Cavendish Experiment

Kadir Simsek

Deparment of Physics

Bogazici University, Istanbul, Turkey

Objective:

In this experiment, we will study radioactive decay through the way of employing the ionizing capability of the charged particles in the decay. We purpose to measure the half-life of the Radon gas.

INTRODUCTION

The **Cavendish experiment**, performed in 1797–98 by British scientist Henry Cavendish, was the first experiment to measure the force of gravity between masses in the laboratory, and the first to yield accurate values for the gravitational constant. Because of the unit conventions then in use, the gravitational constant does not appear explicitly in Cavendish's work. Instead, the result was originally expressed as the specific gravity of the Earth, or equivalently the mass of the Earth; and were the first accurate values for these geophysical constants. The experiment was devised sometime before 1783 by geologist John Michell, who constructed a torsion balance apparatus for it. However, Michell died in 1793 without completing the work, and after his death the apparatus passed to Francis John Hyde Wollaston and then to Henry Cavendish, who rebuilt the apparatus but kept close to Michell's original plan. Cavendish then carried out a series of measurements with the equipment, and reported his results in the *Philosophical Transactions of the Royal Society* in 1798.

EXPERIMENTAL SETUP

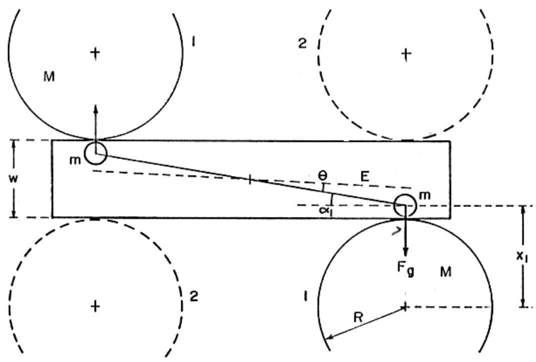

Figure 1: Schematics of Cavendish Apparatus

Apparatus:

- *Low Power Laser*
- *Scale*
- *Cavendish Torsion Balance with large masses*
- *Ruler*
- *A Stopwatch or a Chronometer*
-

The setup consists of two small masses arranged as a dumbbell and hung with a fiber in the form of a torsion balance. There are two more, stationary and heavier, masses placed on the opposite sides of the smaller ones. The system is at an equilibrium to start with. The heavier masses are placed on the opposite side of smaller masses with the help of a revolving fixture on which they are placed. In the new configuration there is a net torque acting on the small masses.

$$\tau = 2Fd$$

Where **F** is the gravitational force between the adjacent small and large masses.

$$F = G\frac{Mm}{r^2}$$

G is the universal gravitational constant which we are trying to determine. This net torque and the opposing torque produced by the torsion balance will result in a damped oscillation of the dumbbell. Hence, the motion of the dumbbell is determined by

$$I\frac{d^2\theta}{d\theta^2} + \kappa\theta = 0.$$

Where **I** is the moment of inertia of the dumbbell system and κ is the torsion constant of the fiber used. Observing the oscillations through the displacement of a laser beam which is reflected off the small mirror mounted in the middle of the dumbbell and falls on a scale at a distance of L, the universal gravitational constant can be determined as

$$G = \frac{\pi^2 b^2 dS}{MT^2 L},$$

where b is the distance between the adjacent small and large masses, M is the large mass, 2d is the length of the dumbbell, and S is the difference between the initial and the final equilibrium positions of the laser beam on the scale. T is the period of the oscillations determined by plotting the position of the laser beam on the scale as a function of time. It is the time difference between the two successive minima or maxima.

