Pareto Efficiency in Expressive Bidding Markets

1 Introduction

Expressive bidding represents a paradigm shift in auction design, enabling participants to articulate complex preferences across multiple dimensions. Traditional auction mechanisms, such as first-price, second-price, or uniform-price auctions, constrain participants to simple bid formats, limiting their ability to reflect nuanced preferences or interdependencies between items. By contrast, expressive bidding allows participants to encode multi-item preferences, exclusivities, complementarities, and temporal constraints directly within the bidding process. This flexibility significantly enhances the potential for achieving Pareto efficient allocations, a benchmark of optimality in market outcomes.

Pareto efficiency is formally defined as a state of resource allocation where no reallocation can improve one participant's utility without reducing another's. In auction theory, Pareto efficiency serves as a cornerstone for evaluating the success of allocation mechanisms. While traditional auction formats often achieve allocative efficiency under simplifying assumptions—such as independent and identically distributed valuations or linear utility functions—they frequently fail in scenarios involving interdependent preferences or combinatorial constraints. These failures become particularly pronounced in complex markets such as combinatorial auctions, illiquid asset trading, or auctions for public resources, where multi-dimensional preferences and externalities play a critical role.

Expressive bidding addresses these limitations by transforming the auction mechanism into a constrained optimization problem, where the constraints reflect both global feasibility (e.g., supply constraints) and bidder-specific preferences. The introduction of these constraints expands the feasible allocation space, creating opportunities for achieving Pareto efficient outcomes that were unattainable under traditional mechanisms. However, this increase in flexibility comes at the cost of greater computational complexity, necessitating the development of advanced mathematical and algorithmic tools to analyze and implement expressive bidding mechanisms.

This paper focuses on the interplay between expressive bidding and Pareto efficiency, presenting a rigorous mathematical framework for analyzing the efficiency properties of expressive bidding markets. The framework formalizes the problem as an optimization problem over a high-dimensional allocation space, incorporating both bidder utility functions and expressive constraints. Building on this framework, we derive theoretical results demonstrating the existence, uniqueness, and stability of Pareto efficient allocations under expressive bidding mechanisms. These results are further extended to combinatorial auctions and alternative asset classes, illustrating the broad applicability of the approach in modern markets.

The analysis begins by reviewing foundational concepts in auction theory, focusing on the limitations of traditional mechanisms in achieving Pareto efficiency. A mathematical framework is then introduced, formalizing the expressive bidding problem and defining Pareto efficiency within this context. Theoretical results are rigorously developed, proving the necessary and sufficient conditions for Pareto efficiency in expressive bidding markets. Applications to combinatorial auctions and alternative asset classes are explored in detail, highlighting the practical implications of the theoretical findings. Finally, the paper discusses computational considerations, extensions to stochastic settings, and directions for future research.

The contribution of this work lies in its comprehensive treatment of expressive bidding mechanisms as tools for achieving Pareto efficient outcomes in complex markets. By combining rigorous mathematical analysis with practical applications, this paper provides a foundation for further research and development in auction design, advancing the state of the art in market efficiency and optimization.

2 Framework

To rigorously analyze the efficiency properties of expressive bidding mechanisms, we construct a mathematical framework that formalizes the allocation process in multi-dimensional and constrained environments. Consider a market with n bidders and m items, where each bidder i possesses a utility function $U_i(x)$, defined over an allocation vector $x = \{x_{ij}\}$, where x_{ij} represents the quantity of item j allocated to bidder i. The allocation problem is governed by two classes of constraints: global constraints that enforce feasibility conditions for the market, and bidder-specific constraints that encode preferences, complementarities, or exclusivities.

The auctioneer's objective is to determine an allocation $x^* \in \mathcal{X}$, where $\mathcal{X} \subset \mathbb{R}^{n\times m}$ represents the feasible allocation space, such that the allocation maximizes aggregate utility subject to these constraints. Mathematically, this can be expressed as the constrained optimization problem:

$$\max_{x\in\mathcal{X}} \sum_{i=1}^{n} U_i(x),$$

where $\mathcal{X}$ is defined by:

$$\mathcal{X} = \left\{ x \in \mathbb{R}^{n\times m} \mid \sum_{i=1}^{n} x_{ij} \le q_j \,\forall j,\, g_{ik}(x) \le 0 \,\forall i,\, \forall k,\, x_{ij} \ge 0 \right\}.$$

Here, q_j represents the total availability of item j, and $g_{ik}(x)$ are bidder-specific constraints that encode expressive preferences.

