Implied Liquidity Barriers in Options Hedging

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1 Introduction

The hedging of options positions requires adjustments to offset directional risk arising from changes in the price of the underlying asset. These adjustments, typically executed through delta hedging, involve trading the underlying asset in proportion to the option's delta. A challenge here is the presence of liquidity barriers which represent thresholds beyond which executing trades incurs disproportionately high costs due to slippage and market impact. When a trader's hedging adjustments approach or exceed these barriers, adverse effects such as widening bid-ask spreads and temporary price dislocations can drive execution costs up.

2 Liquidity Barriers

The barrier L_t represents the upper bound on tradable quantities at time t before incurring disproportionately high market impact costs and depends on the market state $\mathcal{M}_t$. Let

$$d\mathcal{M}_t = \mu_{\mathcal{M}}(\mathcal{M}_t)\, dt + \Sigma_{\mathcal{M}}(\mathcal{M}_t)\, dW_t,$$

where $\mu_{\mathcal{M}}(\mathcal{M}_t)$ is the drift vector governing the deterministic trend in market variables, $\Sigma_{\mathcal{M}}(\mathcal{M}_t)$ is the volatility matrix capturing the stochasticity of the system, and W_t is a multi-dimensional Wiener process.

To encode the dependence of L_t on $\mathcal{M}_t$, we define L_t as:

$$L_t = f(\mathcal{M}_t),$$

where $f : \mathbb{R}^n \to \mathbb{R}_+$ is a smooth, deterministic function. For example, f might reflect the order book imbalance ratio or the inverse of the bid-ask spread.

The trader's position ϕ_t evolves according to delta-hedging principles, where the option's delta Δ_t is:

$$\Delta_t = \frac{\partial V}{\partial S}(S_t, t),$$

with $V(S_t, t)$ being the option price. The underlying asset price S_t follows geometric Brownian motion:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t^S,$$

where μ and σ are the drift and volatility of S_t, respectively. The trader executes trades at rate Q_t, and their position evolves as:

$$d\phi_t = -\Delta_t \, dt + Q_t \, dt.$$

Given the liquidity barrier L_t, the trader incurs additional costs when the trading rate Q_t exceeds L_t. The adjusted liquidity cost function $\ell^*(Q_t, L_t)$ incorporates this penalty:

$$\ell^*(Q_t, L_t) = \ell(Q_t) + \mathbf{1}_{\{Q_t > L_t\}} \kappa (Q_t - L_t)^2,$$

where $\ell(Q_t) = c_1 Q_t + c_2 Q_t^2$ is the base liquidity cost function, and $\kappa > 0$ is a penalty parameter. This quadratic penalty captures the super-linear costs associated with exceeding the liquidity barrier.

The trader seeks to minimize the total expected cost of hedging over a time horizon $[0, T]$:

$$C = \mathbb{E}\left[\int_0^T \ell^*(Q_t, L_t) \, dt\right],$$

subject to the state dynamics:

$$\begin{aligned}
d\phi_t &= -\Delta_t \, dt + Q_t \, dt, \\
d\mathcal{M}_t &= \mu_{\mathcal{M}}(\mathcal{M}_t) \, dt + \Sigma_{\mathcal{M}}(\mathcal{M}_t) \, dW_t, \\
dS_t &= \mu S_t \, dt + \sigma S_t \, dW_t^S.
\end{aligned}$$

The trader's optimization problem is framed as a stochastic control problem with the control variable Q_t and state vector $X_t = (\phi_t, \mathcal{M}_t, S_t)$. The value function $V(X_t, t)$ is defined as the minimum expected cost-to-go from state X_t at time t:

$$V(X_t, t) = \min_Q \mathbb{E}\left[\int_t^T \ell^*(Q_u, L_u) \, du + \Psi(X_T) \middle| X_t = X\right],$$

where $\Psi(X_T)$ is a terminal penalty function enforcing delta-neutrality at maturity. The Hamilton-Jacobi-Bellman (HJB) equation governing this problem is:

$$\frac{\partial V}{\partial t} + \min_Q \left\{\mathcal{L}^X V + \ell^*(Q, L_t)\right\} = 0,$$

where $\mathcal{L}^X$ is the infinitesimal generator associated with the state dynamics:

$$\mathcal{L}^X V = \nabla_X V \cdot b(X_t, Q_t) + \frac{1}{2} \operatorname{Tr}\left(\Sigma_X^\top \nabla_X^2 V \, \Sigma_X\right).$$

Here, $b(X_t, Q_t)$ and Σ_X represent the drift and diffusion coefficients of X_t, respectively. The optimal control Q_t^* satisfies the first-order condition derived from minimizing the integrand:

$$\frac{\partial}{\partial Q} \left(\ell^*(Q, L_t) + \nabla_\phi V \cdot Q\right) = 0.$$

The resulting policy Q_t^* accounts for the liquidity barrier L_t and adjusts trading rates to minimize costs while respecting market constraints.

