nd Law states that the change of *momentum* ρu over an arbitrary open subdomain $\omega \subset \mathbb{R}^n$, is equal to the sum of all forces, including *volume forces*,

$$\int_\omega \rho f \, dx,$$

for a force density $f = f(x,t) = (f_1(x,t), ..., f_n(x,t))$, and *surface forces*,

$$\int_{\partial\omega} n \cdot \sigma \, ds,$$

with the *Cauchy stress tensor* $\sigma = \sigma(x,t) = (\sigma_{ij}(x,t))$, and where we define $n \cdot \sigma = n^T \sigma = (\sigma_{ji} n_j)$. Gauss' theorem gives the total force as

$$\int_\omega \rho f \, dx + \int_{\partial\omega} n \cdot \sigma \, ds = \int_\omega (\rho f + \nabla \cdot \sigma) \, dx.$$

Conservation of momentum

The general continuity equation with $\phi = \rho u$, and the source given by the sum of all forces, leads to the equation for conservation of momentum

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = \rho f + \nabla \cdot \sigma, \tag{5.2}$$

with $u \otimes u = uu^T$, the tensor product of the velocity vector field u. With the help of conservation of mass, we can rewrite the left hand side as

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = u(\dot{\rho} + \nabla \cdot (\rho u)) + \rho(\dot{u} + (u \cdot \nabla)u) = \rho(\dot{u} + (u \cdot \nabla)u),$$

so that

$$\rho(\dot{u} + (u \cdot \nabla)u) = \rho f + \nabla \cdot \sigma. \tag{5.3}$$

We say that (5.2) is an equation on *conservation form*, whereas (5.3) is on *non-conservation form*.

Conservation of momentum

We define the *mechanical pressure* as the mean normal stress,

$$p_{mech} = -\frac{1}{3}\operatorname{tr}(\sigma) = -\frac{1}{3}I_1,$$

and the *deviatoric stress tensor* $\tau = \sigma + p_{mech}I$, with $\operatorname{tr}(\tau) = 0$, such that

$$\sigma = -p_{mech}I + \tau,$$

and we can write conservation of momentum as

$$\rho(\dot{u} + (u \cdot \nabla)u) = \rho f - \nabla p_{mech} + \nabla \cdot \tau.$$

Newtonian flow

To determine the deviatoric stress we need a constitutive model of the fluid. For a Newtonian fluid, the deviatoric stress depends linearly on the *strain rate tensor*

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

with $\tau = 2\mu\epsilon$, where μ is the *dynamic viscosity*, which we here assume to be constant.

The *incompressible Navier-Stokes equations* then takes the form,

$$\dot{u} + (u \cdot \nabla)u + \nabla p - \nu\Delta u = f, \tag{5.4}$$

$$\nabla \cdot u = 0, \tag{5.5}$$

with the *kinematic viscosity* $\nu = \mu/\rho$, and the kinematic pressure $p = p_{mech}/\rho$.

Non-dimensionalization

$$u = Uu_*, \quad p = Pp_*, \quad x = Lx_*, \quad f = Ff_*, \quad t = Tt_*,$$

where U, P, L, T are characteristic scales of the velocity, pressure, force, length and time, respectively. The resulting non-dimensionalized differential operators are scaled as,

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t_*}, \quad \nabla = \frac{1}{L}\nabla_*, \quad \Delta = \frac{1}{L^2}\Delta_*,$$

which gives

$$\frac{U}{T}\frac{\partial}{\partial t_*}u_* + \frac{U^2}{L}(u_* \cdot \nabla_*)u_* + \frac{P}{L}\nabla_* p_* - \frac{\nu U}{L^2}\Delta_* u_* = Ff_*,$$
$$\frac{U}{L}\nabla \cdot u_* = 0,$$

Non-dimensionalization

$$\dot{u} + (u \cdot \nabla)u + \nabla p - Re^{-1}\Delta u = f,$$
$$\nabla \cdot u = 0.$$

Here we have dropped the non-dimensional notation for simplicity, with

$$T = L/U, \quad P = U^2, \quad F = \frac{U^2}{L}, \quad Re = \frac{UL}{\nu},$$

where the *Reynolds number* Re determines the balance between inertial and viscous characteristics in the flow. For low Re linear viscous effects dominate, whereas for high Re we have a flow dominated by nonlinear inertial effect, and turbulence for sufficiently high Reynolds numbers.

Limit cases: Euler and Stokes equations

Formally, in the limit $Re \to \infty$, the viscous term vanishes and we are left with the inviscid *Euler equations*,

$$\begin{aligned} \dot{u} + (u \cdot \nabla)u + \nabla p &= f, \\ \nabla \cdot u &= 0, \end{aligned}$$

traditionally seen as a model for flow at high Reynolds numbers.

In the limit $Re \to 0$, we obtain the *Stokes equations* as a model of viscous flow, now with $P = \nu U/L$ and $F = \upsilon U/L^2$,

$$\begin{aligned} -\Delta u + \nabla p &= f, \\ \nabla \cdot u &= 0 \end{aligned}$$

Anatomy of fluid flow

- Density ρ
- Velocity u
- Pressure p
- Viscosity (dynamic viscosity μ, kinematic viscosity $\upsilon = \frac{\mu}{\rho}$)
- Gravity g
- Surface tension σ
- Speed of sound c
- …

Anatomy of fluid flow

- Mach number $M = \frac{u}{c}$
- Reynolds number Re = $\frac{\rho U L}{\mu} = \frac{UL}{\upsilon}$
- …

Compressibility – Shock waves

- Shock waves appear for M>1
- Flow is compressible for M>0.2
- Flow is incompressible for M<0.2

[https://en.wikipedia.org/wiki/Mach_number#/media/File:FA-18_Hornet_breaking_sound_barrier_(7_July_1999).jpg]

Compressibility – Shock waves

- Shock waves appear for M>1
- Flow is compressible for M>0.2
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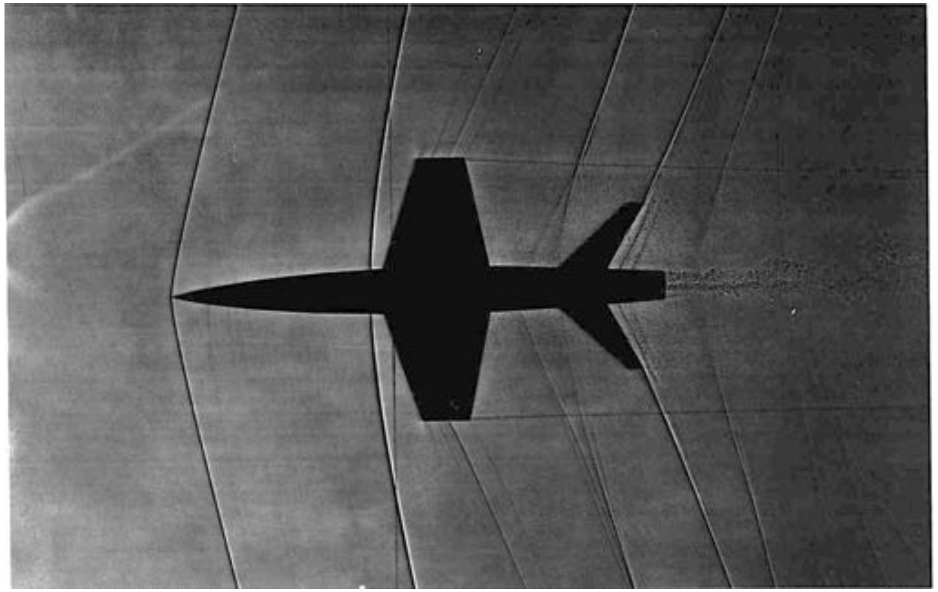
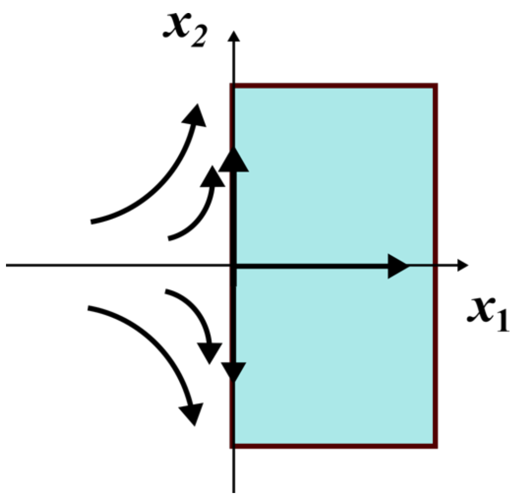
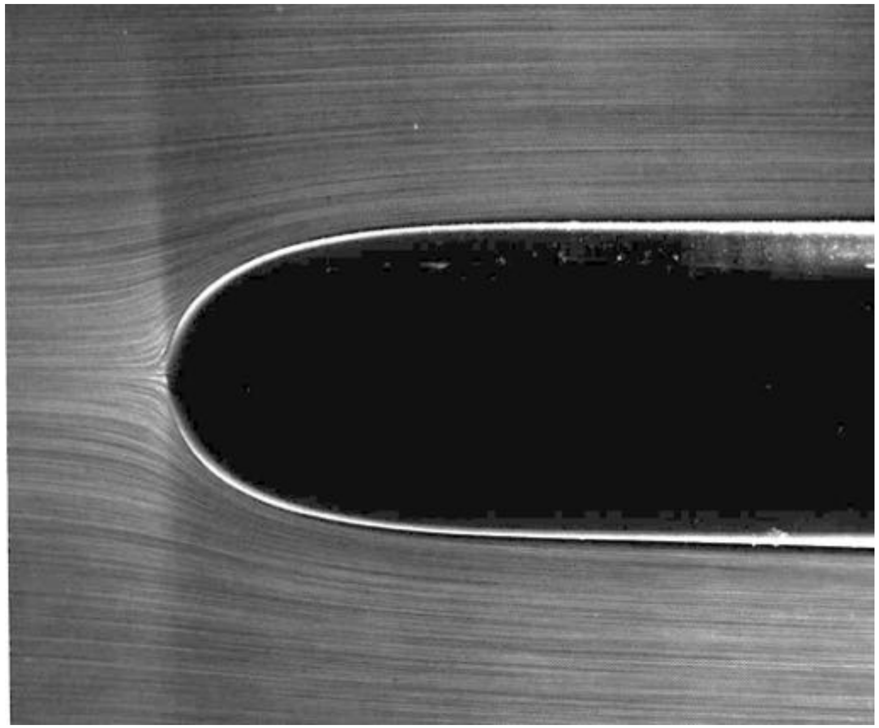


[Shadow graph]