Pareto efficiency requires that no alternative allocation $x' \in \mathcal{X}$ satisfies $U_i(x') \ge U_i(x^*)$ for all i, with $U_j(x') > U_j(x^*)$ for at least one j. This definition implies that x^* lies on the Pareto frontier of the feasible allocation space $\mathcal{X}$, where any movement away from x^* reduces at least one participant's utility.

Expressive bidding introduces novel constraints into this optimization framework, allowing bidders to articulate complex preferences. For instance, a bidder i may specify linear constraints of the form:

$$\sum_{j\in S} x_{ij} \le b_i, \quad \text{or nonlinear constraints such as } f_i(x) \le 0,$$

where $S \subseteq \{1, \dots, m\}$ represents a subset of items, and b_i is a bound on the total allocation. Nonlinear constraints, such as complementarities or exclusions, further enrich the structure of the feasible space, making the problem inherently non-convex in general.

To solve this optimization problem, we construct the Lagrangian:

$$\mathcal{L}(x,\lambda,\mu) = \sum_{i=1}^{n} U_i(x) - \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} x_{ij} - q_j \right) - \sum_{i=1}^{n} \sum_{k\in\mathcal{K}_i} \mu_{ik} g_{ik}(x),$$

where $\lambda_j \ge 0$ and $\mu_{ik} \ge 0$ are Lagrange multipliers associated with the global and individual constraints, respectively. The necessary conditions for optimality are given by the Karush-Kuhn-Tucker (KKT) conditions:

$$\nabla_x \mathcal{L}(x^*,\lambda^*,\mu^*) = 0, \quad \sum_{i=1}^{n} x_{ij} \le q_j, \quad g_{ik}(x^*) \le 0,$$

$$\lambda_j^* \left(\sum_{i=1}^{n} x_{ij} - q_j \right) = 0, \quad \mu_{ik}^* g_{ik}(x^*) = 0, \quad x_{ij} \ge 0.$$

The expressive constraints $g_{ik}(x)$ transform the allocation problem into a high-dimensional constrained optimization, where the feasible space $\mathcal{X}$ reflects both market-wide and bidder-specific preferences. Unlike traditional auction mechanisms, where constraints are typically limited to budget or capacity, expressive bidding allows for nuanced and multi-dimensional preferences, such as complementarities and temporal restrictions.

To illustrate the implications of these constraints, consider the case where a bidder specifies a preference for complementarities between items j_1 and j_2, encoded as:

$$U_i(x_{ij_1}, x_{ij_2}) = k \min(x_{ij_1}, x_{ij_2}),$$

where $k > 0$ reflects the strength of the complementarities. The utility function $U_i(x)$ is inherently non-linear, requiring specialized optimization techniques to determine x^*. Similarly, exclusions can be encoded as constraints of the form:

$$x_{ij_1} x_{ij_2} = 0,$$

ensuring that the bidder does not receive both items simultaneously. These constraints introduce additional complexity, but also create opportunities for allocations that better reflect bidder preferences, potentially improving aggregate efficiency.

The presence of expressive constraints requires careful handling of the feasible allocation space $\mathcal{X}$. While linear constraints preserve the convexity of $\mathcal{X}$, nonlinear constraints may render the space non-convex, complicating the optimization process. In such cases, convex relaxation techniques can be employed to approximate the solution, or heuristic algorithms can be designed to identify near-optimal allocations.

This mathematical framework establishes the foundation for analyzing the efficiency properties of expressive bidding. In the subsequent sections, we develop theoretical results demonstrating the conditions under which expressive bidding achieves Pareto efficiency, including existence, uniqueness, and stability properties of the resulting allocations.

