Market Microstructure Optimization via Entropic Regularization

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1 Introduction

The optimization of trading strategies in fragmented equity markets is a challenge in algorithmic trading. Market participants operating within these environments contend with complexities arising from nonlinear market impacts, adverse selection risks, and the inherent inefficiencies of fragmented liquidity pools. How do we design strategies that minimize execution costs while mitigating these risks?

In this paper I explore entropic regularization as an approach to optimize execution paths. Entropic regularization, originally developed within the context of optimal transport theory, allows us to stabilize execution trajectories while improving computational efficiency and reducing sensitivity to adverse market conditions. The intention is to address the need for stability in execution strategies, which often become overly sensitive to local market irregularities or specific datasets. This sensitivity, left unchecked, can result in erratic trading behavior, suboptimal cost performance, and increased exposure to market impact and adverse selection.

The motivation for this work lies in the decomposition of execution costs into three primary components: the direct cost of market impact, the risk of adverse price movements during execution, and the transactional frictions such as fees and latency. Traditional execution strategies often focus on minimizing impact cost through optimal splitting of orders across venues and time, but such approaches can suffer from overfitting to specific market conditions.

Why entropy? First, it stabilizes execution trajectories by promoting smoothness in the distribution of trading volumes over time. This stabilization reduces the sensitivity of the execution strategy to transient market anomalies. Second, entropy acts as a regularizer, penalizing irregularities and extreme deviations in trading behavior. Formally, for an execution path $X = \{x_t\}_{t=1}^T$, we define entropy as

$$H(X) = -\sum_{t=1}^{T} p_t \log p_t \tag{1}$$

where

$$p_t = \frac{x_t}{\sum_{t=1}^T x_t} \tag{2}$$

represents the normalized trading volume at time t. The entropy term is incorporated into the cost function as $-\epsilon H(X)$, with $\epsilon > 0$ serving as a regularization parameter. This ensures that the solution remains numerically stable and well-posed.

The remainder of this paper explores the theoretical foundations and practical applications of entropic regularization in trading optimization. I begin with a treatment of entropy and its connections to optimal transport theory, establishing a foundational framework for execution optimization. This is followed by a formulation of the execution optimization problem, incorporating entropic penalties into the cost function. Next, I derive efficient numerical methods for solving the entropy-regularized optimization problem, with a particular focus on the Sinkhorn-Knopp algorithm, a computationally efficient iterative method that is well-suited to this class of problems.

2 Entropy and Optimal Transport

Let $P = \{p_i\}_{i=1}^n$ denote a discrete probability distribution over a finite set of states $\{1, 2, \ldots, n\}$, where $p_i \geq 0$ for all i and $\sum_{i=1}^n p_i = 1$. The Shannon entropy of P is defined as:

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i,$$

We adopt the convention $0 \log 0 = 0$, justified by the limit:

$$\lim_{p \to 0^+} p \log p = 0.$$

As a reference, some properties of entropy:

1. For any probability distribution P, $H(P) \geq 0$, with equality if and only if P is degenerate (i.e., $p_i = 1$ for some i and $p_j = 0$ for all $j \neq i$).
2. Among all distributions over n states, the uniform distribution $p_i = \frac{1}{n}$ maximizes entropy, achieving:
$$H(P) = \log n.$$
3. For independent distributions $P = \{p_i\}_{i=1}^n$ and $Q = \{q_j\}_{j=1}^m$, the entropy of their joint distribution satisfies:
$$H(P \otimes Q) = H(P) + H(Q),$$
where $(P \otimes Q)_{ij} = p_i q_j$.

In summary, entropy measures the dispersion of a distribution. A high-entropy distribution spreads probability mass evenly, while a low-entropy distribution concentrates it. In optimization problems, introducing an entropy term penalizes over-concentration.

Entropy-Regularized Optimization

Consider a cost function $C(P) = \sum_{i=1}^n c_i p_i$, where $c_i \geq 0$ represents the cost associated with state i. Directly minimizing $C(P)$ often yields sparse solutions, concentrating probability mass on a few states. To counteract this, we introduce an entropy regularization term:

$$\min_{P \in \Delta} \sum_{i=1}^{n} c_i p_i - \epsilon H(P),$$

where $\Delta = \{P \in \mathbb{R}^n \mid \sum_{i=1}^n p_i = 1, p_i \geq 0\}$ is the probability simplex, and $\epsilon > 0$ is a regularization parameter.