Incompressible flow

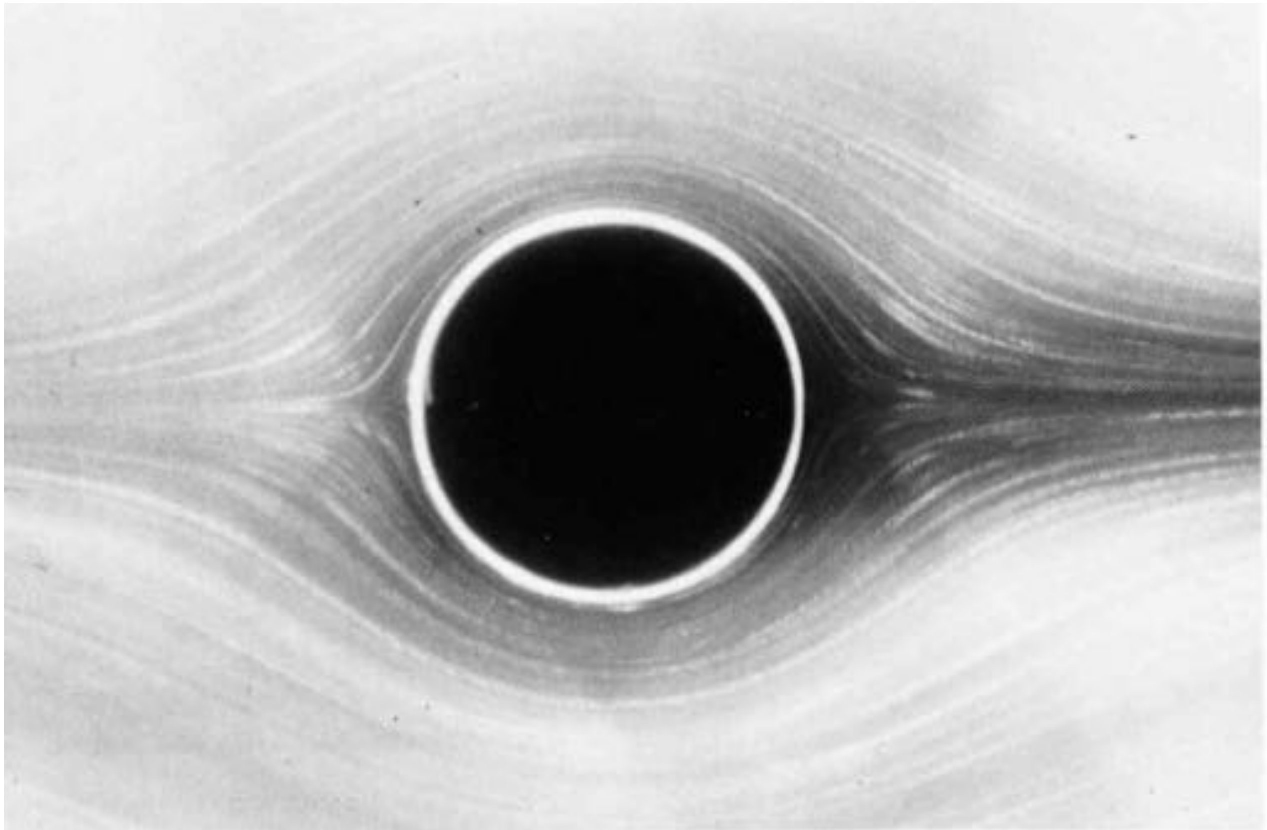
- Approximate small M by M = 0.
- Constant density
- Velocity divergence free: $\nabla \cdot u = 0$
- $\dfrac{\partial u_2}{\partial x_2} = -\dfrac{\partial u_1}{\partial x_1}$





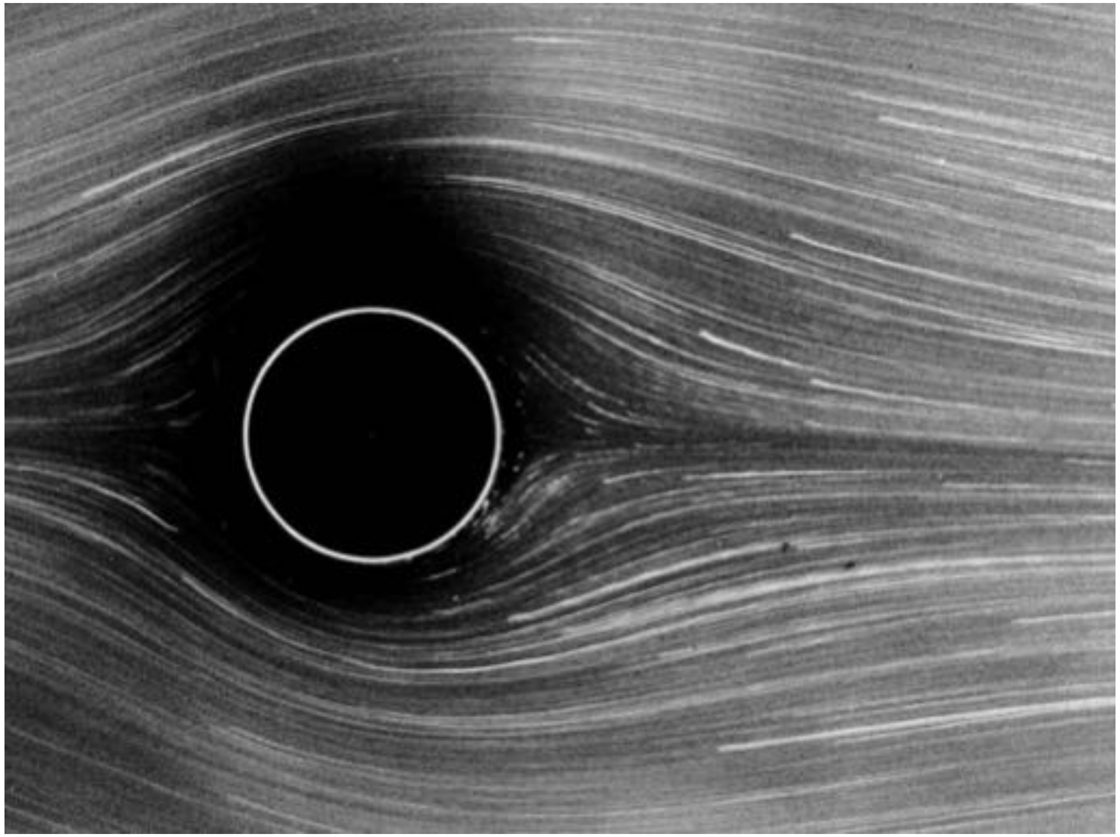
[Water and air bubbles.]

Reynolds number Re = 0.16



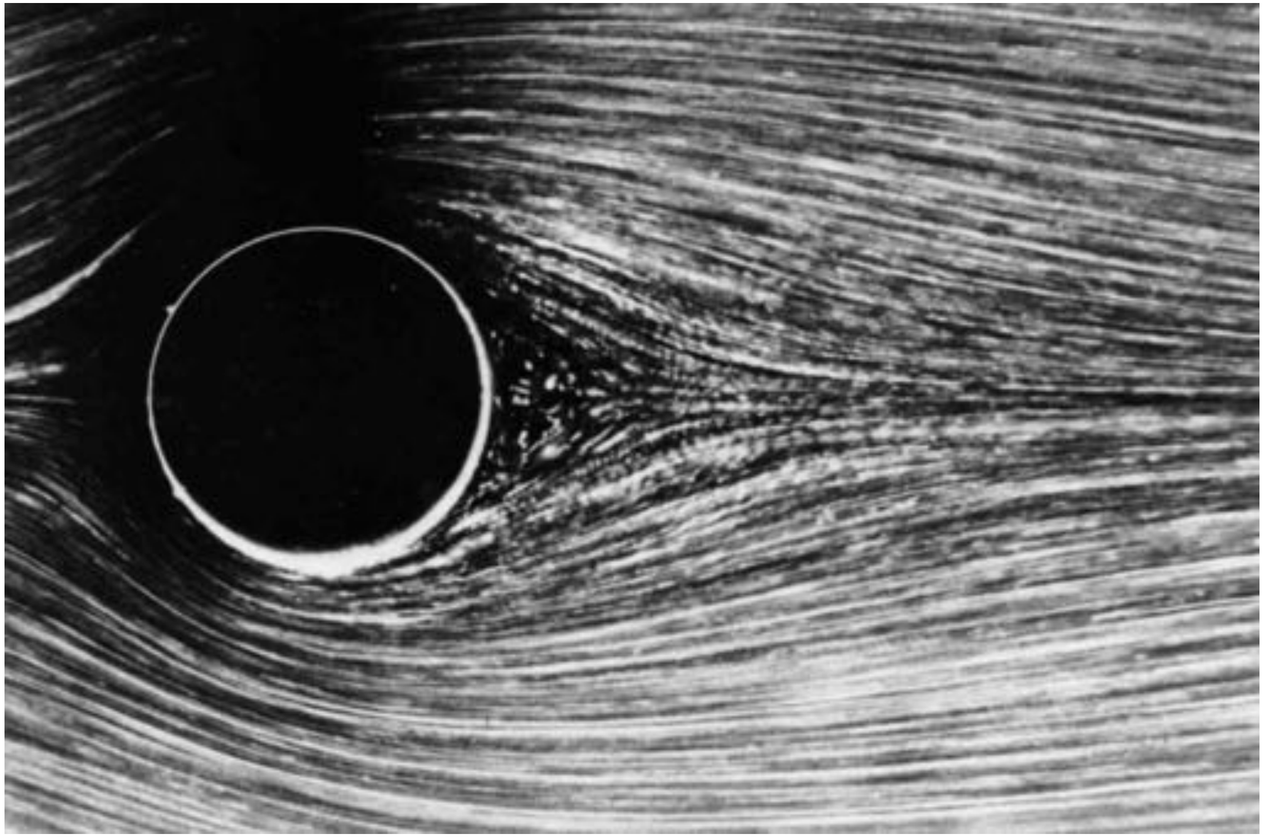
[Water and aluminum dust.]

Reynolds number Re = 1.54



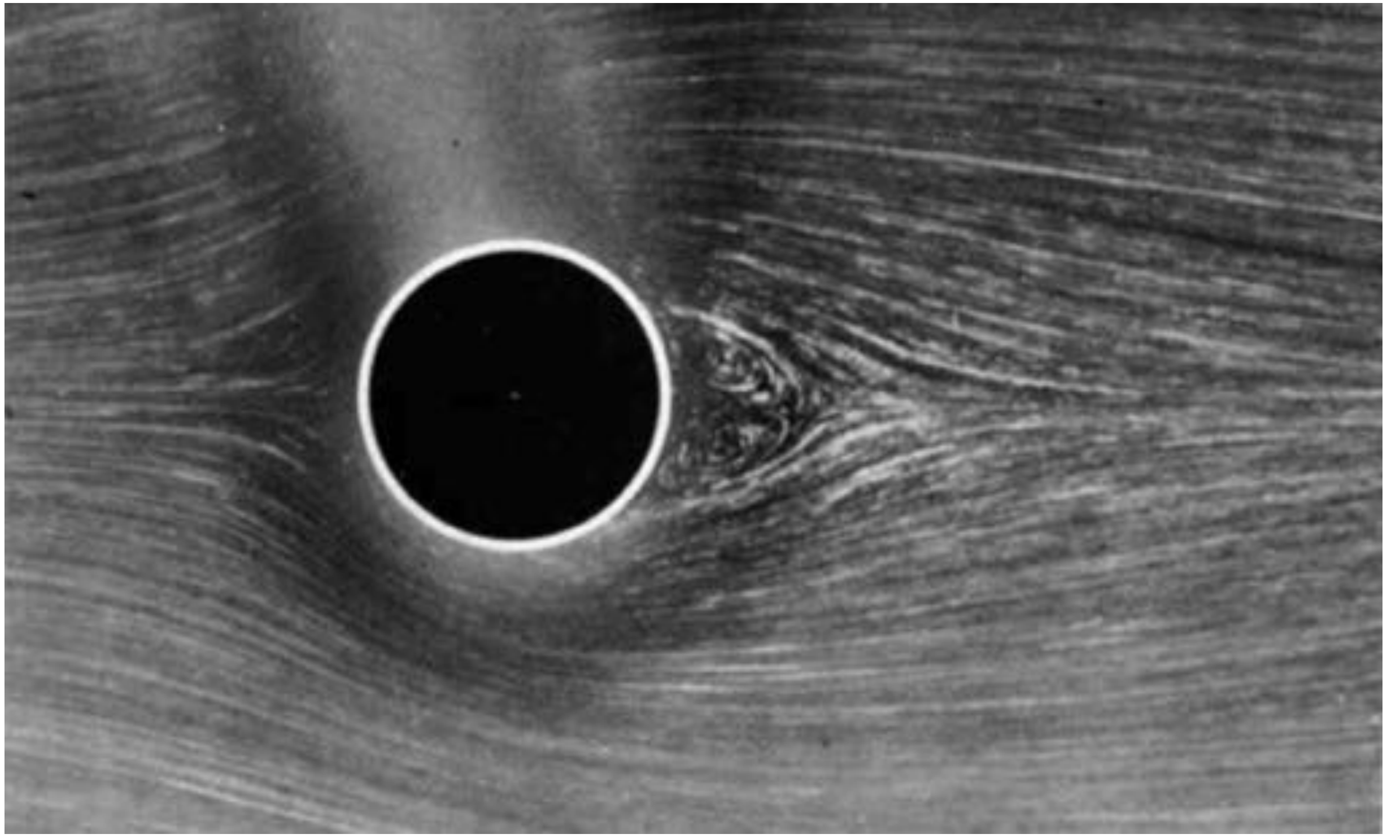
[Water and aluminum dust.]

