B. Sc. Engineering Thesis

On Determining Hamiltonicity of a Graph from its all-pair-shortest-path matrix

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CERTIFICATE

This is to certify that the work presented in this thesis entitled "On Determining Hamiltonicity of a Graph from its all-pair-shortest-path matrix" is the outcome of the investigation carried out by me under the supervision of Professor Dr. M. Kaykobad in the Department of Computer Science and Engineering, Bangladesh University of Engineering and Technology (BUET), Dhaka. It is also declared that neither this thesis nor any part thereof has been submitted or is being currently submitted anywhere else for the award of any degree or diploma.

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I am also grateful to my family for providing me the moral support, like always, to push through the continuous difficulties I was kept facing. During my thesis they also encouraged me to keep working. They are the ones who encouraged me to pursue my interest in theoretical research.

Abstract

In my thesis I solved the problem of determining Hamiltonicity of graph from its all-pair-shortest-path matrix for a special case. This special case is 2-connected, k-regular graphs on utmost $3k$ vertices that are proved to be Hamiltonian by Jackson [3]. I determined the upper bound on the shortest path length between vertices in such graphs. This is how I determined all the possible elements of the all-pair-shortest-path matrix of such graph. I also found how the elements in such a matrix is needed to be arranged. I have proved that a graph with the property that no pair of vertices are at the shortest distance of more than 2 is 2-connected. I have found how 2-connectivity can be determined from the all-pair-shortest-path matrix that works in special cases of k-regular graphs on $3k$ vertices.

Chapter 1

Introduction

1.1 Preliminaries

A graph is a structure that shows the relation between several entities. A graph does so by means of vertex and edge. A vertex can represent existence of an entity and an edge can express relation between two entities. Those two entities can be at any relation and that relation is expressed by that graph.

Graph is a discrete structure. It does not take into account of the facts like continuity. All it does is express relation between object. Graph does not express how the entities represented by its vertices changes. But it can be understood from graph that how different entities are influenced by another. This is how graphs model real life problems and reduce it into problem of graph theory. Knowledge in graph is then applied to solve that problem.

Graph theory is ubiquitous in the field of Computer Science. Everything studied in Computer Science is somehow related to graphs and thus can be modeled using the language of graph theory. As Computer Science deals with discrete entities rather than continuous ones, graph theory proved itself useful in this field.

A graph consists of some vertices and some edges. The smallest possible graph is one vertex with no edge. So the smallest possible graph is a vertex. That vertex represents existence of some entity. There are many special types of graph such as k-regular graph, tree, complete graph, k-partite graph, cycle graph, wheel graph to name a few.

Figure 1.1: The smallest possible graph.

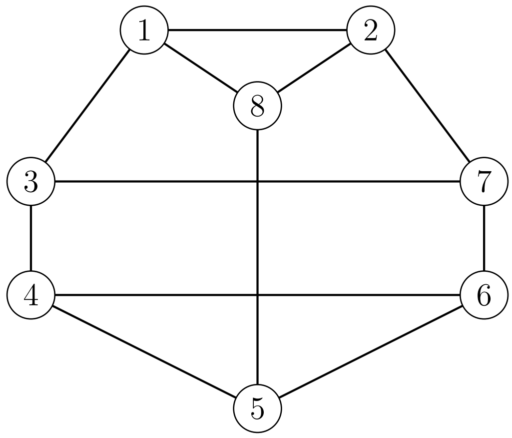

Figure 1.2: A 3-regular graph.

A 3-regular graph. Each vertex of this graph is of degree 3. And hence the name 3-regular. 1.1 shows such a graph. A tree on 11 vertices. There are exactly 10 edges. A tree contains no cycle. A tree on n vertices contains $n - 1$ edges. It is shown in 1.1. A complete graph on n vertices contains $n(n - 1)/2$ edges. This is denoted as K_n. In such graphs all vertices have the property that, rest vertices are adjacent to it. A complete graph on 6 vertices. There are exactly $15 = (6 * (6 - 1)/2)$ edges. A k-partite graph has its vertices partitioned into k-sets. In all such k-sets, no vertex in a particular set is not adjacent to other vertices in that set. Vertices of this graph shown in 1.1 is partitioned into 3 set of vertices, $\{1, 2, 3\}$, $\{4, 5, 6, 7, 8\}$ and, $\{9, 10, 11, 12\}$. All the vertices of C_n has exactly of degree 2. Each vertex v_i of C_n is adjacent to v_{i-1} and v_{i+1}. A cycle graph, C_6 on 6 vertices is shown in 1.1. A wheel graph, W_n consists of a C_n and a vertex in the center. W_6 is shown in 1.1. This consists of a C_6 and another vertex interior to it. To that interior vertex, all other vertices are joined to obtain W_6.

1.1.1 Some Basic Terminology

Loop. A loop in a graph is an edge from a given vertex to itself.

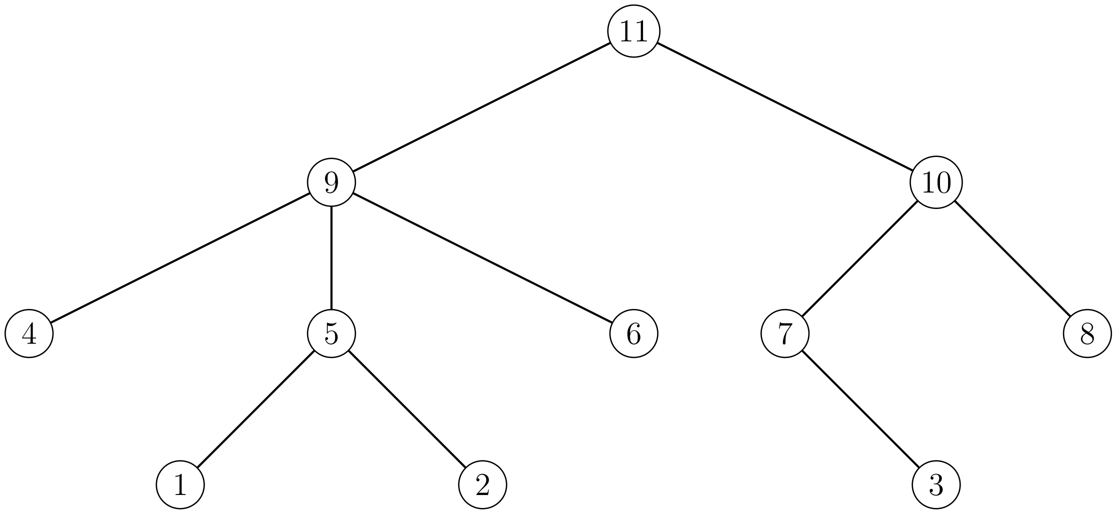

Figure 1.3: A tree.

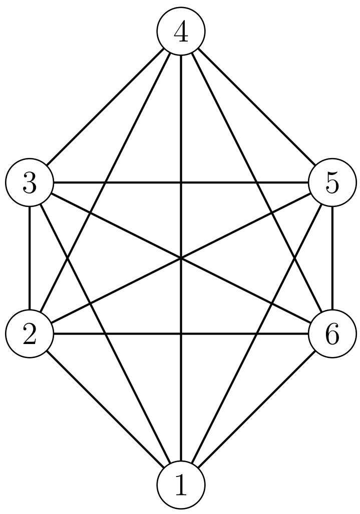

Figure 1.4: K_6.

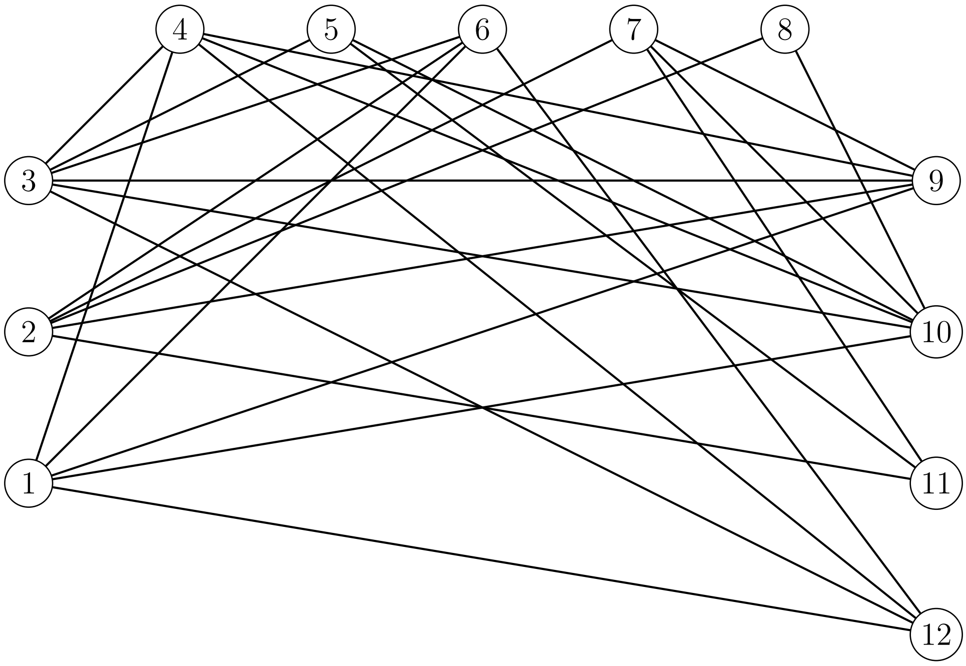

Figure 1.5: A 3-partite graph.

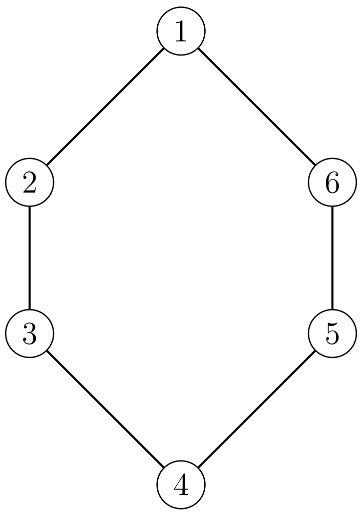

Figure 1.6: C_6.