3 Optimal Control under Liquidity Barriers

The value function $V(X_t, t)$, defined as the minimum expected cost-to-go from the state X_t, satisfies the HJB equation:

$$\frac{\partial V}{\partial t} + \min_{Q} \left\{ \nabla_X V \cdot b(X_t, Q) + \frac{1}{2} \operatorname{Tr}\left(\Sigma_X^\top \nabla_X^2 V \, \Sigma_X \right) + \ell^*(Q, L_t) \right\} = 0,$$

where $X_t = (\phi_t, S_t, \mathcal{M}_t)$ represents the state vector, $b(X_t, Q)$ is the drift term of the state dynamics, and Σ_X is the diffusion matrix. The term $\ell^*(Q, L_t)$ introduces a state-dependent penalty for crossing the liquidity barrier L_t.

Isolate the control-dependent terms from the integrand in the HJB equation:

$$\min_{Q} \left\{ \nabla_\phi V \cdot Q + \ell^*(Q, L_t) \right\}.$$

The effective liquidity cost function $\ell^*(Q, L_t)$ is piecewise:

$$\ell^*(Q, L_t) = \begin{cases} c_1 Q + c_2 Q^2 & \text{if } Q \leq L_t, \\ c_1 Q + c_2 Q^2 + \kappa (Q - L_t)^2 & \text{if } Q > L_t, \end{cases}$$

where $\kappa > 0$ penalizes trades exceeding the liquidity barrier. The first-order condition for optimality is derived by differentiating the control-dependent term:

$$\frac{\partial}{\partial Q} \left(\nabla_\phi V \cdot Q + \ell^*(Q, L_t) \right) = 0.$$

For the case $Q \leq L_t$, the derivative simplifies to:

$$\nabla_\phi V + c_1 + 2c_2 Q = 0.$$

Solving for Q yields:

$$Q_t^* = -\frac{\nabla_\phi V + c_1}{2c_2}, \quad \text{if } Q_t^* \leq L_t.$$

For the case $Q > L_t$, the derivative includes the additional penalty term:

$$\nabla_\phi V + c_1 + 2c_2 Q + 2\kappa (Q - L_t) = 0.$$

Expanding and solving for Q:

$$Q_t^* = \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2 + \kappa)}, \quad \text{if } Q_t^* > L_t.$$

The optimal control Q_t^* is thus:

$$Q_t^* = \begin{cases} -\frac{\nabla_\phi V + c_1}{2c_2}, & \text{if } -\frac{\nabla_\phi V + c_1}{2c_2} \le L_t, \\ \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2+\kappa)}, & \text{otherwise.} \end{cases}$$

i.e trades are executed optimally with respect to the liquidity barrier L_t. The presence of L_t divides the control space into two regimes: one where Q_t^* respects the barrier and another where the penalty term governs the trading behavior.

Substituting Q_t^* back into the HJB equation, we obtain the partial differential equation (PDE) for the value function $V(X_t, t)$:

$$\frac{\partial V}{\partial t} + \nabla_X V \cdot b(X_t, Q_t^*) + \frac{1}{2} \operatorname{Tr}\left(\Sigma_X^\top \nabla_X^2 V \, \Sigma_X\right) + \ell^*(Q_t^*, L_t) = 0.$$

The solution $V(X_t, t)$ depends on the boundary conditions, including the terminal condition at time T:

$$V(X_T, T) = \Psi(X_T),$$

where $\Psi(X_T)$ enforces delta-neutrality by penalizing deviations from $\phi_T = -\Delta_T$.

