Computational Methods for Fluid Mechanics – lecture 3 Error Estimation and Adaptive Methods

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Today

- A posteriori error analysis and adaptive FEM
- Onsager’s conjecture – dissipative weak solutions
- Clay $1 million Prize problem – existence and regularity of NSE
- Singularities – structure of turbulent flow

Elliptic PDEs: existence and uniqueness

For a Hilbert space V consisting of functions with finite norm $\|\cdot\|_V$, we formulate the corresponding variational problem: find $u \in V$, such that

$$a(u,v) = L(v), \quad \forall v \in V, \tag{3.16}$$

with $a : V \times V \to \mathbb{R}$ a bilinear form and $L : V \to \mathbb{R}$ a linear form.

Theorem 5 (Lax-Milgram theorem)**.** *The variational problem (3.16) has a unique solution $u \in V$, if the bilinear form is elliptic and bounded, and the linear form is bounded. That is, there exist constants $\alpha > 0$, $C_1, C_2 < \infty$, such that for $u, v \in V$,*

$$\begin{array}{ll} (i) & a(v,v) \geq \alpha \|v\|_V^2, \\ (ii) & a(u,v) \leq C_1 \|u\|_V \|v\|_V, \\ (iii) & L(v) \leq C_2 \|v\|_V. \end{array}$$

Energy norm and stability of solutions

Partial differential equations rarely admit closed form solutions, but we can still infer some characteristics of the solutions from the weak form (3.16). For an elliptic variational problem, a symmetric bilinear form defines an inner product $(\cdot,\cdot)_E = a(\cdot,\cdot)$ on the Hilbert space V, with an associated *energy norm*

$$\|\cdot\|_E = a(\cdot,\cdot)^{1/2},$$

For the energy norm we can derive the following stability estimate for the solution $u \in V$ to the variational problem (3.16),

$$\|u\|_E^2 = a(u,u) = L(u) \leq C_2\|u\|_V \leq (C_2/\alpha)\|u\|_E,$$

so that

$$\|u\|_E \leq (C_2/\alpha).$$

Optimality of Galerkin's method

In a Galerkin finite element method we seek an approximation $U \in V_h$,

$$a(U, v) = L(v), \quad \forall v \in V_h, \tag{3.19}$$

with $V_h \subset V$ a finite dimensional subspace, which in the case of a finite element method is a piecewise polynomial space. For an elliptic problem, existence and uniqueness of a solution follows from Lax-Milgram's theorem.

Since $V_h \subset V$, the weak form (3.16) is satisfied also for $v \in V_h$, and by subtracting (3.19) from (3.16) we obtain the Galerkin orthogonality property,

$$a(u - U, v) = 0, \quad \forall v \in V_h.$$

Optimality of Galerkin's method

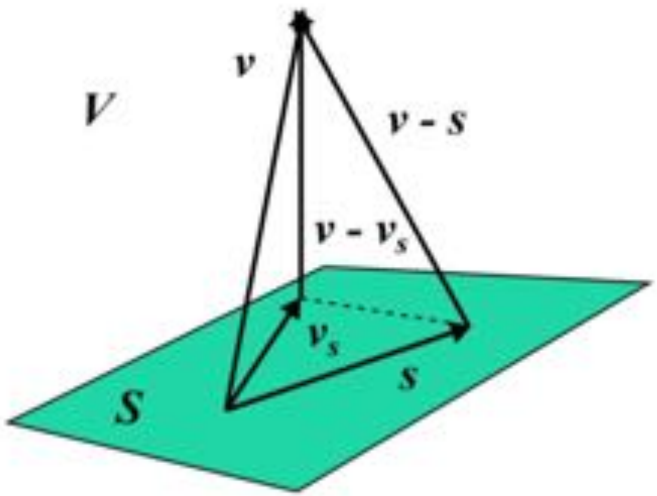
Theorem 1.16 (Optimality of orthogonal projection). *The orthogonal projection $v_s \in S$, defined by*

$$(v - v_s, s) = 0, \quad \forall s \in S,$$

is the optimal approximation of $v \in V$ in $S \subset V$, in the sense that

$$\|v - v_s\| \leq \|v - s\|, \quad \forall s \in S,$$

for $\|\cdot\| = (\cdot,\cdot)^{1/2}$ the norm induced by the inner product in V.



A symmetric bilinear form $a(\cdot,\cdot)$ defines an inner product, hence, the Galerkin orthogonality property represents an orthogonal projection.

Optimality of Galerkin's method

For an elliptic problem with symmetric bilinear form we can show that the Galerkin approximation is optimal in the energy norm, since

$$\begin{aligned}\|u-U\|_E^2 &= a(u-U,u-U) = a(u-U,u-v) + a(u-U,v-U)\\ &= a(u-U,u-v) \le \|u-U\|_E\|u-v\|_E,\end{aligned}$$

and hence

$$\|u-U\|_E \le \|u-v\|_E, \quad \forall v \in V_h.$$

For an elliptic non-symmetric bilinear form, we can prove *Cea's lemma*,

$$\|u-U\|_V \le \frac{C_1}{\alpha}\|u-v\|_V, \quad \forall v \in V, \tag{3.20}$$

which follows from

$$\begin{aligned}\|u-U\|_V^2 &\le (1/\alpha)a(u-U,u-U) = (1/\alpha)a(u-U,u-v)\\ &\le (C_1/\alpha)\|u-U\|_V\|u-v\|_V.\end{aligned}$$

A priori error estimation

For a Galerkin finite element method the approximation space V_h consists of piecewise polynomial functions defined over a mesh that approximates the domain $\Omega \subset \mathbb{R}^n$.

Cea's lemma (3.20) provides an estimate of the Galerkin error in terms of an arbitrary function $v \in V_h$, which we can choose to be an interpolant of the exact solution $v = \mathcal{I}^h u$, with

$$\mathcal{I}^h : V \to V_h$$

an interpolation operator, from which we obtain the *a priori* error estimate

$$\|u - U\|_V \le (C_1/\alpha)\|u - \mathcal{I}^h u\|_V,$$

only in terms of the exact solution to the variational problem.

Interpolation error estimates

$$\left(\sum_K \|v - \mathcal{I}^h v\|^p_{W^{s,p}(K)}\right)^{1/p} \leq Ch^{k-s}|v|_{W^{k,p}(\Omega)}, \quad \forall v \in W^{s,p}(\Omega),$$

where

$$|v|_{W^{k,p}(\Omega)} = \sum_{|\alpha|=k} \|D^\alpha u\|^p_{L^p(\Omega)},$$

A posteriori error estimation

In contrast to an *a priori* error estimate which is expressed in terms of the unknown exact solution $u \in V$, an *a posteriori* error estimate is bounded in terms of a computed approximate solution $U \in V_h$. We define a bounded linear functional

$$M(\cdot) = (\cdot, \psi),$$

with ψ the Riesz representer of the functional $M \in V'$, guaranteed to exist by the Riesz representation theorem. To estimate the error with respect to $M(\cdot)$, we introduce an adjoint problem: find $\varphi \in V$, such that

$$a(v, \varphi) = M(v), \quad \forall v \in V. \tag{3.21}$$

A posteriori error estimation

An *a posteriori* error representation then follows from (3.16) and (3.21),

$$M(u) - M(U) = a(u, \varphi) - a(U, \varphi) = L(\varphi) - a(U, \varphi) = r(U, \varphi), \quad (3.22)$$

with the weak residual functional $r(U, \cdot) = L(\cdot) - a(U, \cdot) \in V'$, acting on the adjoint solution $\varphi \in V$,

$$r(U, \varphi) = L(\varphi) - a(U, \varphi).$$

Adaptive methods

With $U \in V_h$ a finite element approximation computed over a mesh $\mathcal{T}^h$, we can split the integral over the elements K in $\mathcal{T}^h$, so that the a posteriori error representation (3.22) is expressed as

$$M(u) - M(U) = r(U, \varphi) = \sum_{K \in \mathcal{T}^h} r(U, \varphi)|_K = \sum_{K \in \mathcal{T}^h} \mathcal{E}_K,$$

with the local error indicator

$$\mathcal{E}_K = r(U, \varphi)|_K,$$

defined for each element K. To approximate the error indicator we can compute an approximation $\Phi \approx \varphi$ to the adjoint problem (3.21), so that

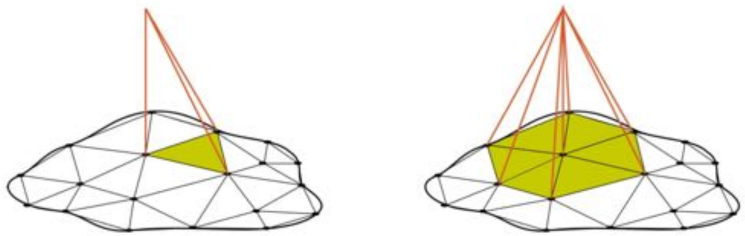
$$\mathcal{E}_K \approx r(U, \Phi)|_K.$$

Finite element method - mesh

For a simplicial mesh $\mathcal{T}^h$, the global approximation space of continuous piecewise polynomial functions V_h is spanned by the global nodal basis $\{\phi_j\}$, where each basis function ϕ_j is associated to a global vertex N_j. Hence with Dirichlet boundary conditions the finite element approximation $U \in V_h$ can be expressed as

$$U(x) = \sum_{N_j \in \mathcal{N}_I} U(N_j)\phi_j(x) + \sum_{N_j \in \mathcal{N}_D} U(N_j)\phi_j(x),$$

with $\mathcal{N}_I$ all internal vertices in the mesh and $\mathcal{N}_D$ all vertices on the Dirichlet boundary, and where $U(N_j)$ is the node which corresponds to function evaluation at the vertex N_j.



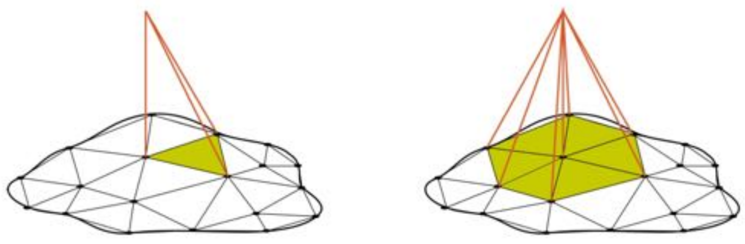
Finite element method - mesh

The finite element method takes the form of a matrix problem

$$Ax = b, \tag{3.24}$$

where $a_{ij} = a(\phi_j, \phi_i)$, $x_j = U(N_j)$ and $b_i = L(\phi_i)$. To compute the Galerkin finite element approximation, we thus have to construct the matrix A and vector b, and then solve the resulting matrix problem (3.24) to obtain the nodal values $U(N_j)$.

$$U(x) = \sum_{N_j \in \mathcal{N}_I} U(N_j)\phi_j(x) + \sum_{N_j \in \mathcal{N}_D} U(N_j)\phi_j(x)$$



