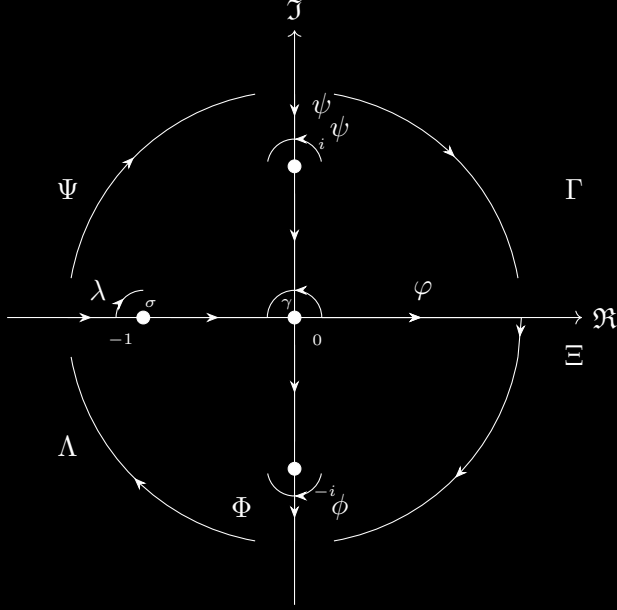


# A CONTOUR WITH FOUR SINGULARITIES

The integral of  $f(z) = \frac{1}{z(z^2 + 1)(z + 1)}$  over  $\mathbb{C}$



## Contour decomposition.

The full contour  $C$  decomposes as:

$$\oint_C = \int_{\gamma} + \int_{\xi} + \int_{\psi} + \int_{\varsigma} + \int_{\Psi} + \int_{\lambda} + \int_{\sigma} + \int_{\Lambda} + \int_{\phi} + \int_{\varphi} + \int_{\Phi} + \int_{\Xi} + \int_{\Gamma}$$

For  $\gamma, \varsigma, \sigma, \varphi, \Gamma \rightarrow 0$  (small indentation radii  $\varepsilon \rightarrow 0$ , large radius  $R \rightarrow \infty$ ):

$$\int_{\gamma} + \int_{\varsigma} + \int_{\sigma} + \int_{\varphi} + \int_{\Gamma} \rightarrow 0$$

Hence the contour reduces to:

$$\oint_C = \int_{\xi} + \int_{\psi} + \int_{\Psi} + \int_{\lambda} + \int_{\Lambda} + \int_{\phi} + \int_{\Phi} + \int_{\Xi}$$

Four singularities inside the contour, therefore by the Residue Theorem:

$$\int_{\xi} + \int_{\psi} + \int_{\Psi} + \int_{\lambda} + \int_{\Lambda} + \int_{\phi} + \int_{\Phi} + \int_{\Xi} = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$$

We isolate  $\int_{\xi}$ :

$$\int_{\xi} = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k) - \left( \int_{\psi} + \int_{\Psi} + \int_{\lambda} + \int_{\Lambda} + \int_{\phi} + \int_{\Phi} + \int_{\Xi} \right)$$

**Residue computation.** The poles of  $f(z) = \frac{1}{z(z^2 + 1)(z + 1)}$  are at  $z = 0, i, -i, -1$ :

$$\text{Res}(f, 0) = \frac{1}{(0^2 + 1)(0 + 1)} = 1$$

$$\text{Res}(f, i) = \frac{1}{i(2i)(i + 1)} = \frac{1}{i \cdot 2i \cdot (i + 1)} = \frac{1}{-2(i + 1)} = \frac{-1 + i}{4}$$

$$\text{Res}(f, -i) = \frac{1}{(-i)(-2i)(-i + 1)} = \frac{1}{-2(1 - i)} = \frac{-1 - i}{4}$$

$$\text{Res}(f, -1) = \frac{1}{(-1)((-1)^2 + 1)} = \frac{1}{-2} = -\frac{1}{2}$$

**Sum of residues:**

$$\sum \text{Res}(f, z_k) = 1 + \frac{-1 + i}{4} + \frac{-1 - i}{4} - \frac{1}{2} = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

$$\boxed{\oint_C \frac{dz}{z(z^2 + 1)(z + 1)} = 0}$$

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*Four singularities. Four residues. One contour.  
The sum is zero : the geometry of  $\mathbb{C}$  balances everything.*