Reynolds number Re = 9.6



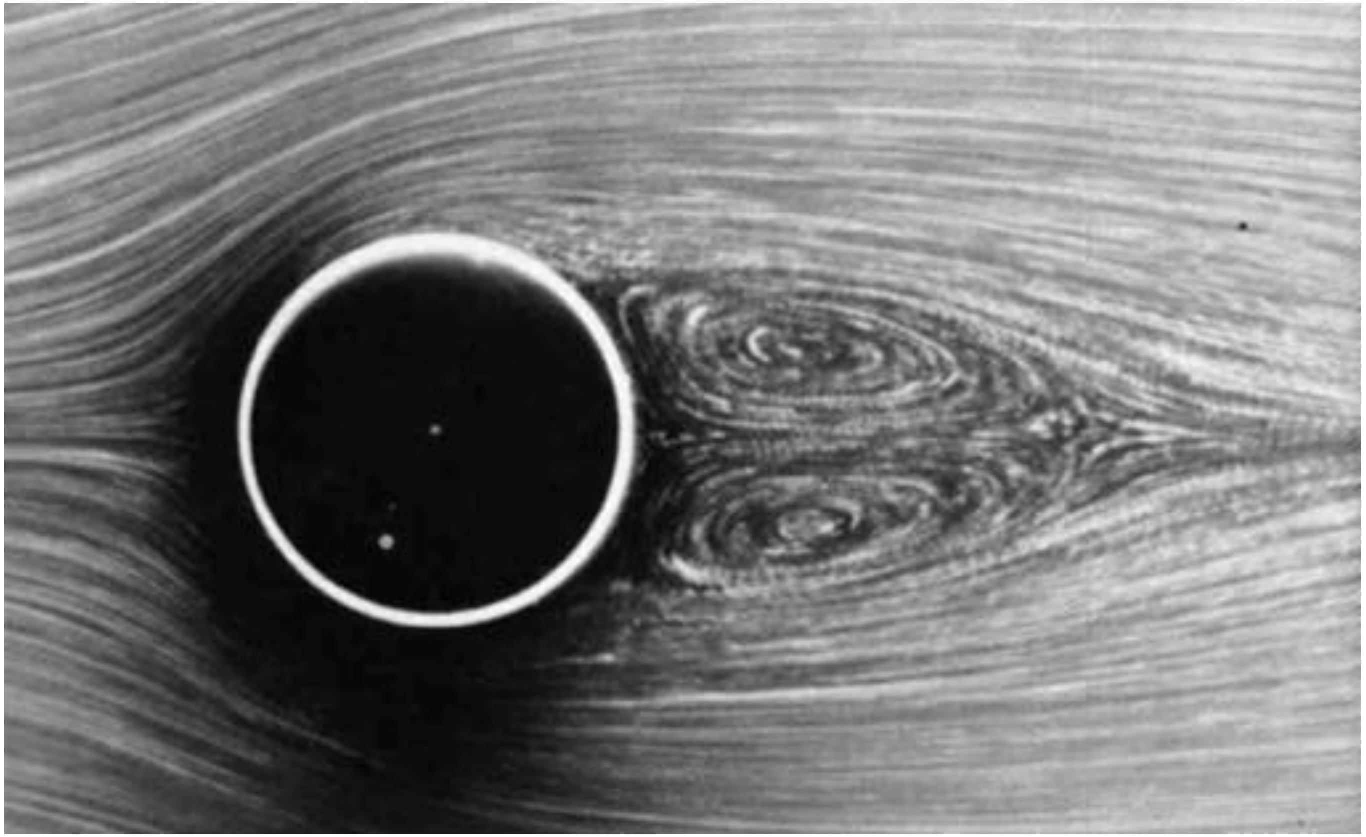
[Water and aluminum dust.]

Reynolds number Re = 9.6



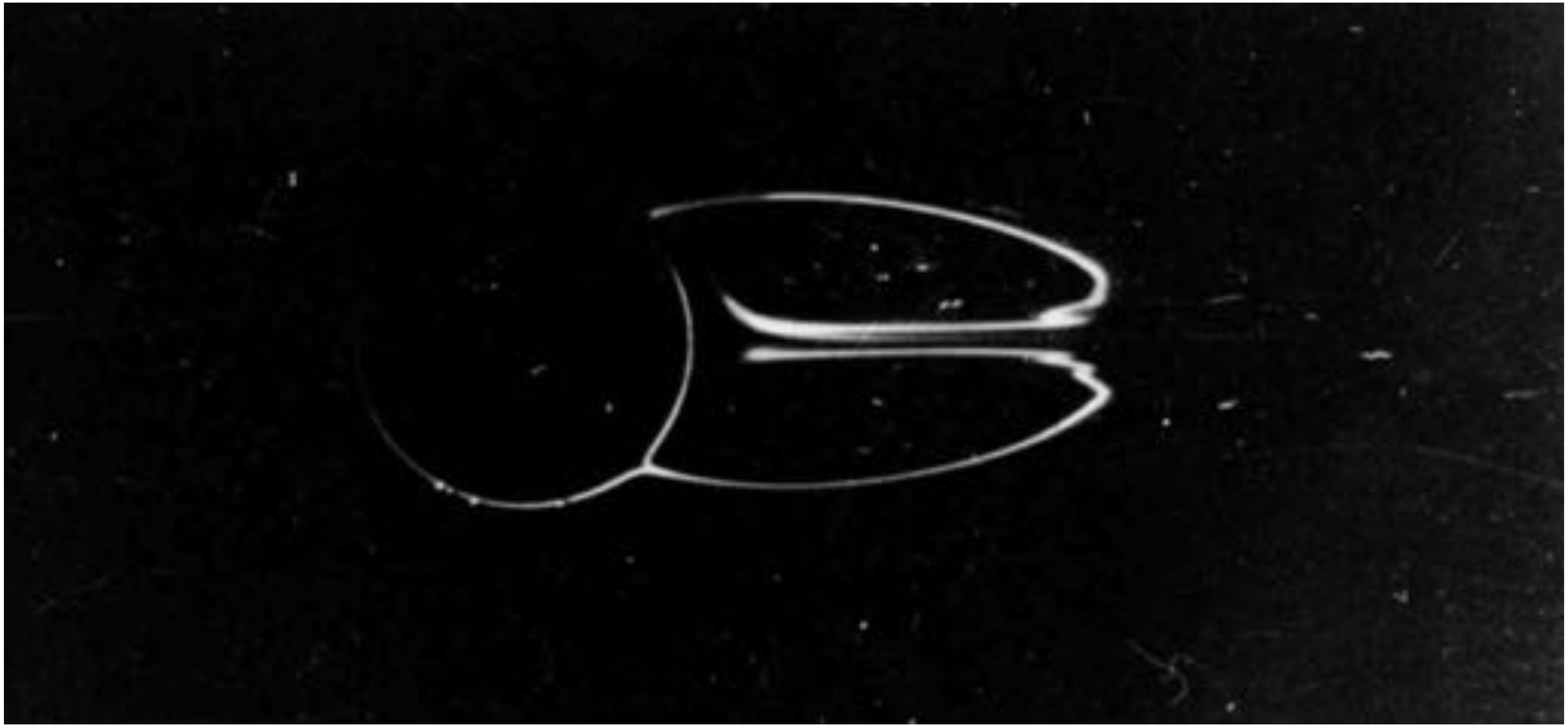
[Water and aluminum dust.]

Reynolds number Re = 26



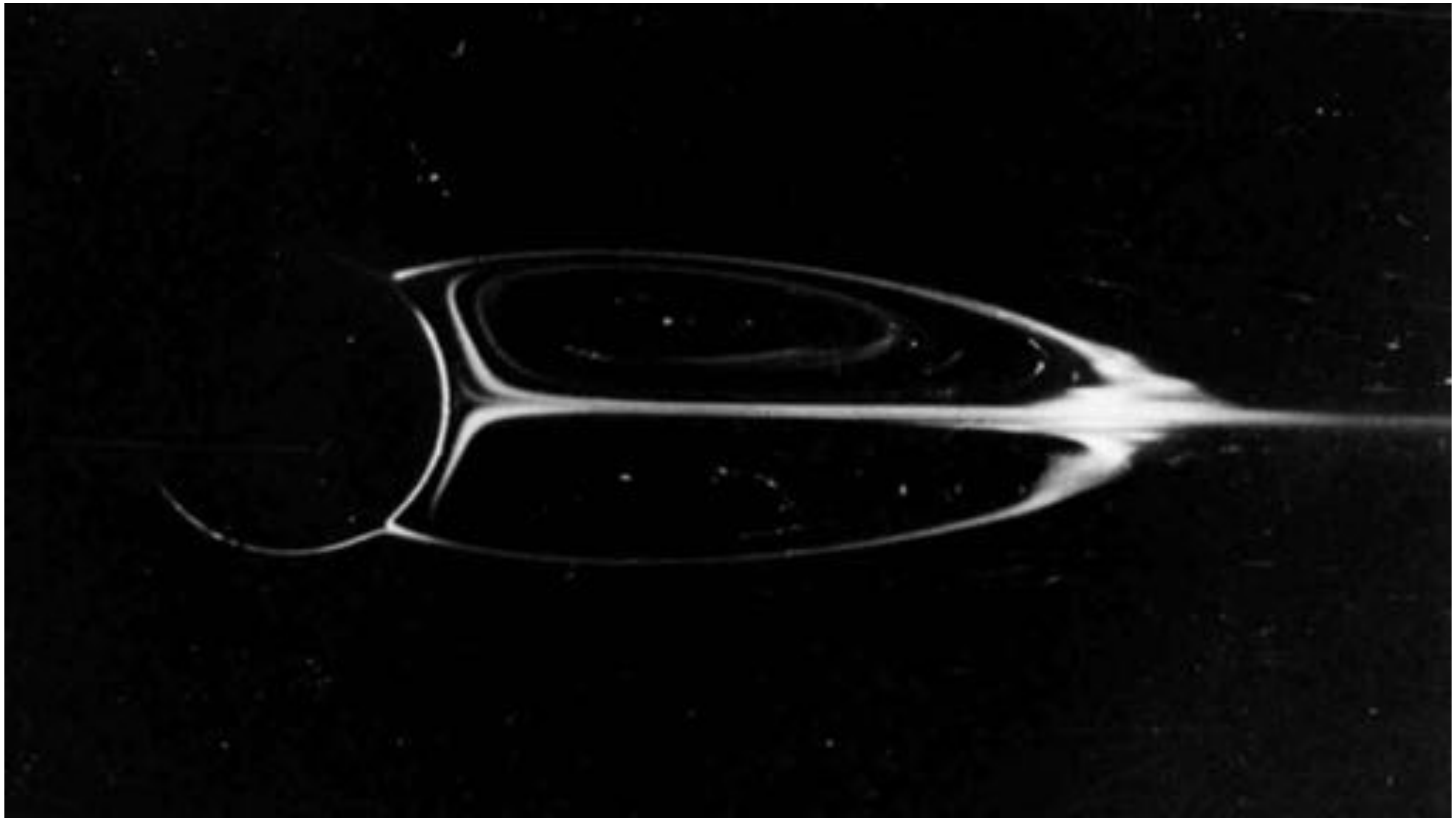
[Oil and magnesium.]