Conforming triangular mesh

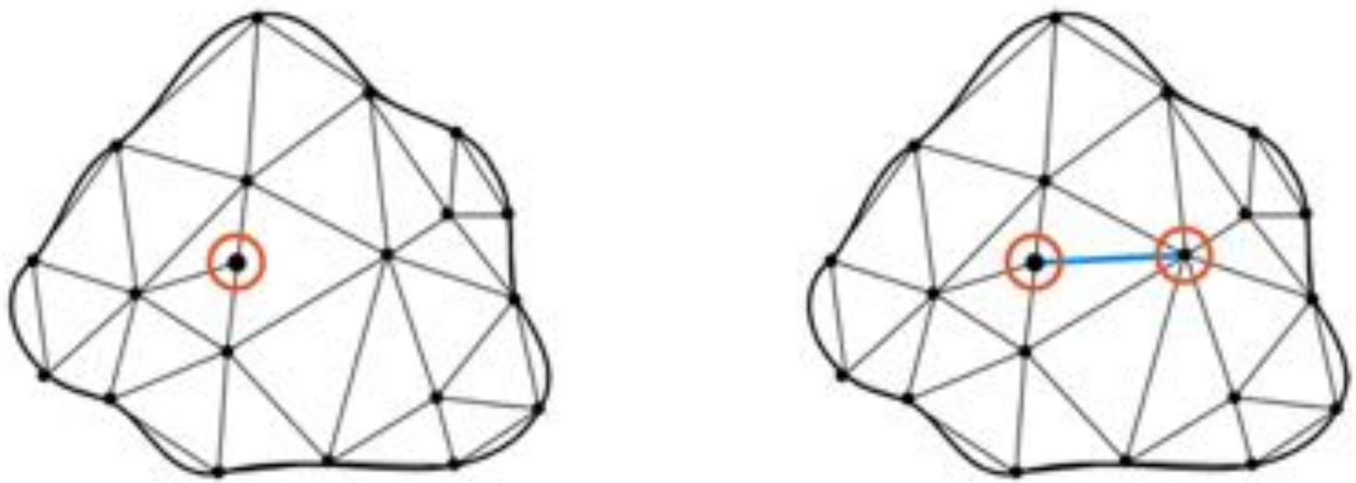
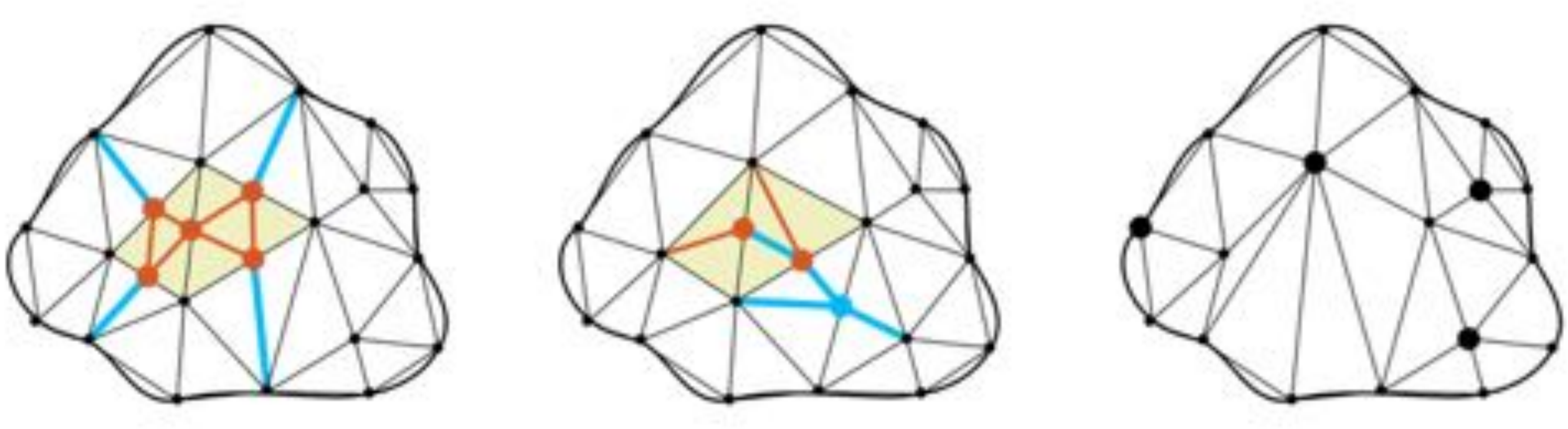


Figure 10.5. *Illustration of a non-conforming triangular mesh with a hanging node (left), which can be made into a conforming mesh by adding a new edge that eliminates the hanging node (right).*

Mesh generation

Figure 10.6. *Mesh generation by a quadtree algorithm (left), by the Delaunay condition (center), and by advancing front mesh generation (right).*

Mesh refinement and coarsening



Stokes equations

The Stokes equations for a domain $\Omega \subset \mathbb{R}^n$ with boundary $\nabla\Omega = \Gamma_D \cup \Gamma_N$, and associated normal n, takes the form

$$\begin{aligned} -\Delta u + \nabla p &= f, & x &\in \Omega, \\ \nabla \cdot u &= 0, & x &\in \Omega, \\ u &= g_D, & x &\in \Gamma_D, \\ -\nabla u \cdot n + pn &= g_N, & x &\in \Gamma_N. \end{aligned}$$

First assume that $\partial\Omega = \Gamma_D$ and $g_D = 0$, that is, homogeneous Dirichlet boundary conditions for the velocity. We then seek a weak solution to the Stokes equations in the following spaces,

$$\begin{aligned} V &= H_0^1(\Omega) \times \ldots \times H_0^1(\Omega) = [H_0^1(\Omega)]^n, \\ Q &= \{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\}, \end{aligned}$$

Stokes equations – variational form

We derive the variational formulation by taking the inner product of the momentum equation with a test function $v \in V$, and the inner product of the continuity equation with a test function $q \in Q$. By Green's formula and the homogeneous Dirichlet boundary condition, we obtain the variational formulation as: find $(u, p) \in V \times Q$, such that

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v), \quad \forall v \in V, && (5.6 \\ -b(u,q) &= 0, \quad \forall q \in Q, && (5.7 \end{aligned}$$

$$a(v,w) = (\nabla v, \nabla w) = \int_\Omega \nabla v : \nabla w \, dx, \qquad \nabla v : \nabla w = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}$$

$$b(v,q) = -(\nabla \cdot v, q) = -\int_\Omega (\nabla \cdot v) q \, dx,$$

Stokes equations – finite element method

We seek an approximation $(U, P) \in V_h \times Q_h$, such that,

$$
\begin{aligned}
a(U, v) + b(v, P) &= (f, v), &(5.11)\\
-b(U, q) &= 0, &(5.12)
\end{aligned}
$$

for all $(v, q) \in V_h \times Q_h$, where V_h and Q_h are finite element approximation spaces. There exists a unique solution to (5.11)-(5.12), under similar conditions as for the continuous variational problem.

A posteriori error estimation

The Stokes equations take the form

$$\nabla p - \Delta u = f, \quad \nabla \cdot u = 0,$$

together with boundary conditions for $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_F$

$$u = g_D, \quad x \in \Gamma_D$$

$$u = 0, \quad x \in \Gamma_F$$

$$-\nabla u \cdot n + pn = 0, \quad x \in \Gamma_N$$

Here Γ_D is the part of the boundary where we prescribe Dirichlet boundary conditions, Γ_N a part of the boundary where we apply a homogeneous Neumann boundary condition, and Γ_F the part of the boundary over which we want to compute the force.

We seek a finite element approximation $(U, P) \in V_h \times Q_h$ such that

$$-(P, \nabla \cdot v) + (\nabla U, \nabla v) + (\nabla \cdot U, q) = (f, v)$$

for all test functions $(v, q) \in \hat{V}_h \times Q_h$, where $\hat{V}_h$ are the test functions v such that $v = 0$ for $x \in \Gamma_D$. Here $V_h \subset V, Q_h \subset Q, \hat{V}_h \subset \hat{V}$ are finite dimensional subspaces defined over the computational mesh by finite element basis functions.

A posteriori error estimation

We consider the linear functional $F : V \times Q \to \mathbb{R}$,

$$F(v, q) = (v, \psi_1)_\Omega + (q, \psi_2)_\Omega + \langle \nabla v \cdot n - pn, \psi_3 \rangle_{\Gamma_F}$$

corresponding to weighted mean values of v and q, and the force on the surface $\Gamma_F \subset \partial\Omega$, which generates the adjoint Stokes equations

$$-\nabla\theta - \Delta\varphi = \psi_1, \quad -\nabla \cdot \varphi = \psi_2,$$

together with boundary conditions that reflect the primal equations and the chosen functional.

$$\varphi = 0, \quad x \in \Gamma_D$$

$$\varphi = \psi_3, \quad x \in \Gamma_F$$

$$-\nabla\varphi \cdot n - \theta n = 0, \quad x \in \Gamma_N$$

The weak form of the adjoint Stokes equations take the form: find $(\varphi, \theta) \in \hat{V} \times Q$ such that

$$-(q, \nabla \cdot \varphi) + (\nabla v, \nabla\varphi) + (\nabla \cdot v, \theta) = (v, \psi_1)_\Omega + (q, \psi_2)_\Omega + \langle \nabla v \cdot n - pn, \psi_3 \rangle_{\Gamma_F} = F(v, q)$$

for all test functions $(v, q) \in V \times Q$.

A posteriori error estimation

Since the Stokes equations are linear we can express the error in the linear functional with respect to an approximation $(u, p) \approx (U, P)$ as

$F(u, p) - F(U, P) = (f, \varphi) + (P, \nabla \cdot \varphi) - (\nabla U, \nabla \varphi) - (\nabla \cdot U, \theta) = r(U, P; \varphi, \theta) = \sum_K \mathcal{E}_K$

where we used that $F(u, p) = (f, \varphi)$ since $\varphi \in \hat{V}$, with the error indicator

$\mathcal{E}_K = r(U, P; \varphi, \theta)|_K$,

which is the local residual on weak form with the solution to the adjoint equation as test function. The error indicator $\mathcal{E}_K$ can be used as an indicator for where to refine the mesh to reduce the global error as efficiently as possible.

A posteriori error estimation

Note however that since $(U, P) \in V_h \times Q_h$ is the solution of a Galerkin finite element method, if we use the approximation $(\varphi, \theta) \approx (\varphi_h, \theta_h) \in V_h \times Q_h$, the error indicators sum to zero. Hence, this sum cannot be used as a stopping criterion for an adaptive algorithm. Instead we may use error estimates of the type

$$\mathcal{E}_K \leq Ch_K(\|\nabla\varphi_h\|_K + \|\nabla\theta_h\|_K)\|R(U, P)\|_K$$

where $R(U, P) = (R_1(U, P), R_2(U))$ is the residual of the equations in strong form, with

$$R_1(U, P) = f + \Delta U - \nabla P$$

$$R_2(U) = \nabla \cdot U$$

$$\begin{aligned} |F(u) - F(U)| &= (R(U), \varphi) = (R(U), \varphi - \pi_h\varphi) \\ &= \sum_{i=1}^{n+1} \int_{I_i} R(U)(\varphi - \pi_h\varphi)\, dx \leq \sum_{i=1}^{n+1} C_i \|h_i^2 R(U)\| \|\varphi''\|. \end{aligned}$$

Demo Lab 3

Navier-Stokes equations

The *incompressible Navier-Stokes equations* then takes the form,

$$\dot{u} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = f,$$
$$\nabla \cdot u = 0,$$

with the *kinematic viscosity* $\nu = \mu/\rho$

No slip boundary condition: $u = 0$

Slip boundary conditions: $u \cdot n = 0$

Friction boundary conditions: $n^T \sigma t_i = \beta u \cdot t_i$

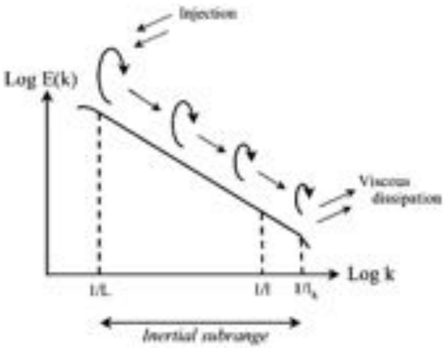
Outflow boundary conditions: $n^T \sigma = 0$

The turbulence problem(s)

Turbulent flow at high Re/in the inviscid limit

- Turbulence simulation (resolution, efficiency)
- Predictability and computability (chaos, perturbation growth)
- Non-viscous dissipation? (Onsager’s conjecture 1949)
- Singularities or not? ($1 million Clay Prize 2000)
- Turbulent flow separation? (d’Alembert’s paradox 1752)

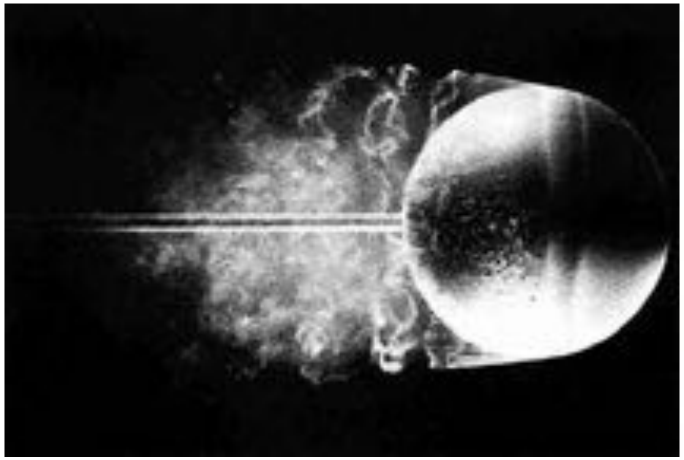
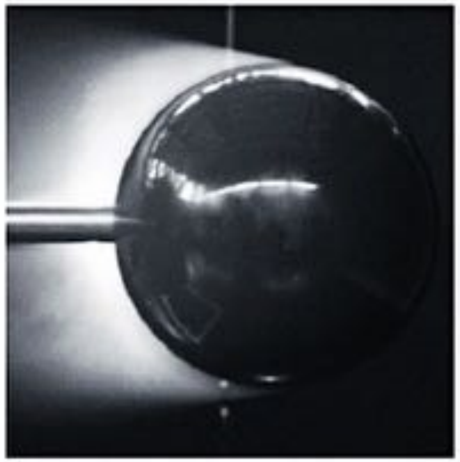
Direct Numerical Simulation (DNS)




- Energy dissipation to heat at smallest scale in turbulent flow (Kolmogorov scale) ~ $Re^{-3/4}$
- Cost of Direct Numerical Simulation (DNS) ~ $Re^{9/4}$
- $Re > 10^6$: DNS impossible! Need cheaper models!

