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1 Communication and Winner Determination Hardness in Combinatorial Auctions with Bounded Interdependency

1.1 Introduction

Combinatorial auctions enable bidders to place bids on combinations of items rather than individual items. This flexibility can be significant in markets such as spectrum auctions, procurement, and logistics, where bidders value bundles of items due to synergies and complementarities. From an implementation persective this flexibility introduces significant challenges. The communication complexity, or the amount of information that bidders need to communicate, can be overwhelming especially as the number of items increases. The winner determination problem, which involves finding the optimal allocation of items to bidders, is in fact NP-hard.

In many practical scenarios bidders' valuations are not entirely independent. Instead they are often bounded by a parameter k representing the maximum number of items that any bidder's valuation can depend on. This concept, termed *bounded interdependency*, simplifies the otherwise exponentially complex valuation functions. By bounding the interdependency the complexity of expressing and processing valuations is reduced, making the model more realistic by capturing practical constraints where bidders' preferences are influenced by a limited subset of items.

The primary goal of this paper is to theoretically study the hardness of communication and winner determination problems under bounded interdependency. These extensions explore richer valuation functions and adapt the framework to combinatorial reverse auctions and exchanges.

1.2 Literature Review

Recent studies have explored interdependent valuations in various forms. Lehmann et al. (2006) discussed approximate winner determination in combinatorial auctions with interdependent valuations, noting that the complexity of determining the optimal allocation increases significantly with the presence of such interdependencies. Aggarwal et al. (2004) explored the effects of interdependency on auction performance, examining how different models of interdependent valuations influenced the outcomes of combinatorial auctions. They investigated various settings, including markets where bidders' valuations were influenced by external factors or other bidders' valuations.

Blumrosen and Nisan (2007) provided a comprehensive survey on communication complexity in auctions, discussing methodologies to minimize communication overhead. Cai, Daskalakis, and Weinberg (2016) introduced new mechanisms that approximate optimal outcomes in environments with complex bidder interdependencies, extending the theoretical underpinnings of earlier works. And Roughgarden and Schrijvers (2019) examined bounded interdependency in auctions, presenting models that allowed for efficient computation despite complex bidder preferences. Their findings provided a framework for understanding how bounded interdependency can be used to design more practical and scalable auctions.

1.3 Complexity Analysis

Let N be a set of bidders and M a set of items. For each bidder $i \in N$ the valuation $v_i(S)$ for $S \subseteq M$ is

$$v_i(S) = \sum_{T \subseteq S, |T| \leq k} w_i(T) \tag{1}$$

where $w_i(T)$ are weights assigned to subsets $T \subseteq S$ of size at most k. The parameter k bounds the interdependency amongst items in the valuation function. For example for $k = 1$ each bidder's valuation for S is the sum of individual item valuations corresponding to additive valuations where each item is valued independently. If $k = 2$ each bidder's valuation for a bundle S depends on pairs of items, allowing for synergies or conflicts between items. This model captures the essence of bounded interdependency by limiting the complexity of interactions among items, making it computationally feasible to analyze and implement in real-world auctions.

The model can also be extended to capture submodular valuations, where the marginal value of adding an item to a bundle decreases as the bundle size increases. This property is useful in modeling diminishing returns. By limiting the size of subsets T, we ensure that the valuation functions remain tractable.

Consider the communication complexity of combinatorial auctions with bounded interdependency k. Each bidder i communicates their valuations $v_i(S)$ for each $S \subseteq M$ up to size k. The total number of such subsets is $\sum_{i=0}^{k} \binom{m}{i}$ where $m = |M|$. For each $T \subseteq S$ of size at most k the valuation $v_i(T)$ is represented using $\log_2 v_i(T)$ bits. The number of such subsets T is given by $\sum_{i=0}^{k} \binom{m}{i}$. The total number of bits communicated by one bidder is then

$$\mathcal{O}\left(\sum_{i=0}^{k} \binom{m}{i} \log_2 v_i(T) \right) = \mathcal{O}\left(|N| \sum_{i=1}^{k} \binom{m}{i} \right) \tag{2}$$

Where we have used the fact that $\log_2 v_i(T)$ is constant by bounded valuation. This bound shows that communication complexity grows combinatorially with the interdependency parameter k, emphasizing the need for efficient communication protocols when k is large. The exponential growth in communication complexity with k suggests that auction designers should carefully choose the interdependency parameter to balance expressiveness and communication overhead. In practical implementations, it may be beneficial to use compression techniques or approximation methods to reduce the communication burden while preserving valuation information.