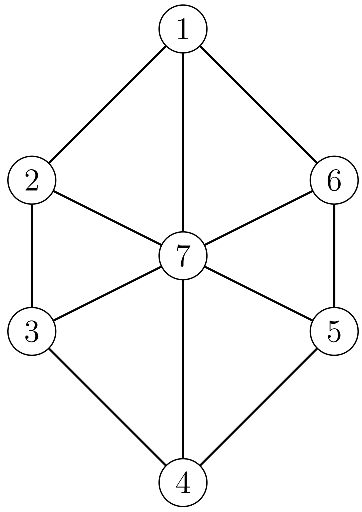

Figure 1.7: W_6.

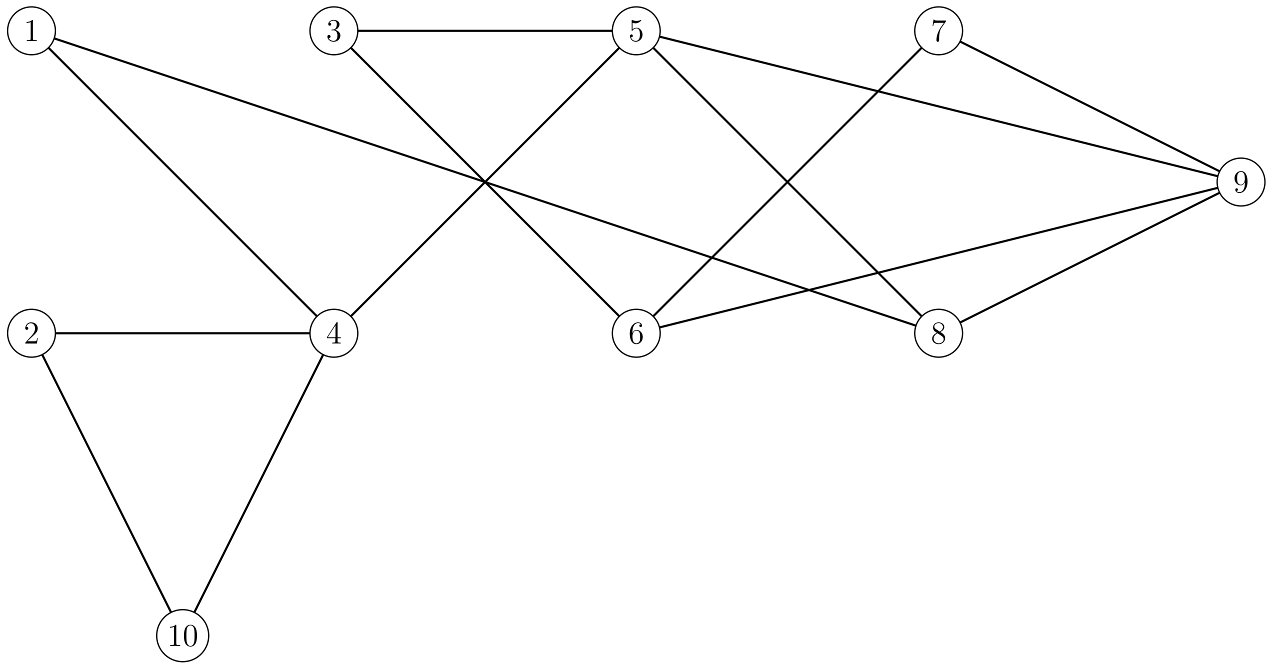

Figure 1.8: A graph with a Hamiltonian path but not a Hamiltonian graph.

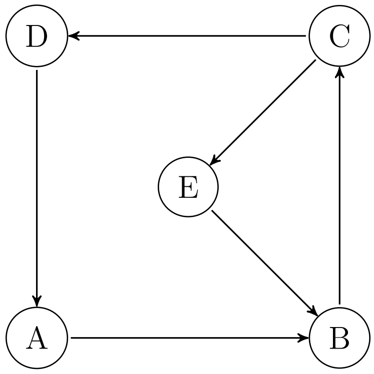

Figure 1.9: A directed graph.

Traveling along loop results in returning back to the vertex where the travel began.

Simple Graph. A graph with no loop is a simple graph.

Directed Graph. A graph is called Directed graph if it has such edges traveling along which is permitted in only one direction.

Directed graph is also known as Digraph. Each edge in a digraph is associated with a direction. For instance, in the graph $G = (V, E)$, an edge $(u, v) \in E$ allows traveling from the vertex u to the vertex v along the edge (u, v). Along the edge (u, v) traveling from the vertex v to u is not possible. A Directed Graph is shown in 1.1.1. Here traveling from the vertex C to E is possible along the edge CE but not possible to return to C from E along the edge CE.

Undirected Graph. A graph is called undirected graph if its edges allow traveling in both direction along it.

In the graph $G = (V, E)$, an edge $(u, v) \in E$ allows traveling from the vertex u to the vertex v as well as traveling from the vertex v to u along the edge (u, v). both endpoints can be reached along edges of undirected graph. An undirected graph is shown in 1.1.1. Here traveling from the vertex B to E is possible along the edge BE and it is also possible to return to B from E along the edge BE.

Weighted Graph. If a value is associated with each edges of a graph then the graph is a weighted graph. An weighted graph is shown in 1.1.1. Here traveling from the vertex B to E

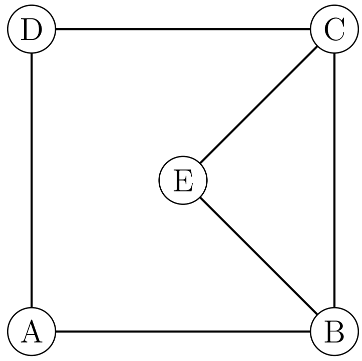

Figure 1.10: An undirected graph.

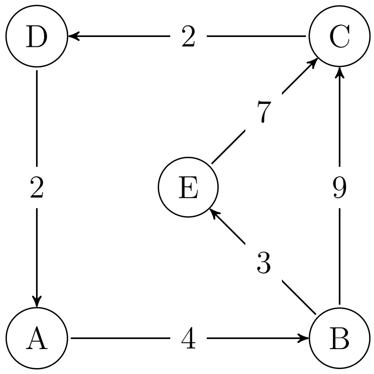

Figure 1.11: An weighted graph.

costs 3. Again traveling from the vertex C to D costs 2. A weighted graph is not necessarily a directed graph.

Unweighted Graph. If no value is associated with any edge of a graph then the graph is a weighted graph.

An example of unweighted graph is shown in 1.1.1. Unit cost is charged for traveling along any edge and any direction.

Connected Graph. A connected graph is one in which any vertex of the graph is reachable from a vertex given.

For example, the graph 1.1.1 is a connected graph.

Cut Point. Vertex in a graph removal of which makes the graph disconnected is a cut point of that graph.

For example, the vertex 4 is a cut point of the graph 1.1.1.

Connectivity. Connectivity of a graph is the number of vertex required to remove to make that graph non-connected.

Path. A path in a graph is the sequence of vertices that are followed on after another to reach a vertex from another.

A path in a graph can begin and end at any vertex. There is no bound on how many times a particular vertex can be entered and left.

Hamiltonian Path. A Hamiltonian path is a path in a graph consisting of all the vertices of the graph and all the vertices but the start and end vertex is entered and left exactly once while the start vertex is left only once and end vertex is entered only once.

For example, the path $2 \to 10 \to 4 \to 1 \to 8 \to 9 \to 7 \to 6 \to 3 \to 5$ is a Hamilton path in the graph 1.1.1. The start vertex is the vertex 1 which is left once but never entered and the end vertex is 5 which is once entered but never left.

Hamiltonian Cycle. A Hamiltonian cycle of a graph is a cycle where each vertex is entered exactly once and left once.

In case of Hamiltonian cycle each vertex is exactly once entered and left once.

Hamiltonian Graph. A graph that contains at least one Hamiltonian cycle is a Hamiltonian graph.

A Hamiltonian graph with a Hamiltonian cycle $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 1$ is shown in1.1.1.

1.1.2 Some Known Results on Hamiltonian Graph

Hamiltonian graph is a graph with at least one Hamiltonian cycle. Finding Hamiltonian cycle is an NP-hard problem. Unless $P = NP$, we can expect no such algorithm that can determine any Hamiltonian graph.

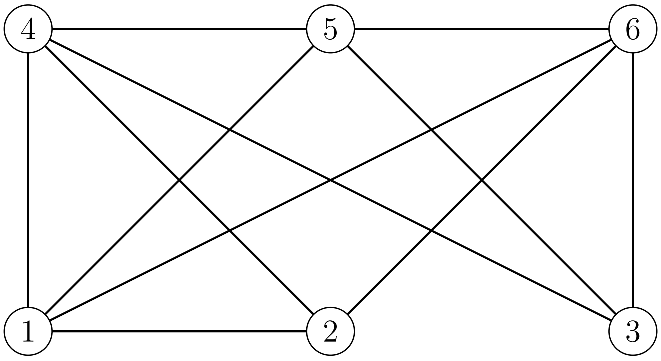

Figure 1.12: A Hamiltonian graph.

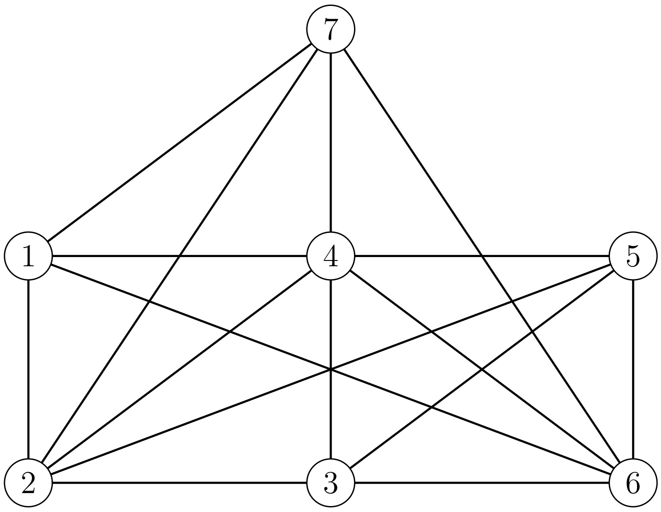

Figure 1.13: A non-regular graph with a Hamiltonian cycle.