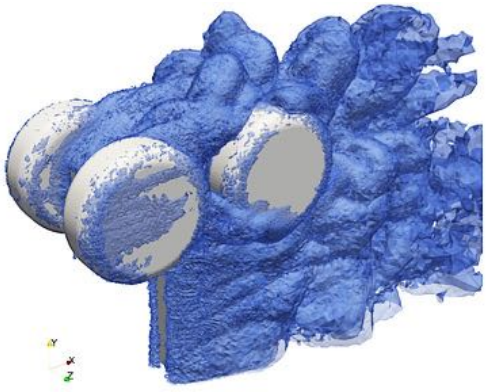
Reynolds Averaged NSE (RANS)

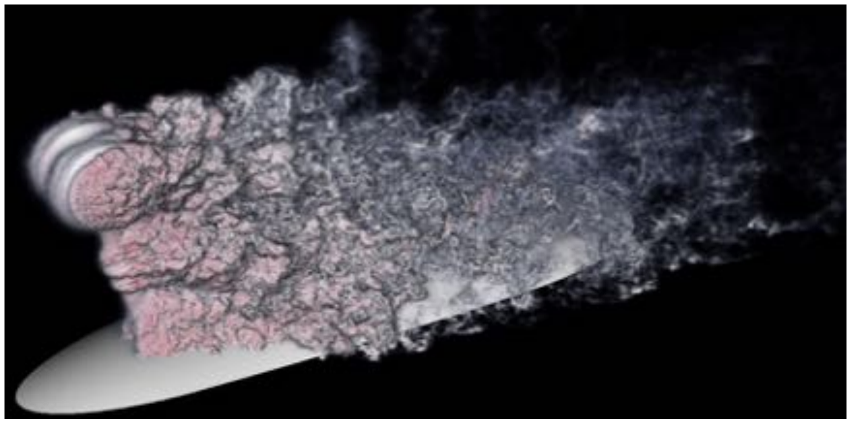
- Compute statistical ensemble average of NSE solutions
- Introduces Reynolds stresses that need to be modeled (turbulent eddy viscosity) : the closure problem
- Calibration of RANS model parameters is a challenge

Large Eddy Simulation (LES)

- Compute only the largest scales of the flow
- Filter NSE (spatial average): model subgrid scales
- Near walls “Resolved LES” impractical for high Re
- Wall-layer models: e.g., assume RANS near the wall (DES)





Direct FEM Simulation of Turbulence (DFS)

- Direct FEM approximation of the Navier-Stokes equations
  - No RANS/LES averaging/filtering (only the mesh scale)
  - No explicit turbulence/subgrid model (no closure problem)
  - Automatic turbulence model based on NSE residual
    (cf. Implicit LES, MILES [Fureby/Grinstein AIAA 99],
    VMM-LES [Bazilevs et.al. 07, Principe/Codina/Henke 10,…])

- Automatic mesh resolution : adaptive algorithm with a posteriori error control in output of interest (drag, lift,…)
- Cheap wall-layer model : slip (friction) boundary condition
  (no boundary layer resolution : no boundary layer mesh)

Direct FEM Simulation of Turbulence (DFS)

- For (v,q) in W_h : find (U,P) in V_h such that

$$(D_t U + (U\cdot\nabla)U,v) + (\nu\nabla U,\nabla v) – (P,\nabla\cdot v) + (q,\nabla\cdot U) + (\delta R(U,P),R(v,q)) = (f,v)$$

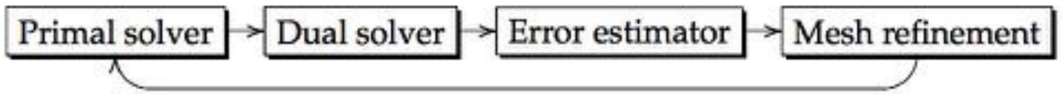
- Slip (no penetration) velocity: $u\cdot n = 0$
- Wall shear stress: $\tau = n^T\sigma t = \beta(u\cdot t)$ (β skin friction coefficient)

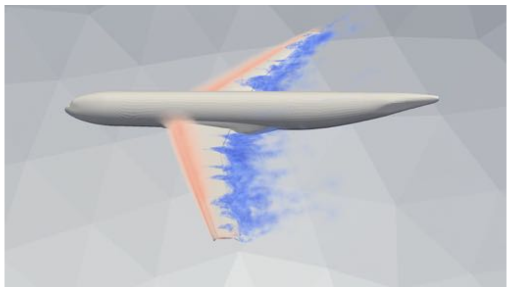
- Least squares stabilization of residual: R(U,P), with $\delta \sim h$
- $R(v,q) = [D_t v + (U\cdot\nabla)v + \nabla q- \nu\Delta v - f, \nabla\cdot v]^T$
- No explicit subgrid model of unresolved scales
- Dissipation: $- dK/dt = ||\beta^{1/2}u\cdot t||^2 + ||\nu^{1/2}\nabla U||^2 + ||\delta^{1/2}R(U,P)||^2$

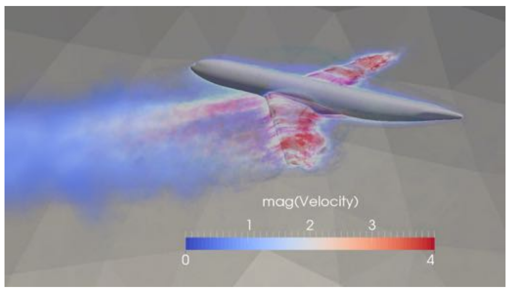
Adjoint based adaptive FEM for turbulence

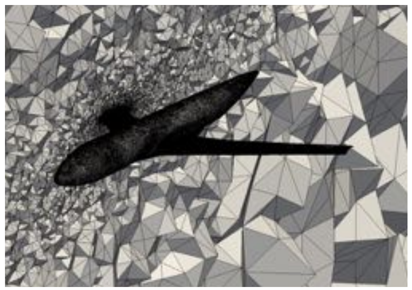
- A posteriori error estimate: $|M(u) - M(U)| \leq \sum_K E_K$
- Error indicator $E_K \sim h_K\, R_K(U)\, S_K$ ($R_K(U)$ residual, h_K mesh size)
- Output sensitivity by adjoint equation: stability weight S_K

$$-\frac{\partial \varphi}{\partial t} - (u \cdot \nabla)\varphi + \nabla U^T \varphi + \nabla \theta - \upsilon \Delta \varphi = M(\cdot),$$
$$\nabla \cdot \varphi = 0$$

