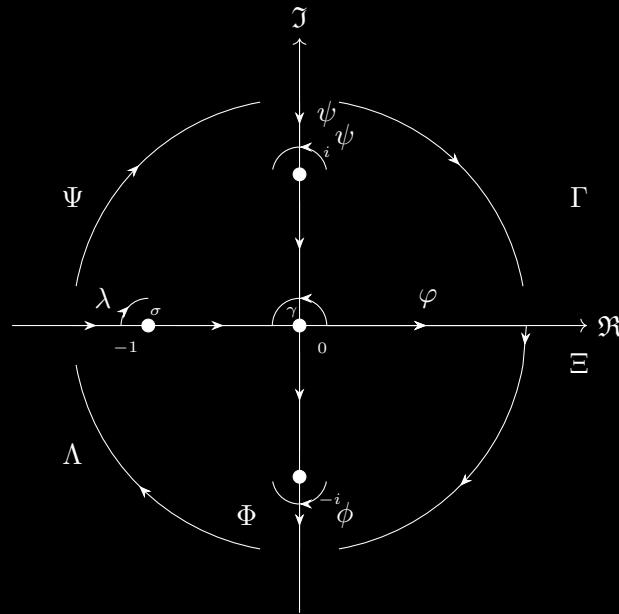


A CONTOUR WITH FOUR SINGULARITIES

The integral of $f(z) = \frac{1}{z(z^2+1)(z+1)}$ over \mathbb{C}



Contour decomposition.

The full contour C decomposes as:

$$\oint_C = \int_\gamma + \int_\xi + \int_\varsigma + \int_\psi + \int_\Psi + \int_\lambda + \int_\sigma + \int_\Lambda + \int_\phi + \int_\varphi + \int_\Phi + \int_\Xi + \int_\Gamma$$

For $\gamma, \varsigma, \sigma, \varphi, \Gamma \rightarrow 0$ (small indentation radii $\varepsilon \rightarrow 0$, large radius $R \rightarrow \infty$):

$$\int_\gamma + \int_\varsigma + \int_\sigma + \int_\varphi + \int_\Gamma \rightarrow 0$$

Hence the contour reduces to:

$$\oint_C = \int_\xi + \int_\psi + \int_\Psi + \int_\lambda + \int_\Lambda + \int_\phi + \int_\Phi + \int_\Xi$$

Four singularities inside the contour, therefore by the Residue Theorem:

$$\int_\xi + \int_\psi + \int_\Psi + \int_\lambda + \int_\Lambda + \int_\phi + \int_\Phi + \int_\Xi = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$$

We isolate \int_{ξ} :

$$\int_{\xi} = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k) - \left(\int_{\psi} + \int_{\Psi} + \int_{\lambda} + \int_{\Lambda} + \int_{\phi} + \int_{\Phi} + \int_{\Xi} \right)$$

Residue computation. The poles of $f(z) = \frac{1}{z(z^2+1)(z+1)}$ are at $z = 0, i, -i, -1$:

$$\begin{aligned} \text{Res}(f, 0) &= \frac{1}{(0^2+1)(0+1)} = 1 \\ \text{Res}(f, i) &= \frac{1}{i(2i)(i+1)} = \frac{1}{i \cdot 2i \cdot (i+1)} = \frac{1}{-2(i+1)} = \frac{-1+i}{4} \\ \text{Res}(f, -i) &= \frac{1}{(-i)(-2i)(-i+1)} = \frac{1}{-2(1-i)} = \frac{-1-i}{4} \\ \text{Res}(f, -1) &= \frac{1}{(-1)((-1)^2+1)} = \frac{1}{-2} = -\frac{1}{2} \end{aligned}$$

Sum of residues:

$$\sum \text{Res}(f, z_k) = 1 + \frac{-1+i}{4} + \frac{-1-i}{4} - \frac{1}{2} = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

$$\boxed{\oint_C \frac{dz}{z(z^2+1)(z+1)} = 0}$$

*Four singularities. Four residues. One contour.
The sum is zero : the geometry of \mathbb{C} balances everything.*