There are few cases of Hamiltonian graph that can be found in linear time. These linear time algorithms are based on some results that provides some criteria sufficient for a graph to possess to be Hamiltonian. These results are mostly related to the degree of vertices in graph.

One of earliest sufficiency result is due to Dirac. He proved that if a graph posses such property that each of its vertex has degree no less than $|V|/2$ then the graph is Hamiltonian [1]. He proved this result by method of contradiction. Such a graph is shown in figure 1.1.2. This is a non-regular graph with each vertex having degree no less than $|V|/2$. This is a graph on 7 vertices with 4 as minimum degree of vertex. This is a Hamiltonian graph with a Hamiltonian cycle $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 1$.

Later Ore proved, in 1960, that in a simple graph $G = (V, E)$ if sum of degree of two non-

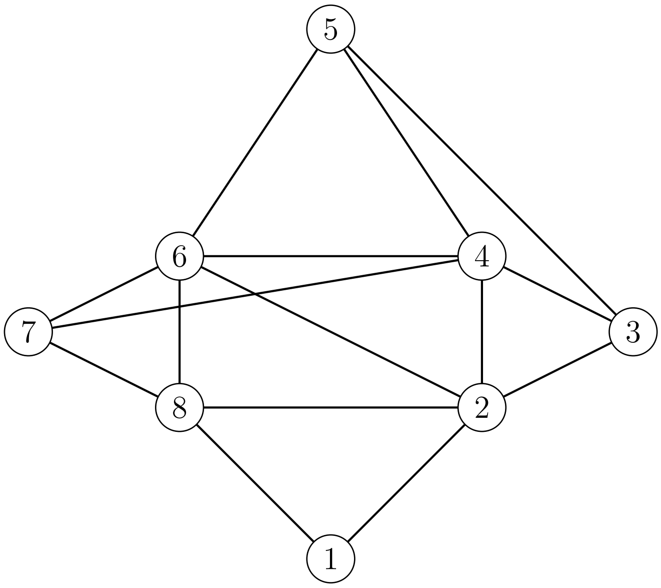

Figure 1.14: A graph on 8 vertices.

adjacent vertices, say u and v, in G exceeds $|V|$ then G is Hamiltonian given $G + uv$ is too Hamiltonian[4]. Ore eliminated the minimum degree condition, as imposed by Dirac, by taking $deg(u) + deg(v) \geq |V|$.

Pósa provided a theorem that generalizes theorem of Dirac[1] and Ore[4]. He proved this theorem by method of contradiction.

Theorem 1.1. *The graph G has $p \geq 3$ vertices. If for every n, $1 \leq n \leq (p-1)/2$, the number of vertices of degree not exceeding n is less than n and if, for odd p, the number of vertices of degree at most $(p-1)/2$, then G is Hamiltonian.*

Chvátal and Erdős provided a sufficient condition on Hamiltonicity based on connectivity and independence number of a graph[6]. Independence number is the cardinality of the largest independent set of a graph. This is also known as vertex independence number. Independent set of a vertex is the set of vertices of a graph such that no two vertices in the set are adjacent. this can be better illustrated from an example 1.1.2. This is a 2-connected graph and the independence number of this graph is 2. Such a graph is Hamiltonian as result provided by Chvátal and Erdős.

Another result worth mentioning is due to Jackson [3]. Jackson proved that a k-regular, 2-connected graph on utmost $3k$ vertices is Hamiltonian. Most of the results in this thesis is based upon this result.

1.2 Motivation of the Problem

A graph can be represented by means of adjacency matrix. When adjacency matrix of a graph is given we can find degree of each vertex and we also can know neighbors of a vertex. Finding Hamiltonian cycle in a graph is a NP-hard problem. So it is unlikely that there would be a polynomial time algorithm that can detect any case of Hamiltonian graph from its adjacency matrix.

The all-pair-shortest-path matrix of a graph contains not only the information that adjacency matrix contains but also information on the shortest distance between any pair of vertices. When the degree of vertices of a graph is fixed or bounded the length of the shortest path between the pairs of vertices also is bounded. The upper bound and lower bound of the length of the shortest path between vertices reveal a great deal of the structure of the graph.

The distance information that the all-pair-shortest-path matrix provides actually reveals the how vertices are arranged in contrast to only letting know of how they are connected to one another as the adjacency matrix does. It would be better understood from figures provided in 2 and 3. Especially it is better understood for graphs on large number of vertices as shown in 2.3.2 and 3.1. This distance information can be used to determine properties like 2-connectivity as we would be seeing later in this thesis.

1.3 Definition of the Problem

The problem we would be exploring in this thesis is to determine if a graph is Hamiltonian or not from its all-pair-shortest-path matrix. The maximum length of the shortest path between pairs of vertices plays a great role in revealing information on how the graph is constructed. Formally the problem can be stated as following

Problem 1.1. To determine the Hamiltonicity of a graph from its all-pair-shortest-path matrix.

1.4 Our Contribution

We analyze the structure of the k-regular, 2-connected graph on utmost $3k$ vertices in 2. We denote such a graph by G_J from this point on. We also consider all the graphs to be connected from now on unless explicitly mentioned. We also find the upper bound of the length of the shortest path between pair of vertices in such graphs there. But this upper bound does not guarantee 2-connectivity. We explore the structure of such graph in depth more in 3 in search of 2-connectivity. In that chapter chapter we have find necessary conditions for 2-connectivity in such a graph that can be found easily from its all-pair-shortest-path matrix.

Chapter 2

Structure of the graph G_J

In this chapter we study the structure of k-regular graph on utmost $3k$ vertices. We would be studying how such graph can be formed and how this formation affects the maximum length of the shortest path between pair of vertices. In the first section we define some terms to ease our studying such structures. In the second section we would be discussing of such graphs where the maximum length of the shortest path between pair of vertices is as large as 2. In the last section we would be seeing how such a graph is structured that causes the maximum length of the shortest path between vertices to be as large as 3, 4 and 5.

2.1 Some Definitions

Notation 2.1. G_J is a 2-connected, k-regular graph on utmost $3k$ vertices.

Notation 2.2. $\mathfrak{V}_x$ is the set of vertices adjacent to the vertex v_x. Here $\mathcal{V}_x \subseteq V$ and $v_x \in V$.

Notation 2.3. $\delta(G) = min_{v \in V} deg(v)$. In other words, $\delta(G)$ is the minimum degree of a vertex in G.

Notation 2.4. $V_{rest} = V - \{\mathfrak{V}_i \cup \mathfrak{V}_j \cup \{v_i, v_j\}\}$.

2.2 Graphs with Shortest Distance between Pairs of Vertices 2 or Less

We first consider of such G_J that $\delta(G) \geq |V|/2$. In this case each vertex would have no less than $n/2$ neighbors. But unlike in G_J we can drop the assumption of 2-connectivity. Because such a graph is Hamiltonian [1]. But first we would prove that in such a graph there can be no pair of vertices with the shortest distance of more than 2 between them.

Lemma 2.1. *If a graph on n vertices has minimum degree of vertices to be $n/2$, then the maximum length of the shortest path between any pair of vertices is no more than 2.*

Proof. We consider a graph $G = (V, E)$ with $\delta(G) \geq |V|/2$. Here $|V| = n$. Now we consider any two nonadjacent vertices v_i and v_j. Now, $|\mathfrak{V}_i| = |\mathfrak{V}_j| = k \geq n/2$. Therefore, $|V_{rest}| < 0$. So there exists some vertices that are both in $\mathfrak{V}_i$ and $\mathfrak{V}_j$. We say such a vertex is v_a. So the shortest path from v_i to v_j is $v_i \to v_a \to v_j$. It is of length 2. This is true for all nonadjacent pairs of vertices in G.

Therefore, in such a graph the maximum length of the shortest path between any pair of vertices is no more than 2. □

Corollary 2.2. *If a k-regular graph on utmost $3k$ vertices has the property of $k \geq n/2$, then the maximum length of the shortest path between any pair of vertices is no more than 2.*

Lemma 2.3. *A graph on n vertices with each vertex having minimum $n/2$ neighbors and the maximum length of the shortest path between any pair of vertices is no more than 2 is Hamiltonian*

Proof. A graph $G = (V, E)$ on n vertices with each neighbor having minimum $n/2$ neighbors means that each vertex has a minimum degree of $n/2$. Such a graph G is known to be Hamiltonian[1]. 2.1 shows that in G, no pair of vertices can have shortest distance of more than 2 between them. □

Now we would look at a lemma stating that any k-regular graph with the property that no pair of vertices in that graph are at the shortest distance of no more than 2 is 2-connected. This would enable us to reach decision that when we have a k-regular graph on utmost $3k$ vertices it is Hamiltonian. We no more need to assume that it is already 2-connected, for 2.4 proves it.

Lemma 2.4. *If the maximum length of the shortest path between any pair of vertices is at most* 2 *in a simple undirected unweighted k-regular graph with $k \geq 2$ then the graph is* 2*-connected.*

Proof. We consider a simple undirected unweighted k-regular graph $G = (V, E)$ where $\forall v_a, v_b \in V, sd(v_a, v_b) \leq 2$.

We choose two arbitrary vertices $v_i, v_j \in V$ such that $sd(v_i, v_j) = 1$. We say, the vertex v_f is a neighbor of the vertex v_i. Now, G being k-regular, $|\mathfrak{V}_f| = k$. If $v_j \notin \mathfrak{V}_f$, we go to the vertex $v_g \in \mathfrak{V}_f$ and so on until we reach a vertex v_h such that $v_j \in \mathfrak{V}_h$. This is possible due to the fact that $k \geq 2$, that is $deg(u) \geq 2, \forall u \in V$. So there exists at least two vertex-disjoint path between the vertex v_i and v_j namely, $v_i \to v_j$ and $v_i \to v_f \to v_g \to \ldots \to v_h \to v_j$.

We now choose any two arbitrary vertex $v_i, v_j \in V$ such that $sd(v_i, v_j) = 2$. We note that $\mathfrak{V}_i \cap \mathfrak{V}_j \neq \emptyset$. Otherwise the condition that $\forall v_a, v_b \in V, sd(v_a, v_b) \leq 2$ would violate. We say $\{v_c\} \subseteq \mathfrak{V}_i \cap \mathfrak{V}_j$. So v_c can have exactly $k-2$ number of neighbors other than v_i and v_j. These other neighbors are the elements of the set $V - \{v_i, v_j, v_c\}$, that is $\mathfrak{V}_c \subseteq V - \{v_i, v_j, v_c\}$.