To solve the problem, we define the Lagrangian:

$$\mathcal{L}(P, \lambda) = \sum_{i=1}^{n} c_i p_i - \epsilon \sum_{i=1}^{n} p_i \log p_i + \lambda \left(\sum_{i=1}^{n} p_i - 1 \right),$$

where λ enforces the normalization constraint $\sum_{i=1}^n p_i = 1$.

Differentiating with respect to p_i gives:

$$\frac{\partial \mathcal{L}}{\partial p_i} = c_i - \epsilon(1 + \log p_i) + \lambda = 0.$$

Rearranging, we find:

$$p_i = \exp\left(-\frac{c_i}{\epsilon}\right)\exp\left(-\frac{\lambda+\epsilon}{\epsilon}\right).$$

Enforcing normalization, $\sum_{i=1}^n p_i = 1$, determines λ:

$$\lambda = -\epsilon \log\left(\sum_{i=1}^n \exp\left(-\frac{c_i}{\epsilon}\right)\right).$$

Substituting back, the optimal solution is the Gibbs distribution:

$$p_i = \frac{\exp\left(-\frac{c_i}{\epsilon}\right)}{\sum_{j=1}^n \exp\left(-\frac{c_j}{\epsilon}\right)}.$$

This result demonstrates how entropy regularization smooths the solution by distributing probability mass across states according to their costs.

Optimal Transport and Entropic Regularization

We now extend these principles to the optimal transport problem. Let $P = \{p_i\}_{i=1}^n$ and $Q = \{q_j\}_{j=1}^m$ be discrete probability distributions over supports $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$, respectively. The goal is to find a coupling $\pi = \{\pi_{ij}\}$ minimizing the transport cost:

$$\min_\pi \sum_{i=1}^n \sum_{j=1}^m \pi_{ij} c_{ij},$$

subject to:

$$\sum_{j=1}^m \pi_{ij} = p_i, \quad \sum_{i=1}^n \pi_{ij} = q_j, \quad \pi_{ij} \geq 0.$$

To enhance stability, we introduce an entropic penalty:

$$\min_\pi \sum_{i=1}^n \sum_{j=1}^m \pi_{ij} c_{ij} - \epsilon \sum_{i=1}^n \sum_{j=1}^m \pi_{ij} \log \pi_{ij}.$$

Let u and v be dual potentials associated with the marginal constraints. The dual problem is:

$$\max_{u,v} \sum_{i=1}^n u_i p_i + \sum_{j=1}^m v_j q_j - \epsilon \sum_{i=1}^n \sum_{j=1}^m \exp\left(\frac{u_i + v_j - c_{ij}}{\epsilon}\right).$$

The primal solution is recovered as:

$$\pi_{ij} = u_i v_j \exp\left(-\frac{c_{ij}}{\epsilon}\right),$$

where u_i and v_j are scaling factors adjusted to satisfy the marginal constraints.

We invoke the Sinkhorn-Knopp algorithm for an iterative solution, which alternates updates for u_i and v_j:

$$u_i^{(k+1)} = \frac{p_i}{\sum_{j=1}^m v_j^{(k)} \exp\left(-\frac{c_{ij}}{\epsilon}\right)}, \quad v_j^{(k+1)} = \frac{q_j}{\sum_{i=1}^n u_i^{(k+1)} \exp\left(-\frac{c_{ij}}{\epsilon}\right)}.$$

3 The Execution Optimization Problem

Execution optimization in algorithmic trading requires determining the trajectory $X = \{x_t\}_{t=1}^T$ that minimizes execution costs while adhering to constraints imposed by market conditions and risk management principles. This trajectory governs how an order of size Q is distributed across T discrete time intervals, balancing market impact, execution risk, and frictional costs. The challenge lies in accounting for dynamic price evolution and liquidity constraints while ensuring that the execution strategy is robust to irregularities or adverse selection.