Reynolds number Re = 28.4



[Water and condensed milk.]

Reynolds number Re = 41



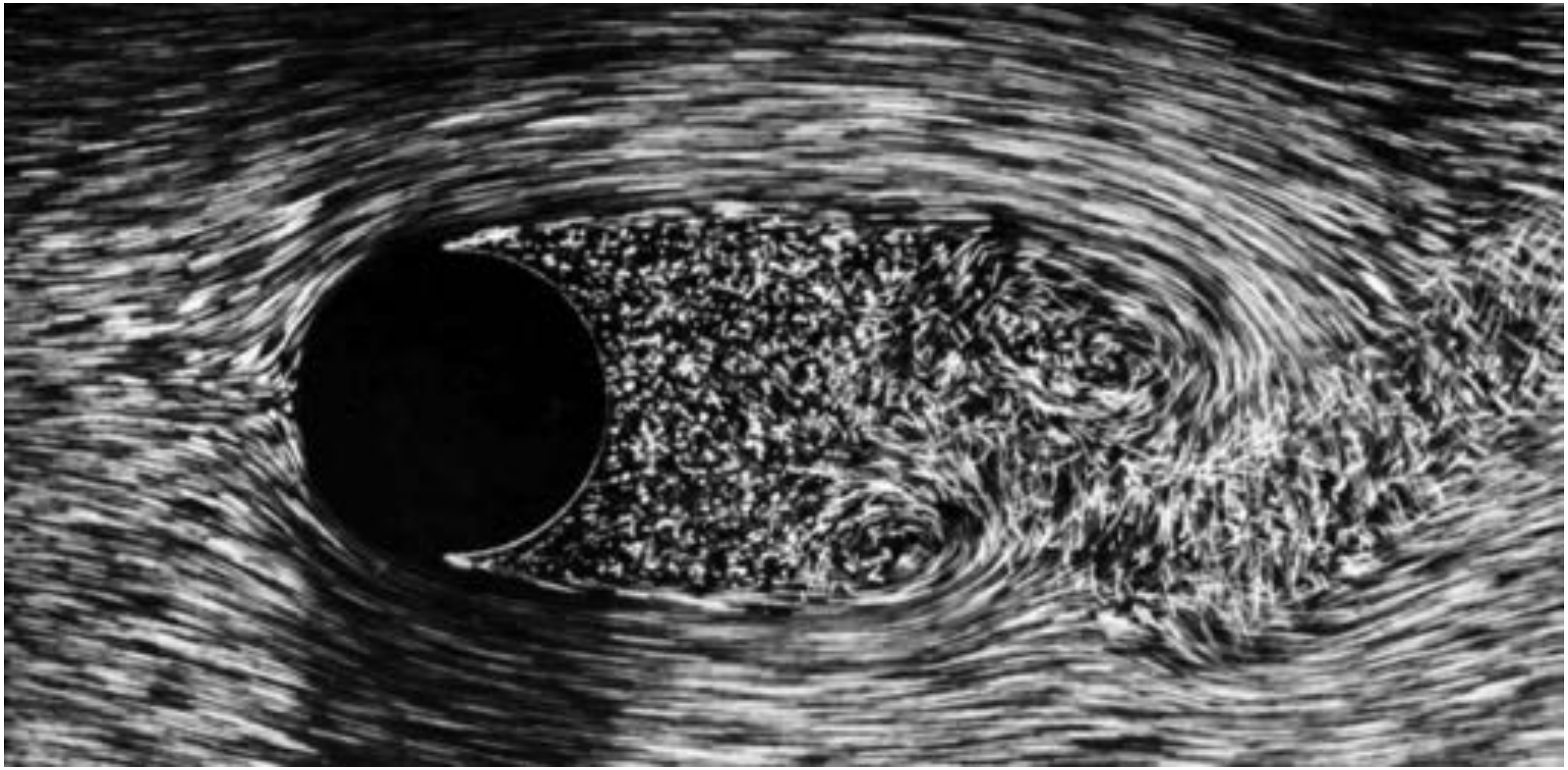
[Water and condensed milk.]

Reynolds number Re = 300

Karman vortex street

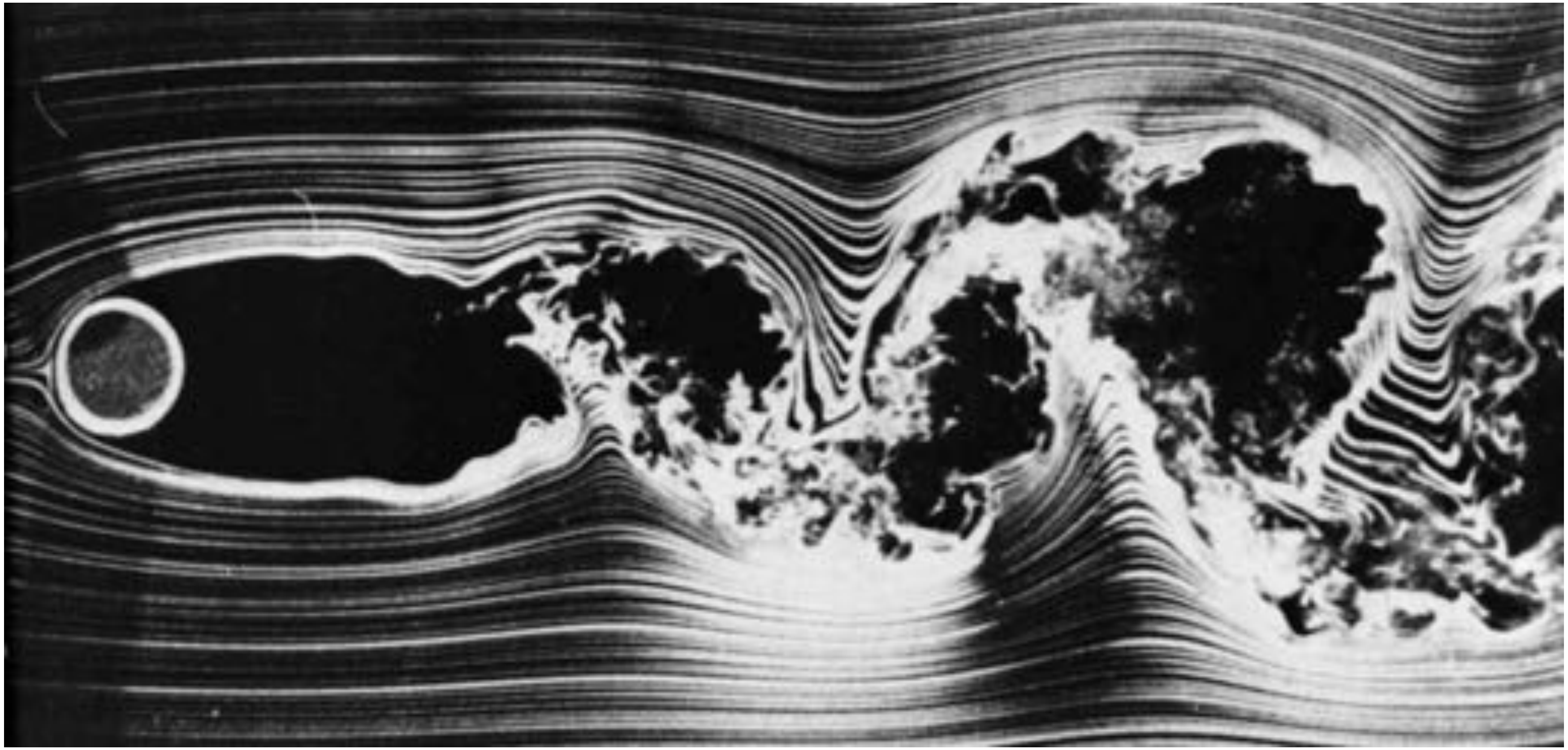
[Wind and smoke.]

Reynolds number Re = 2000



[Water and air bubbles.]

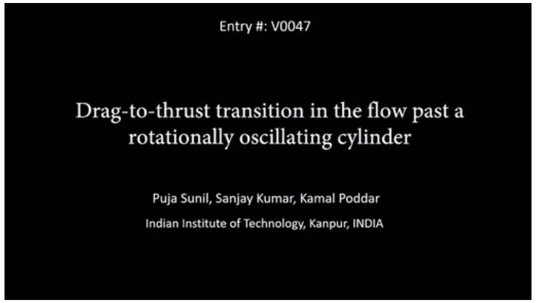
Reynolds number Re = 10 000



[Water and air bubbles.]

Vortex shedding

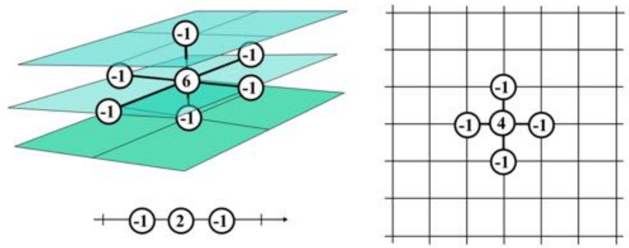
- https://www.youtube.com/watch?v=9FRTj6_1J2k



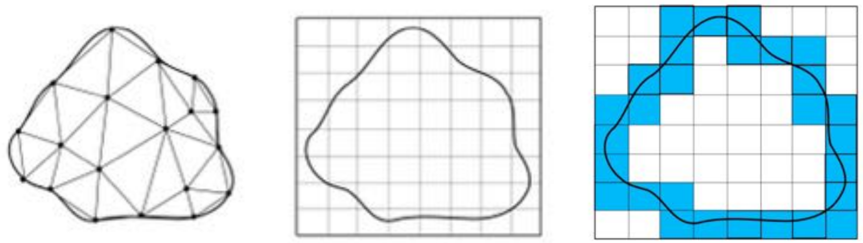
- https://gfm.aps.org/meetings/dfd-2020/5f5f0056199e4c091e67bd9e

[Water and air bubbles.]