Since the graph is k-regular, $|\mathfrak{V}_i| = k$. It might be the case that v_c is adjacent to $k-2$ number of elements of $\mathfrak{V}_i$. So there must be at least $k - (k-2) = 2$ elements of $\mathfrak{V}_i$ those are not adjacent to v_c. We consider that, those two elements of $\mathfrak{V}_i$ are v_d, v_e.

We assumed that, in the graph G, $\forall v_a, v_b \in V, sd(v_a, v_b) \leq 2$. So, $sd(v_d, v_j) \leq 2$. Now, when $sd(v_d, v_j) = 1$, we have two vertex-disjoint paths from the vertex v_i to v_j namely, $v_i \to v_c \to v_j$ and $v_i \to v_d \to v_j$.

When $sd(v_d, v_j) = 2$, we note that, to reach v_j from v_d by traveling a distance of 2, the vertex v_i is not used. Otherwise the length of the path from v_d to v_j would be greater than 2. Moreover v_d is not adjacent to v_c, that is, $v_d \notin \mathfrak{V}_c$. So there exists at least one vertex v_x such that $\{v_x\} \subseteq \mathfrak{V}_i \cap \mathfrak{V}_j$ and v_x is adjacent to v_d. So in this case there exists at least two vertex-disjoint paths between v_i and v_j namely, $v_i \to v_c \to v_j$ and $v_i \to v_d \to v_x \to v_j$. Both $v_d = v_x$ and $v_d \neq v_x$ are possible cases. And for both the cases, there exists at least two vertex-disjoint paths between the vertex v_i and v_j.

So for any pair of vertices $v_i, v_j \in V$ we have at least two vertex-disjoint path between them.

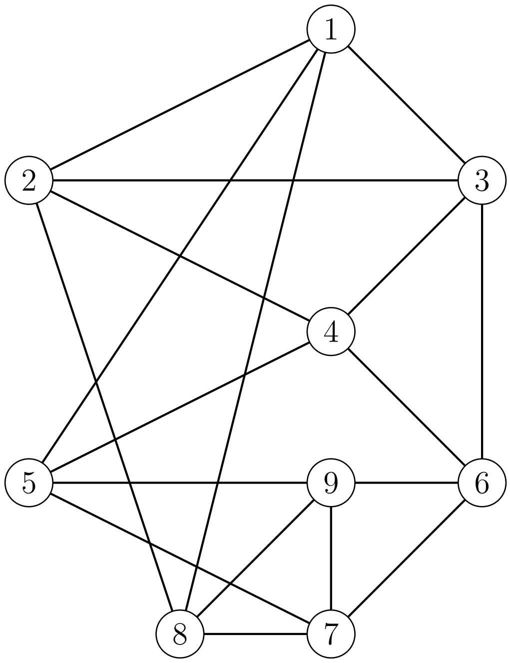

Figure 2.1: A 4-regular graph on 9 vertices.

But when there are two vertex-disjoint path between any pair of vertices in a graph, then the graph is 2-connected [2]. Hence G is 2-connected. $\square$

We see an evidence of such a graph in 2.2. This is a 4-regular graph on 9 vertices. This is a Hamiltonian graph. The Hamiltonian cycle is $5 \to 4 \to 2 \to 1 \to 3 \to 6 \to 9 \to 8 \to 7 \to 5$. Now $9 < 3 * 4$. So this example justifies that we can drop of the assumption of 2-connectivity to prove k-regular graphs on $3k$ vertices to be Hamiltonian due to 2.4, given the graph contains no pair of vertices with the maximum shortest distance between them to be greater than 2.

Corollary 2.5. *A k-regular graph on $3k$ vertices with the property that the maximum shortest distance between any pair of vertices is no more than 2 is Hamiltonian.*

Proof. We know that such a graph is 2-connected as in 2.4 being k-regular and due to absence of pair of vertices with the maximum shortest distance more than 2. Now we have that the graph under discussion is a 2-connected, k-regular graph on $3k$ vertices. Such a graph is Hamiltonian as proved by Jackson[3]. $\square$

There are non-regular graph with Hamiltonian cycle where the maximum length of the shortest distance between pair of vertices is at most 2. Such a graph is shown in 2.2. This is

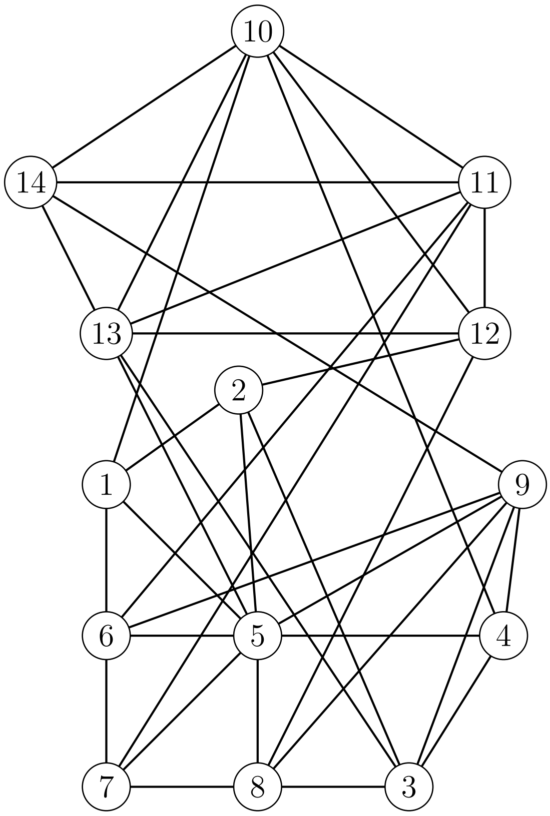

Figure 2.2: A non-regular 2-connected graph.

a non-regular 2-connected graph with the maximum length of the shortest path between any pair of vertices is 2. The Hamiltonian cycle in this graph is $9 \to 14 \to 10 \to 11 \to 12 \to 13 \to 2 \to 1 \to 6 \to 7 \to 8 \to 5 \to 4 \to 3 \to 9$.

2.3 Structure of G_J under Various Maximum Shortest Path Length

2.3.1 The Maximum Length of the Shortest Path between Pairs of Vertices is 3

When 3 is the largest shortest path distance between vertices it is not needed to reach the fourth vertex on a path from some start vertex, for there already is a path from starting vertex to it. Using it would yield the shortest path. If a is the start and b is the end vertex then

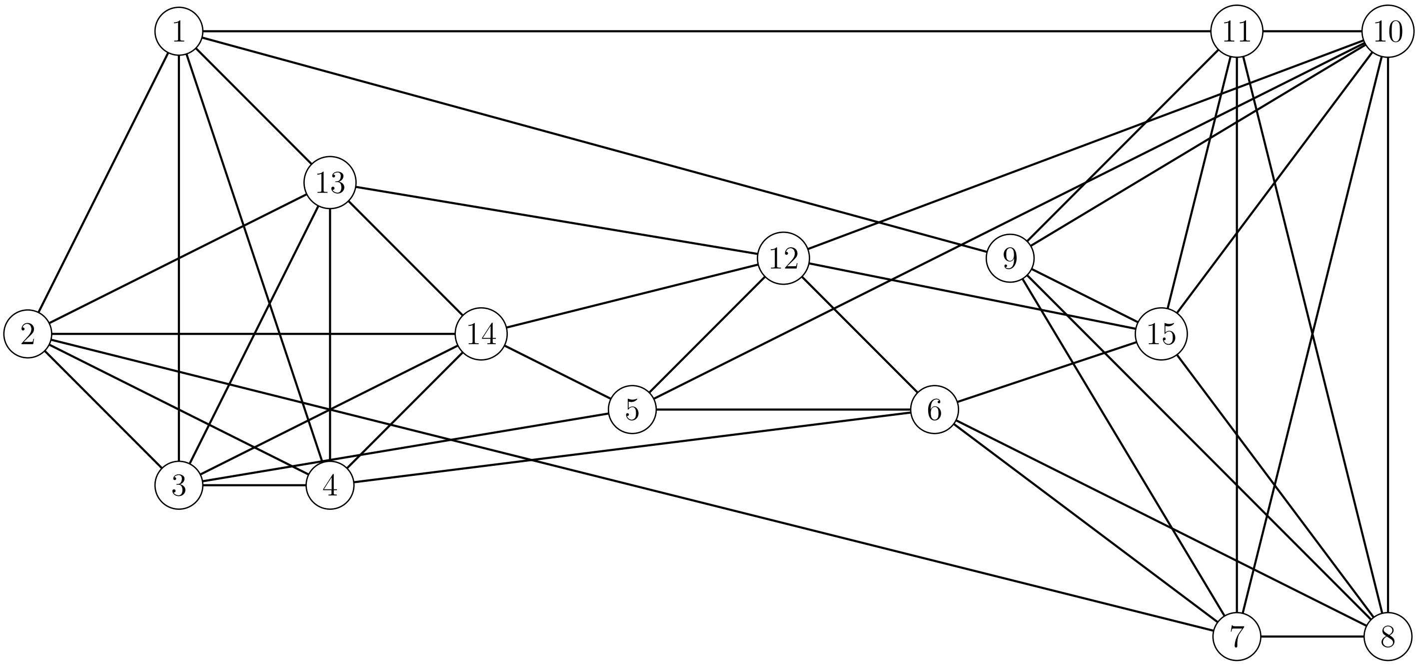

Figure 2.3: A 6-regular graph on 15 vertices.

b must be adjacent to a given it is the fourth vertex on some path. Otherwise it is the third vertex on some path from a to or beyond b.

If $sd(a,b) = 3$ then b is the third vertex on some path from a to b. We say the path be $a \to c \to d \to b$. So we have that $c \in \mathfrak{V}_a$ and $d \in \mathfrak{V}_b$. Unless there is an edge joining c and d we do not have b as third vertex from a. This is shown in 2.3.1 where, for instance, we see that $sd(2,10) = 3$ and $14 \in \mathfrak{V}_2$, $12 \in \mathfrak{V}_1 0$. This graph is a 6-regular graph on 15 vertices. The maximum length of the shortest path in this graph is 3. This a Hamiltonian graph with a Hamiltonian cycle $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 15 \to 12 \to 14 \to 13 \to 1$.