4 Analysis

To gain deeper intuition about the optimal control Q_t^*, we analyze the two trading regimes defined by the liquidity barrier L_t and the implications for the trader's strategy. The control splits into two distinct regions: the *liquidity-constrained region*, where $Q_t^* \le L_t$, and the *liquidity-penalized region*, where $Q_t^* > L_t$.

For $Q_t^* \le L_t$, the control is governed solely by the gradient of the value function $\nabla_\phi V$ and the base liquidity costs:

$$d\phi_t = -\Delta_t \, dt - \frac{\nabla_\phi V + c_1}{2c_2} \, dt.$$

The term $-\frac{\nabla_\phi V + c_1}{2c_2}$ represents a continuous adjustment in trading rate driven by both the hedging demand (Δ_t) and the marginal cost of execution. This regime is optimal when the liquidity barrier is **not binding.**

For $Q_t^* > L_t$, the penalty term dominates the control, leading to the modified optimal trading rate:

$$Q_t^* = \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2 + \kappa)}.$$

The penalty coefficient κ force the trading rate to asymptotically approach the barrier L_t as $\kappa \to \infty$. Intuitively, the strategy prioritizes minimizing the incremental cost of trading beyond the barrier which grows quadratically (at best) with $Q_t - L_t$.

The corresponding inventory evolution in this regime is:

$$d\phi_t = -\Delta_t \, dt + \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2 + \kappa)} \, dt.$$

Here L_t effectively serves as a dynamic constraint on the trader's adjustments.

The two regimes highlight a fundamental trade-off between achieving delta neutrality and respecting liquidity constraints. This trade-off can be quantified by examining the marginal cost of deviating from the optimal hedge, $\nabla_\phi V$, against the incremental penalty incurred by exceeding L_t. Formally, the condition for transitioning between regimes is given by:

$$-\frac{\nabla_\phi V + c_1}{2c_2} = L_t,$$

which defines the boundary separating the liquidity-constrained and liquidity-penalized regions.

Further consider the sensitivity of Q_t^* to changes in L_t. Differentiating the optimal control with respect to L_t in the penalized region yields:

$$\frac{\partial Q_t^*}{\partial L_t} = \frac{2\kappa}{2(c_2 + \kappa)}.$$

As $\kappa \to \infty$, the sensitivity $\frac{\partial Q_t^*}{\partial L_t} \to 1$, indicating that the optimal trading rate becomes tightly constrained by the liquidity barrier.

The stability of the optimal policy is determined by the feedback between the inventory ϕ_t, the market state $\mathcal{M}_t$, and the liquidity barrier L_t. Consider the impact of large deviations in $\mathcal{M}_t$, such as a sudden widening of the bid-ask spread, which reduces L_t. The resulting contraction in the liquidity barrier forces Q_t^* to decrease, slowing the rate of hedging adjustments and potentially delaying convergence to delta neutrality.

The convergence of ϕ_t to the target delta-neutral state is governed by the interplay of $\nabla_\phi V$ and L_t. If L_t fluctuates frequently due to market microstructure noise, the trajectory of ϕ_t exhibits stochastic oscillations around the target state. The expected deviation from delta neutrality at time t is

$$\mathbb{E}[|\phi_t + \Delta_t|] \leq \mathbb{E}\left[\frac{|\nabla_\phi V + c_1|}{2c_2}\mathbf{1}_{\{Q_t^* \leq L_t\}} + \frac{|\nabla_\phi V + c_1 - 2\kappa L_t|}{2(c_2 + \kappa)}\mathbf{1}_{\{Q_t^* > L_t\}}\right].$$

Note the dependence on the gradient of the value function.

5 Asymptotic Properties

Recall that the liquidity barrier L_t is a deterministic function $f(\mathcal{M}_t)$ of the market state $\mathcal{M}_t$, which evolves stochastically according to:

$$d\mathcal{M}_t = \mu_{\mathcal{M}}(\mathcal{M}_t)\,dt + \Sigma_{\mathcal{M}}(\mathcal{M}_t)\,dW_t.$$