M	*d*	*b*
1.038 kg	*0.05m*	*0.0465 m*

In the derivation above, the influence of the other large mass is ignored. If this is not taken into account, there will be a systematic error in the measurement. However, since the existence of such a systematic effect is known, it should definitely be corrected for. Hence, the correction factor for the universal gravitational constant is given as

$$f_{corr} = \left[1 - \left(\frac{b}{\sqrt{b^2 + 4d^2}}\right)^3\right]^{-1}$$

Then the corrected expression for the universal gravitational constant is

$$G = f_{corr}G$$

DATA & ANALYSIS

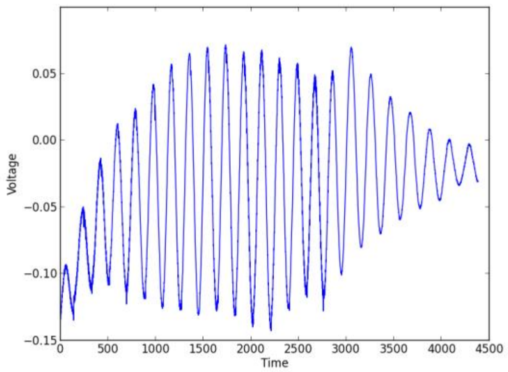

Figure 2: The oscillation Graph (Voltage vs Time)

The driven resonance method of determining G has the advantage that the experimental data can be collected in a short time since one does not have to wait for the oscillations of the balance to damp away. Measurements can begin at any time the balance reaches a turning point. The large balls are rotated back and forth between the two extreme positions so that the force of gravity between the large balls and the boom is always doing positive work on the balance, and the amplitude builds up until the energy loss from damping is equal to the work done by the gravitational force. Thus, determining G requires knowledge of the damping coefficient of the balance. This is most easily determined by measuring the amplitude decay as the balance is freely oscillating.

When freely oscillating, the angle of the boom as a function of time is given by

$$\theta(t) = \theta_e + Ae^{-bt}\cos(\omega_1 + \delta) \quad (1)$$

Where

θ_e = equilibrium angle of the balance,
A = oscillation amplitude at t = 0,
b^{-1} = time for the amplitude to decay to 1/e of the initial value,
ω_1 = oscillation frequency; $\omega_1^2 = \omega_o^2 - b^2$, $\omega_1 = 2\pi/T$;
ω_o = oscillation frequency in the absence of damping; $\omega_o^2 = K/I$,
T = oscillation period
K = torsion constant of the suspension fiber,
I = moment of inertia of the boom,
δ = phase of the oscillation at the time t = 0,

and where we have made the standard assumption that the damping torque is directly proportional to the angular velocity of the boom.

Since the large masses are rotated at turning points of the oscillation, it is convenient to define the zero of time to occur at a turning point. In this case, the phase δ is specified by the requirement $d\theta/dt=0$ at t=0, and Equation 1 can be rewritten as

$$\theta(t) = \theta_e + Ae^{-bt}\left[\cos(\omega_1 + \delta) + \frac{b}{\omega_1}\sin(\omega_1 t)\right] \quad (2)$$

In what follows, we will concentrate on the turning points of the motion.

Let t_n be the time of the nth turning point ($t_n=(n-1)T/2$, the first turning point occurs at t=0), and let θ_n be the boom angle at the nth turning point, $\theta_n = \theta(t_n)$. The initial amplitude A is then just $\theta_1-\theta_e$. From Equation 2 we find

$$\boldsymbol{\theta_n = \theta_e + (\theta_1 - \theta_e)(e^{-(n-1)bT/2})(-1)^{n-1}} \quad \textit{(3)}$$

since $\omega_1 t_n = (n-1)\pi$. The factor $e^{-bT/2}$ occurs so often in the formulas below that it is convenient to define a separate symbol for it; let's call it x ($x \equiv e^{-bT/2}$). With this definition, Equation 3 becomes

$$\boldsymbol{\theta}_n - \boldsymbol{\theta}_e = (-x)^{n-1}(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_e) \quad (4)$$

which can also be written in the form:

$$(\boldsymbol{\theta}_{n+1} - \boldsymbol{\theta}_e) = -x(\boldsymbol{\theta}_n - \boldsymbol{\theta}_e) \quad (5)$$