Lemma 2.6. *If in the graph $G = (V,E)$ we have two such vertices v_i and v_j such that $\mathfrak{V}_i$ and $\mathfrak{V}_j$ are disjoint but there exists at least one vertex in $\mathfrak{V}_i$ which is a neighbor of at least one vertex in $\mathfrak{V}_j$, then $sd(v_i, v_j) = 3$*

Proof. Under such conditions, there can be no path of length 2 from the vertex v_i to the vertex v_j. If there were then there would be at least one vertex in $\mathfrak{V}_i$ such that it also in $\mathfrak{V}_j$, for it were not the case we must enter and leave one vertex in V_{rest} to reach vertices in $\mathfrak{V}_j$ increasing path length by 1.

Now, we first note that, $|\mathfrak{V}_i| = |\mathfrak{V}_j| = k$. $|V_{rest}| \le k-2$. We let $v_a \in \mathfrak{V}_i$ and $v_b \in \mathfrak{V}_j$. Therefore, vertices in V_{rest} must have neighbors in $\mathfrak{V}_i$ or $\mathfrak{V}_j$ or both. So to reach v_j from v_i, we have a path $v_i \to v_a \to v_b \to v_j$. This is a path of length 3.

Therefore, under conditions stated we have a path of length 3. And this is the shortest path available to reach v_j from v_i. □

Now we see that vertices in $\mathfrak{V}_j$ needs $k-1$ more neighbors to fulfill k-degree condition. To remain 2-connected, one neighbor is needed in $\mathfrak{V}_i$. This leaves $k-2$ degree left unfilled. This can be filled by vertices in $V_1 = V - \{\mathfrak{V}_i \cup \mathfrak{V}_j \cup \{a, b\}\}$. Again $|V_1| \le k-2$.

Corollary 2.7. *If such a graph is* 2*-connected as described in lemma 2.6, then there can be at most* $k-1$ *vertices at distance* 3 *from a particular vertex* v_i

Proof. We consider there are two paths from to reach v_j from v_i namely, $v_i \to v_a \to v_b \to v_j$ and $v_i \to v_c \to v_d \to v_j$. Now we define a set of vertices $V_1 = V_{rest}$. We note that, $|V_1| \le k-2$. For all vertices $v_t \in V_1$, we might have $\mathfrak{V}_t = \mathfrak{V}_j$. In such case, the vertices in V_1 are at the shortest distance of 3 from the vertex v_i.

Now vertices in $\mathfrak{V}_j$ can be adjacent to at least 1 more vertices apart from vertices in $V_1 \cup \{v_j\}$, for $k-(k-2)-1 \ge 1$. So the condition of k-regularity is still held in such case, for no vertices in $\mathfrak{V}_j$ needs to be neighbor of more than k vertices. Condition of connectivity also holds similarly.

Now we have that these at most $k-2$ vertices from V_1 and 1 vertex of $\{v_j\}$ are at the shortest distance of 3 from v_i. Similarly we can prove the same result for the case where vertices in V_1 are adjacent to v_i.

Now we clearly see that $k-1$ is the upper bound on such vertices. Therefore, there can be no more than $k-1$ vertices in such a graph G that these vertices are at the shortest distance 3 from a particular vertex v_i. □

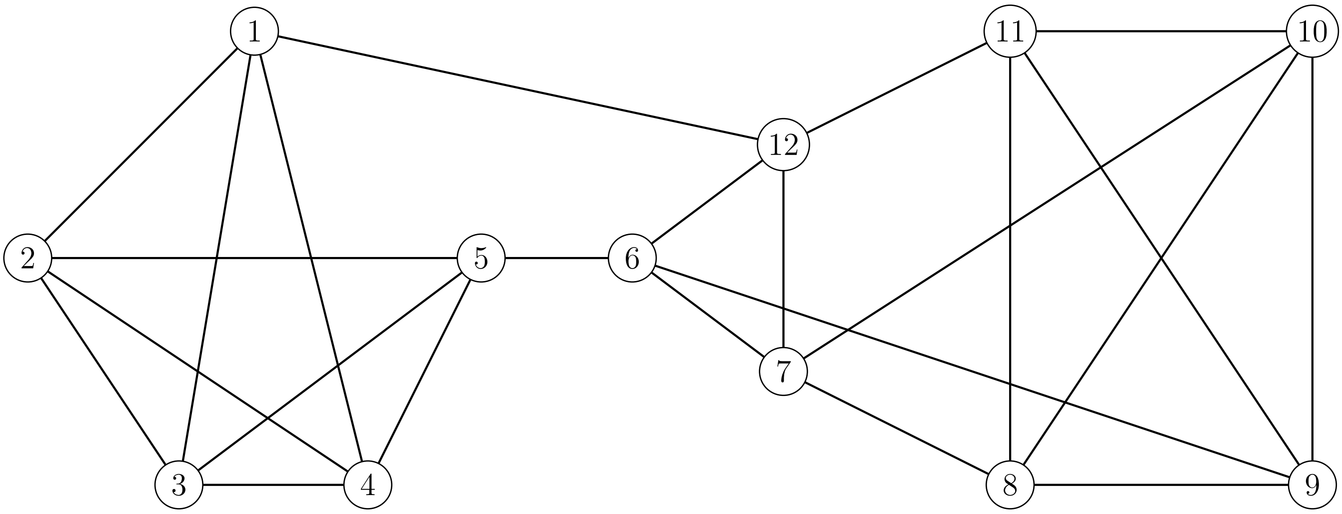

Figure 2.4: A 4-regular graph on 12 vertices.

2.3.2 The Maximum Length of the Shortest Path between Pairs of Vertices is 4

A path from the vertex a to b of the shortest length of 4 can be $a \to c \to d \to e \to b$. Clearly $c \in \mathfrak{V}_a$ and $e \in \mathfrak{V}_b$. Now d is such a vertex that is both neighbor to vertices in $\mathfrak{V}_a$ and vertices in $\mathfrak{V}_b$. In 2.3.2 we see that the vertex 2 and 10 are at distance of 4 with 6 being neighbor vertices in $5 \in \mathfrak{V}_2$ and $7 \in \mathfrak{V}_10$. This graph is a 4-regular graph on 12 vertices. The maximum length of the shortest path between any pair of vertices is 4. This is a Hamiltonian graph. The Hamiltonian path is $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 12 \to 1$.

Lemma 2.8. *If in the graph $G = (V, E)$ we have two such vertices v_i and v_j such that $\mathfrak{V}_i$ and $\mathfrak{V}_j$ are disjoint and no vertex in $\mathfrak{V}_i$ is a neighbor of vertex in $\mathfrak{V}_j$ but there exists at least one vertex in V_{rest} such that it has neighbor in both $\mathfrak{V}_i$ and $\mathfrak{V}_j$, then $sd(v_i, v_j) = 4$.*

Proof. Under such conditions, there can be no path of length 3 from the vertex v_i to the vertex v_j. If it were then there would be at least one vertex in $\mathfrak{V}_i$ such that it has neighbor in $\mathfrak{V}_j$, for it were not the case we must enter and leave one vertex in V_{rest} to reach vertices in $\mathfrak{V}_j$ increasing path length by 1.

Now, we first note that, $|\mathfrak{V}_i| = |\mathfrak{V}_j| = k$. Therefore, $|V_{rest}| \leq k - 2$. We let v_p is adjacent to both $v_a \in \mathfrak{V}_i$ and $v_b \in \mathfrak{V}_j$. So to reach v_j from v_i, we have a path $v_i \to v_a \to v_p \to v_b \to v_j$.

This is a path of length 4.

Therefore, under conditions stated we have a path of length 4. And this is the shortest path available to reach v_j from v_i. □

Corollary 2.9. *If such a graph is* 2*-connected as described in lemma 2.8, then there can be at most* $k-3$ *vertices at distance* 4 *from a particular vertex* v_i

Proof. We consider there are two paths from to reach v_j from v_i namely, $v_i \to v_a \to v_p \to v_b \to v_j$ and $v_i \to v_c \to v_q \to v_d \to v_j$. Now we define a set of vertices $V_1 = V_{rest} - \{v_p\} - \{v_q\}$. We note that, $|V_1| \leq k-4$. For all vertices $v_t \in V_1$, we might have $\mathfrak{V}_t = \mathfrak{V}_j$. In such case, the vertices in V_1 are at the shortest distance of 4 from the vertex v_i.

Now vertices in $\mathfrak{V}_j$ can be adjacent to 3 more vertices apart from vertices in $V_1 \cup \{v_j\}$, for $k-(k-4)-1 \geq 3$. So the condition of k-regularity is still held in such case, for no vertices in $\mathfrak{V}_j$ needs to be neighbor of more than k vertices.

Now we have that these at most $k-4$ vertices from V_1 and 1 vertex of $\{v_j\}$ are at the shortest distance of 4 from v_i. Similarly we can prove the same result for the case where vertices in V_1 are adjacent to v_i.

Now we prove that $k-3$ is the upper bound on such vertices. For proving that we reach a contradiction. We suppose that we pick one more vertex, say v_p from V_{rest} and add it to the set $V_1 \cup \{v_j\}$. But then there would be one less path from the vertex v_i to the vertex v_j. We note that this violates the hypothesis.

Therefore, there can be no more than $k-3$ vertices in such a graph G that these vertices are at the shortest distance 4 from a particular vertex. □

The evidence of such a graph is shown in graph 2.3.2 where the vertices 18 and 22 are at the shortest distance of 5 from the vertex 1. This is an 8-regular 2-connected graph on 23 vertices with the maximum length of the shortest path between any pair of vertices be 4.