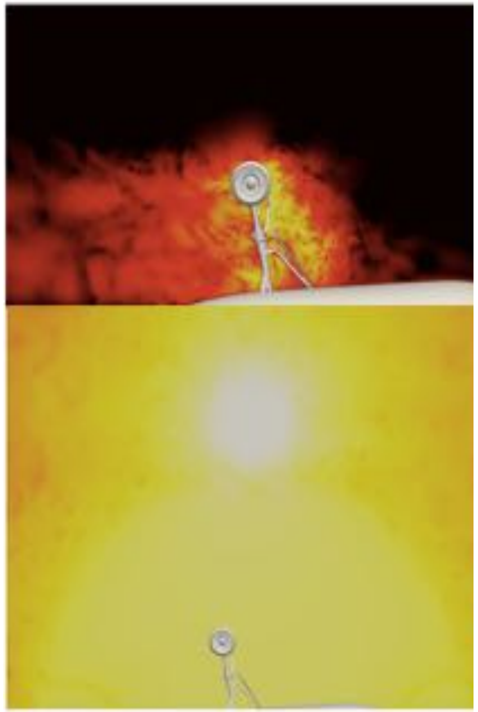


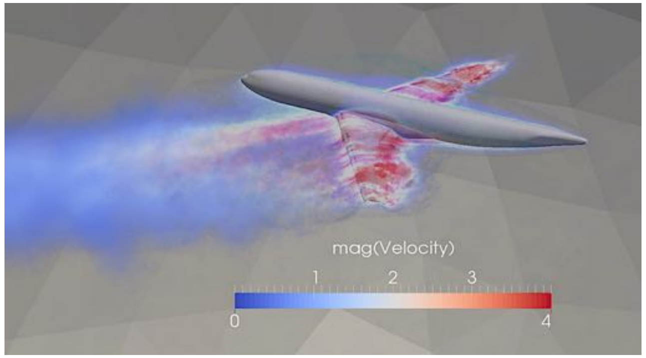






FEM computation of adjoint fields



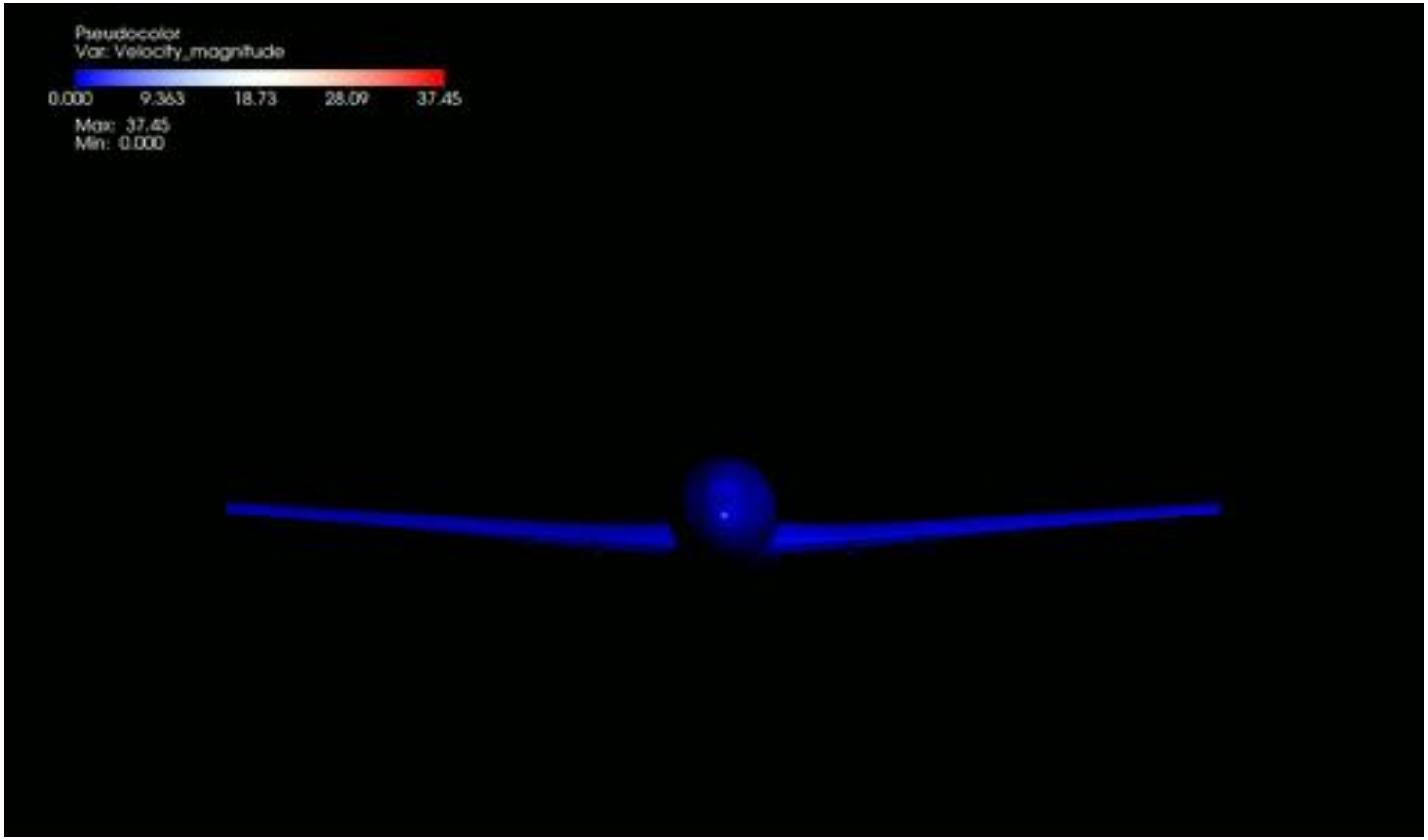


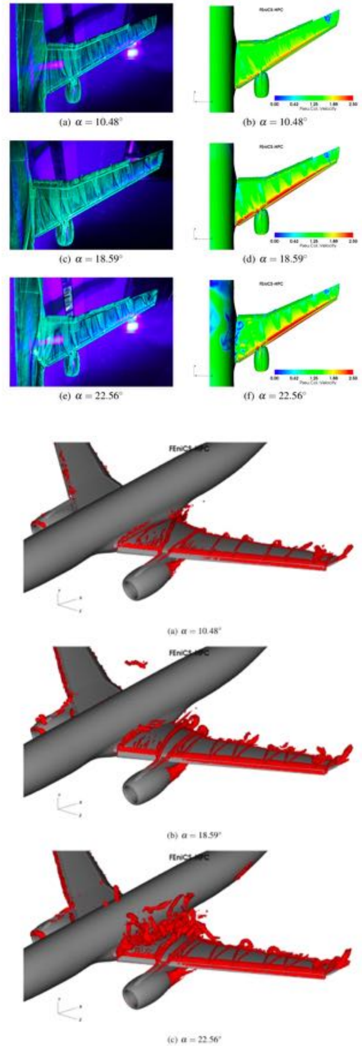
- Adjoint problem solved backwards in time
- Need to store or recompute primal field

Examples

- Acoustic sources
- Aerodynamic forces
- Aerodynamic drag

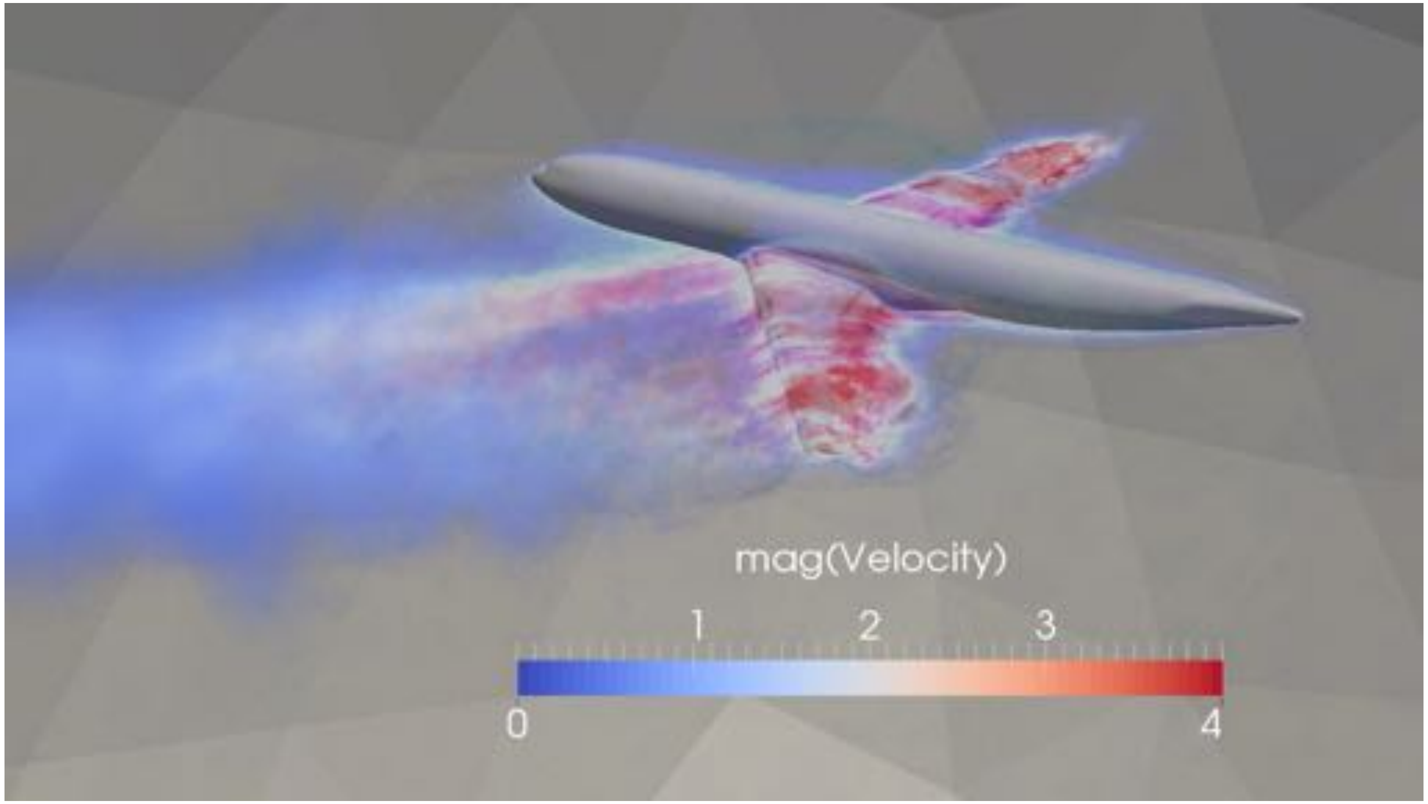
Simulation of airflow past airplane

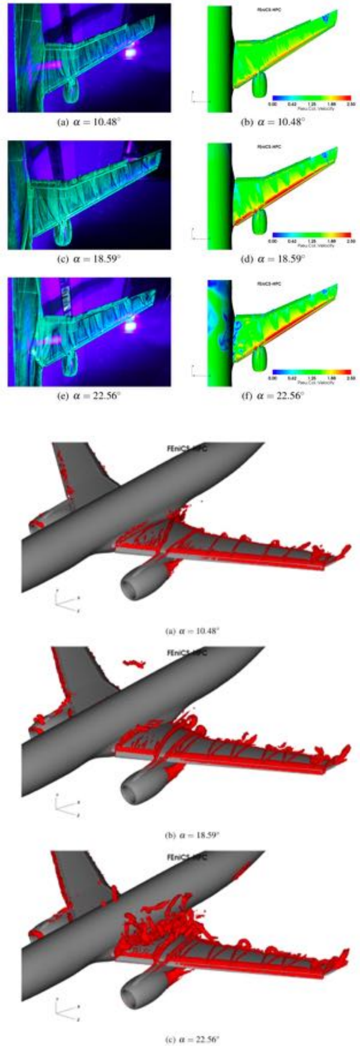




[Jansson et al., Springer, 2018]

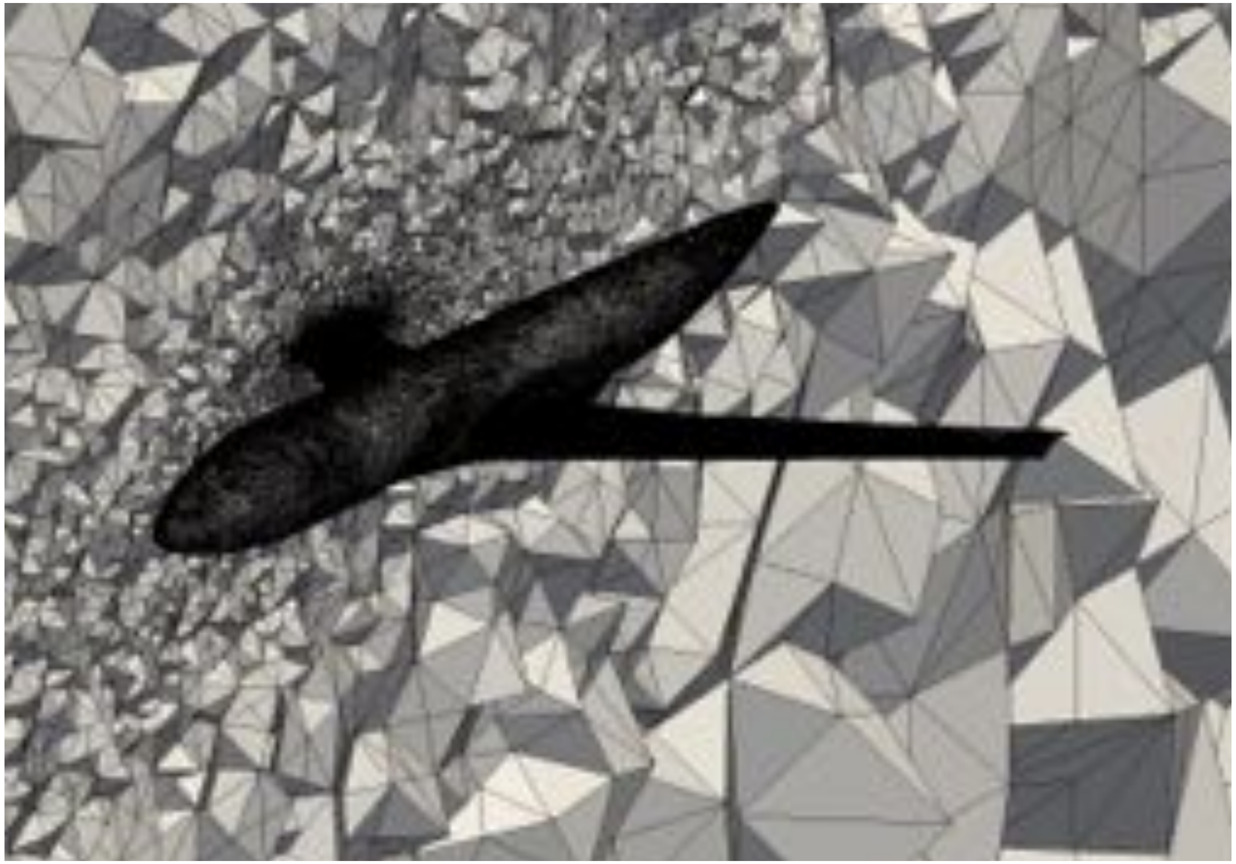
Simulation of airflow past airplane

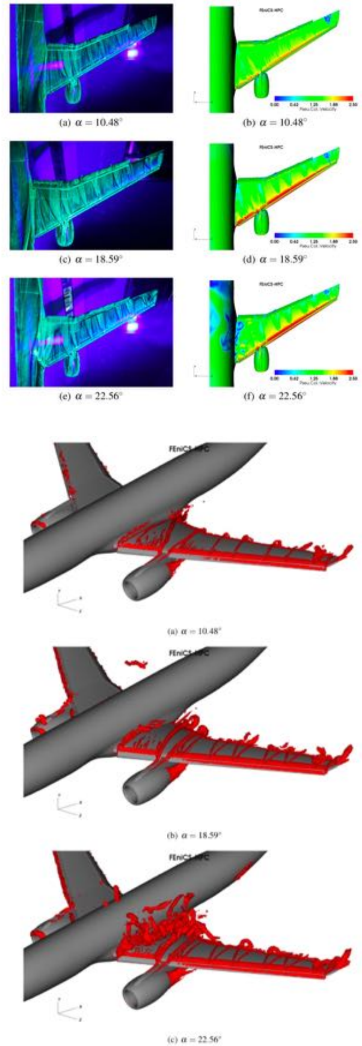




[Jansson et al., Springer, 2018]