Discretization of NSE: $R(u) = 0 \longrightarrow Ax = b$

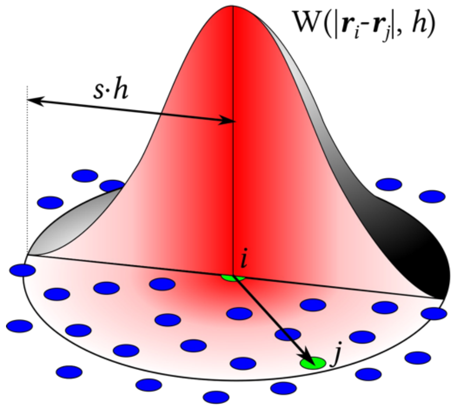


- Structured grid: stencil
- Unstructured mesh: mesh-based basis function
- Particle system: mesh-free radial basis function



- Minimization: $min \parallel R(u) \parallel$
- Collocation: $R\big(u(\mathrm{x_i})\big) = 0$, for all i
- Projection: $(R(u), v) = 0$, for all v

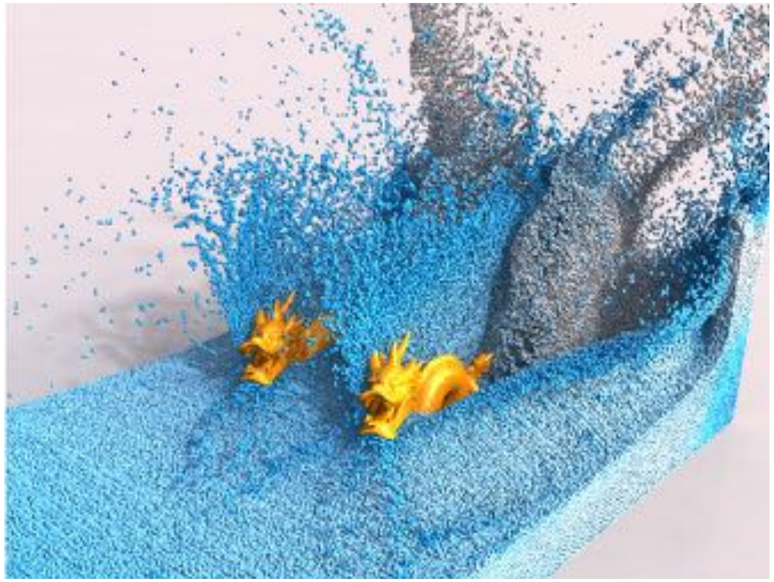
Smooth particle hydrodynamics (SPH)



- Particle system $\{x_i\}$
- Kernel function W_h
- Smoothing length h
- Provides a field representation:

$$A(x) = \sum_i A_i W_h(\|x - x_i\|)$$

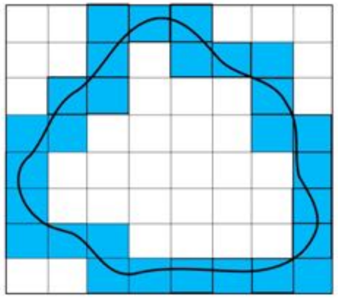
$$\frac{\partial}{\partial x_i} A(x) = \sum_i A_i \frac{\partial}{\partial x_i} W_h(\|x - x_i\|)$$

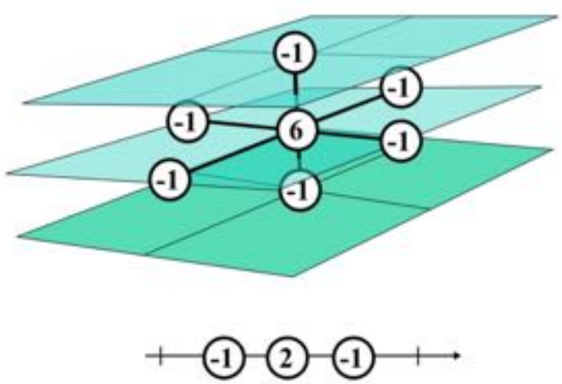


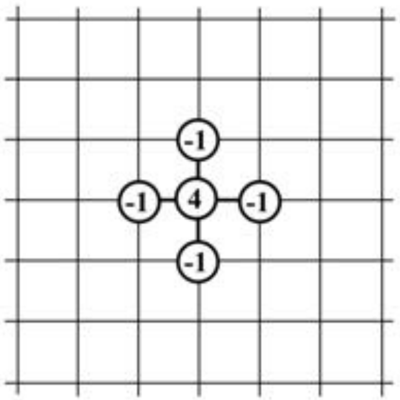
[Bender, Koschier, SIGGRAPH, 2015]

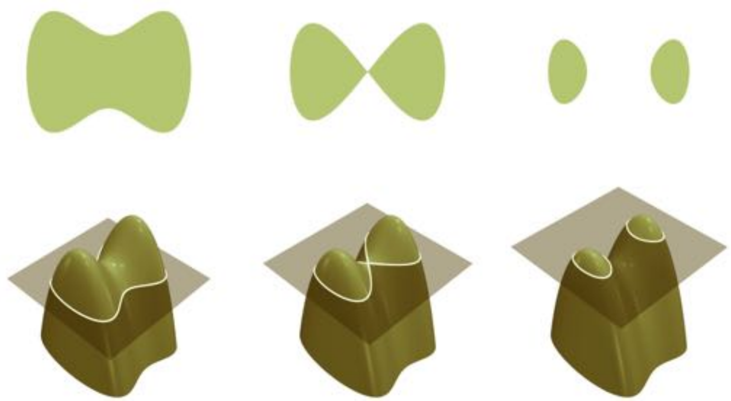
Finite difference method (FDM)

- Structured grid: stencil (finite difference operator)
- Collocation $R\left(u(\mathrm{x_i})\right) = 0$, for all i
- Complex geometry: use e.g. level set function









[https://en.wikipedia.org/wiki/Level-set_method#/media/File:Level_set_method.png]

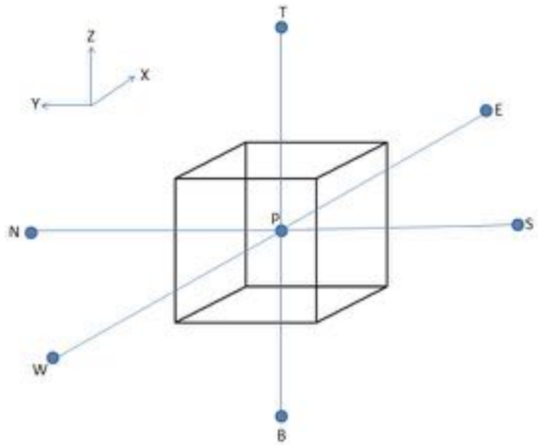
Finite volume method (FVM)

- Based on local conservation laws over grid cells using Gauss theorem.
- Computes average quantities from fluxes over cell faces

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}\left(\mathbf{u}\right) = \mathbf{0}.$$

$$\int_{v_i} \frac{\partial \mathbf{u}}{\partial t}\, dv + \int_{v_i} \nabla \cdot \mathbf{f}\left(\mathbf{u}\right)\, dv = \mathbf{0}.$$

$$\frac{d\bar{\mathbf{u}}_i}{dt} + \frac{1}{v_i}\oint_{S_i} \mathbf{f}\left(\mathbf{u}\right) \cdot \mathbf{n}\, dS = \mathbf{0}.$$

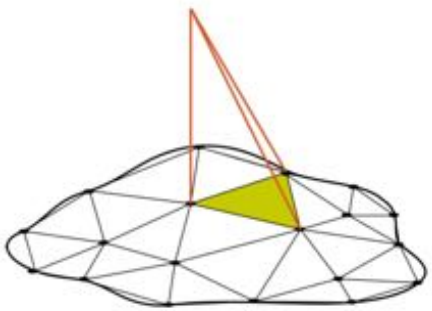


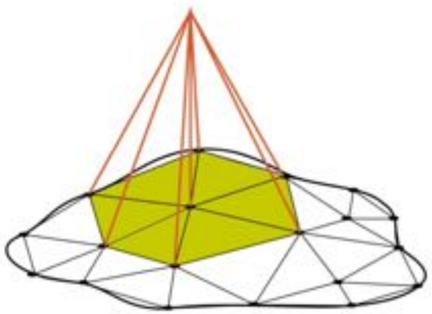
[https://en.wikipedia.org/wiki/Finite_volume_method]

Finite element method (FEM)

- Unstructured mesh: basis function
- Fixed or deforming mesh
- Projection $(R(u), v) = 0$, for all test functions v

