# Summer school: Computational methods for fluid mechanics

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The summer school consists of four days focused on computational methods for fluid mechanics. After a brief overview of different computational methods, we will focus on variational methods (finite element methods) for incompressible flow, which fit well with the dominant mathematical theory of the Navier-Stokes equations. Lectures will be mixed with laboratory work using the open-source software FEniCS and the open cloud computing platform Google Colab based on Jupyter notebooks.

https://fenicsproject.org
https://colab.research.google.com/
https://jupyter.org

Day 1 (Monday Sep 11): Viscous incompressible flow

Introduction to conservation laws and incompressible flow, Navier-Stokes equations, and different computational methods used in science and industry. Mathematical theory and finite element methods for viscous flow (Stokes equations).

Day 2 (Tuesday Sep 12): The Navier-Stokes equations

The Navier-Stokes equations as a dynamical system, stability analysis, and the structure of turbulent flow. Semi-discretization by a finite element method and time stepping, residual-based stabilization, and simulation of turbulence.

Day 3 (Thursday Sep 14): Error estimation and adaptive methods

Existence and uniqueness of the Navier-Stokes equations, weak solutions, and a $1 million Clay Prize problem. Adaptive finite element methods and a posteriori error analysis, dissipative weak solutions, and the Onsager conjecture.

Day 4 (Friday Sep 15): Fluid-structure interaction

Boundary layers, flow separation, and wall modelling. The structure of turbulent flow separation, analytical models, and the d'Alembert's paradox. Deforming domains and fluid-structure interaction, mesh and meshless computational methods, finite element methods.

**Lab exercise 1 - Viscous incompressible flow**

**Material**

1. Lecture notes: Section 5.3.

2. Template file from GitHub: template-report-Stokes.ipynb

**Assignments**

1. The domain

Change the computational domain into a channel with only one circular hole at the center of the channel.

2. The mesh

Change the resolution of the mesh to a uniform mesh size of h=1/32. Then locally refine the mesh once in a circle of radius 1 centered at the circular hole.

3. The inf-sup condition

Verify that the Taylor-Hood mixed element is stable, whereas equal order interpolation of the velocity and pressure is unstable. Describe how the instability manifests itself.

**Extra assignments**

4. Boundary conditions

Switch the  inflow and outflow boundaries so that the fluid flows from right to left. Describe how the pressure changes.

5. Stabilization

Implement Brezzi-Pitkäranta stabilization and show that equal order interpolation is then stable.

**Lab exercise 2 - The Navier-Stokes equations**

**Material**

1. Lecture notes: section 5.4-5.5.

2. Template file from GitHub: template-report-Navier-Stokes.ipynb

**Assignments**

1. Reynolds number

Compute the Reynolds number based on the inflow velocity U and the diameter D of the cylinder. Compare simulations for Reynolds numbers Re = 1, 10, 100, 1000. For which Reynolds numbers is the flow unsteady and steady, respectively?

2. Drag and lift coefficients

Compute the drag and lift coefficients, and the Strouhal number, for the cylinder at the different Reynolds numbers (Re = 1, 10, 100, 1000). Verify that the von Karman vortex street has a Strouhal number of St≈0.2.

3. Mesh resolution

Compare the solution for Re=100 under the different mesh resolutions h=1/8, 1/16, 1/32. Describe how the simulations change with respect to drag and lift coefficients and the Strouhal number.

**Extra assignments**

1. Stabilization

Modify the stabilization terms to the "Streamline diffusion" stabilization in the lecture notes section 5.5. Compare the solution at Re=100 with the previous stabilization, does the solution change for different mesh resolutions h=1/8, 1/16, 1/32?

2. Paraview visualization

Create movies of the simulation for Re=100 and h=1/32, by using the open source software Paraview. To download the needed files, uncomment the following two lines in "template-report-Navier-Stokes.ipynb":

```
#!tar -czvf results-NS.tar.gz results-NS
#files.download('results-NS.tar.gz')
```

**Lab exercise 3 - Error estimation and adaptive methods**

**Material**

1. Lecture notes: Chapters 1-5 (specifically Section 3.2)

2. Template file from GitHub: template-report-Stokes-AMR.ipynb

**Assignments**

1. The domain

Use the data to the adjoint problem which defines the functional equal to the drag force:

psi_1 = (0,0), psi_2 = 0 and psi_3 = (1,0)

Then change the height of the domain to H=L, and move the center of the circle over the domain. Describe how the adjoint solution and the local mesh refinement changes as a result.

2. The functional

Change the data to the adjoint problem (psi_1, psi_2, psi_3) and describe how the adjoint solution and the local mesh refinement change as a result.

**Extra assignments**

1. The total error

With the same approximation spaces for the primal and the adjoint problem, note that the total error appears to be close to zero. This is a false conclusion, and a consequence of the fact that we approximate the exact adjoint solution with a finite element function in the test space of the primal problem, for which the weak form is zero (modulo errors from solving the algebraic system). Change the mixed finite element space of the adjoint equation such that the total error is not zero, for example, increase the polynomial order of the adjoint finite element spaces. Then verify that the total error is reduced when you use a mesh with finer resolution.

2. Adaptive algorithm

Extend the template file into an adaptive method for recursive local mesh refinement based on the error indicator from the adjoint solution, with a stopping criterion based on the total error.

**Literature**

Lecture notes

**Research articles related to the methods in this course:**


Hoffman, Johan, et al. "Towards a parameter-free method for high reynolds number turbulent flow simulation based on adaptive finite element approximation." *Computer Methods in Applied Mechanics and Engineering* 288 (2015): 60-74.

Hoffman, Johan et al. "An interface-tracking unified continuum model for fluid-structure interaction with topology changed full-friction contact with application to aortic valves", International Journal for Numerical Methods in Engineering (2020).

Hoffman, Johan et al. "New theory of flight." *Journal of Mathematical Fluid Mechanics* 18.2 (2016): 219-241.

Jansson, Niclas et al. "Framework for massively parallel adaptive finite element computational fluid dynamics on tetrahedral meshes." *SIAM Journal on Scientific Computing* 34.1 (2012): C24-C41.

Hoffman, Johan et al., "Computational Turbulent Incompressible Flow", Springer, 2007.

Fuchsberger, Jana et al. "On the incorporation of obstacles in a fluid flow problem using Navier-Stokes-Brinkman penalisation approach, arXiv 2021.

Hoffman, Johan, ”Energy stability analysis of turbulent incompressible flow based on the triple decomposition of the velocity gradient tensor.”, Physics of Fluids, Vol.33, 2021.

Kronborg, Joel et al., “The triple decomposition of the velocity gradient tensor as a standardized real Schur form”, Physics of Fluids, Vol.35, 2023.