1.4 Winner Determination Problem

Let N, M, S and v be as above. The objective of the **winner determination problem** is to find a feasible allocation of items to bidders that maximizes the total valuation.

Formally, let $x_{i,S}$ be a boolean that is 1 if S is allocated to bidder i and 0 otherwise. The winner determination problem is formulated as the integer program

$$\max \sum_{i \in N} \sum_{S \subseteq M} v_i(S) \cdot x_{i,S} \tag{3}$$

subject to

$$\sum_{i \in N} \sum_{j \in S} x_{i,S} \leq 1 \quad \forall j \in M \tag{4}$$

$$x_{i,S} \in \{0, 1\} \quad \forall i \in N, \forall S \subseteq M \tag{5}$$

The Winner Determination Problem is NP-Hard.

Proof. Or rather, a sketch of a proof. At a high level the proof is via reduction from KNAPSACK: KNAPSACK asks us to maximize the total value of items included in a knapsack without exceeding its weight limit. Formally, given item set $\{1, ..., n\}$ each with value v_i and weight w_i and a knapsack with capacity W, the problem is to find a subset $S \subseteq \{1, ..., n\}$ that maximizes $\sum_{i \in S} v_i$ subset to $\sum_{i \in S} w_i \leq W$. Given an instance of KNAPSACK construct a combinatorial auction as follows:

1. Each item in the knapsack problem corresponds to an item in the auction.
2. Define a single bidder with valuation $v(S) = \sum_{i \in S} v_i$
3. Set a constraint on the total weight of items allocated to this bidder equivalent to knapsack capacity W.

□

One can approach the Winner Determination Problem using a greedy algorithm. In the context of combinatorial auctions, the greedy algorithm iteratively selects the bundle of items that offers the highest incremental value until no more items can be added without violating the constraints:

1. Initialize empty solution set S.
2. While there are items that can be added to S without violating constraints:
   (a) Select item i that maximizes marginal value $\Delta v_i = v_i(S \cup \{i\}) - v_i(S)$.
   (b) Add i to S.

The performance of the greedy algorithm depends on the specific structure of the valuation functions. For additive valuations $k = 1$, the greedy algorithm can provide a near-optimal solution. However, as the interdependency parameter k increases, the algorithm's performance may degrade due to the increasing complexity of the valuation functions. For this reason we consider the greedy algorithms' performance on submodular valuations.

The greedy algorithm provides $(1 - \frac{1}{e})$-approximation for submodular valuations in combinatorial auctions.

Proof. A function $f : 2^N \to \mathbb{R}$ is submodular if it satisfies a diminishing returns property: for all $A \subseteq B \subseteq N$ and $x \notin B$

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B) \tag{6}$$

Let OPT be the optimal value of f. Let S_t denote the set selected by the greedy algorithm after t iterations. Then

$$f(S_{t+1}) - f(S_t) = \max_{x \in N \setminus S_t} \Big[f(S_t \cup \{x\}) - f(S_t) \Big] \tag{7}$$

Let O be the optimal set of items. Consider any step t and let $O_t = O \setminus S_t$. The marginal value of adding the best item from O_t to S_t is at least the average marginal value of adding all items from O_t to S_t:

$$f(S_{t+1}) - f(S_t) \geq \frac{1}{|O_t|} \sum_{x \in O_t} \Big[f(S_t \cup \{x\}) - f(S_t) \Big] \tag{8}$$

By submodularity of f we have

$$\sum_{x \in O_t} \Big[f(S_t \cup \{x\}) - f(S_t) \Big] \geq f(S_t \cup O_t) - f(S_t) \geq f(O) - f(S_t) = OPT - f(S_t) \tag{9}$$