Discretization by a mesh



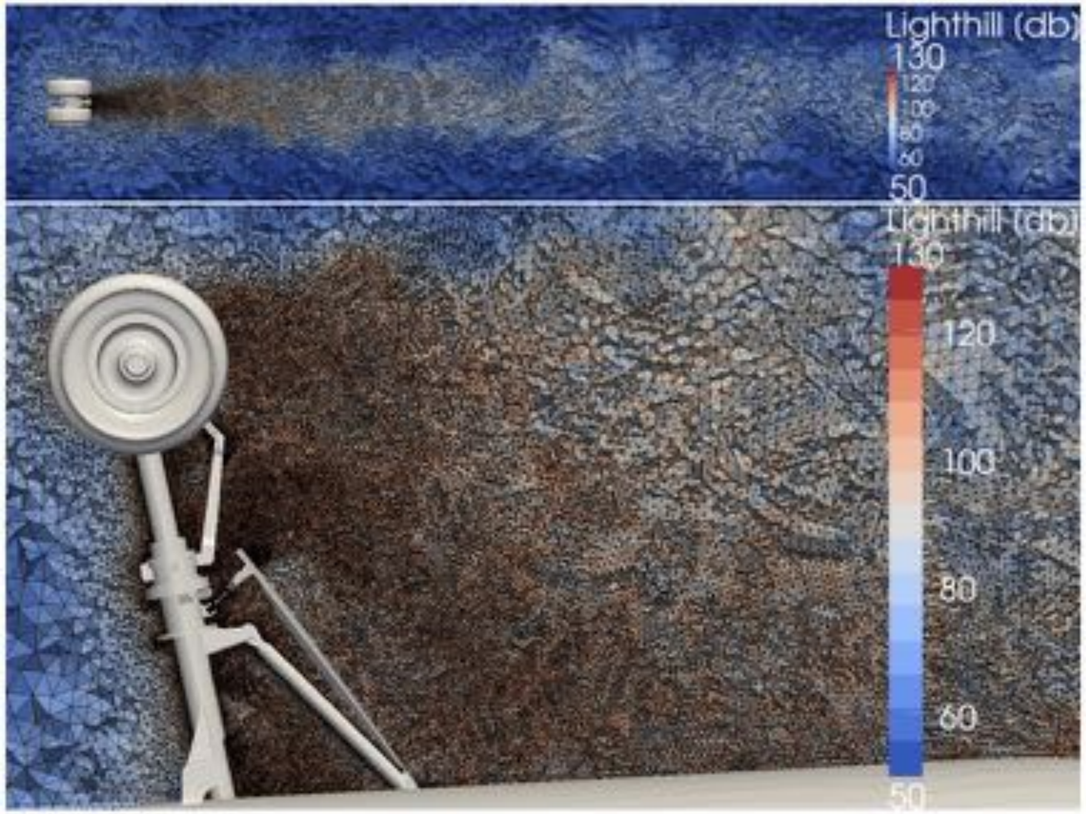


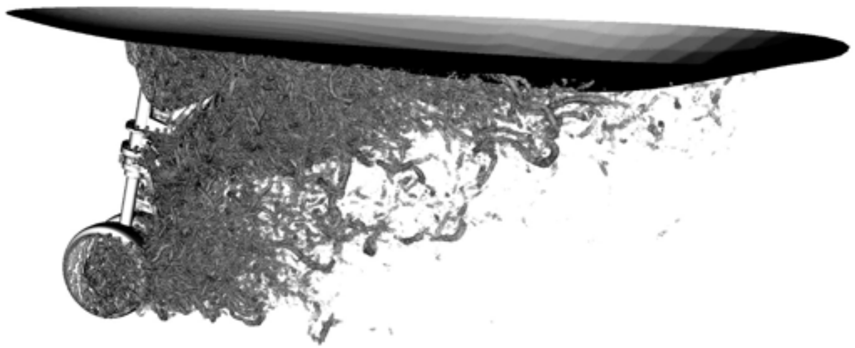
[Jansson et al., Springer, 2018]

Simulation of airflow past landing gear

[De Abreu et al., Computers and Fluids, 2016]

Adaptively refined mesh





[De Abreu et al., Computers and Fluids, 2016]

Turbulent incompressible flow

- Turbulent flow at high Reynolds number $Re = \frac{UL}{\upsilon}$ (and low Mach number)
- Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \upsilon \Delta u = f, \qquad \nabla \cdot u = 0$$

- Euler equations (“$Re = \infty$”)

$$\partial_t u + (u \cdot \nabla)u + \nabla p = f, \qquad \nabla \cdot u = 0$$

- Leray (1934): at least one weak solution to NSE exists.
- Uniqueness of weak NSE solutions - open problem
- Existence of strong (classical) solution – open problem ($1 million Clay Prize problem)
- Existence of weak/strong Euler solutions - open problem

Inertial energy dissipation - NSE

- Local energy equation for weak NSE solution (distributions)

$$\partial_t\left(\frac{1}{2}u^2\right)+div\left(u\left(\frac{1}{2}u^2+p\right)\right)-\upsilon\Delta\frac{1}{2}u^2+\upsilon(\nabla u)^2+D(u)=0$$

- Viscous dissipation intensity: $\upsilon(\nabla u)^2\sim\upsilon\frac{U^2}{L^2}$

- Inertial dissipation intensity (independent of υ):

$$D(u)=\lim_{\epsilon\to 0}D_\varepsilon(u)=\lim_{\epsilon\to 0}\frac{1}{4}\int\nabla\varphi^\varepsilon(\xi)\cdot\delta u(\delta u)^2d\xi\sim\frac{U^3}{L}$$

$$\delta u=u(x+\xi)-u(x)$$

- Dissipative (suitable) weak NSE solution: $D(u)\geq 0$

Inertial energy dissipation - Euler

- Local energy equation for weak Euler solution (distributions)

$$\partial_t\left(\frac{1}{2}u^2\right)+div\left(u\left(\frac{1}{2}u^2+p\right)\right)+D(u)=0$$

- Dissipative weak solution: $D(u)\geq 0$ (cf. entropy condition)

- Onsager's conjecture (cf. Hölder continuity $\alpha > 1/3$)

$$\int|u(x+\xi,t)-u(x)|^3dx\leq C(t)|\xi|\,\sigma(|\xi|)\Rightarrow D(u)=0$$
$$\int_0^T C(t)\,dt<\infty,\ \lim_{a\to 0}\sigma(a)=0$$

- Singularities exist s.t. $D(u)>0$ (Hölder continuity $\alpha < 1/3$)

Inertial energy dissipation – FEM/DFS

- FEM Galerkin least squares stabilization (GLS), linear approximation functions, tetrahedral mesh
- Local energy equation over time interval $I_n = (t_{n-1}, t_n)$ (in a weak sense, up to a factor $Ch^{1/2}$)

$$\partial_t \left(\frac{1}{2}|U^n|^2\right) + div\left(\bar{U}^n\left(\frac{1}{2}|\bar{U}^n|^2 + P^n\right)\right) + D^n = 0$$

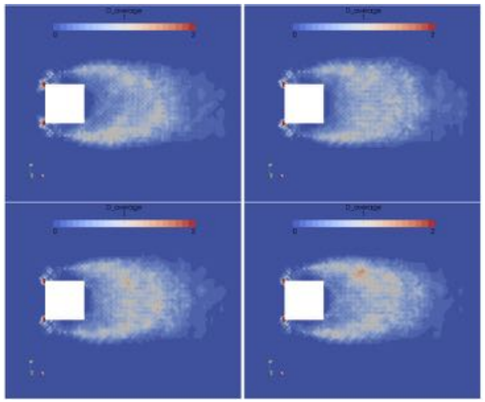
- Residual based numerical dissipation

$$D^n = \delta_1 |R_1(\bar{U}^n, P^n)|^2 + \delta_2 |R_2(\bar{U}^n\ )|^2 \sim \frac{U^3}{L}$$

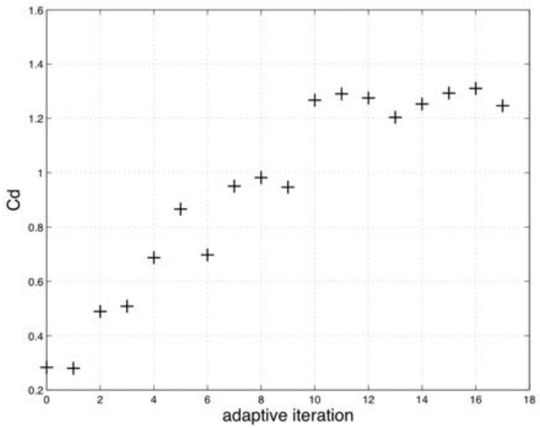
- Computation of dissipative weak solutions by GLS?

Inertial energy dissipation – FEM/DFS

- Numerical evidence for a law of finite dissipation: $D^n > 0$



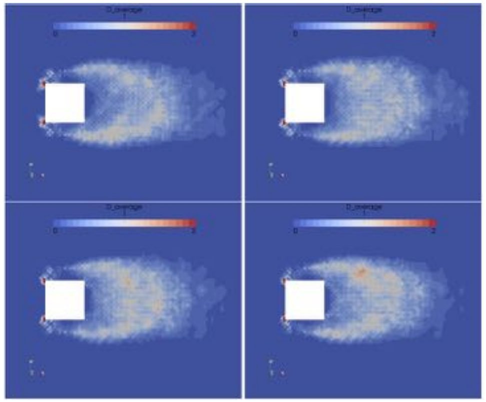
Dissipation intensity $D_h = |\delta^{1/2}R(U)|^2$ under mesh refinement



Convergence in drag for high Re turbulent flow

Inertial energy dissipation – FEM/DFS

- Numerical evidence for a law of finite dissipation: $D^n > 0$



Dissipation intensity $D_h = |\delta^{1/2}R(U)|^2$ under mesh refinement

iteration	mean D^n
3	0.0050
5	0.0252
7	0.0732
10	0.1156
12	0.1282
13	0.1158
14	0.1230
15	0.1479
16	0.1407
17	0.1447

Clay $1 million Prize problem

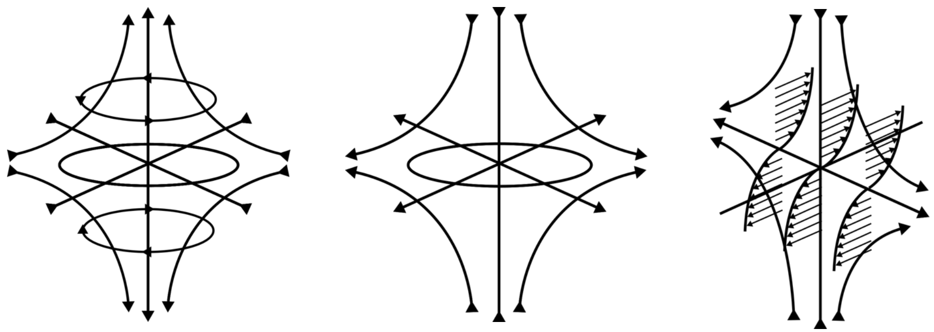
- Existence and smoothness of Navier–Stokes solutions on R^3. Take $\nu > 0$ and $n = 3$. Let $u_\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u(x,t)$ on $R^3 \times [0, \infty)$.
- Breakdown of Navier–Stokes solutions on R^3. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u_\circ(x)$ on R3 and a smooth $f(x, t)$ on $R^3 \times [0, \infty)$, for which there exist no solutions (p, u) of the Navier-Stokes equations.

Clay $1 million Prize problem

- If there is breakdown/blowup of Navier-Stokes equations – what do such singularities look like?
- What is the structure of turbulent flow?