$$u(x,t) \approx \sum_{i=1}^{N} U_i(t)\phi_i(x)$$

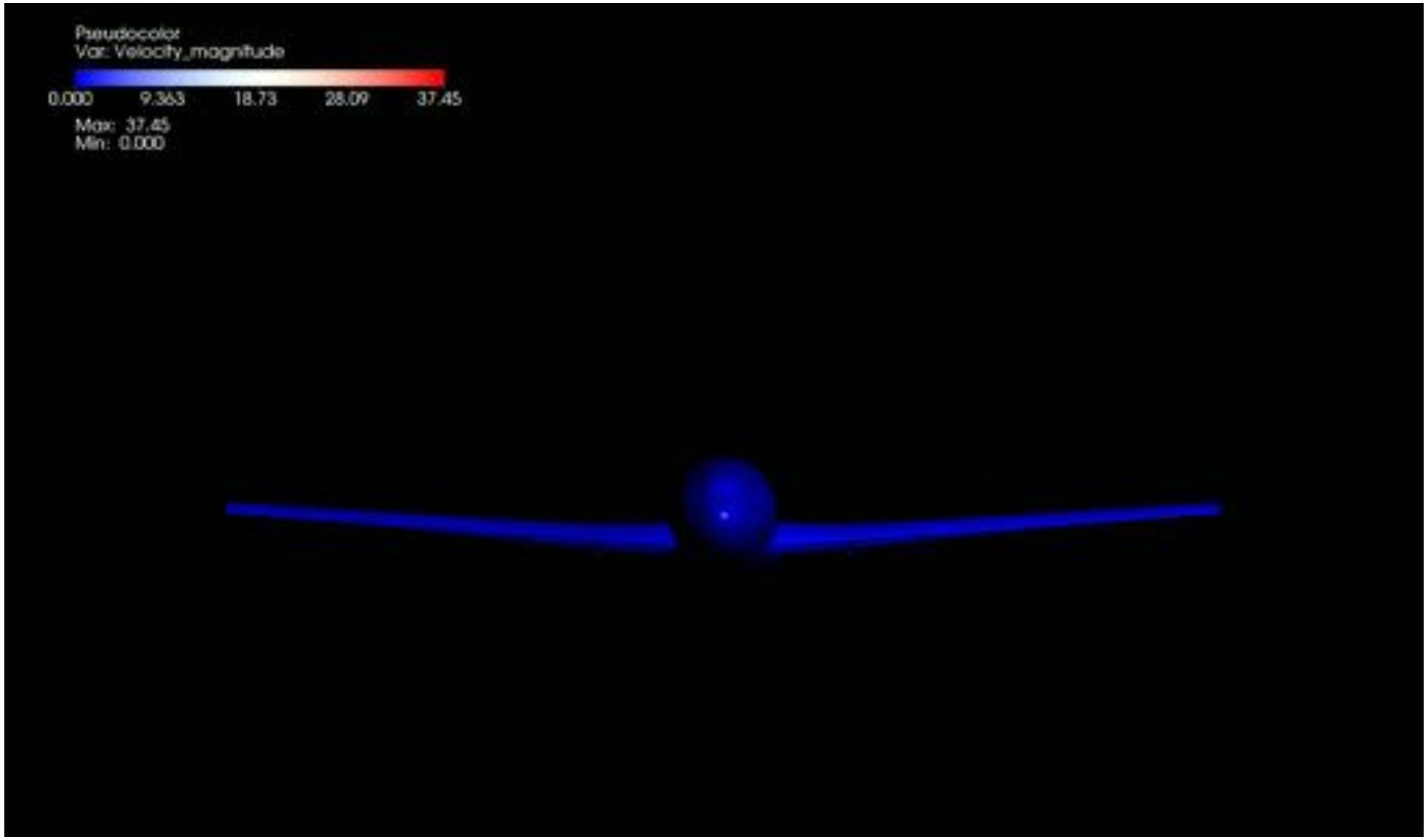


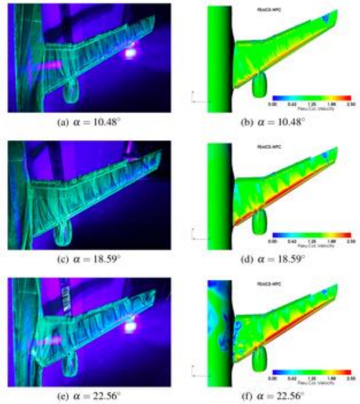


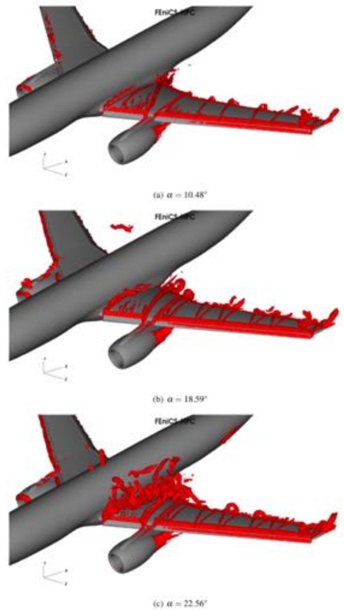
Algorithm 9.2. **(A, b) = assemble_system(f).**
Input: function **f**
Output: assembled matrix **A** and vector **b**.

```
for k=0:no_elements-1 do
    q = get_no_local_shape_functions(k)
    loc2glob = get_local_to_global_map(k)
    for i=0:q do
        b[i] = integrate_vector(f, k, i)
        for j=0:q do
            a[i,j] = integrate_matrix(k, i, j)
        end for
    end for
    add_to_global_vector(b, loc2glob)
    add_to_global_matrix(a, loc2glob)
end for
return A, b
```

FEM simulation of air past airplane

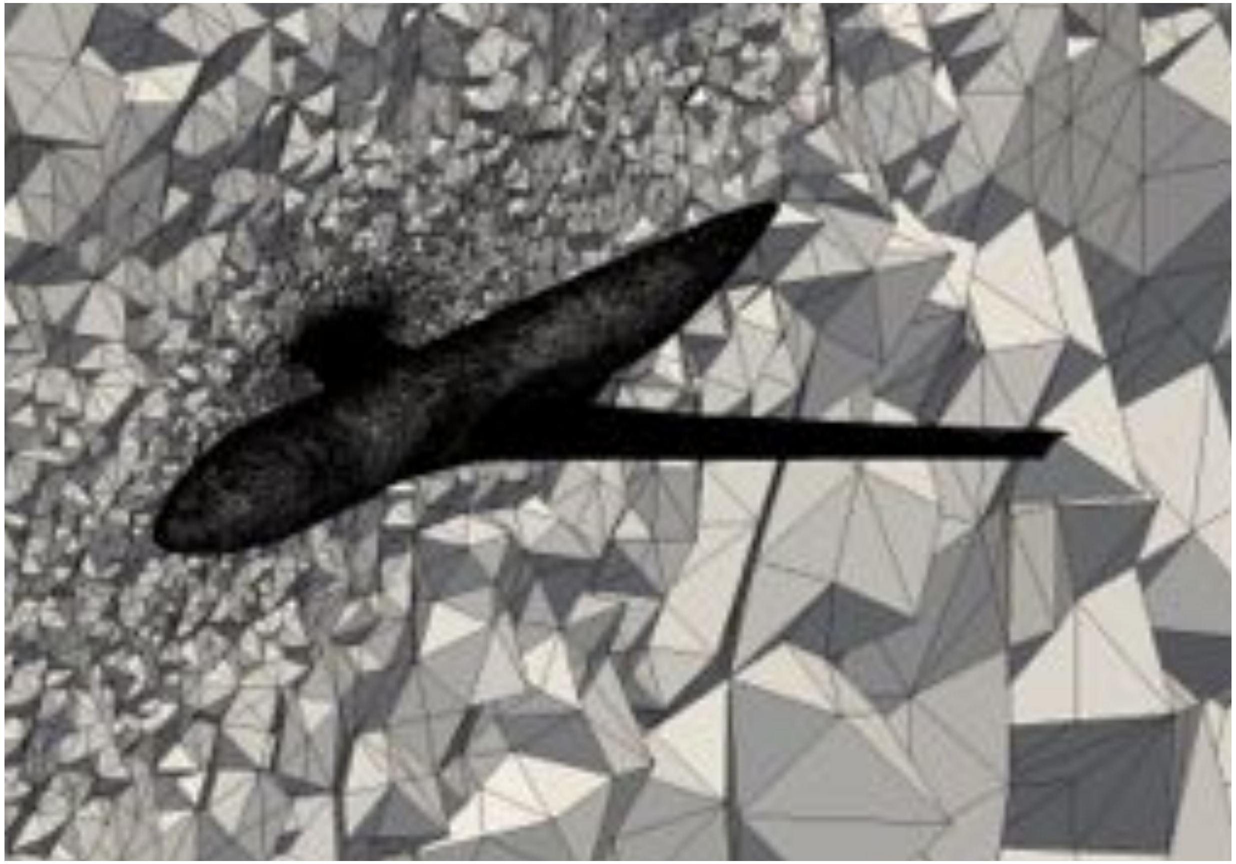


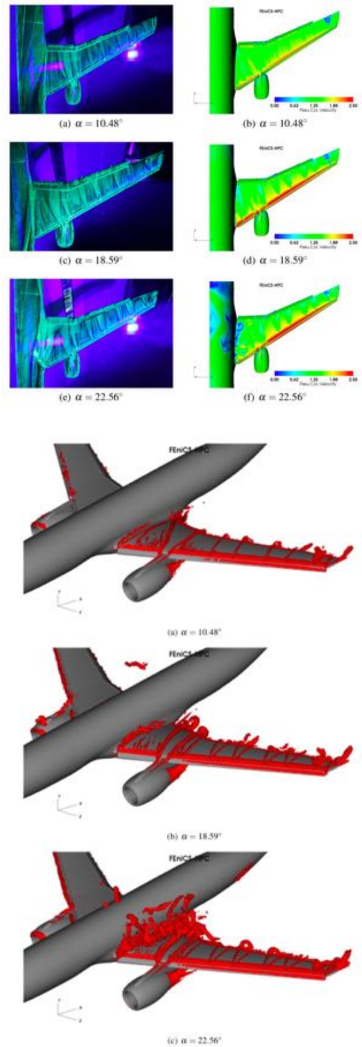




[Jansson et al., Springer, 2018]

Discretization by a mesh



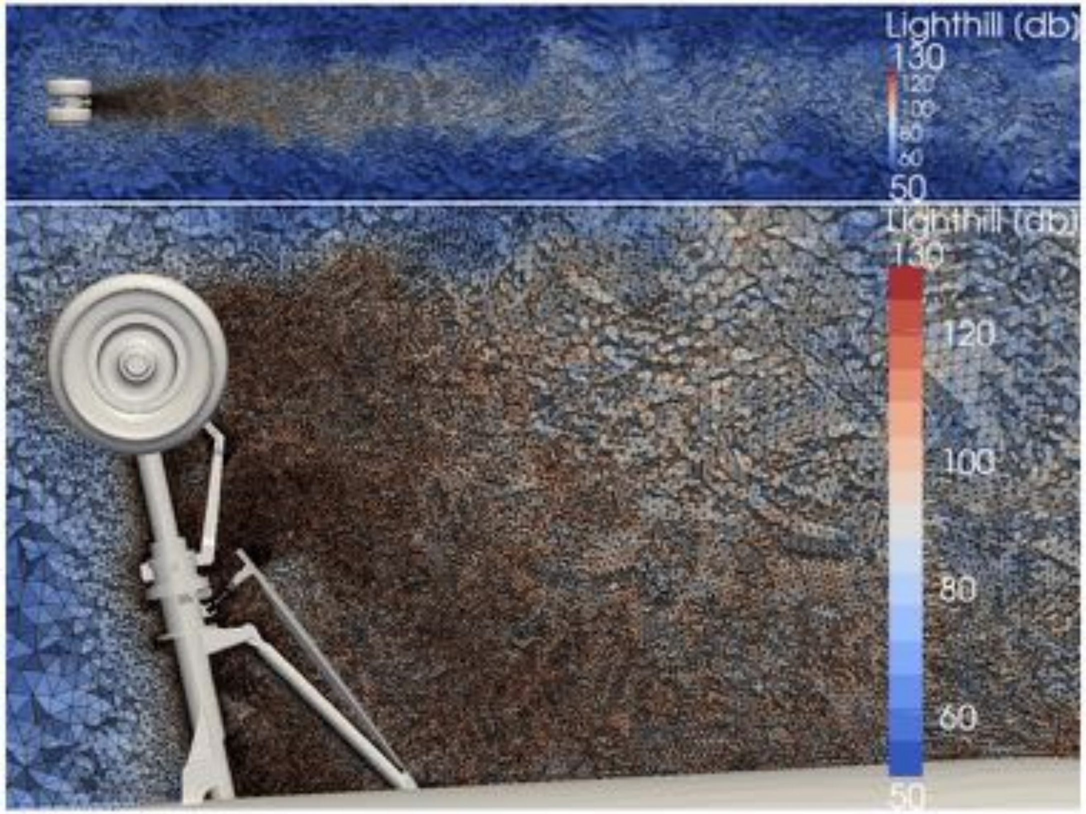


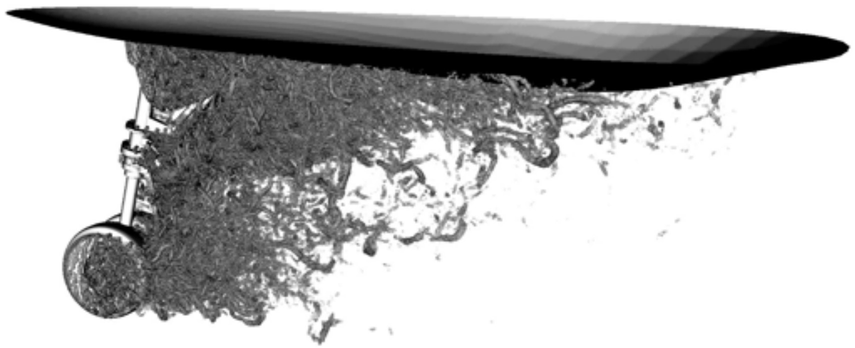
[Jansson et al., Springer, 2018]

FEM simulation of airflow past landing gear

[De Abreu et al., Computers and Fluids, 2016]