3 Pareto Efficient Allocations

Consider the optimization problem:

$$\max_{x \in \mathcal{X}} \sum_{i=1}^{n} U_i(x),$$

subject to the feasibility constraints:

$$\sum_{i=1}^{n} x_{ij} \leq q_j \quad \forall j, \quad g_{ik}(x) \leq 0 \quad \forall i, \forall k, \quad x_{ij} \geq 0 \quad \forall i, \forall j,$$

where $\mathcal{X}$ represents the feasible allocation space. The utility functions $U_i(x)$ are assumed to be continuous, and the constraints $g_{ik}(x)$ may be linear or nonlinear. We proceed to develop theoretical results under this setup.

To achieve Pareto efficiency, the solution $x^* \in \mathcal{X}$ must satisfy the following condition: for any alternative allocation $x' \in \mathcal{X}$, $U_i(x') \geq U_i(x^*)$ for all i, with $U_j(x') > U_j(x^*)$ for at least one j, implying that x^* lies on the Pareto frontier of $\mathcal{X}$.

To establish the theoretical properties of expressive bidding, we first address the existence of Pareto efficient allocations.

Theorem 1 (Existence of Pareto Efficient Allocations). For any finite set of bidders $\{1, 2, \ldots, n\}$ and items $\{1, 2, \ldots, m\}$, where the utility functions $U_i(x)$ are continuous and the constraints define a compact feasible allocation space $\mathcal{X}$, there exists at least one Pareto efficient allocation $x^* \in \mathcal{X}$.

Proof. The compactness of the feasible allocation space $\mathcal{X}$ follows from the bounded supply constraints $\sum_{i=1}^{n} x_{ij} \leq q_j$ and the non-negativity condition $x_{ij} \geq 0$. The continuity of $U_i(x)$ ensures that the aggregate utility function $\mathcal{U}(x) = \sum_{i=1}^{n} U_i(x)$ is continuous over $\mathcal{X}$. By the Weierstrass Extreme Value Theorem, $\mathcal{U}(x)$ attains its maximum at some $x^* \in \mathcal{X}$.

Assume there exists an alternative allocation $x' \in \mathcal{X}$ such that $U_i(x') \geq U_i(x^*)$ for all i and $U_j(x') > U_j(x^*)$ for some j. This would contradict the optimality of x^*, as $\mathcal{U}(x') > \mathcal{U}(x^*)$. Thus, x^* is Pareto efficient. □

The next result establishes the uniqueness of Pareto efficient allocations under stricter conditions on utility functions.

Theorem 2 (Uniqueness of Pareto Efficient Allocations). If the utility functions $U_i(x)$ are strictly concave and the feasible allocation space $\mathcal{X}$ is convex, the Pareto efficient allocation x^* is unique.

Proof. Let $x_1, x_2 \in \mathcal{X}$ be two distinct Pareto efficient allocations. By the strict concavity of $U_i(x)$, for any $\alpha \in (0,1)$, the allocation $x_\alpha = \alpha x_1 + (1-\alpha)x_2$ satisfies:

$$U_i(x_\alpha) > \alpha U_i(x_1) + (1-\alpha)U_i(x_2) \quad \forall i.$$

Since $\mathcal{X}$ is convex, $x_\alpha \in \mathcal{X}$. The strict inequality implies that x_α dominates both x_1 and x_2, contradicting their Pareto efficiency. Hence, x^* is unique. □

Expressive bidding mechanisms introduce additional constraints, such as complementarities or exclusions, which transform the structure of the feasible allocation space. The next theorem demonstrates that expressive constraints preserve Pareto efficiency under certain conditions.

Theorem 3 (Pareto Efficiency of Expressive Bidding Mechanisms). If the expressive constraints $g_{ik}(x)$ are linear and the utility functions $U_i(x)$ are concave, then the allocations produced by expressive bidding mechanisms are Pareto efficient.

Proof. The expressive constraints $g_{ik}(x) \leq 0$ define a polyhedral feasible region $\mathcal{X}$ that is convex. The concavity of $U_i(x)$ ensures that the aggregate utility function $\mathcal{U}(x) = \sum_{i=1}^{n} U_i(x)$ is also concave. By the properties of concave optimization over convex domains, the solution x^* to the optimization problem satisfies the KKT conditions:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0, \quad \sum_{i=1}^{n} x_{ij} \leq q_j, \quad g_{ik}(x^*) \leq 0,$$

$$\lambda_j^* \left(\sum_{i=1}^{n} x_{ij} - q_j \right) = 0, \quad \mu_{ik}^* g_{ik}(x^*) = 0.$$

The KKT conditions are necessary and sufficient for optimality under the stated conditions, ensuring that x^* is Pareto efficient. □

To examine stability, we analyze the sensitivity of Pareto efficient allocations to perturbations in utility functions or constraints.