Some structures of incompressible flow

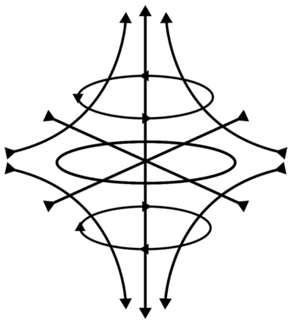
Vortex tube | Pancake | Shear layer



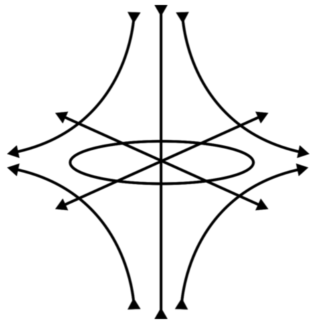
$$\nabla u \quad \begin{bmatrix} \lambda_R & 0 & 0 \\ 0 & \alpha & -\beta \\ 0 & \beta & \alpha \end{bmatrix} \quad \begin{bmatrix} 2\lambda_R & 0 & 0 \\ 0 & -\lambda_R & 0 \\ 0 & 0 & -\lambda_R \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \zeta \\ 0 & \lambda_R & 0 \\ 0 & 0 & -\lambda_R \end{bmatrix}$$

Some structures of incompressible flow

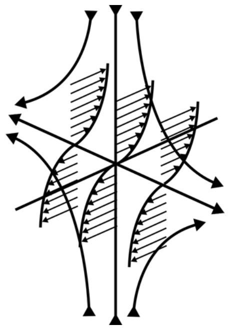
Vortex tube



Pancake



Shear layer



$$\nabla u \qquad \mathrm{V}\begin{bmatrix} \lambda_R & 0 & 0 \\ 0 & \lambda_C & 0 \\ 0 & 0 & \overline{\lambda_C} \end{bmatrix} V^* \quad \begin{bmatrix} 2\lambda_R & 0 & 0 \\ 0 & -\lambda_R & 0 \\ 0 & 0 & -\lambda_R \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \zeta \\ 0 & \lambda_R & 0 \\ 0 & 0 & -\lambda_R \end{bmatrix}$$

$$\lambda_C, \overline{\lambda_C} = \alpha \pm i\beta$$

Analysis of flow structures

- Vorticity $\omega = \nabla\times u$
- Double decomposition of velocity gradient tensor (VGT) into a strain rate tensor $S(u)$ and a spin tensor $\Omega(u)$ (used for Q-criterion, etc.)

$$\nabla u = \frac{1}{2}(\nabla u + \nabla u^T) + \frac{1}{2}(\nabla u - \nabla u^T) = S(u) + \Omega(u)$$

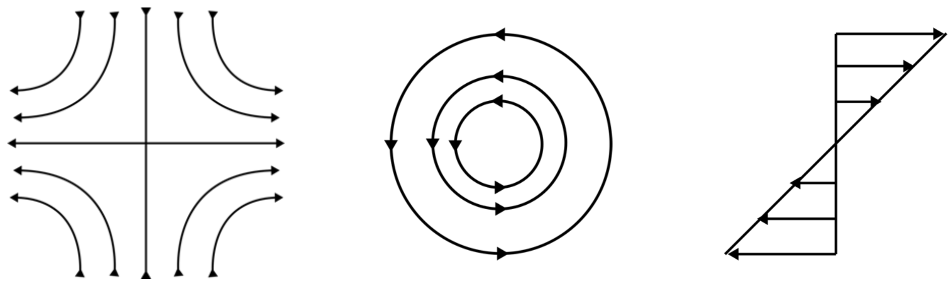
- These methods have a weakness: they do not distinguish shear flow from rotational flow and straining flow.
- Pointed out by V. Kolár [2007] in the context of vortex identification. Suggested a triple decomposition of VGT.
- Followed up by others [e.g. Liu et al. 2018, Keylock 2019, Nagata et al. 2020].

Triple decomposition of VGT

- Triple decomposition of the velocity gradient tensor

$$\nabla u = \nabla u_{strain} + \nabla u_{rotation} + \nabla u_{shear}$$

- Structure of fluid flow: strain + rotation + shear



Algebraic derivation

- Any square matrix A has a (non-unique) Schur factorization

$$A = UTU^*$$

U is unitary, T upper triangular with eigenvalues of A on the diagonal.

- Any normal square matrix is unitary diagonalizable (spectral theorem)

$$A = UDU^*$$

U is unitary with eigenvectors as columns, D diagonal with eigenvalues.

- Any square matrix A is decomposed into sum of a normal and a non-normal part.

$$A = UTU^* = UDU^* + U(T - D)U^*, \qquad \|T - D\|_F^2 = \sum_i \sigma_i^2 - \sum_i |\lambda_i|^2$$

Algebraic derivation

- If A is a real matrix it has a real Schur form

$$A = QBQ^T$$

Q real and orthogonal, B real upper quasi-triangular with possible pairs of conjugate complex eigenvalues $(\lambda, \bar{\lambda})$ represented by real 2x2 block matrices on the diagonal with the same eigenvalues.
[Golub/Van Loan 1996]

$$\begin{bmatrix} \lambda & * \\ 0 & \bar{\lambda} \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- If $a = d$ and $cb < 0$, then the real Schur form is said to be in standardized form, which can be computed by standard methods. [Bai/Demmel 1993]

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \alpha \end{bmatrix}, \qquad \beta\gamma < 0, \qquad \lambda, \bar{\lambda} = \alpha \pm \sqrt{\beta\gamma} = \alpha \pm i\sqrt{|\beta\gamma|}$$

Triple decomposition of VGT

- For the real 3x3 VGT we have a standardized real Schur form

$$\nabla u = Q\overline{\nabla u}Q^T = Q\left(\overline{\nabla u}_{diag} + \overline{\nabla u}_{skew} + \overline{\nabla u}_{nn}\right)Q^T = \nabla u_{sym} + \nabla u_{skew} + \nabla u_{nn}$$

- Assume $|\beta| > |\gamma|$, then with $\lambda_R + 2\alpha = 0$,

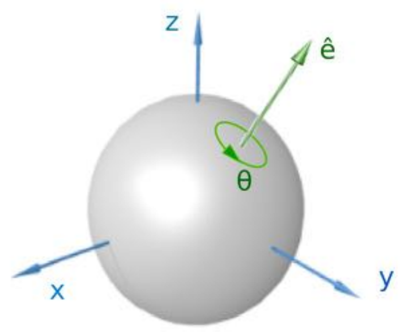
$$\overline{\nabla u} = \begin{bmatrix} \lambda_R & \varepsilon & \zeta \\ 0 & \alpha & \beta \\ 0 & \gamma & \alpha \end{bmatrix} = \begin{bmatrix} \lambda_R & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} + \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -\gamma \\ 0 & \gamma & 0 \end{vmatrix} + \begin{vmatrix} 0 & \varepsilon & \zeta \\ 0 & 0 & \beta + \gamma \\ 0 & 0 & 0 \end{vmatrix}$$

$$\left\|\nabla u_{sym}\right\|_F^2 = \lambda_R^2 + 2\alpha^2, \qquad \|\nabla u_{skew}\|_F^2 = 2\gamma^2, \qquad \|\nabla u_{nn}\|_F^2 = \sum_i \sigma_i^2 - \sum_i |\lambda_i|^2$$

- This algebraic decomposition can be identified with a triple decomposition of VGT into strain, rotation and shear flow.
- Q represents a change of basis – how to interpret Q?

Euler's rotation theorem

- "When a sphere is moved around its center it is always possible to find a diameter whose direction in the displaced position is the same as in the initial position" (Euler, 1776)



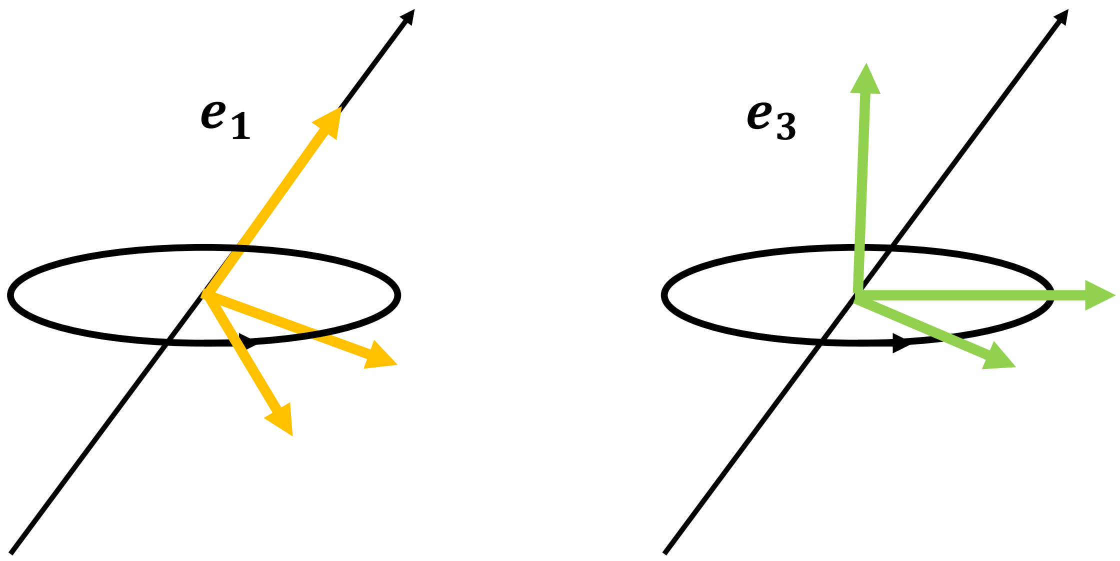
[Wikipedia]

- Rigid body rotational transformation represents the eigenvalue problem:

  $Rv = \lambda v$ (with $\lambda = 1$)

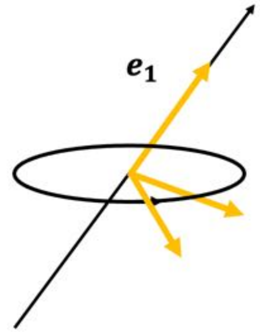
- Here it is natural to align Q with v.
- What about a general linear transformation?

How to choose Q with shear?

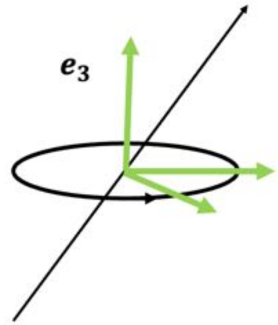


How to choose Q with shear?

$$\nabla u = Q\overline{\nabla u}Q^T, \qquad \overline{\nabla u} = \begin{bmatrix} \lambda_R & \varepsilon & \zeta \\ 0 & \alpha & \beta \\ 0 & \gamma & \alpha \end{bmatrix}$$



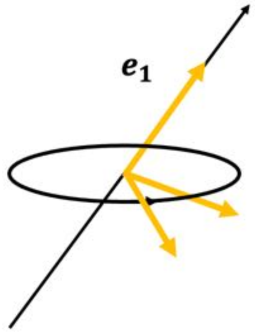
$$\nabla u = \hat{Q}\widehat{\nabla u}\hat{Q}^T, \qquad \widehat{\nabla u} = \begin{bmatrix} \alpha & \hat{\beta} & \hat{\zeta} \\ \hat{\gamma} & \alpha & \hat{\varepsilon} \\ 0 & 0 & \lambda_R \end{bmatrix}$$



How to choose Q with shear?

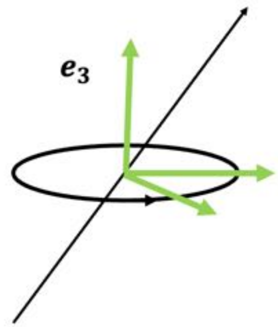
$$v_R = Qe_1, \qquad Q^T v_R = e_1$$

$$\nabla u v_R = Q\overline{\nabla u}Q^T v_R = Q\overline{\nabla u}e_1 = Q\lambda_R e_1 = \lambda_R v_R$$



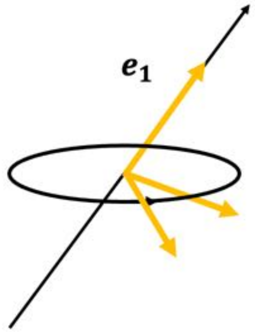
$$\hat{v}_R = \hat{Q}e_3, \qquad \hat{Q}^T \hat{v}_R = e_3$$

$$\nabla u \hat{v}_R = \hat{Q}\widehat{\nabla u}\hat{Q}^T \hat{v}_R = \hat{Q}\widehat{\nabla u}e_3 = \hat{Q}\begin{bmatrix}\hat{\zeta}\\ \hat{\varepsilon}\\ \lambda_R\end{bmatrix} = \hat{Q}\begin{bmatrix}\hat{\zeta}/\lambda_R\\ \hat{\varepsilon}/\lambda_R\\ 1\end{bmatrix}$$

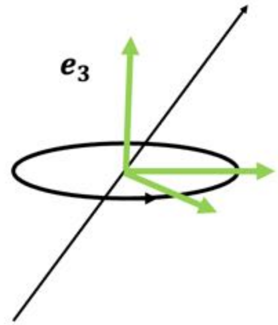


How to choose Q with shear?