In free decay, x can be measured using any two adjacent turning points:

$$x = -(\theta_{n+1} - \theta_e)/(\theta_n - \theta_e). \quad (6)$$

One drawback to using Equation 6 to measure x is that it requires knowledge of the equilibrium angle, θ_e. By using three adjacent turning points, only differences in the turning point angles need be measured. Using Equation 5 twice, we find

$$\boldsymbol{x = -(\theta_{n+2} - \theta_{n+1})/(\theta_{n+1} - \theta_n).} \quad (7)$$

Equation 7 is a very useful method to determine x. To reduce the measurement error on x, more turning points can be measured. If an odd number N of adjacent turning points are measured, multiple use of Equation 5 gives

$$\boldsymbol{x = 1 - (\theta_1 - \theta_N)/(\theta_1 - \theta_2 + \theta_3 - \theta_4 + \ldots - \theta_{N-1}).} \quad (8)$$

NOTE: For mechanical oscillators in free decay, the positive and negative turning points can correspond to measurably different decay constants. Therefore, to improve accuracy in general, it is recommended that the results from Eq. 8 be averaged with the following:

$$\boldsymbol{x' = 1 - (\theta_2 - \theta_{N-1})/(\theta_2 - \theta_3 + \theta_4 - \theta_5 + \ldots - \theta_{N-2})} \quad (8a)$$

If the measurement error on each turning point is $\delta\theta$, the error on x can be shown to be

$$\delta x = \delta\theta\,(1-x)[(N-1)(1-x)^2 + 2x]^{1/2}/|\theta_1-\theta_N|. \qquad (9)$$

which has a broad minimum beginning around N=11 for x values typical of the balance.

Now let's consider the balance response to a resonant square-wave drive. We shall assume the gravitational torque exerted by the large masses on the boom when they are rotated from the center to the extreme positions is a constant, i.e. it is not appreciably changed by the small movements of the boom. With this assumption, the effect of the large masses is just to change the equilibrium angle of the boom. Before the drive is applied, the balance will either be at rest at the equilibrium angle θ_e, or else will be freely oscillating.

Let $\pm\theta_D$ be the change in the equilibrium angle when the large masses are rotated from the center position to either of the (symmetrically located) extreme positions. Suppose at time t=0 (a turning point if the balance is oscillating) the large masses are rotated to the extreme position where the new equilibrium angle is θ_e-θ_D. Then from Equation 2, the time dependence of the boom angle is

$$\theta(t) = (\theta_e-\theta_D) + (\theta_1-(\theta_e-\theta_D))\, e^{-bt}\,[\cos(\omega_1 t)+b/\omega_1 \sin(\omega_1 t)] \qquad (10)$$

where θ_1 is the angle of the boom at t=0 (the first turning point). The boom angle at the second turning point is

$$\theta_2 = (\theta_e-\theta_D) - x\,(\theta_1-(\theta_e-\theta_D)). \qquad (11)$$

At the second turning point (t=T/2) the large masses are quickly rotated so that the new balance equilibrium angle becomes θ_e+θ_D. Thus, the boom angle at the third turning point (t=T) is

$$\theta_3 = (\theta_e+\theta_D) - x\,(\theta_2-(\theta_e+\theta_D)). \qquad (12)$$

Each pair of adjacent turning points can be used to measure θ_D from which the gravitational constant G can be determined. It can be seen from Equations 11 and 12 that the solution for θ_D in terms of the two turning points θ_n and θ_{n+1} is

$$\theta_D = (-1)^n[(\theta_{n+1} -\theta_e)+x\,(\theta_n -\theta_e)]/(1+x). \qquad (13)$$

The equilibrium angle θ_e can be eliminated from the measurement process if the results of two measurements of θ_D using three adjacent turning points are averaged:

$$\theta_D = (-1)^n[x\,\theta_n + (1-x)\,\theta_{n+1} -\theta_{n+2}] / [2\,(1+x)]. \qquad (14)$$