Execution Path as a Discrete Probability Distribution

The execution trajectory can be represented as a discrete probability distribution over the intervals $\{1, 2, \dots, T\}$. Let x_t denote the volume traded at time t, with the total volume satisfying:

$$\sum_{t=1}^{T} x_t = Q, \quad x_t \geq 0 \text{ for all } t.$$

To normalize this trajectory, we define the probabilities $P = \{p_t\}_{t=1}^T$, where:

$$p_t = \frac{x_t}{Q}, \quad \sum_{t=1}^{T} p_t = 1.$$

Each p_t represents the fraction of the total order executed during interval t. The probabilistic interpretation is essential for introducing entropy as a regularization term, which penalizes over-concentration of execution volumes, thereby promoting smoother trajectories.

The total cost of execution, $C(X)$, is decomposed into three principal components: market impact, execution risk, and frictional costs.

Market Impact Cost. Market impact reflects the adverse price movement caused by executing a large order. Let S_t denote the mid-price at time t, which evolves as a function of x_t. A common model for instantaneous impact is:

$$\phi(x_t) = k x_t^{\alpha},$$

where $k > 0$ is a liquidity scaling parameter, and $\alpha \in (0, 1]$ determines the nonlinearity of the impact. For time-varying liquidity, this generalizes to:

$$\phi_t(x_t) = k_t x_t^{\alpha},$$

where k_t captures the variation in market depth or resilience across intervals.

Execution Risk. Execution risk arises from the uncertainty in price movements during the execution period. If the price S_t evolves as a stochastic process, the risk is modeled as the variance of the execution trajectory relative to the volume-weighted average price (VWAP), $\bar{S}$:

$$C_{\text{risk}}(X) = \lambda \sum_{t=1}^{T} \mathbb{E}\left[(S_t - \bar{S})^2\right],$$

where $\lambda > 0$ is a risk aversion parameter. The VWAP is defined as:

$$\bar{S} = \frac{\sum_{t=1}^{T} S_t x_t}{\sum_{t=1}^{T} x_t}.$$

Friction Costs. Frictional costs include transaction fees, latency penalties, and other operational inefficiencies. These are modeled as a linear term:

$$C_{\text{friction}}(X) = \sum_{t=1}^{T} f_t x_t,$$

where f_t denotes the fee per unit volume at time t.

Total Cost Function. Combining these components, the total cost function is:

$$C(X) = \sum_{t=1}^{T} \phi_t(x_t) + \lambda \sum_{t=1}^{T} \mathbb{E}\left[(S_t - \bar{S})^2\right] + \sum_{t=1}^{T} f_t x_t.$$

Entropy-Augmented Optimization Framework

The entropy of the normalized execution path $P = \{p_t\}_{t=1}^{T}$ is given by:

$$H(P) = -\sum_{t=1}^{T} p_t \log p_t.$$

In other words, entropy penalizes irregular or concentrated trajectories, introducing robustness and stability to the execution strategy. The entropy-augmented cost function is:

$$C_{\text{reg}}(X) = C(X) - \epsilon H(P),$$

where $\epsilon > 0$ is the regularization parameter controlling the trade-off between cost minimization and trajectory smoothness.

High entropy enforces a more uniform distribution of trading volumes, mitigating risks associated with over-concentration at specific times. This is particularly valuable in fragmented markets where liquidity is unpredictable.

Stochastic Price Dynamics and Control Formulation

The mid-price S_t evolves according to a stochastic differential equation:

$$dS_t = \mu_t dt + \sigma_t dW_t,$$

where μ_t is the drift term capturing deterministic price trends, σ_t is the volatility, and W_t is a standard Brownian motion. The execution trajectory $X = \{x_t\}_{t=1}^{T}$ is treated as a control variable in the stochastic optimization problem:

$$\min_X \mathbb{E}\left[C_{\text{reg}}(X)\right],$$

subject to the constraints:

$$\sum_{t=1}^{T} x_t = Q, \quad x_t \geq 0 \text{ for all } t.$$

The value function $V(S_t, t)$ represents the minimum expected cost from time t onward:

$$V(S_t, t) = \min_{x_t} \mathbb{E}\left[\int_t^T C_{\text{reg}}(X) dt \mid S_t\right].$$