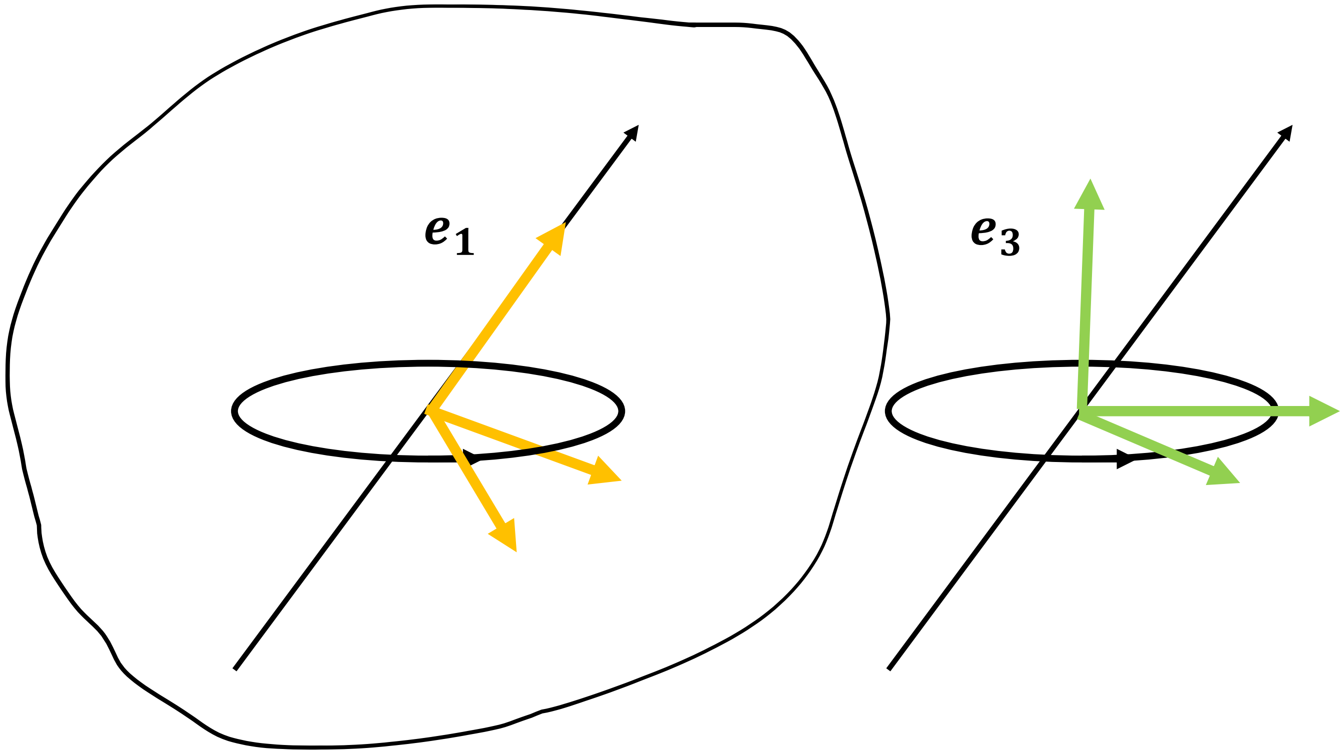
$$\widehat{\nabla u}\begin{bmatrix}\hat{\xi}\\ \hat{\nu}\\ 0\end{bmatrix}=\begin{bmatrix}\alpha & \hat{\beta} & \hat{\zeta}\\ \hat{\gamma} & \alpha & \hat{\varepsilon}\\ 0 & 0 & \lambda_R\end{bmatrix}\begin{bmatrix}\hat{\xi}\\ \hat{\nu}\\ 0\end{bmatrix}=\begin{bmatrix}\alpha\hat{\xi}+\hat{\beta}\hat{\nu}\\ \hat{\gamma}\hat{\xi}+\alpha\hat{\nu}\\ 0\end{bmatrix}$$



$$\overline{\nabla u}\begin{bmatrix}0\\ \xi\\ \nu\end{bmatrix}=\begin{bmatrix}\lambda_R & \varepsilon & \zeta\\ 0 & \alpha & \beta\\ 0 & \gamma & \alpha\end{bmatrix}\begin{bmatrix}0\\ \xi\\ \nu\end{bmatrix}=\begin{bmatrix}\varepsilon\xi+\zeta\nu\\ \alpha\xi+\beta\nu\\ \gamma\xi+\alpha\nu\end{bmatrix}$$



For consistency – go with Euler!



Stability analysis of NSE

- Navier-Stokes equations

$$
\begin{gathered}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \upsilon \Delta u = f \\
\nabla \cdot u = 0
\end{gathered}
$$

- Perturbation equation, adjoint equation, vorticity equation

$$
\begin{gathered}
\frac{\partial u'}{\partial t} + (u \cdot \nabla)u' + (u' \cdot \nabla)U + \nabla p' - \upsilon \Delta u' = P(\cdot), \qquad \nabla \cdot u' = 0 \\
-\frac{\partial \varphi}{\partial t} - (u \cdot \nabla)\varphi + \nabla U^T \varphi + \nabla \theta - \upsilon \Delta \varphi = M(\cdot), \qquad \nabla \cdot \varphi = 0 \\
\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \upsilon \Delta \omega = \Omega(\cdot), \qquad \nabla \cdot \omega = 0
\end{gathered}
$$

Stability analysis of NSE

- All have the same type of stability property ($\phi = \varphi, u', \omega$)

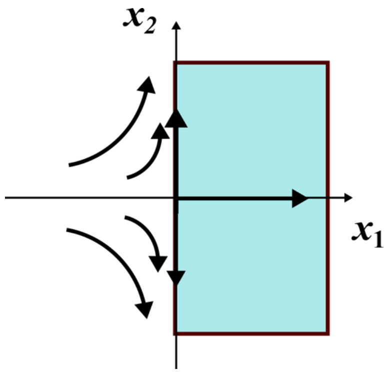
$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2 \pm \int_\Omega \phi^T \nabla u \, \phi \, dx = -\upsilon\|\nabla\phi\|^2 + S(\phi)$$

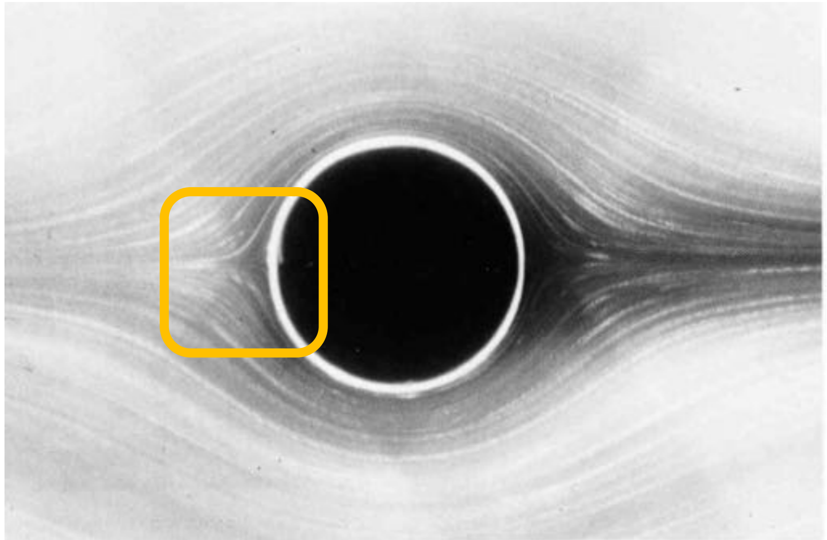
- The key term which determines the stability is the integral

$$\int_\Omega \phi^T \nabla u \, \phi \, dx = \int_\Omega \phi^T (\nabla u_{sym} + \nabla u_{skew} + \nabla u_{nn}) \, \phi \, dx = \int_\Omega \phi^T (\nabla u_{sym} + \nabla u_{nn}) \, \phi \, dx$$

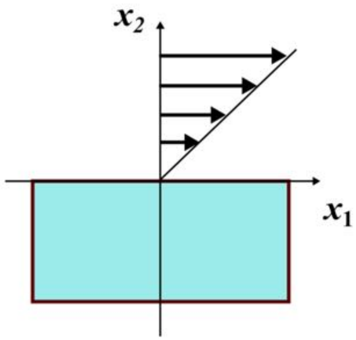
- Rigid body rotational flow is stable (stable vortices)
- Shear flow is linearly unstable (e.g. shear layer roll-up)
- Straining flow exponentially unstable (e.g. vortex stretching)