To reduce errors, the results of an odd number N of adjacent turning points can be averaged:

$$\theta_D = [(1-x)(\theta_1-\theta_2+\theta_3-\theta_4+\dots\theta_N) -\theta_1+ x\,\theta_N] / [(N-1)(1+x)] \qquad (15)$$

where the error on θ_D is has contributions from the measurement error on the in
dividual turning points ($\delta\theta_{D_\theta}$) and the error on x ($\delta\theta_{D_x}$):

$$\delta\theta_D = [\, \delta\theta_{D_\theta}{}^2 + \delta\theta_{D_x}{}^2\,]^{1/2} \qquad (16a)$$

$$\delta\theta_{D_\theta} = \delta\theta\,[(N-1)(1-x)^2 + 2x]^{1/2}/[(N-1)(1+x)] \qquad (16b)$$

$$\delta\theta_{D_x} = \delta x\,[2(\theta_1-\theta_2+\theta_3-\dots-\theta_{N-1}) + (\theta_N-\theta_1)] / [(N-1)(1+x)^2]$$
(16c)

assuming the measurements errors on the individual turning points are uncorrelated and equal to $\delta\theta$.

The minimum time required to collect the data needed to determine G is very short. Using Equation 7 to determine x and Equation 14 to determine θ_D, only 3 adjacent turning points when the balance is being resonantly driven and

3 adjacent turning points when the balance is in free decay need be measured. Each set of 3 measurements require the balance to oscillate through one complete cycle which, depending on the length of the tungsten fiber, is about 4 minutes or less. If more accuracy is desired, any odd number N of adjacent data points can be used to determine x and θ_D using Equations 8 and 14.

Once θ_D has been measured, the gravitational constant G can be determined. The torque exerted by the gravitational force of the large balls on the boom is balanced by a restoring torque of the tungsten fiber when it is rotated by the angle θ_D.

$$\tau_n = 2\, G\, M\, m\, d / R^2 \qquad (17)$$

Equating the gravitational torque to the restoring torque of the fiber ($K\theta_D$) gives

$$G = K\, \theta_D\, R^2 / (2\, M\, m\, d). \qquad (18)$$

The torsion constant K can be determined by

$$K = (4\pi^2/T^2 + b^2)\, I. \qquad (19)$$

For this balance the b^2 term is very small and can be ignored. The moment of inertia is the sum of the moment of inertia of the two small spheres (I_s) plus the moment of inertia of the aluminium beam (I_b):

$$I_s = 2(m\, d^2 + 2/5\, mr^2) \quad I_b = m_b\, (l_b^2 + w_b^2)/12 \qquad (20)$$

where r = radius of the small sphere, and the aluminum beam is assumed to be a uniform rectangle rotated about its center with mass m_b, length l_b, and width w_b.

Datasets:

L	***T***	I_b	I_s	θ_d
2.18	*254*	*1.26x10^-5^*	*1.25x10^-5^*	*0.17*
K	f_{corr}	***I***	***G***	
*3.79x10^-7^*	*0.036*	*1.39x10^-5^*	*7.42x10^-11^*	

Other data is taken from the "Cavendish RP2111.doc" file.

Calculations are done in python code.

RESULTS

G_T =6,673 x $10^{-11} m^3 kg^{-1} s^{-2}$

We calculated G=4,6 $\times 10^{-11} m^3 kg^{-1} s^{-2}$

$$Error = \frac{G_T - G_E}{G_T} \times 100 = \frac{7.42 \times 10^{-11} - 6.67 \times 10^{-11}}{6{,}67 \times 10^{-11}} = 11\%$$

The error can be derived from too many reasons, first of all our masses, little vibrations caused from wind, temperature, etc. To minimize the error we need an isolated system.

REFERENCES

- E. Gulmez, ”Advanced Physics Experiment”, Istanbul Bogazici University Publication, 1999
- “Cavendish RP2111.doc”