To examine the long-term behavior of L_t, consider the stationary distribution $\pi(\mathcal{M})$ of the market state $\mathcal{M}_t$. Assuming ergodicity, $\mathcal{M}_t$ converges in distribution to $\pi(\mathcal{M})$ as $t \to \infty$. Consequently, $L_t = f(\mathcal{M}_t)$ inherits the stationary distribution:

$$\pi_L(L) = \int_{\mathbb{R}^n} \delta(L - f(\mathcal{M}))\,\pi(\mathcal{M})\,d\mathcal{M},$$

where $\delta(\cdot)$ is the Dirac delta function. The moments of $\pi_L(L)$ determine the expected range and variability of L_t over time. For example, the expected barrier is:

$$\mathbb{E}[L_t] = \int_{\mathbb{R}^n} f(\mathcal{M})\, \pi(\mathcal{M})\, d\mathcal{M}.$$

The inventory ϕ_t evolves according to:

$$d\phi_t = -\Delta_t\, dt + Q_t^*\, dt,$$

where Q_t^* is the optimal control. Substituting Q_t^* from the two regimes, the inventory dynamics in the liquidity-constrained region ($Q_t^* \leq L_t$) become:

$$d\phi_t = -\Delta_t\, dt - \frac{\nabla_\phi V + c_1}{2c_2}\, dt.$$

In the liquidity-penalized region ($Q_t^* > L_t$):

$$d\phi_t = -\Delta_t\, dt + \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2 + \kappa)}\, dt.$$

Over time, ϕ_t converges to a stationary distribution if L_t and Δ_t fluctuate within bounded regions. Let $\psi(\phi)$ represent the stationary distribution of ϕ_t. Using Itô's lemma the steady-state probability density function satisfies:

$$0 = -\frac{\partial}{\partial \phi}\left(A(\phi)\psi(\phi)\right) + \frac{1}{2}\frac{\partial^2}{\partial \phi^2}\left(B(\phi)\psi(\phi)\right),$$

where $A(\phi)$ is the drift term and $B(\phi)$ is the diffusion term associated with the inventory dynamics.

The value function $V(X_t, t)$ captures the cost-to-go from the current state and satisfies the HJB equation:

$$\frac{\partial V}{\partial t} + \nabla_X V \cdot b(X_t, Q_t^*) + \frac{1}{2}\operatorname{Tr}\left(\Sigma_X^\top \nabla_X^2 V\, \Sigma_X\right) + \ell^*(Q_t^*, L_t) = 0.$$

Differentiate V with respect to L_t:

$$\frac{\partial V}{\partial L_t} = \frac{\partial \ell^*}{\partial L_t} + \nabla_X V \cdot \frac{\partial b}{\partial L_t} + \frac{1}{2}\operatorname{Tr}\left(\Sigma_X^\top \frac{\partial^2 V}{\partial X \partial L_t}\Sigma_X\right).$$

The first term, $\frac{\partial \ell^*}{\partial L_t}$, represents the direct sensitivity of the trading cost to L_t. The second term captures the indirect effect of L_t on the state dynamics, while the third term quantifies second-order interactions.

In the liquidity-penalized regime, where $Q_t^* > L_t$, the sensitivity simplifies to:

$$\frac{\partial V}{\partial L_t} = \frac{-2\kappa(Q_t^* - L_t)}{2(c_2 + \kappa)}.$$

As $\kappa \to \infty$, the sensitivity $\frac{\partial V}{\partial L_t} \to -(Q_t^* - L_t)$, indicating that the value function penalizes deviations from the liquidity barrier linearly in this limit.

The convergence of the inventory ϕ_t to the delta-neutral state $-\Delta_t$ is governed by the interaction between Q_t^*, L_t, and $\nabla_\phi V$. To quantify the asymptotic deviation from delta neutrality define the deviation:

$$\epsilon_t = \phi_t + \Delta_t.$$

The dynamics of ϵ_t are given by:

$$d\epsilon_t = Q_t^* \, dt - \frac{\partial \Delta_t}{\partial S} \, dS_t - \frac{\partial \Delta_t}{\partial t} \, dt.$$

Substituting Q_t^* from the optimal control, the steady-state expected deviation $\mathbb{E}[\epsilon_t]$ satisfies:

$$\mathbb{E}[\epsilon_t] = \int \left(Q_t^* - \frac{\partial \Delta_t}{\partial S} \mu S - \frac{\partial \Delta_t}{\partial t} \right) \psi(\phi) \, d\phi.$$