The HJB equation for $V(S_t, t)$ is:

$$\frac{\partial V}{\partial t} + \min_{x_t}\left[\phi_t(x_t) + \lambda \sigma_t^2 \frac{\partial^2 V}{\partial S_t^2} + \mu_t \frac{\partial V}{\partial S_t} - \epsilon H(P) + f_t x_t\right] = 0.$$

Derivation of Optimal Execution Strategy

The Lagrangian for the entropy-regularized optimization is:

$$L(x_t) = \phi_t(x_t) + f_t x_t - \epsilon \left(\frac{1}{Q} + \log \left(\frac{x_t}{Q} \right) \right).$$

Taking the derivative with respect to x_t and setting it to zero, we obtain:

$$\frac{\partial L}{\partial x_t} = \phi_t'(x_t) + f_t - \frac{\epsilon}{Q} \left(1 + \log \left(\frac{x_t}{Q} \right) \right) = 0.$$

Solving for x_t, the optimal trading volume at time t is:

$$x_t = Q \exp \left(-\frac{1}{\epsilon} \left(\phi_t'(x_t) + f_t \right) \right).$$

he exponential form of x_t reflects the balancing act between market impact, friction costs, and entropy. A larger ϵ results in smoother trajectories, while smaller ϵ prioritizes cost minimization.

4 Numerical Methods for Entropy-Regularized Execution Optimization

The entropy-regularized execution optimization problem seeks to minimize a cost function over a transport plan $\pi = \{\pi_{tj}\}$ that encodes the allocation of execution volumes x_t over time intervals t and trading venues j.

The problem is to minimize the entropy-regularized transport cost:

$$\min_{\pi} \sum_{t=1}^{T} \sum_{j=1}^{m} \pi_{tj} c_{tj} - \epsilon \sum_{t=1}^{T} \sum_{j=1}^{m} \pi_{tj} \log \pi_{tj},$$

subject to the marginal constraints:

$$\sum_{j=1}^{m} \pi_{tj} = p_t, \quad \sum_{t=1}^{T} \pi_{tj} = q_j, \quad \pi_{tj} \geq 0,$$

where p_t is the normalized execution trajectory and q_j represents venue-specific demand or cost constraints. The term c_{tj} encodes the execution cost at time t and venue j, incorporating market impact, risk, and frictional effects.

The entropic term $-\epsilon \sum_{t,j} \pi_{tj} \log \pi_{tj}$ ensures the convexity of the objective function. Convexity follows from the fact that both the linear term $\sum_{t,j} \pi_{tj} c_{tj}$ and the entropy term are convex in π (the entropy term is strictly convex over the interior of the feasible set).

The existence of a solution follows directly from the Weierstrass theorem: the objective function is continuous and coercive (entropy penalizes arbitrarily large values of π_{tj}), while the feasible set defined by the linear constraints is compact. Uniqueness is guaranteed by the strict convexity of the entropy term.

The dual formulation is essential for efficient computation. To derive the dual formulation, introduce dual variables u_t and v_j for the marginal constraints, leading to the Lagrangian:

$$\mathcal{L}(\pi, u, v) = \sum_{t=1}^{T} \sum_{j=1}^{m} \pi_{tj} c_{tj} - \epsilon \sum_{t=1}^{T} \sum_{j=1}^{m} \pi_{tj} \log \pi_{tj} + \sum_{t=1}^{T} u_t \left(p_t - \sum_{j=1}^{m} \pi_{tj} \right) + \sum_{j=1}^{m} v_j \left(q_j - \sum_{t=1}^{T} \pi_{tj} \right).$$