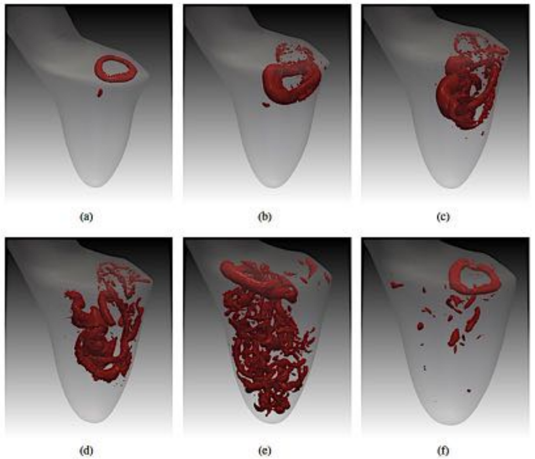
Acoustic sources and turbulent vortices

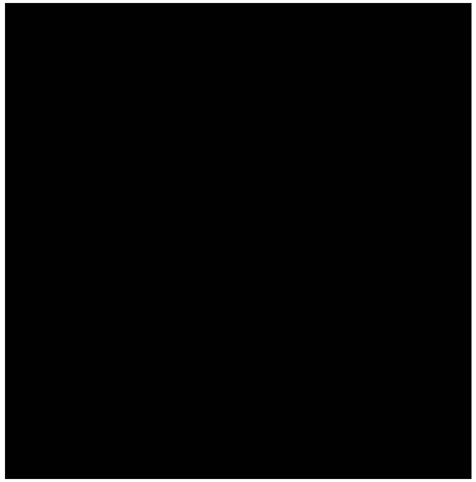


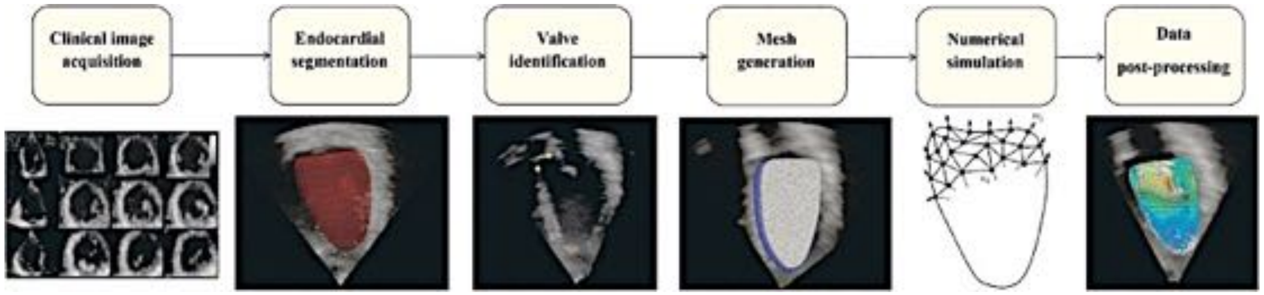


[De Abreu et al., Computers and Fluids, 2016]

Heart (deforming mesh) simulation







[Spühler et al., 2017, 2020]

Finite element method: Poisson equation

We now consider the *Poisson equation* for a function $u : \mathbb{R}^n \to \mathbb{R}$,

$$-\Delta u = f, \quad \text{in } \Omega, \tag{3.1}$$

with the domain $\Omega \subset \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ given data.

For the equation to have a unique solution we need to specify boundary conditions. We may prescribe *Dirichlet boundary conditions*,

$$u|_{\partial\Omega} = g_D, \tag{3.2}$$

Neumann boundary conditions,

$$\nabla u \cdot n|_{\partial\Omega} = g_N, \tag{3.3}$$

or a linear combination of the two, referred to as a *Robin boundary condition*,

$$\nabla u \cdot n|_{\partial\Omega} = \alpha(u|_{\partial\Omega} - g_D) + g_N, \tag{3.4}$$

with $\alpha(x)$ a given weight function.

Homogeneous Dirichlet bc

With homogeneous Dirichlet boundary conditions, we have the problem

$$
\begin{aligned}
-\Delta u &= f, \qquad \text{in } \Omega, \\
u &= 0, \qquad \text{on } \partial\Omega.
\end{aligned} \tag{3.5}
$$

To make the problem statement precise, let the trial and test functions belong to a certain function space V. With $V = H_0^1(\Omega)$ we obtain the following variational formulation: find $u \in V$, such that

$$
(\nabla u, \nabla v) = (f, v), \quad \forall v \in V, \tag{3.6}
$$

since the boundary term vanishes as the test function is an element of the vector space $H_0^1(\Omega)$.

Homogeneous Neumann bc

$$
\begin{aligned}
-\Delta u &= f, && \text{in } \Omega, \\
\nabla u \cdot n &= 0, && \text{on } \partial\Omega.
\end{aligned} \tag{3.7}
$$

With $V = H^1(\Omega)$ we have the following variational formulation: find $u \in V$, such that

$$
(\nabla u, \nabla v) = (f, v), \quad \forall v \in V \tag{3.8}
$$

However, it turns out that the variational problem (3.8) has no unique solution, since for any solution $u \in V$, also $u + C$ is a solution, with $C \in \mathbb{R}$ any constant. To ensure a unique solution, we need an extra condition for the solution which determines the arbitrary constant, for example, we may change the trial space to

$$
V = \{u \in H^1(\Omega) : \int_\Omega u(x)\, dx = 0\}. \tag{3.9}
$$

Galerkin Finite Element Method

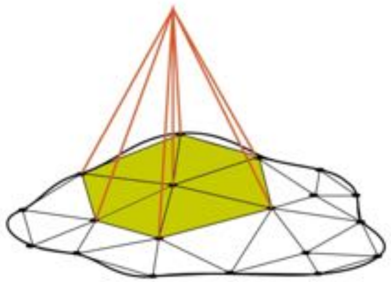
To formulate a Galerkin method for the Poisson equation we replace the Hilbert space V by a finite dimensional subspace $V_h \subset V$ in the variational formulation of the equation. We hence seek $U \in V_h$, such that

$$(\nabla U, \nabla v) = (f, v), \quad \forall v \in V_h. \tag{3.13}$$

For a simplicial mesh $\mathcal{T}^h$, the global approximation space of continuous piecewise polynomial functions V_h is spanned by the global nodal basis $\{\phi_j\}$, where each basis function ϕ_j is associated to a global vertex N_j. Hence with Dirichlet boundary conditions the finite element approximation $U \in V_h$ can be expressed as

$$U(x) = \sum_{N_j \in \mathcal{N}_I} U(N_j)\phi_j(x) + \sum_{N_j \in \mathcal{N}_D} U(N_j)\phi_j(x),$$

with $\mathcal{N}_I$ all internal vertices in the mesh and $\mathcal{N}_D$ all vertices on the Dirichlet boundary, and where $U(N_j)$ is the node which corresponds to function evaluation at the vertex N_j.



FEM for general variational problem

For a Hilbert space V consisting of functions with finite norm $\|\cdot\|_V$, we formulate the corresponding variational problem: find $u \in V$, such that

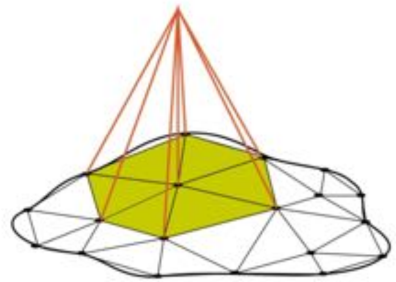
$$a(u,v) = L(v), \quad \forall v \in V,$$

with $a : V \times V \to \mathbb{R}$ a bilinear form and $L : V \to \mathbb{R}$ a linear form.

The finite element method takes the form of a matrix problem

$$Ax = b, \tag{3.24}$$

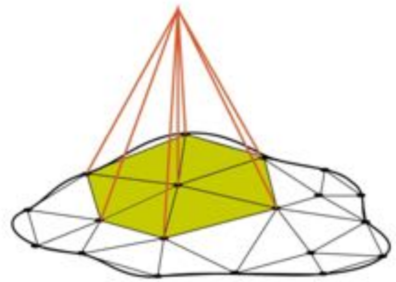
where $a_{ij} = a(\phi_j, \phi_i)$, $x_j = U(N_j)$ and $b_i = L(\phi_i)$. To compute the Galerkin finite element approximation, we thus have to construct the matrix A and vector b, and then solve the resulting matrix problem (3.24) to obtain the nodal values $U(N_j)$.



FEM for general variational problem

In the case of Dirichlet boundary conditions, the rows in the matrix corresponding to boundary nodes $N_j \in \mathcal{N}_D$ are replaced by a row with one on the diagonal and with all other components zero. To enforce the Dirichlet boundary condition, each corresponding vector component is then set to the interpolated Dirichlet boundary value $b_j = \mathcal{I}^h g_D(N_j)$.

$$\left[\begin{array}{c|c} A_{II} & A_{ID} \\ \hline 0_{DI} & I_{DD} \end{array}\right] \left[\begin{array}{c} x_I \\ \hline x_D \end{array}\right] = \left[\begin{array}{c} b_I \\ \hline b_D \end{array}\right],$$



where A_{II} is a square $n_I \times n_I$ matrix, with n_I the number of internal nodes, A_{ID} an $n_I \times n_D$ matrix, with n_D the number of boundary nodes, I_{DD} an $n_D \times n_D$ identity matrix, 0_{DI} an $n_D \times n_I$ zero matrix, and b_D is an n_D vector with components $(b_D)_j = \mathcal{I}^h g_D(N_j)$.

Assembly algorithm

The matrix and vector are constructed by an assembly algorithm, which loops over all elements K in the mesh to compute the local element matrices $A^K = (a_{ij}^K)$, with

$$a_{i,j}^K = a(\lambda_j, \lambda_i)|_K,$$

and the local element vector

$$b_i^K = L(\lambda_i)|_K,$$

with $a(\cdot,\cdot)|_K$ and $L(\cdot)|_K$ the bilinear and linear forms restricted to element K, and with $\{\lambda_i\}_{i=1}^{n_q-1}$ the element shape functions, for example, local Lagrange basis functions over K. The integrals are often approximated by quadrature over a reference element $\hat{K}$, based on a map $F_K : \hat{K} \to K$.

Algorithm 1: Assembly of matrix $A = (a_{i,j})$ and vector $b = (b_i)$

for $K \in \mathcal{T}^h$ **do**
  for $i = 0, 1..., n_q - 1$ **do**
    $b_i^K = L(\lambda_i)|_K$ ▷ compute element vector
    $b_{loc2glob(i)} \mathrel{+}= b_i^K$ ▷ add to global vector
  end
  for $i = 0, 1..., n_q - 1$ **do**
    for $j = 0, 1, ..., n_q - 1$ **do**
      $a_{i,j}^K = a(\lambda_j, \lambda_i)|_K$ ▷ compute element matrix
      $a_{loc2glob(i),loc2glob(j)} \mathrel{+}= a_{i,j}^K$ ▷ add to global matrix
    end
  end
end

Assembly algorithm

To add the local element matrix and element vector to the global matrix and vector, we use an index map

$$loc2glob : i_K \to i_A,$$

which maps the index of each local degree of freedom $i \in i_K$, to the corresponding index in the global matrix $loc2glob(i) \in i_A$.