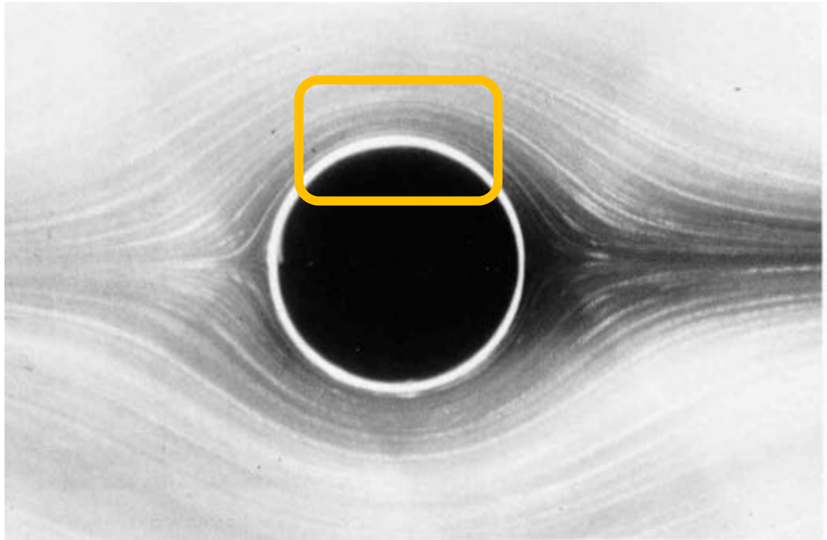
Cylinder (Re = 0.16) – attachment point



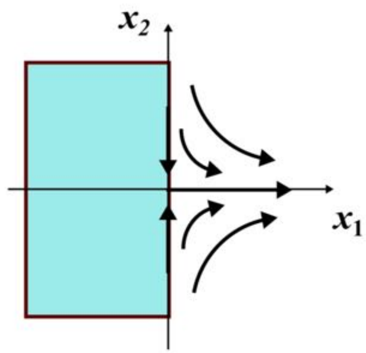


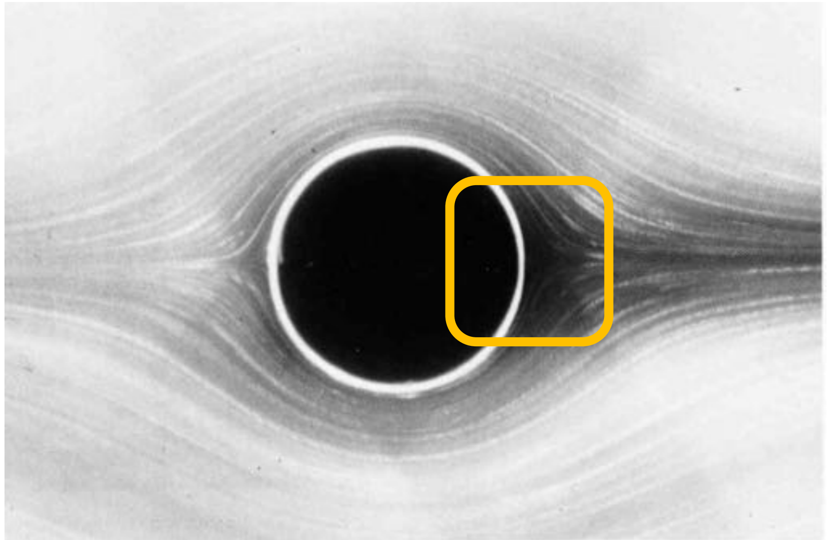
Cylinder (Re = 0.16) – boundary layer



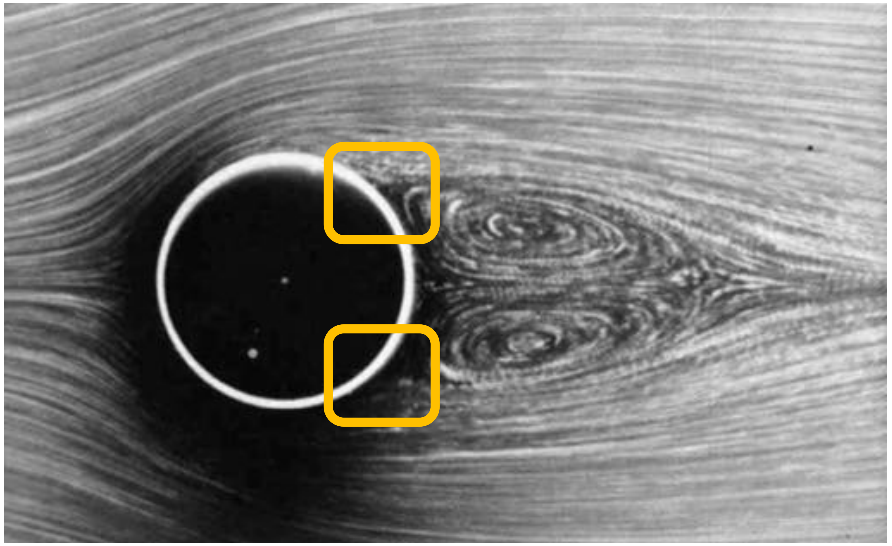


Cylinder (Re = 0.16) – separation point

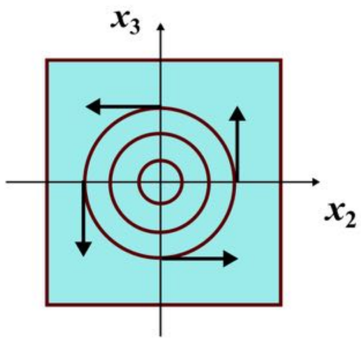


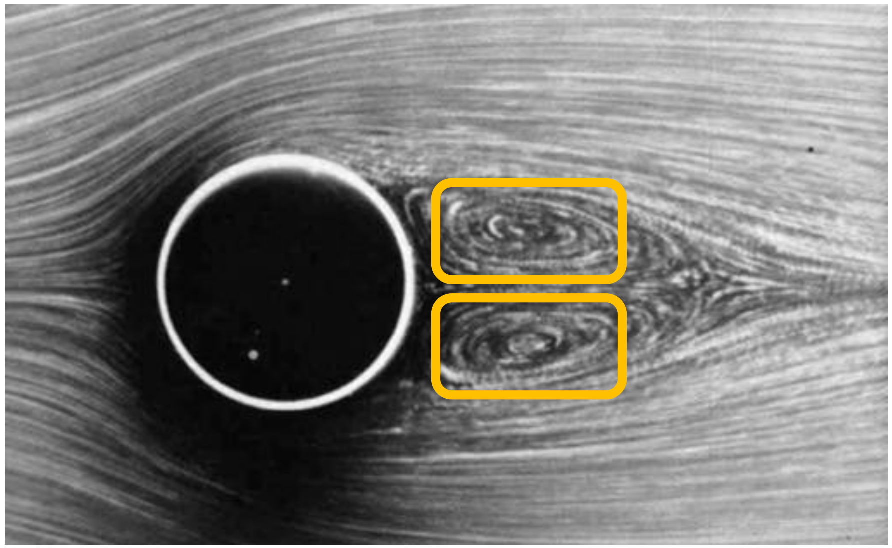


Cylinder (Re = 26) – 2 separation points



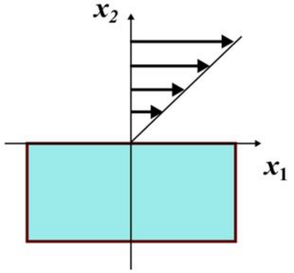
Cylinder (Re = 26) – 2 vortices

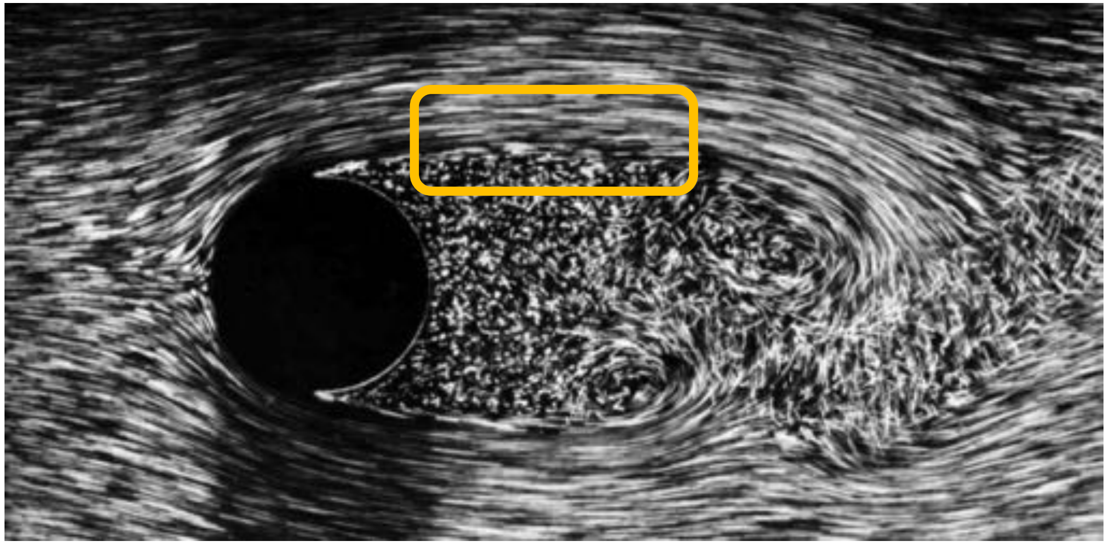




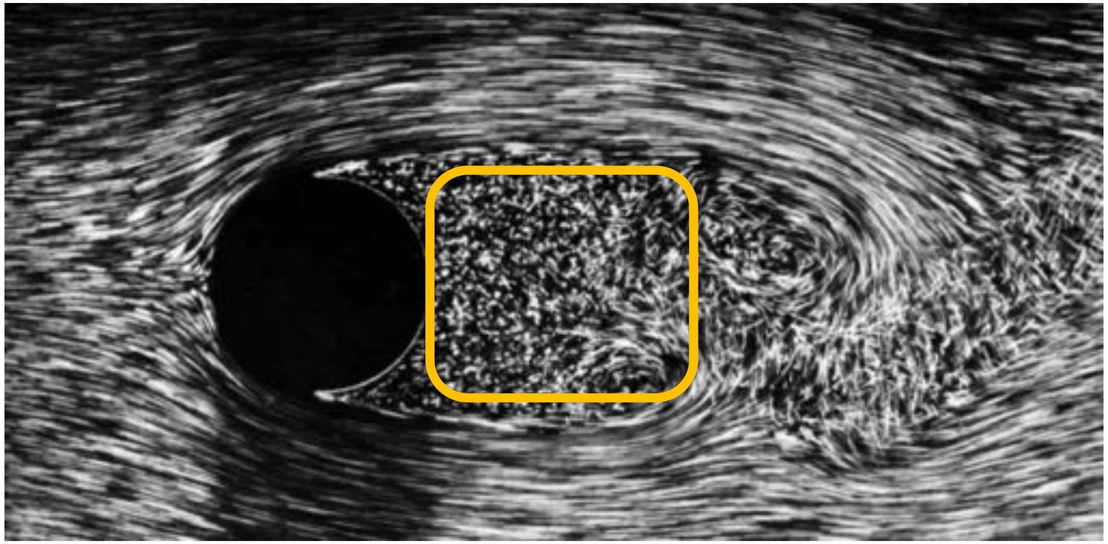
Cylinder (Re = 300) – Karman vortex street

Cylinder (Re = 2000) – shear layer

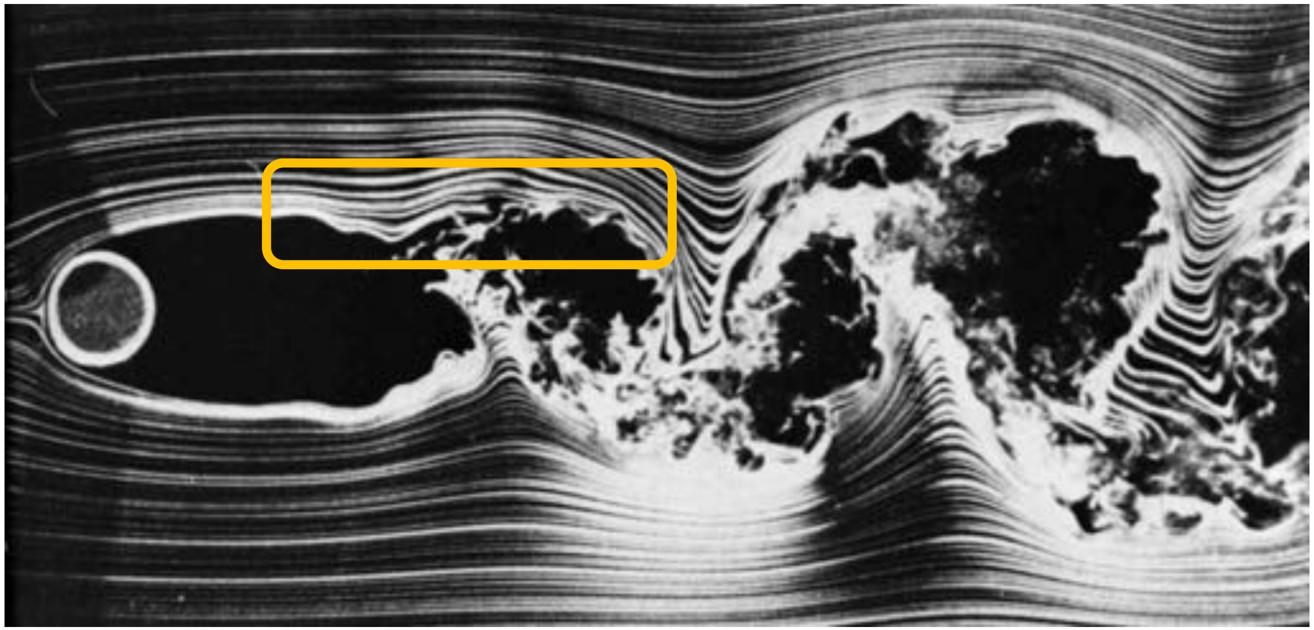




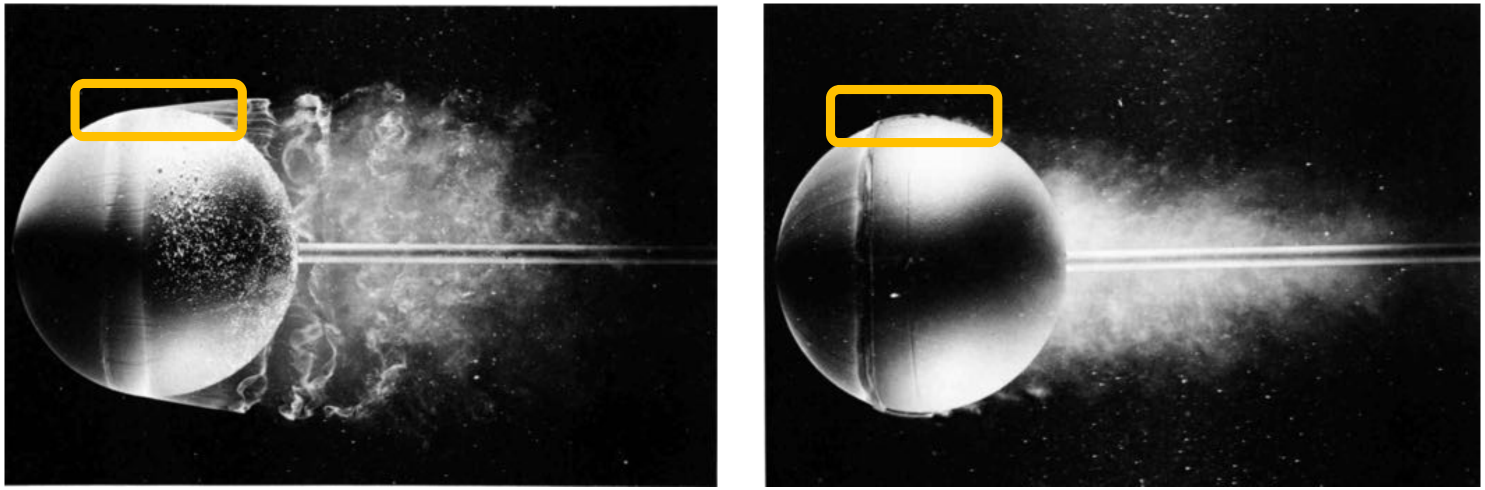
Cylinder (Re = 2000) – 3D turbulent wake



Re = 10 000 – turbulent shear layers



Re = 30 000 - turbulent boundary layer



Simulation of airflow past landing gear

Turbulent vortices (lambda2 criterion)