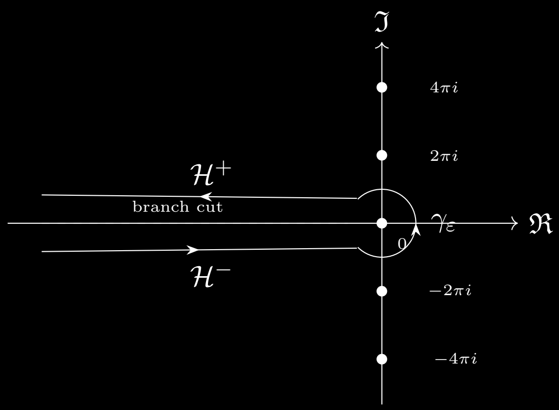
The Mellin Transform of the Bose-Einstein Kernel

The identity $\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \Gamma(s)\, \zeta(s)$



Step 1 : Geometric series expansion.

For $\mathrm{Re}(s) > 1$ and $x > 0$, we expand the kernel:

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}$$

Substituting into the integral and using Fubini (justified for $\mathrm{Re}(s) > 1$):

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \sum_{n=1}^{\infty} \int_0^\infty x^{s-1} e^{-nx}\, dx$$

Step 2 : Change of variable.

For each $n \geq 1$, set $u = nx$, so $x = u/n$, $dx = du/n$:

$$\int_0^\infty x^{s-1} e^{-nx}\, dx = \int_0^\infty \left(\frac{u}{n}\right)^{s-1} e^{-u} \frac{du}{n} = \frac{1}{n^s} \int_0^\infty u^{s-1} e^{-u}\, du = \frac{\Gamma(s)}{n^s}$$

Step 3 : Summing over n.

Summing back over all $n \geq 1$:

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\boxed{\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \Gamma(s)\,\zeta(s)}$$

Step 4 : Analytic continuation via Hankel contour.

The formula above holds for $\mathrm{Re}(s) > 1$. To extend to all $s \in \mathbb{C} \setminus \{1\}$, consider the Hankel contour $\mathcal{H} = \mathcal{H}^- \cup \gamma_\varepsilon \cup \mathcal{H}^+$ and the integral:

$$I(s) = \int_{\mathcal{H}} \frac{(-z)^{s-1}}{e^z - 1}\, dz$$

where $(-z)^{s-1} = e^{(s-1)\log(-z)}$ with the branch cut along $\mathbb{R}_{>0}$.

On $\mathcal{H}^- : z = xe^{-i\pi}$, $x > 0$, so $(-z)^{s-1} = x^{s-1}e^{-i\pi(s-1)}$.

On $\mathcal{H}^+ : z = xe^{+i\pi}$, $x > 0$, so $(-z)^{s-1} = x^{s-1}e^{+i\pi(s-1)}$.

Combining both branches as $\varepsilon \to 0$:

$$I(s) = \left(e^{i\pi(s-1)} - e^{-i\pi(s-1)}\right) \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = 2i\sin(\pi(s-1))\,\Gamma(s)\,\zeta(s)$$

Using $\sin(\pi(s-1)) = -\sin(\pi s)$ and the reflection formula $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$:

$$\Gamma(s)\,\zeta(s) = \frac{I(s)}{-2i\sin(\pi s)} = \frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{H}} \frac{(-z)^{s-1}}{e^z - 1}\, dz$$

This representation is entire except at $s = 1$, completing the analytic continuation of $\zeta(s)$.

A sum over integers. A product of two transcendental functions.
The Hankel contour reveals that Γ and ζ were always one.