Differentiating with respect to π_{tj}, the first-order optimality condition is:

$$\frac{\partial \mathcal{L}}{\partial \pi_{tj}} = c_{tj} - \epsilon(1 + \log \pi_{tj}) - u_t - v_j = 0,$$

which simplifies to:

$$\pi_{tj} = \exp\left(-\frac{c_{tj} - u_t - v_j}{\epsilon}\right).$$

Substituting this expression into the marginal constraints yields the dual problem:

$$\max_{u,v} \sum_{t=1}^{T} u_t p_t + \sum_{j=1}^{m} v_j q_j - \epsilon \sum_{t=1}^{T} \sum_{j=1}^{m} \exp\left(\frac{u_t + v_j - c_{tj}}{\epsilon}\right).$$

The relationship between the primal solution π_{tj} and the dual potentials u_t and v_j is given explicitly as:

$$\pi_{tj} = u_t v_j \exp\left(-\frac{c_{tj}}{\epsilon}\right),$$

where u_t and v_j are scaling factors determined iteratively to enforce the marginal constraints:

$$\sum_{j=1}^{m} \pi_{tj} = p_t, \quad \sum_{t=1}^{T} \pi_{tj} = q_j.$$

The Sinkhorn-Knopp algorithm solves for u_t and v_j iteratively. Starting with $u_t^{(0)} = 1$ and $v_j^{(0)} = 1$, the updates are:

$$v_j^{(k+1)} = \frac{q_j}{\sum_{t=1}^{T} u_t^{(k)} \exp\left(-\frac{c_{tj}}{\epsilon}\right)},$$
$$u_t^{(k+1)} = \frac{p_t}{\sum_{j=1}^{m} v_j^{(k+1)} \exp\left(-\frac{c_{tj}}{\epsilon}\right)}.$$

This procedure alternates between scaling rows and columns of the matrix $\exp(-C/\epsilon)$ to match the marginals p_t and q_j.

The Sinkhorn-Knopp algorithm converges exponentially under the conditions $\epsilon > 0$ and C bounded. Convergence follows from the contraction mapping principle, with the contraction constant determined by the entropy regularization strength ϵ.

In the context of execution optimization, c_{tj} is parameterized as:

$$c_{tj} = \phi_t(x_t) + \lambda \sigma_t^2 + f_t,$$

where $\phi_t(x_t) = k_t x_t^{\alpha}$ models market impact, $\lambda \sigma_t^2$ reflects execution risk, and f_t captures friction costs. The entropy-regularized objective balances these costs with stability induced by entropy.

The algorithm computes the optimal π_{tj}, determining how trading volume x_t is distributed across time intervals and venues.

The proof of the Sinkhorn-Knopp algorithm's optimality and stability relies on three aspects: (1) the primal-dual relationship, (2) the strict convexity induced by the entropy term, and (3) numerical stability under entropy regularization.

The entropy-regularized optimal transport problem is defined as:

$$\min_{\pi \in \mathbb{R}_+^{T \times m}} \sum_{t=1}^{T} \sum_{j=1}^{m} \pi_{tj} c_{tj} - \epsilon \sum_{t=1}^{T} \sum_{j=1}^{m} \pi_{tj} \log \pi_{tj},$$

subject to the marginal constraints:

$$\sum_{j=1}^{m} \pi_{tj} = p_t, \quad \sum_{t=1}^{T} \pi_{tj} = q_j, \quad \pi_{tj} \geq 0.$$

Let $u = \{u_t\}_{t=1}^T$ and $v = \{v_j\}_{j=1}^m$ be dual potentials corresponding to the row and column constraints. The dual formulation is:

$$\max_{u,v} \sum_{t=1}^{T} u_t p_t + \sum_{j=1}^{m} v_j q_j - \epsilon \sum_{t=1}^{T} \sum_{j=1}^{m} \exp\left(\frac{u_t + v_j - c_{tj}}{\epsilon}\right).$$

The primal-dual relationship is established through complementary slackness. For any primal-dual feasible pair (π, u, v), the coupling matrix $\pi = \{\pi_{tj}\}$ satisfies:

$$\pi_{tj} = \exp\left(-\frac{c_{tj} - u_t - v_j}{\epsilon}\right).$$

Substituting π_{tj} into the marginal constraints:

$$\sum_{j=1}^{m} \pi_{tj} = \sum_{j=1}^{m} u_t v_j \exp\left(-\frac{c_{tj}}{\epsilon}\right) = p_t,$$

$$\sum_{t=1}^{T} \pi_{tj} = \sum_{t=1}^{T} u_t v_j \exp\left(-\frac{c_{tj}}{\epsilon}\right) = q_j.$$

These equations implicitly define u_t and v_j as scaling factors ensuring marginal feasibility.