Algorithm 1: Assembly of matrix $A = (a_{i,j})$ and vector $b = (b_i)$

```
for K ∈ T^h do
    for i = 0, 1..., n_q − 1 do
        b_i^K = L(λ_i)|_K                          ▷ compute element vector
        b_{loc2glob(i)} += b_i^K                   ▷ add to global vector
    end
    for i = 0, 1..., n_q − 1 do
        for j = 0, 1, ..., n_q − 1 do
            a_{i,j}^K = a(λ_j, λ_i)|_K             ▷ compute element matrix
            a_{loc2glob(i),loc2glob(j)} += a_{i,j}^K   ▷ add to global matrix
        end
    end
end
```

Existence and uniqueness

Theorem 5 (Lax-Milgram theorem). *The variational problem (3.16) has a unique solution $u \in V$, if the bilinear form is elliptic and bounded, and the linear form is bounded. That is, there exist constants $\alpha > 0$, $C_1, C_2 < \infty$, such that for $u, v \in V$,*

$$
\begin{aligned}
&(i) && a(v,v) \geq \alpha \|v\|_V^2, \\
&(ii) && a(u,v) \leq C_1 \|u\|_V \|v\|_V, \\
&(iii) && L(v) \leq C_2 \|v\|_V.
\end{aligned}
$$

Energy norm and stability of solutions

For an elliptic variational problem, a symmetric bilinear form defines an inner product $(\cdot,\cdot)_E = a(\cdot,\cdot)$ on the Hilbert space V, with an associated *energy norm*

$$\|\cdot\|_E = a(\cdot,\cdot)^{1/2},$$

which is equivalent to the norm $(\cdot,\cdot)_V$, since

$$\alpha\|\cdot\|_V^2 \le (\cdot,\cdot)_E \le C_1\|\cdot\|_V^2.$$

For the energy norm we can derive the following stability estimate for the solution $u \in V$ to the variational problem (3.16),

$$\|u\|_E^2 = a(u,u) = L(u) \le C_2\|u\|_V \le (C_2/\alpha)\|u\|_E,$$

so that

$$\|u\|_E \le (C_2/\alpha).$$

Optimality of Galerkin's method

In a Galerkin finite element method we seek an approximation $U \in V_h$,

$$a(U, v) = L(v), \quad \forall v \in V_h, \tag{3.19}$$

with $V_h \subset V$ a finite dimensional subspace, which in the case of a finite element method is a piecewise polynomial space. For an elliptic problem, existence and uniqueness of a solution follows from Lax-Milgram's theorem.

Since $V_h \subset V$, the weak form (3.16) is satisfied also for $v \in V_h$, and by subtracting (3.19) from (3.16) we obtain the Galerkin orthogonality property,

$$a(u - U, v) = 0, \quad \forall v \in V_h.$$

Optimality of Galerkin's method

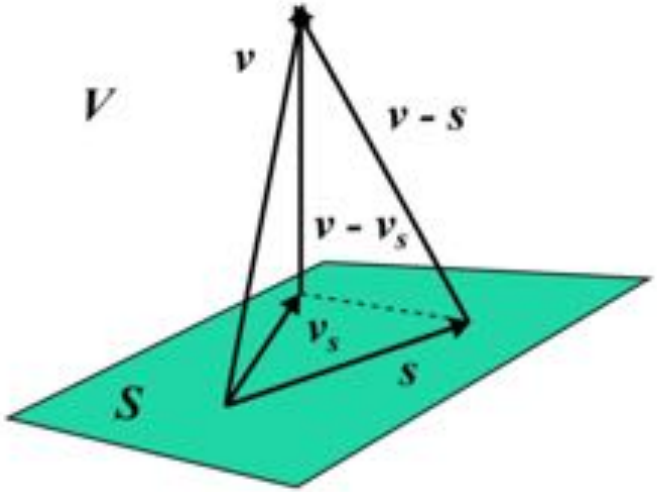
Theorem 1.16 (Optimality of orthogonal projection). *The orthogonal projection $v_s \in S$, defined by*

$$(v - v_s, s) = 0, \quad \forall s \in S,$$

is the optimal approximation of $v \in V$ in $S \subset V$, in the sense that

$$\|v - v_s\| \leq \|v - s\|, \quad \forall s \in S,$$

for $\|\cdot\| = (\cdot,\cdot)^{1/2}$ the norm induced by the inner product in V.



A symmetric bilinear form $a(\cdot,\cdot)$ defines an inner product, hence, the Galerkin orthogonality property represents an orthogonal projection.

Optimality of Galerkin's method

For an elliptic problem with symmetric bilinear form we can show that the Galerkin approximation is optimal in the energy norm, since

$$\begin{aligned} \|u-U\|_E^2 &= a(u-U,u-U) = a(u-U,u-v) + a(u-U,v-U) \\ &= a(u-U,u-v) \le \|u-U\|_E \|u-v\|_E, \end{aligned}$$

and hence

$$\|u-U\|_E \le \|u-v\|_E, \quad \forall v \in V_h.$$

For an elliptic non-symmetric bilinear form, we can prove *Cea's lemma*,

$$\|u-U\|_V \le \frac{C_1}{\alpha}\|u-v\|_V, \quad \forall v \in V,$$

which follows from

$$\begin{aligned} \|u-U\|_V^2 &\le (1/\alpha)a(u-U,u-U) = (1/\alpha)a(u-U,u-v) \\ &\le (C_1/\alpha)\|u-U\|_V \|u-v\|_V. \end{aligned}$$

Stokes equations

The Stokes equations for a domain $\Omega \subset \mathbb{R}^n$ with boundary $\nabla\Omega = \Gamma_D \cup \Gamma_N$, and associated normal n, takes the form

$$\begin{aligned} -\Delta u + \nabla p &= f, && x \in \Omega, \\ \nabla \cdot u &= 0, && x \in \Omega, \\ u &= g_D, && x \in \Gamma_D, \\ -\nabla u \cdot n + pn &= g_N, && x \in \Gamma_N. \end{aligned}$$

Stokes equations – mixed function spaces

First assume that $\partial\Omega = \Gamma_D$ and $g_D = 0$, that is, homogeneous Dirichlet boundary conditions for the velocity. We then seek a weak solution to the Stokes equations in the following spaces,

$$V = H_0^1(\Omega) \times ... \times H_0^1(\Omega) = [H_0^1(\Omega)]^n,$$
$$Q = \{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\},$$

where the extra condition in the vector space Q is needed to assure uniqueness of the pressure, which otherwise is undetermined up to a constant.

Stokes equations – variational formulation

We derive the variational formulation by taking the inner product of the momentum equation with a test function $v \in V$, and the inner product of the continuity equation with a test function $q \in Q$. By Green's formula and the homogeneous Dirichlet boundary condition, we obtain the variational formulation as: find $(u, p) \in V \times Q$, such that

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v), \quad \forall v \in V, && (5.6)\\ -b(u,q) &= 0, \qquad\quad \forall q \in Q, && (5.7) \end{aligned}$$

$$a(v,w) = (\nabla v, \nabla w) = \int_\Omega \nabla v : \nabla w \, dx, \tag{5.8}$$

$$b(v,q) = -(\nabla \cdot v, q) = -\int_\Omega (\nabla \cdot v) q \, dx, \tag{5.9}$$

Stokes equations – inf-sup condition

Theorem 7. *The variational problem (5.6)-(5.7) has a unique weak solution $(u, p) \in V \times Q$, which satisfies the following stability inequality,*

$$\|u\|_V + \|q\|_Q \leq C\|f\|_{-1},$$

if the following conditions hold,

(i) $a(\cdot,\cdot)$ *is bounded and coercive, i.e. that exists a constant* $\alpha > 0$,

$$a(v, v) \geq \alpha\|v\|_V^2,$$

for all $v \in Z = \{v \in V : b(v, q) = 0, \forall q \in Q\}$,

(ii) $b(\cdot,\cdot)$ *is bounded and satisfies the inf-sup condition, i.e. there exists a constant* $\beta > 0$,

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta.$$

Stokes equations – mixed FEM

We now formulate a finite element method for solving Stokes equations. Since we use different approximation spaces for the velocity and the pressure, we refer to the method as a mixed finite element method.

We seek an approximation $(U, P) \in V_h \times Q_h$, such that,

$$a(U, v) + b(v, P) = (f, v), \tag{5.11}$$

$$-b(U, q) = 0, \tag{5.12}$$

for all $(v, q) \in V_h \times Q_h$, where V_h and Q_h are finite element approximation spaces. There exists a unique solution to (5.11)-(5.12), under similar conditions as for the continuous variational problem.

Stokes equations – mixed FEM

Theorem 8. *The mixed finite element problem (5.11)-(5.12) has a unique solution* $(U, P) \in V_h \times Q_h$, *if*

(i) $a(\cdot,\cdot)$ *is coercive, i.e. that exists a constant* $\alpha_h > 0$, *such that*

$$a(v, v) \geq \alpha_h \|v\|_V,$$

for all $v \in Z_h = \{v \in V_h : b(v, q) = 0, \forall q \in Q_h\}$,

(ii) $b(\cdot,\cdot)$ *satisfies the inf-sup condition, i.e. there exists a constant* $\beta_h > 0$,

$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta_h,$$

and this unique solution satisfies the following error estimate,

$$\|u - U\|_V + \|p - P\|_Q \leq C \left(\inf_{v \in V_h} \|u - v\| + \inf_{q \in Q_h} \|p - q\| \right),$$

for a constant $C > 0$.

Stokes equations – Taylor-Hood elements

The pair of approximation spaces must be chosen to satisfy the inf-sup condition, with the velocity space sufficiently rich compared to the pressure space. For example, continuous piecewise quadratic approximation of the velocity and continuous piecewise linear approximation of the pressure, referred to as the Taylor-Hood elements. On the other hand, continuous piecewise linear approximation of both velocity and pressure is not inf-sup stable.

Stokes equations – discrete system

We seek finite element approximations in the following spaces,

$$V_h = \{v = (v_1, v_2, v_3) : v_k(x) = \sum_{j=1}^{N} v_k^j \phi_j(x), k = 1, 2, 3\}$$

$$Q_h = \{q : q(x) = \sum_{j=1}^{M} q^j \psi_j(x)\},$$

which leads to a discrete system in matrix form,

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

with u and p vectors holding the coordinates of U and P in the respective bases of V_h and Q_h.

Stokes equations – Schur complement

The matrix A is symmetric positive definite and thus invertible, so we can express

$$u = A^{-1}(f - Bp),$$

and since $B^T u = 0$,

$$B^T A^{-1} B p = B^T A^{-1} f,$$

which is the *Schur complement* equation. If $\mathrm{null}(B) = \{0\}$, then the matrix $S = B^T A^{-1} B$ is symmetric positive definite and can also be inverted.

Schur complement methods take the form

$$p_k = p_{k-1} - C^{-1}(B^T A^{-1} B p_{k-1} - B^T A^{-1} f),$$

where C^{-1} is a preconditioner for $S = B^T A^{-1} B$. The Usawa algorithm is based on C^{-1} as a scaled identity matrix, which gives

1. Solve $Au_k = f - Bp_{k-1}$,
2. Set $p_k = p_{k-1} + \alpha B^T u_k$.

Stokes equations - stabilization

Approximation spaces of equal order is possible, by stabilization of the standard Galerkin finite element method: find $(U, P) \in V_h \times Q_h$, such that,

$$\begin{aligned} a(U,v) + b(v,P) &= (f,v), \\ -b(U,q) + s(P,q) &= 0, \end{aligned}$$

for all $(v,q) \in V_h \times Q_h$, where $s(P,q)$ is a pressure stabilization term. The resulting discrete system takes the form,

$$\begin{bmatrix} A & B \\ B^T & S \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

where the stabilization term is chosen so that the matrix S is invertible.

For example, the *Brezzi-Pitkäranta* stabilization takes the form,

$$s(P,q) = C \int_\Omega h^2 \nabla P \cdot \nabla q \, dx,$$

with $C > 0$ a constant.

Credits

Album of fluid flow (Milton Van Dyke)

- https://en.wikipedia.org/wiki/An_Album_of_Fluid_Motion
- https://www.abebooks.com/9780915760022/Album-Fluid-Motion-Milton-Dyke-0915760029/plp

SIAM book (2021)