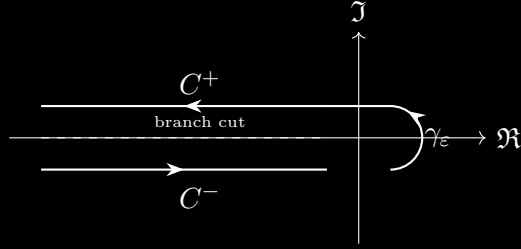


THE ANALYTIC CONTINUATION OF $\zeta(s)$

and the identity $\zeta(-1) = -\frac{1}{12}$



Step 1 : Hankel integral representation.

For $\text{Re}(s) > 1$, the Riemann zeta function admits the integral representation via the Hankel contour \mathcal{H} :

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \oint_{\mathcal{H}} \frac{(-z)^{s-1}}{e^z - 1} dz$$

where $(-z)^{s-1} = e^{(s-1)\log(-z)}$ with the branch cut along $\mathbb{R}_{>0}$.

Step 2 : Decomposition of the contour.

The Hankel contour decomposes as $\mathcal{H} = C^- \cup \gamma_\varepsilon \cup C^+$, giving:

$$\oint_{\mathcal{H}} = \int_{C^-} + \int_{\gamma_\varepsilon} + \int_{C^+}$$

As $\varepsilon \rightarrow 0$, for $\text{Re}(s) < 0$ the small circle contribution vanishes:

$$\int_{\gamma_\varepsilon} \frac{(-z)^{s-1}}{e^z - 1} dz \longrightarrow 0$$

Step 3 : Meromorphic continuation.

On C^\pm , setting $z = xe^{\pm i\pi}$ on the two branches:

$$\oint_{\mathcal{H}} \frac{(-z)^{s-1}}{e^z - 1} dz = (e^{i\pi(s-1)} - e^{-i\pi(s-1)}) \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

$$= 2i \sin(\pi(s-1)) \cdot \Gamma(s) \zeta(s)$$

Using $\sin(\pi(s-1)) = -\sin(\pi s)$ and the reflection formula $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, one obtains the *functional equation*:

$$\boxed{\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)}$$

This provides the analytic continuation to all $s \in \mathbb{C} \setminus \{1\}$.

Step 4 : Evaluation at $s = -1$.

Apply the functional equation at $s = -1$:

$$\zeta(-1) = 2^{-1} \pi^{-2} \sin\left(-\frac{\pi}{2}\right) \Gamma(2) \zeta(2)$$

We substitute the known values:

$$\sin\left(-\frac{\pi}{2}\right) = -1, \quad \Gamma(2) = 1! = 1, \quad \zeta(2) = \frac{\pi^2}{6}$$

Therefore:

$$\zeta(-1) = \frac{1}{2} \cdot \frac{1}{\pi^2} \cdot (-1) \cdot 1 \cdot \frac{\pi^2}{6}$$

$$\boxed{\zeta(-1) = -\frac{1}{12}}$$

*The sum $1 + 2 + 3 + \dots$ does not converge in \mathbb{R} .
Yet the universe of analytic continuation assigns it a finite value.
This is not magic — it is the geometry of \mathbb{C} .*