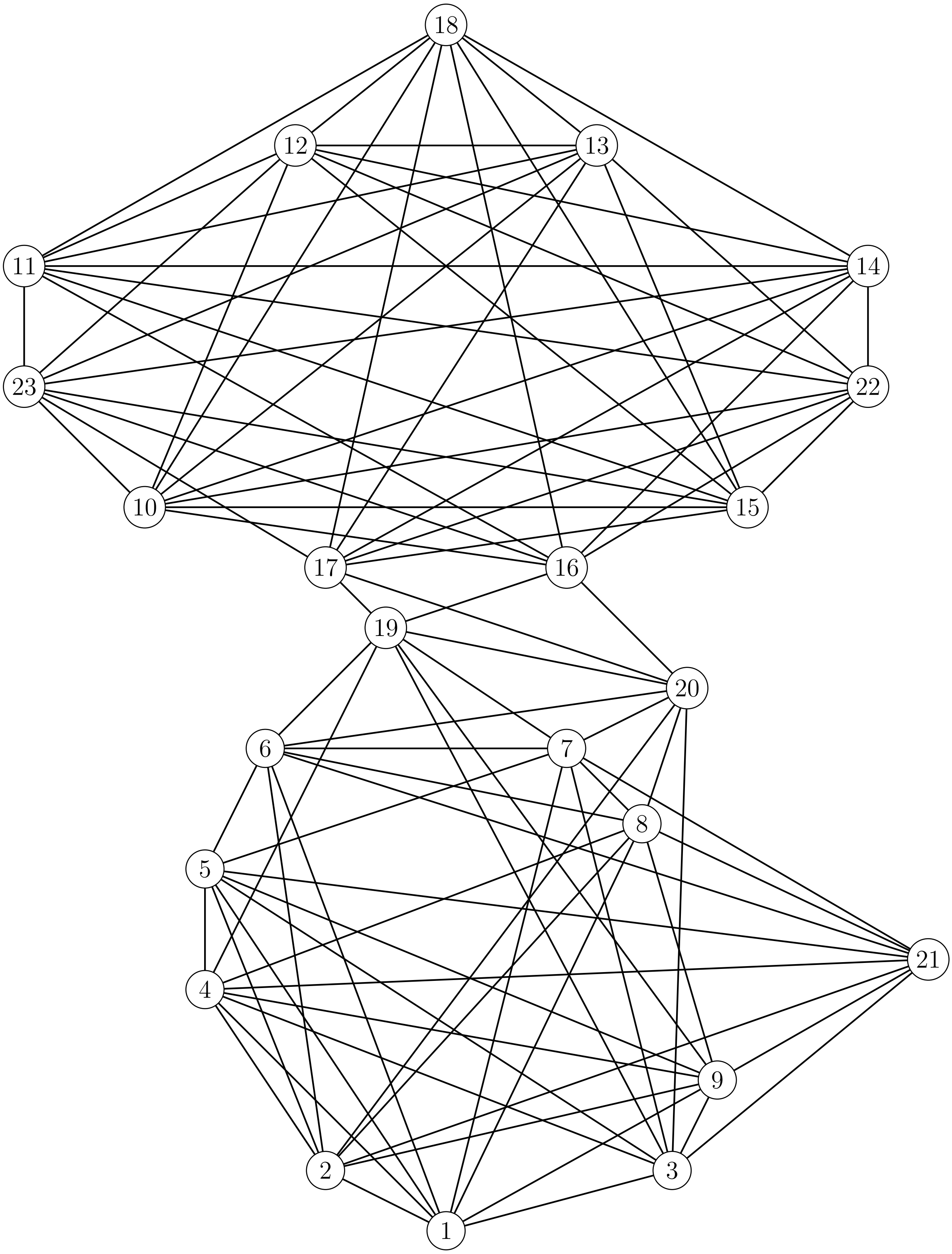

Figure 2.5: A 8-regular 2-connected graph on 23 vertices.

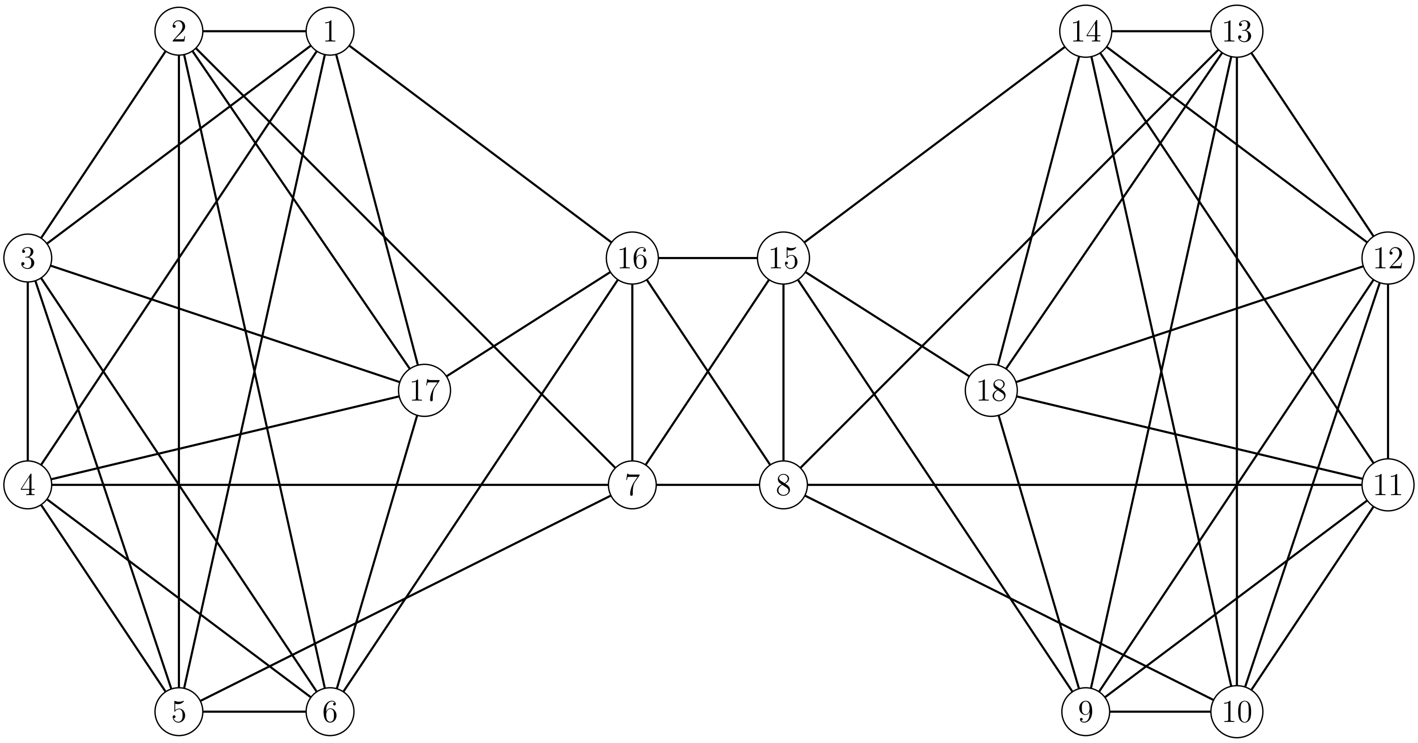

Figure 2.6: A 6-regular graph on 18 vertices.

2.3.3 The Maximum Length of the Shortest Path between Pairs of Vertices is 5

It can be the fact that vertices in $V_1 = V - \{\mathfrak{V}_i \cup \mathfrak{V}_j \cup \{a, b\}\}$ can be divided into two subsets V_2 and V_3. This would arise shortest path of length 5 between two vertex in G_J. In 2.3.3 we have that both the vertices $15, 16 \in V_1$. But the vertex 15 is not a neighbor of any vertex in $\mathfrak{V}_1 2$ and vertex 16 is not a neighbor of any vertex in $\mathfrak{V}_3$. This is a 6-regular graph on 18 vertices where distance between any pair of vertices is at most 5. This is a Hamiltonian graph. A Hamiltonian cycle of this graph is $1 \to 2 \to 3 \to 4 \to 5 \to 7 \to 8 \to 10 \to 9 \to 11 \to 12 \to 13 \to 18 \to 14 \to 15 \to 16 \to 6 \to 17 \to 1$. We observe that removing the vertices 7 and 16 or 8 and 15 is needed to make the graph non-connected. Removing vertices 1 and 3, 3 and 6, 5 and 6 or, 11 and 10 etc. does not make this graph non-connected.

Lemma 2.10. *In the graph G, if we have $\mathfrak{V}_i$ and $\mathfrak{V}_j$ disjoint, and there does not exist such a vertex $v_p \in V_{rest}$ that v_p is neighbor of at least one vertex of $\mathfrak{V}_i$ and at least one vertex of $\mathfrak{V}_j$,* *then the maximum length of the shortest path from v_i to v_j is* 5.

Proof. We first note that, in such G, there can be no path of length 4 from v_i to v_j. A path of length 4 would require that, there exists a vertex, say v_x, that is neighbor of both $v_a \in \mathfrak{V}_i$ and $v_b \in \mathfrak{V}_j$. But such a vertex v_x does not exist by hypothesis.

Now we note that, $|\mathfrak{V}_i| = |\mathfrak{V}_j| = k$. Therefore, $|V_{rest}| \leq k - 2$. We let v_p is adjacent to $v_a \in \mathfrak{V}_i$. Since the graph is connected, we can consider without loss of generality that v_p is adjacent to $v_q \in V_{rest}$ and v_q is adjacent to $v_b \in \mathfrak{V}_j$. So the path from the vertex v_i to the vertex v_j is given by, $v_i \to v_a \to v_p \to v_q \to v_b \to v_j$. This is a path of length 5. □

Corollary 2.11. *If such a graph is* 2*-connected as described in lemma 2.10, then there can be at most* $k-5$ *vertices at distance* 5 *from a particular vertex* v_x

Proof. We consider there are two paths from to reach v_j from v_i namely, $v_i \to v_a \to v_p \to v_r \to v_b \to v_j$ and $v_i \to v_c \to v_q \to v_s \to v_d \to v_j$. Now we define a set of vertices $V_1 = V_{rest} - \{v_p, v_q, v_r, v_s\}$. We note that, $|V_1| \leq k - 6$. For all vertices $v_t \in V_1$, we might have $\mathfrak{V}_t = \mathfrak{V}_j$. In such case, the vertices in V_1 are at the shortest distance of 5 from the vertex v_i.