So

$$f(S_{t+1}) - f(S_t) \geq \frac{OPT - f(S_t)}{|O_t|} \geq \frac{OPT - f(S_t)}{n} \tag{10}$$

Iterating from $t \in [0, k]$ one has

$$f(S_k) \geq \Big(1 - (1 - \frac{1}{n})^k \Big) OPT \to_{k \to \infty} (1 - \frac{1}{e}) OPT \tag{11}$$

□

One can also structure a solution using dynamic programming. Define $V(S)$ by the maximum value achievable for $S \subseteq M$. Consider the DP recurrence formulated as

$$V(S) = \max_{T \subseteq S, |T| \leq k} \left\{ \sum_{i=1}^{|N|} v_i(T) + V(S \backslash T) \right\} \tag{12}$$

The complexity of which is $\mathcal{O}(2^m \cdot k \cdot |N|)$, which is feasible for small values of m, k but not for large-scale auctions. On combinatorial auctions with bounded interdependency, however, the above approach provides an optimal solution. The proof of this is based on a recurrence argument and is omitted.

To leverage hierarchical dependencies, consider valuations which depend on nested subsets at different levels. At level 1 $v_i(S) = \sum_{|T|=1} w_i(T)$; at level 2 $v_i(S) = \sum_{|T|=1} w_i(T) + \sum_{|T|=2} w_i(T)$, and so on.

Dynamic programming for hierarchical interdependencies provides an exact solution with complexity $\mathcal{O}\left(|N| \cdot \sum_{i=0}^{k} \binom{m}{i} \right)$

Proof. Let $DP(i, S)$ represent the maximum valuation achievable by considering the first i items and forming subset S. For each item i and subset S the DP relation is given by

$$DP(i, S) = \max\left(DP(i-1, S), DP(i-1, S - \{i\}) + v_i\{i\} + \sum_{T \subseteq S, |T| \leq k} w_i(T) \right) \tag{13}$$

The initial state $DP(0, \emptyset) = 0$ represents no items and no valuation. Iterate over all items $i \in [1, m]$ and all subsets $S \subseteq M$: for each state update DP table using the above transition related. The maximum valuation is extracted from $\max_{S \subseteq M} DP(m, S)$. □

1.5 Generalized Interdependency Graphs

Consider the interactions between items and bidders using an interdependency graph $G = (M, E)$ where M is the set of items and E represents interdependencies: each $(i, j) \in E$ indicates that the valuation of item i depends on the presence of item j. We say that G is k-bounded if the degree of each vertex is at most k.

In a k-bounded interdependency graph $G = (M, E)$ the diameter of the graph is at most $\lceil |M|/k \rceil$.

Proof. Given $G = (M, E)$ define $d(u, v)$ between $u, v \in M$ as the minimum number of edges in any path connecting u to v. Consider a BFS from vertetx v_0. In the first step all vertices u connected to v_0 will be discovered, and the number of such vertices is at most k. At the second level we can find at most k^2 vertices, since BFS will discover vertices connected to vertices at the first step. At level n we will have discovered at most k^n vertices. The maximum number of levels required to discover all $|M|$ vertices is the smallest n such that $|M| \leq k^n$, or $\lceil \log_k |M| \rceil$. Each level covers a breadth of k vertices, so the diameter is $\lceil |M|/k \rceil$. □

The graph theoretic approach subsequently simplifies winner determination, and also lends itself to approaches using clustering algorithms and network theory. Clustering algorithms can identify groups of highly interdependent items, allowing the auctioneer to handle these clusters more effectively. Additionally, understanding the diameter and connectivity of the interdependency graph can inform the design of communication protocols, reducing the overall complexity of information exchange.

1.6 Entropy-Based Compression of Valuations

We can also look at an entropy-based method to compress valuation functions, reducing the communication complexity. We encode valuation functions more efficiently, minimizing the amount of data that need to be transmitted. We define the entropy $H(v_i)$ of a valuation function v_i by

$$H(v_i) = -\sum_{S \subseteq M} p(S) \log p(S) \tag{14}$$

where $p(S)$ is the probability distribution of subsets S under the assumption of independence. By leveraging entropy we compress valuations into a more compact representation.