6 Inventory Dynamics

Again using Itô's lemma, the dynamics of L_t can be expressed as:

$$dL_t = \frac{\partial f}{\partial \mathcal{M}}^\top \mu_{\mathcal{M}}(\mathcal{M}_t) \, dt + \frac{1}{2} \operatorname{Tr} \left(\Sigma_{\mathcal{M}}^\top \frac{\partial^2 f}{\partial \mathcal{M}^2} \Sigma_{\mathcal{M}} \right) dt + \frac{\partial f}{\partial \mathcal{M}}^\top \Sigma_{\mathcal{M}}(\mathcal{M}_t) \, dW_t.$$

This decomposition highlights three sources of variability in L_t: the drift term (first-order sensitivity of f to $\mathcal{M}_t$), the diffusion term (stochastic fluctuations in $\mathcal{M}_t$), and the curvature term (second-order sensitivity of f to $\mathcal{M}_t$).

The variance of L_t at any time t is:

$$\operatorname{Var}(L_t) = \mathbb{E} \left[\left(\frac{\partial f}{\partial \mathcal{M}}^\top \Sigma_{\mathcal{M}} W_t \right)^2 \right].$$

If $\mathcal{M}_t$ reaches a stationary distribution $\pi(\mathcal{M})$, the long-term variance of L_t is determined by:

$$\operatorname{Var}(L) = \int_{\mathbb{R}^n} \left(\frac{\partial f}{\partial \mathcal{M}}^\top \Sigma_{\mathcal{M}} \Sigma_{\mathcal{M}}^\top \frac{\partial f}{\partial \mathcal{M}} \right) \pi(\mathcal{M}) \, d\mathcal{M}.$$

The trader's inventory ϕ_t evolves as:

$$d\phi_t = -\Delta_t \, dt + Q_t^* \, dt,$$

where Q_t^* depends explicitly on L_t. Substituting Q_t^* in the liquidity-penalized regime:

$$Q_t^* = \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2 + \kappa)},$$

the inventory dynamics become:

$$d\phi_t = -\Delta_t \, dt + \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2 + \kappa)} \, dt.$$

The joint dynamics of (ϕ_t, L_t) can be written as:

$$
\begin{aligned}
d\phi_t &= -\Delta_t \, dt + \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2 + \kappa)} \, dt, \\
dL_t &= \frac{\partial f}{\partial \mathcal{M}}^\top \mu_\mathcal{M}(\mathcal{M}_t) \, dt + \frac{1}{2} \operatorname{Tr}\left(\Sigma_\mathcal{M}^\top \frac{\partial^2 f}{\partial \mathcal{M}^2} \Sigma_\mathcal{M} \right) \, dt + \frac{\partial f}{\partial \mathcal{M}}^\top \Sigma_\mathcal{M}(\mathcal{M}_t) \, dW_t.
\end{aligned}
$$

Which form a coupled system that can be represented compactly as:

$$
d \begin{bmatrix} \phi_t \\ L_t \end{bmatrix} = \begin{bmatrix} -\Delta_t + \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2+\kappa)} \\ \frac{\partial f}{\partial \mathcal{M}}^\top \mu_\mathcal{M} + \frac{1}{2} \operatorname{Tr}\left(\Sigma_\mathcal{M}^\top \frac{\partial^2 f}{\partial \mathcal{M}^2} \Sigma_\mathcal{M} \right) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \frac{\partial f}{\partial \mathcal{M}}^\top \Sigma_\mathcal{M} \end{bmatrix} dW_t.
$$

In the stationary regime, where $\mathcal{M}_t \sim \pi(\mathcal{M})$, the expected drift of L_t is zero:

$$
\mathbb{E}\left[\frac{\partial f}{\partial \mathcal{M}}^\top \mu_\mathcal{M}(\mathcal{M}_t) \right] = 0.
$$

For ϕ_t stability requires that the effective drift in ϕ_t asymptotically dampens deviations from the delta-neutral target $-\Delta_t$. This can be formalized by examining the Lyapunov function $\mathcal{V}(\phi) = (\phi + \Delta_t)^2$, whose time derivative satisfies:

$$
\frac{d\mathcal{V}}{dt} = 2(\phi + \Delta_t) \left(-\Delta_t + \frac{-\nabla_\phi V - c_1 + 2\kappa L_t}{2(c_2 + \kappa)} \right).
$$

In the stationary regime of L_t, $\mathbb{E}\left[\frac{d\mathcal{V}}{dt}\right] < 0$ guarantees mean-reversion of ϕ_t toward $-\Delta_t$.