Now vertices in $\mathfrak{V}_j$ can be adjacent to 5 more vertices apart from vertices in $V_1 \cup \{v_j\}$, for $k - (k-6) - 1 \geq 5$. So the condition of k-regularity is still held in such case, for no vertices in $\mathfrak{V}_j$ needs to be neighbor of more than k vertices.

Now we have that these at most $k-6$ vertices from V_1 and 1 vertex of $\{v_j\}$ are at the shortest distance of 5 from v_i. Similarly we can prove the same result for the case where vertices in V_1 are adjacent to v_i.

Now we prove that $k-5$ is the upper bound on such vertices. For proving that we reach a contradiction. We suppose that we pick one more vertex, say v_y from V_{rest} and add it to the set $V_1 \cup \{v_j\}$. But then there would be one less vertex-disjoint path from the vertex v_i to the vertex v_j. We note that this violates the hypothesis.

Therefore, there can be no more than $k-5$ vertices in such a graph G that these vertices are at the shortest distance 5 from a particular vertex. □

Chapter 3

Main Results

In this chapter we look at the properties of the all-pair-shortest-path matrix of G_J. We apply the lemmas and corollaries seen in 2 to learn the elements of such matrices.

No possible arrangement of G_J is there for which the largest shortest distance of 6 or more would exist. If so happens then some vertex would lack k neighbors. This is proved in the first theorem3.1 discussed in this chapter.

Theorem 3.1. *The length of the shortest path of a* 2*-connected,* k*-regular graph on* $n \leq 3k$ *vertices never exceeds* 5*.*

Proof. We prove this theorem by method of contradiction. Since in any graph, to reach a vertex at the shortest distance m from a particular vertex v_r, such a vertex must be reached that is at the shortest distance $m-1$ from the vertex v_r. It is sufficient to prove that any vertex can be reached from a given particular vertex using a path of length less than 6. We assume that there is a vertex $v_j \in V$ which cannot be reached from the vertex v_i using a path of length 5. We say, using a path of length 6, v_j can be reached from v_i. We let, the path is $v_i \to v_a \to v_b \to v_c \to v_d \to v_e \to v_j$. Here $v_a \in \mathfrak{V}_i$, $v_b, v_c, v_d \in V_{rest}$, $v_e \in \mathfrak{V}_j$.

Since there is no path of length 5 from v_i to v_j, v_b is not adjacent to $\mathfrak{V}_j$ and v_d is not adjacent to $\mathfrak{V}_i$. v_b satisfies the k-regularity condition by being adjacent to vertices of $\mathfrak{V}_i \cup V_{rest}$ and v_d satisfies the k-regularity condition by being adjacent to vertices of $\mathfrak{V}_j \cup V_{rest}$. But there exists no vertex $v_s \in V_{rest}$ such that v_s is adjacent to both v_b and v_d, for it would imply that there exists a shortest path of length 5 from v_i to v_j. Now v_c can also not be adjacent to any vertex vertex in $\mathfrak{V}_i$ or $\mathfrak{V}_j$. If it were then v_c would have been chosen and v_b would have been discarded while finding the shortest path from v_i to v_j. This leaves $deg(v_c) < k$. Hence, our assumption was wrong. This completes the proof. □

Theorem 3.2. *If S is the all-pair-shortest-path matrix of a 2-connected graph $G = (V, E)$ and there are exactly k number of 1s in each row, diagonal elements are 0, and the maximum element in S does not exceed 5, then G is a Hamiltonian graph. Here S is a square matrix with dimension $n \times n$ with n not exceeding $3k$.*

Proof. Since there are exactly k number of 1's in each row we consider that G is k-regular. Again since S is a square matrix with dimension $n \times n$ with n not exceeding $3k$, we have that $|V| = n \leq 3k$. G is already 2-connected as assumed. Diagonal elements of S to be 0 admits the fact that there is no loop in the graph. Hence G is a simple graph.

A 2-connected, k-regular graph on utmost $3k$ vertices is Hamiltonian [3]. Therefore, G is Hamiltonian. According to 3.1 such a graph have the property that the shortest distance between any pair of vertices never exceeds 5. So no element in S is larger than 5. □

3.1 Special Cases

There are special cases for which we can drop out the assumption that the given all-pair-shortest-path matrix is of a 2-connected graph. Properties of the all-pair-shortest-path matrix is to be checked to know 2-connectivity. In other words, we can know of a graphs' being 2-connected by seeing its elements.

The idea is splitting edges in two subgraphs to join them. It should be joined in such a way that no bridge is created. Splitting only one edge thwarts off this possibility due to the fact that if such cut point would have created then on vertex would be there with $k - 1$ degree and another with $k + 1$ degree.

The all-pair-shortest-path matrix of the 4-regular graph on 9 vertices shown in previous chapter 2.2 is following

$$S = \begin{pmatrix} 0 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\ 1 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 1 & 0 & 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 & 2 & 0 & 1 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

No where in this matrix we see elements as large as 3. So we can reach to the decision that this represents the all-pair-shortest-path of a Hamiltonian graph based on lemmas and corollaries we have seen in earlier chapter.

Corollary 3.3. *If S is the all-pair-shortest-path matrix of the graph $G = (V, E)$ with the property that, the maximum element in S is 2, the diagonal elements are 0, n is as large as $3k$, and there are exactly k number of 1s in each row, then G is a Hamiltonian graph.*

Proof. Each row has exactly k number of 1's implies that the graph G is k-regular. Diagonal elements of S are 0 refers to the fact that G is a simple graph. And the maximum element in S is 2 means that there is no pair of vertices with shortest distance greater than 2 between them. Such a graph G is 2-connected as admitted by 2.4. Now we have that G is a k-regular, 2-connected graph on utmost $3k$ vertices. Such a graph is Hamiltonian [3]. □

Lemma 3.4. *If S is the all-pair-shortest-path matrix of a graph G such that, $S(a, b) = 3$, both k and $|V_1|$ is even, dimension of S is no larger than $3k$, diagonal elements are 0, exactly there are k number of 1's in each row, and if there are vertices u, v such that $\mathfrak{V}_u = \mathfrak{V}_a$ and $\mathfrak{V}_v = \mathfrak{V}_b$ then number of both such u and v is even, then G is 2-connected. Here, $V_1 = V - \{\mathfrak{V}_a \cup \mathfrak{V}_b \cup \{a, b\}\}$.*

Proof. We pick a vertex arbitrarily, say it is $a \in V$. We now join k number of other vertices to it. These vertices are the members of the set $\mathfrak{V}_a$. We now take such a vertex $b \in V$ that we have $sd(a, b) = 3$. We consider that the path from the vertex a to the vertex b is $a \to c \to d \to b$.

Here $c \in \mathfrak{V}_a$ and $d \in \mathfrak{V}_b$. We note that $|V_1| \leq k-2$. A vertex $u \in V_1$ must have neighbor in $\mathfrak{V}_a$ or $\mathfrak{V}_b$ or both. Otherwise k-regularity condition would not be satisfied.

Now there can be three cases. First, more than one vertices in V_1 is adjacent to vertices in $\mathfrak{V}_a$ and $\mathfrak{V}_b$. Second, all the vertices in V_1 is adjacent to vertices in $\mathfrak{V}_a$ but not adjacent to vertices in $\mathfrak{V}_b$. Third, some vertices in V_1 is only adjacent to vertices in $\mathfrak{V}_a$ and some are only adjacent to vertices in $\mathfrak{V}_b$.

In first case, we clearly see that G is 2-connected. In second case, $|V_1|$ is even. So we have that at least two vertices are there in $\mathfrak{V}_a$ that they are neighbor of vertices in $\mathfrak{V}_b$ otherwise k-regularity condition fails. And in the third case, for $|V_1|$ is even we must have G to be 2-connected. $\square$

$$S = \begin{pmatrix}
0 & 1 & 1 & 1 & 2 & 3 & 2 & 3 & 4 & 3 & 2 & 1 \\
1 & 0 & 1 & 1 & 1 & 2 & 3 & 4 & 3 & 4 & 3 & 2 \\
1 & 1 & 0 & 1 & 1 & 2 & 3 & 4 & 3 & 4 & 4 & 3 \\
1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 3 & 4 & 3 & 2 \\
2 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 \\
3 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 1 & 2 & 3 & 1 \\
2 & 3 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 1 & 2 & 1 \\
3 & 4 & 4 & 4 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 2 \\
4 & 3 & 3 & 3 & 2 & 1 & 2 & 1 & 0 & 1 & 1 & 2 \\
3 & 4 & 4 & 4 & 3 & 2 & 1 & 1 & 1 & 0 & 1 & 2 \\
2 & 3 & 4 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 1 \\
1 & 2 & 3 & 2 & 2 & 1 & 1 & 2 & 3 & 2 & 1 & 0
\end{pmatrix}$$

This is the all-pair-shortest-path matrix of the 4-regular graph on 12 vertices shown in 2.3.2. Based on lemmas and corollaries so far we have seen we can reach to the decision in 3.5.

Lemma 3.5. *If S is the all-pair-shortest-path matrix of a graph G such that, $S(a,b) = 4$, both k and $|V_1|$ is even, dimension of S is no larger than $3k$, diagonal elements are 0, exactly there are k number of 1's in each row, and if there are vertices u, v such that $\mathfrak{V}_u = \mathfrak{V}_a$ and $\mathfrak{V}_v = \mathfrak{V}_b$ then number of both such u and v is even, then G is 2-connected. Here, $V_1 = V - \{\mathfrak{V}_a \cup \mathfrak{V}_b \cup \{a,b\}\}$.*

Proof of this lemma 3.5 is as same as 3.4.

$$S = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 & 4 & 4 & 3 & 3 & 2 & 1 & 1 & 3 \\
1 & 0 & 1 & 2 & 1 & 1 & 1 & 2 & 4 & 4 & 3 & 5 & 4 & 3 & 3 & 2 & 1 & 4 \\
1 & 1 & 0 & 1 & 1 & 1 & 2 & 3 & 4 & 4 & 4 & 5 & 4 & 4 & 3 & 2 & 1 & 4 \\
1 & 2 & 1 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 & 3 & 3 & 2 & 2 & 1 & 4 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 4 & 3 & 3 & 4 & 3 & 3 & 2 & 2 & 2 & 4 \\
2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 3 & 3 & 4 & 4 & 3 & 3 & 2 & 1 & 1 & 3 \\
2 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 2 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 2 & 2 \\
2 & 2 & 3 & 2 & 2 & 2 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 \\
3 & 4 & 4 & 3 & 4 & 3 & 2 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 3 & 1 \\
4 & 4 & 4 & 3 & 3 & 3 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 4 & 2 \\
4 & 3 & 4 & 3 & 3 & 4 & 2 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 4 & 1 \\
4 & 5 & 5 & 4 & 4 & 4 & 3 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 3 & 4 & 1 \\
3 & 4 & 4 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 2 & 3 & 1 \\
3 & 3 & 4 & 3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 1 \\
2 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 1 \\
1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 2 & 1 & 0 & 1 & 2 \\
1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 4 & 4 & 4 & 3 & 3 & 2 & 1 & 0 & 3 \\
3 & 4 & 4 & 4 & 4 & 3 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 0
\end{pmatrix}$$

This is the all-par-shortest-path matrix of the 6-regular graph on 18 vertices shown in 2.3.3. Several pair of vertices there are with the shortest distance of 5 between them.