The dual objective function is strictly concave because the entropy term $-\epsilon \sum_{t,j} \exp((u_t + v_j - c_{tj})/\epsilon)$ is convex with respect to u and v. The first-order optimality conditions are:

$$\frac{\partial}{\partial u_t}\left(\sum_{t=1}^{T} u_t p_t - \epsilon \sum_{t=1}^{T} \sum_{j=1}^{m} \exp\left(\frac{u_t + v_j - c_{tj}}{\epsilon}\right)\right) = 0,$$

yielding:

$$p_t = \sum_{j=1}^{m} v_j \exp\left(-\frac{c_{tj}}{\epsilon}\right).$$

Similarly, differentiating with respect to v_j:

$$\frac{\partial}{\partial v_j}\left(\sum_{j=1}^{m} v_j q_j - \epsilon \sum_{t=1}^{T} \sum_{j=1}^{m} \exp\left(\frac{u_t + v_j - c_{tj}}{\epsilon}\right)\right) = 0,$$

yields:

$$q_j = \sum_{t=1}^{T} u_t \exp\left(-\frac{c_{tj}}{\epsilon}\right).$$

Thus, the dual optimality conditions correspond precisely to the iterative updates in the Sinkhorn-Knopp algorithm.

The entropy-regularized objective function:

$$\mathcal{L}(\pi) = \sum_{t=1}^{T} \sum_{j=1}^{m} \pi_{tj} c_{tj} - \epsilon \sum_{t=1}^{T} \sum_{j=1}^{m} \pi_{tj} \log \pi_{tj},$$

is strictly convex over the feasible set $\{\pi \mid \sum_j \pi_{tj} = p_t, \sum_t \pi_{tj} = q_j, \pi_{tj} \geq 0\}$.

1. The term $\sum_{t,j} \pi_{tj} c_{tj}$ is linear in π, hence convex.

2. The entropy term $-\epsilon \sum_{t,j} \pi_{tj} \log \pi_{tj}$ is strictly convex over $\mathbb{R}_+^{T\times m}$ because its Hessian matrix is diagonal with positive entries:

$$\frac{\partial^2}{\partial \pi_{tj}^2} \left(-\pi_{tj} \log \pi_{tj}\right) = \frac{\epsilon}{\pi_{tj}},$$

   which is positive for $\pi_{tj} > 0$.

Since the feasible set is convex and the objective function is strictly convex, any local minimum is the global minimum, ensuring unique optimal solutions.

Entropy regularization ensures numerical stability by preventing degeneracy in the coupling matrix π. Without regularization ($\epsilon = 0$), the optimal transport plan may concentrate all mass on a small subset of entries, leading to numerical instability. With $\epsilon > 0$:

1. The entropy penalty bounds π_{tj} away from zero, as:

$$\pi_{tj} = \exp\left(-\frac{c_{tj} - u_t - v_j}{\epsilon}\right) > 0.$$

   This prevents issues arising from underflow in numerical computations.

2. The iterative updates for u_t and v_j:

$$u_t^{(k+1)} = \frac{p_t}{\sum_{j=1}^m v_j^{(k)} \exp\left(-\frac{c_{tj}}{\epsilon}\right)}, \quad v_j^{(k+1)} = \frac{q_j}{\sum_{t=1}^T u_t^{(k+1)} \exp\left(-\frac{c_{tj}}{\epsilon}\right)},$$

   are continuous mappings, ensuring stable convergence to the unique fixed point.

Note that Sinkhorn-Knopp algorithm converges geometrically with a rate dependent on ϵ and the sparsity of $C = \{c_{tj}\}$. The contraction constant ρ is proportional to:

$$\rho \sim \frac{\epsilon}{\lambda_{\min}(\text{Hessian})},$$

where $\lambda_{\min}$ is the smallest eigenvalue of the Hessian matrix of the entropy-regularized objective function.