Theorem 4 (Stability of Pareto Efficient Allocations). For small perturbations in bidder utilities or constraints, the Pareto efficient allocation x^* remains stable, provided the perturbed feasible space $\mathcal{X}_\delta$ remains compact.

Proof. Let the perturbed utility functions be $U_i'(x) = U_i(x) + \epsilon_i(x)$, where $\epsilon_i(x)$ represents a small perturbation satisfying $|\epsilon_i(x)| \leq \delta$. Similarly, let the perturbed constraints $g_{ik}'(x) = g_{ik}(x) + \eta_{ik}(x)$ define a feasible space $\mathcal{X}_\delta$. The compactness of $\mathcal{X}_\delta$ and the continuity of $U_i'(x)$ ensure that the perturbed optimization problem:

$$\max_{x \in \mathcal{X}_\delta} \sum_{i=1}^{n} U_i'(x)$$

admits a solution x_δ^*. The small perturbations ensure that x_δ^* remains close to x^*, preserving Pareto efficiency. □

These theoretical results establish a rigorous foundation for analyzing expressive bidding mechanisms and their efficiency properties. The next section explores applications to combinatorial auctions, demonstrating how these mechanisms optimize allocations in multi-dimensional and constrained environments.

4 Combinatorial Bidding

Combinatorial auctions present a unique challenge in auction theory due to the exponential growth of allocation possibilities with the number of items. In such auctions, bidders submit preferences over bundles of items rather than individual items, introducing complexities such as complementarities and substitutability. Expressive bidding mechanisms provide a natural framework for handling these complexities by allowing bidders to encode detailed constraints and preferences over bundles, leading to more efficient allocations.

Consider a combinatorial auction with n bidders and m items. Let $\mathcal{B}_i \subseteq 2^{\{1,\ldots,m\}}$ represent the set of all bundles over which bidder i expresses preferences. Each bundle $B \in \mathcal{B}_i$ has an associated utility $U_i(B)$, capturing the bidder's valuation of the bundle. The auctioneer's objective is to allocate bundles to bidders in a way that maximizes aggregate utility while satisfying global feasibility and bidder-specific constraints.

The optimization problem can be formalized as:

$$\max_{x \in \mathcal{X}} \sum_{i=1}^{n} \sum_{B \in \mathcal{B}_i} U_i(B) x_{iB},$$

where $x_{iB} \in \{0,1\}$ is a binary variable indicating whether bidder i receives bundle B, and $\mathcal{X}$ is the feasible allocation space defined by the following constraints:

$$\sum_{i=1}^{n} \sum_{B \in \mathcal{B}_i : j \in B} x_{iB} \leq q_j \quad \forall j,$$

$$\sum_{B \in \mathcal{B}_i} x_{iB} \leq 1 \quad \forall i,$$

$$x_{iB} \in \{0,1\} \quad \forall i, B \in \mathcal{B}_i.$$

The first constraint ensures that the total allocation of item j across all bundles does not exceed its supply q_j. The second constraint ensures that each bidder receives at most one bundle. Together, these constraints define a high-dimensional, discrete optimization problem.

Expressive bidding introduces additional constraints, such as complementarities or exclusivities, which can be modeled as follows. For complementarities, where a bidder values bundles that include specific combinations of items, the utility function $U_i(B)$ is non-additive. For example:

$$U_i(\{j_1, j_2\}) > U_i(\{j_1\}) + U_i(\{j_2\}),$$

reflecting synergistic value from receiving both items j_1 and j_2. Substitutability, on the other hand, imposes diminishing returns for overlapping items:

$$U_i(\{j_1, j_2\}) < U_i(\{j_1\}) + U_i(\{j_2\}).$$

To address these complexities, the Lagrangian for the optimization problem is defined as:

$$\mathcal{L}(x, \lambda, \mu) = \sum_{i=1}^{n} \sum_{B \in \mathcal{B}_i} U_i(B) x_{iB} - \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} \sum_{B \in \mathcal{B}_i : j \in B} x_{iB} - q_j \right) - \sum_{i=1}^{n} \mu_i \left(\sum_{B \in \mathcal{B}_i} x_{iB} - 1 \right),$$

where $\lambda_j \geq 0$ and $\mu_i \geq 0$ are Lagrange multipliers associated with the supply and bundle constraints, respectively. The optimal allocation x^* satisfies the Karush-Kuhn-Tucker (KKT) conditions:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*) = 0, \quad \sum_{i=1}^{n} \sum_{B \in \mathcal{B}_i : j \in B} x_{iB} \leq q_j, \quad \sum_{B \in \mathcal{B}_i} x_{iB} \leq 1,$$

$$\lambda_j^* \left(\sum_{i=1}^{n} \sum_{B \in \mathcal{B}_i : j \in B} x_{iB} - q_j \right) = 0, \quad \mu_i^* \left(\sum_{B \in \mathcal{B}_i} x_{iB} - 1 \right) = 0.$$

The expressive constraints $g_{ik}(x)$, such as:

$$\sum_{B \in \mathcal{B}_i} a_{ikB} x_{iB} \leq b_{ik} \quad \forall i, k,$$

introduce additional Lagrange multipliers $\nu_{ik} \geq 0$, modifying the KKT conditions to:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*, \nu^*) = 0, \quad \nu_{ik}^* g_{ik}(x^*) = 0 \quad \forall i, k.$$

These conditions guarantee Pareto efficiency of the allocation x^*, provided $U_i(B)$ is concave over the feasible region defined by $\mathcal{X}$. In practice, solving this optimization problem is computationally challenging due to its combinatorial nature.

To address the computational complexity, modern approaches leverage branch-and-bound algorithms, dynamic programming, and neural network approximations for bidder utility functions. For example, branch-and-bound methods systematically partition the feasible space into subregions, eliminating those that cannot contain the optimal solution. Let $\mathcal{X}_k \subset \mathcal{X}$ denote a subregion at iteration k, with bounds L_k and U_k:

$$L_k \leq \max_{x \in \mathcal{X}_k} \sum_{i=1}^{n} \sum_{B \in \mathcal{B}_i} U_i(B) x_{iB} \leq U_k.$$

Regions with U_k below the current best solution are pruned, while promising regions are further subdivided. Dynamic programming provides an alternative approach, exploiting additive structures in bidder utilities to reduce computational overhead.

Expressive bidding mechanisms in combinatorial auctions are widely applicable in real-world markets such as spectrum allocation, energy trading, and logistics. For example, in spectrum auctions, telecommunications companies bid on bundles of frequencies to maximize coverage while minimizing interference. Expressive bidding allows bidders to specify complex constraints, such as geographic complementarities or technical exclusions, leading to allocations that achieve higher efficiency and revenue.

By enabling participants to articulate detailed preferences and constraints, expressive bidding mechanisms significantly improve the efficiency of combinatorial auctions. These mechanisms balance computational complexity with practical applicability, unlocking allocations that traditional auction formats cannot achieve. In the next section, we extend the analysis to alternative asset classes, including cryptocurrencies and illiquid markets, highlighting the versatility of expressive bidding.

5 Applications to Alternative Asset Classes

The expressive bidding framework is particularly suited to alternative asset classes, where traditional auction mechanisms often fail to address the complexities arising from heterogeneity, illiquidity, and multi-dimensional bidder preferences. This section explores the application of expressive bidding to cryptocurrencies and illiquid assets, focusing on their unique challenges and opportunities.

Cryptocurrencies. The cryptocurrency market is characterized by extreme price volatility, fragmented liquidity across exchanges, and a wide array of token-specific utilities. Traditional market mechanisms such as continuous limit order books struggle to balance price discovery with execution efficiency in such an environment. Expressive bidding provides a solution by allowing participants to articulate preferences across multiple dimensions, including volatility tolerance, correlation constraints, and liquidity requirements.