Lemma 3.6. *If S is the all-pair-shortest-path matrix of the graph $G = (V, E)$ with the property that, the maximum element in S is 5, the diagonal elements are 0, n is as large as $3k$, k is even, and there are exactly k number of $1s$ in each row, then G is a Hamiltonian graph.*

Proof. We consider that the vertices $a, b \in V$ are at the shortest distance of 5, that is $sd(a, b) = 5$. We intuitively see that the graph G can be divided into two subgraphs with set of vertices respectively,

$$V_x = \{a\} \cup \mathfrak{V}_a \cup V_2 \tag{3.1}$$

$$V_y = \{b\} \cup \mathfrak{V}_b \cup V_3 \tag{3.2}$$

Here the set V_1, V_2 and V_3 is defined as following:

$$V_1 = V - \{\mathfrak{V}_a \cup \mathfrak{V}_b \cup \{a, b\}\} \tag{3.3}$$

$$V_2 = \{x \mid x \in V_1, y \in \mathfrak{V}_a, (x, y) \in E\} \tag{3.4}$$

$$V_3 = \{x \mid x \in V_1, y \in \mathfrak{V}_b, (x, y) \in E\} \tag{3.5}$$

We now see that we can have two k-regular subgraph $G_x = (V_x, E_x)$ and $G_y = (V_y, E_y)$. We can take one edge in G_x and split it. Then we can plug those two vertices two two other vertices in G_y that have lost k-degree property after splitting. Preferably these edges join respectively vertices in V_2 and V_3.

Unless k is even, we might have both k and $k + |\{V_x - \mathfrak{V}_a - \{a\}\}| + 1$ odd. Giving rise to fractional number of edges which causes either more or less than k neighbor for some vertex. □

Such an example of either more or less than k neighbor for some vertex owing to both k and $k + |\{V_x - \mathfrak{V}_a - \{a\}\}| + 1$'s being odd is provided in 3.1. In that graph the shortest distance from the vertex 1 to 11 is 5. But here we have a vertex 16 that lacks 9 neighbors thus violates 9-regularity. This deficiency of neighbor is due to the fact that both 9 and 13 is odd. Hence we would be having fraction number of edges. This is an example of the reason why k needs to be even. In this graph both 9 and $9 + (13 - 9 - 1) + 1 = 13$ are odd. Here 16 has one less neighbor to reach 9 neighbor.

Corollary 3.7. *If both $k + |\{V_x - \mathfrak{V}_a - \{a\}\}| + 1$ and $k + |\{V_y - \mathfrak{V}_b - \{b\}\}| + 1$ are even though k is odd, then graph with such property, except even k, can be obtained.*

3.7 can be proved similarly to the proof of 3.6. 3.7 is stronger in the sense that it eliminates the requirement that k always needs to be even.

In the cases where the maximum length of the shortest distance between pair of vertices is as large as 3 or 4, such condition is not seen. In these cases vertices in V_{rest} are free to have neighbors in $\mathfrak{V}_a$ and $\mathfrak{V}_b$. Thus when forming G_x or G_y, regularity of vertex property would not come useful as it was in case of 3.7.

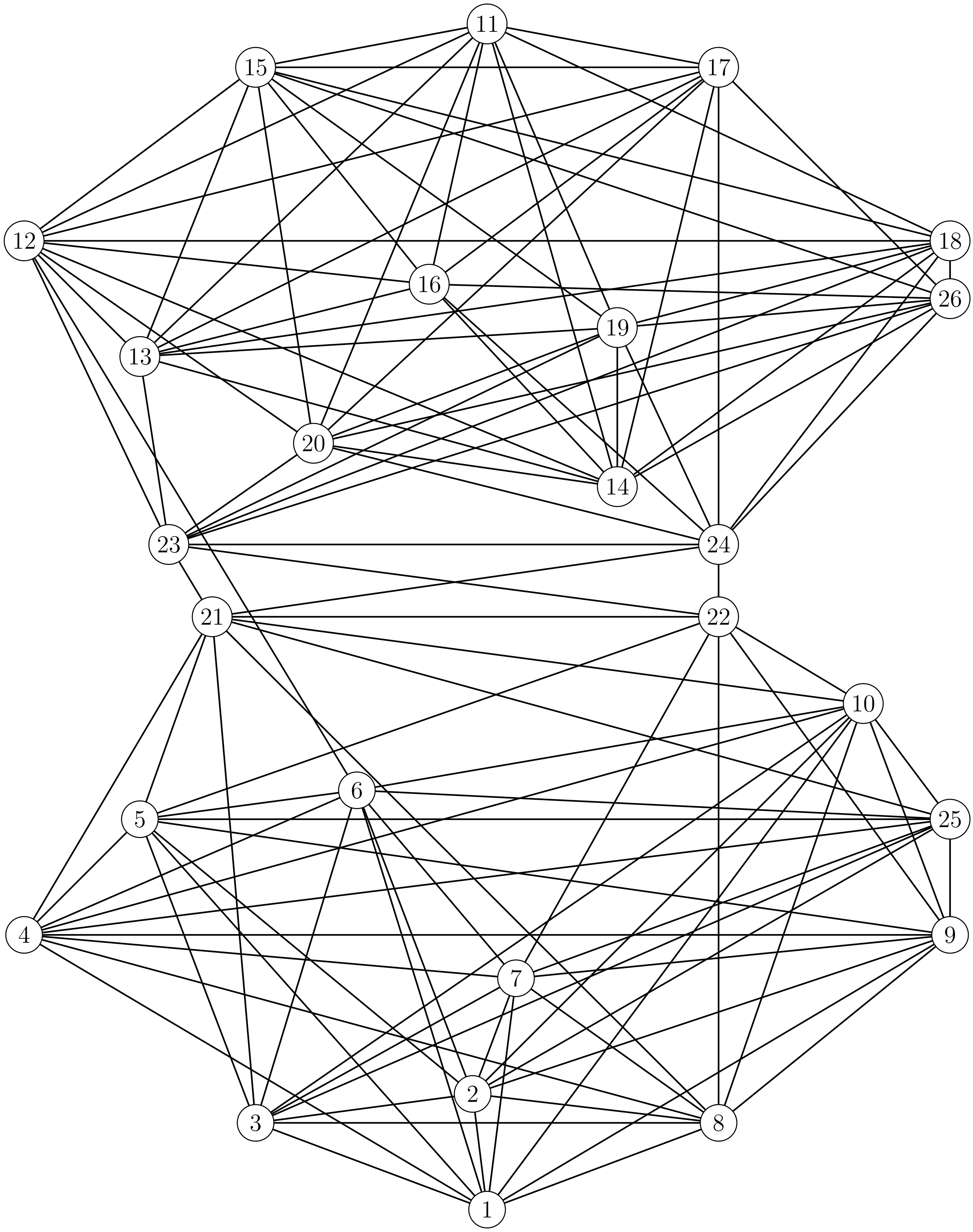

Figure 3.1: An example showing the reason why k needs to be even.

Note 1. Both $|\{V_x - \mathfrak{V}_a - \{a\}\}|$ and $|\{V_y - \mathfrak{V}_b - \{b\}\}|$ can be determined in $O(n^2)$ time. So in polynomial time we can detect 2-connectivity by seeing elements of the all-pair-shortest-path matrix.

Chapter 4

Conclusion

4.1 Future Works

One question naturally arises that what would be the properties of a 2-connected k-regular graph on utmost $3k$ vertices where k is odd. We have seen an answer in case the largest element in S is 5 in 3.7. But question remains unanswered when the largest element in S is 3 or 4.

Another question that naturally arise is what are the properties of a non-regular 2-connected graph. In other words, what are the criteria we would be looking for to determine whether a given all-pair-shortest-path matrix is of a 2-connected graph or not. Techniques used in this thesis can be used to determine 2-connectivity of k-regular graphs when k is even. But yet questions left unanswered for odd k.

Another related question is colorability of such graphs. Though it is obvious that the upper bound is $k+1$ and lower one to be 2, we are looking for tighter bounds. It is possible that $k-1$ color is needed the evidence is graph shown in 2.3.3 can be colored with 5 colors. It is a 6-regular graph.

An question, though not directly related, now can arise. It can be asked that if we can determine a given m-colored graph is Hamiltonian or not from its all-pair-shortest-path matrix. Coloring of vertices influences how vertices are connected. In case of G_J we cannot have v_i and $u \in \mathfrak{V}_i$ colored same. Again $w \in V_{rest}$ can be colored with any given $sd \geq 4$. But when $sd \leq 3$

the choice of color gets more restricted because there is no more V_{rest}.

Various other properties like independence number of a graph expressed in terms of shortest path lengths between vertices would be interesting. This would enable more cases of Hamiltonian graph to be detected from all-pair-shortest-path matrices in polynomial time.

4.2 Concluding Remarks

In this thesis interest is kept focused on G_J. The structure of G_J is analyzed in depth and revealed the properties related to the shortest path length between vertices. Determining Hamiltonicity from the all-pair-shortest-path matrix depends mostly on shortest path length between vertices but the number of different elements of matrix also plays a great role on the structure of the graph.

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