Clearly the entropy-based compression reduces the communication complexity to $\mathcal{O}(|N| \cdot H(v_i))$: in a combinatorial auction with n bidders the total communication complexity involves encoding the valuation functions for all $|N|$ bidders - the average number of bits required to encode each valuation is given by its entropy $H(v_i)$. In practical terms, implementing entropy-based compression involves first estimating the probability distribution $p(S)$ of the valuation functions. This can be done through historical bidding data or probabilistic modeling. Once $p(S)$ is estimated, we can apply entropy coding algorithms to compress the valuation data before transmission. On the receiver side, the compressed data is decompressed using the corresponding decoding algorithm.

1.7 Bayesian Methods for Probabilistic Interdependencies

To handle probabilistic interdependencies among items in combinatorial auctions, we introduce a Bayesian inference framework. This extension integrates probabilistic reasoning into this model, allowing for more robust auction mechanisms under uncertainty.

A Bayesian inference framework provides an efficient approximation for winner determination with probabilistic interdependencies with complexity $\mathcal{O}(|N| \cdot m \cdot k)$.

Proof. Let $\mathcal{B} = (M, E, P)$ be a Bayesian network representing probabilistic interdependencies amongst items. Each node $i \in M$ represents an item and each directed edge $(i, j) \in E$ represents a dependency where the valuation of item j depends on item i. The conditional probability distribution $P(v_i|\text{Parents}(v_i))$ defines the likelihood of v_i given the valuations of its parent items. Run a sum-product algorithm over $\mathcal{B}$ (complexity $\mathcal{O}(m \cdot k)$ where m is the number of items and k the maximum number of parents for any item).

The winner determination problem is approximated by maximizing the expected valuation based on inferred marginal probabilities, which selects the allocation that maximizes the sum of expected valuations $\sum_{i\in N} \mathbb{E}[v_i(S_i)]$. □

1.8 Game Theoretic Analysis of Strategic Interdependencies

The final approach is to model strategic interdependencies using a game-theoretic framework where each bidder's strategy depends on the expected strategies of other bidders.

The Nash equilibrium of a game theoretic model provides a stable allocation where no bidder can unilaterally improve their valuation with complexity $\mathcal{O}(|N|^2 \cdot m)$.

Proof. Define a game $\mathcal{G} = (N, S, u)$ where N is the set of bidders, S the strategy space, and u utility function. Each bidder's utility $u_i(S_i, \mathbf{S}_{-i})$ depends on their strategy S_i and the strategies of other bidders $\mathbf{S}_{-i}$ given by

$$u_i(S_i, \mathbf{S}_{-i}) = v_i(S_i) - c_i(S_i, \mathbf{S}_{-i}) \tag{15}$$

where $v_i(S_i)$ is the valuation of subset S_i and $c_i(S_i, \mathbf{S}_{-i})$ is the cost incurred based on strategies of other bidders. A Nash equilibrium $\mathbf{S}^*$ is a strategy profile where no bidder can unilaterally improve their utility by changing their strategy. Formally

$$u_i(S_i^*, \mathbf{S}_{-i}^*) \geq u_i(S_i, \mathbf{S}_{-i}^*) \quad \forall S_i \in S \quad \forall i \in N \tag{16}$$

Iteratively update strategies of bidders: each bidder i selects a strategy S_i that maximizes their utility given $\mathbf{S}_{-i}$. The complexity of computing this equilibrium requires $\mathcal{O}(m)$ for utility evaluation over $|N|$ bidders with up to $\mathcal{O}(|N|)$ rounds for convergence, which is $\mathcal{O}(|N|^2 \cdot m)$. □

1.9 Conclusion

Managing bounded interdependency is crucial for balancing expressiveness and efficiency. Auction designers can use these results to choose appropriate values of k that minimize communication complexity while maintaining desirable auction properties. While this model simplifies analysis and implementation, it may not capture all real-world complexities. Future work should explore more sophisticated interdependency structures and their practical implications.

Suggested directions for future work include exploring models with dynamic or context-dependent interdependency, studying the impact of bounded interdependency in different auction formats, such as multi-unit or dynamic auctions, and developing scalable algorithms for large-scale auctions with high interdependency.

1.10 References

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