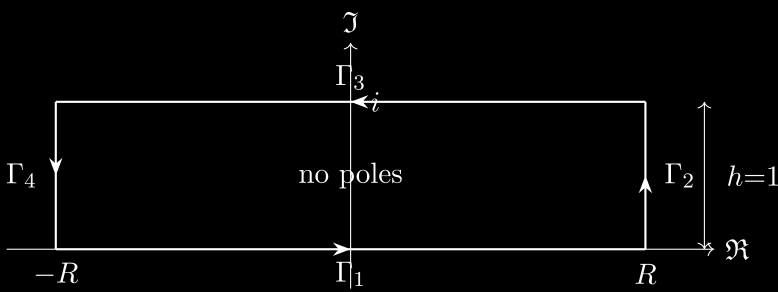
The Gaussian Integral via a Rectangular Contour

The identity $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$, seen through $\mathbb{C}$



Step 1 : Setup.

Consider the function $f(z) = e^{-z^2}$, which is entire (no poles anywhere in $\mathbb{C}$). Integrate over the rectangle $\mathcal{R}$ with vertices $\pm R$ and $\pm R + i$, height $h = 1$. By Cauchy's theorem, since f has no singularities inside $\mathcal{R}$:

$$\oint_{\mathcal{R}} e^{-z^2}\,dz = 0$$

Step 2 : Decomposition.

$$\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} = 0$$

Step 3 : Vertical edges vanish as $R \to \infty$.

On $\Gamma_2 :\ z = R + it,\ t \in [0,1]$, so $|e^{-z^2}| = e^{-(R^2 - t^2)} \le e^{-(R^2-1)} \to 0.$

By the same argument on Γ_4. Therefore:

$$\int_{\Gamma_2} e^{-z^2}\,dz \to 0 \qquad \text{and} \qquad \int_{\Gamma_4} e^{-z^2}\,dz \to 0$$

Step 4 : Bottom edge Γ_1.

On $\Gamma_1 : z = x \in \mathbb{R}$, x from $-R$ to R:

$$\int_{\Gamma_1} e^{-z^2}\,dz \xrightarrow{R\to\infty} \int_{-\infty}^{+\infty} e^{-x^2}\,dx =: I$$

Step 5 : Top edge Γ_3.

On $\Gamma_3 : z = x + i$, x from R to $-R$:

$$\int_{\Gamma_3} e^{-z^2}\,dz = -\int_{-\infty}^{+\infty} e^{-(x+i)^2}\,dx$$

Expand: $-(x+i)^2 = -x^2 + 1 - 2ix$, so:

$$\int_{\Gamma_3} e^{-z^2}\,dz = -e^1 \int_{-\infty}^{+\infty} e^{-x^2} e^{-2ix}\,dx = -e \cdot \hat{f}(2)$$

where $\hat{f}$ denotes the Fourier transform of e^{-x^2}.

Step 6 : Key observation.

From Step 2 and Steps 3, 4, 5, we get:

$$I - e \cdot \hat{f}(2) = 0 \implies \hat{f}(2) = \frac{I}{e}$$

But we also know independently that the Fourier transform of e^{-x^2} satisfies:

$$\int_{-\infty}^{+\infty} e^{-x^2} e^{-2i\xi x}\,dx = \sqrt{\pi}\, e^{-\xi^2}$$

At $\xi = 1 :\ \hat{f}(2) = \sqrt{\pi}\, e^{-1} = \dfrac{\sqrt{\pi}}{e}$.

Step 7 : Conclusion.

Substituting back:

$$\frac{I}{e} = \frac{\sqrt{\pi}}{e}$$

$$\boxed{\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

Every proof of this identity uses a different miracle.
The polar proof sees a circle. The contour proof sees a rectangle.
The answer is always the same.