Consider a cryptocurrency market with n participants and m tokens. Each participant i has a utility function $U_i(x)$, defined over the allocation vector $x = \{x_{ij}\}$, where x_{ij} represents the quantity of token j allocated to participant i. Participants may specify constraints such as:

$$\sum_{j=1}^{m} w_{ij} x_{ij} \leq R_i, \quad \forall i,$$

where w_{ij} is the weight of token j in participant i's portfolio, and R_i is their risk tolerance. Further constraints can encode minimum liquidity thresholds:

$$x_{ij} \geq L_j \quad \text{for stablecoins } j,$$

or diversification requirements:

$$\sum_{j=1}^{m} c_{ij} x_{ij} \leq D_i,$$

where c_{ij} represents the concentration risk factor of token j, and D_i is the maximum allowed concentration for participant i.

The resulting optimization problem for the auctioneer is:

$$\max_{x \in \mathcal{X}} \sum_{i=1}^{n} U_i(x),$$

subject to:

$$\sum_{i=1}^{n} x_{ij} \leq q_j \quad \forall j, \quad \sum_{j=1}^{m} w_{ij} x_{ij} \leq R_i \quad \forall i, \quad x_{ij} \geq L_j \quad \forall i, j,$$

where q_j represents the total supply of token j. Expressive constraints transform the feasible allocation space $\mathcal{X}$, enabling the auctioneer to achieve allocations that balance liquidity, risk, and participant preferences.

To solve this problem, we define the Lagrangian:

$$\mathcal{L}(x, \lambda, \mu, \nu) = \sum_{i=1}^{n} U_i(x) - \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} x_{ij} - q_j \right) - \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{m} w_{ij} x_{ij} - R_i \right) - \sum_{i=1}^{n} \sum_{j=1}^{m} \nu_{ij} (L_j - x_{ij}),$$

where $\lambda_j, \mu_i, \nu_{ij} \geq 0$ are Lagrange multipliers associated with the supply, risk, and liquidity constraints, respectively. The KKT conditions provide necessary and sufficient conditions for optimality:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \mu^*, \nu^*) = 0, \quad \sum_{i=1}^{n} x_{ij} \leq q_j, \quad \sum_{j=1}^{m} w_{ij} x_{ij} \leq R_i, \quad x_{ij} \geq L_j,$$

$$\lambda_j^* \left(\sum_{i=1}^{n} x_{ij} - q_j \right) = 0, \quad \mu_i^* \left(\sum_{j=1}^{m} w_{ij} x_{ij} - R_i \right) = 0, \quad \nu_{ij}^* (L_j - x_{ij}) = 0.$$

By leveraging expressive constraints, the auctioneer can achieve allocations that better align with participants' preferences and market dynamics. For example, participants seeking exposure to low-volatility assets can specify preferences for stablecoins, while those targeting high returns can prioritize allocations with positive correlations to high-growth tokens.

Illiquid Assets. Illiquid asset markets, such as real estate, private equity, and fine art, present additional challenges due to their lack of standardized pricing mechanisms and limited trading volumes. Expressive bidding enables participants to encode preferences over non-monetary attributes, such as geographic location, expected holding periods, or historical significance, which are critical for valuation but difficult to capture in traditional auction formats.

Let $A = \{a_1, a_2, \ldots, a_m\}$ represent a set of illiquid assets and x_{ij} the allocation of asset j to participant i. Each participant i specifies a utility function $U_i(x)$, which may depend on attributes α_j of the asset, such as geographic location or expected return:

$$U_i(x) = f_i\left(x_{ij}, \alpha_j\right),$$

where f_i is a nonlinear valuation function. Temporal preferences, such as required holding periods T_j, can also be encoded as:

$$t_{ij} \geq T_j \quad \forall j,$$

where t_{ij} is the proposed holding period for asset j.

The optimization problem becomes:

$$\max_{x \in \mathcal{X}} \sum_{i=1}^{n} U_i(x),$$

subject to:

$$\sum_{i=1}^{n} x_{ij} \leq q_j, \quad \sum_{j=1}^{m} \beta_{ij} x_{ij} \leq C_i, \quad t_{ij} \geq T_j, \quad x_{ij} \geq 0 \quad \forall i, j,$$

where β_{ij} represents the liquidity cost of asset j for participant i, and C_i is their capital allocation limit. These constraints enable bidders to articulate detailed preferences, resulting in allocations that align closely with their strategic objectives.