

1 Feynman Path Integral.

The version of quantum mechanics:

1. Schrödinger's wavefunction (operator form):

$$\hat{H}|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle \quad (1)$$

2. Feynman's Path Integral (Common number form):

$$iG(\text{Green's function}) \propto \int \mathcal{D}(x, t) e^{i \int \mathcal{L} dt} \quad (2)$$

There are many advantages of Feynman Path Integral:

1. Make the double-slit experiment more understandable.
2. The classical limit " $\hbar \rightarrow 0$ " is "tractable": quantum $\xrightarrow{\hbar \rightarrow 0}$ classical.
3. Provide a semi-classical picture. for. quantum mechanics.
4. "Quantum fluctuations" are more "understandable".
5. A natural route. to low energy effective theory of quantum many-body systems.
6. A natural language for describing topological properties of quantum many-body systems.

But the practical calculation in the path-integral representation of simple quantum mechanical problem many be notoriously difficult and lengthy.

1.1 Propagators.

Consider a quantum particle confined in a one-dimensional space:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad (3)$$

and the canonical pair: $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$

The Schrödinger's equation is:

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle \quad (4)$$

This first-order nature allows us to define a time evolution operator $\hat{U}(t, t_0)$ which propagates the state vector from an initial time t_0 to a final time t :

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \quad (5)$$

Assuming the Hamiltonian H is not explicitly depend on time, the formal solution of \hat{U} is:

$$\hat{U} = e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} \quad (6)$$

A crucial property of \hat{U} is the “chain-like” rule, or composition property. For any intermediate time t' such that $t > t' > t_0$:

$$\hat{U}(t, t_0) = \hat{U}(t, t')\hat{U}(t', t_0) \quad (7)$$

This property is the key to the entire path integral derivation. And $\hat{U}(t, t_0)$ is unitary:

$$\hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = \hat{\mathbb{I}} \quad (8)$$

where $\hat{\mathbb{I}}$ is the identity operator.

In the position representation, we can obtain matrix elements:

$$\begin{aligned} U(x, t; x_0, t_0) &= \langle x | \hat{U}(t, t_0) | x_0 \rangle \\ &= \langle x | e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} | x_0 \rangle \end{aligned} \quad (9)$$

We can define a propagators (Green’s function) of the quantum system by using the matrix elements:

$$iG(x, t; x_0, t_0) = U(x, t; x_0, t_0) \quad (10)$$

Using the matrix elements, $\psi(x, t)$ can be reformulated as:

$$\begin{aligned} \psi(x, t) &= \langle x | \hat{U}(t, t_0) | \psi(t_0) \rangle \\ &= \int dx_0 \langle x | \hat{U}(t, t_0) | x_0 \rangle \langle x_0 | \psi(t_0) \rangle \\ &= \int dx_0 U(x, t; x_0, t_0) \psi(x_0, t_0) \end{aligned} \quad (11)$$

Also, the propagator also satisfies the Schrödinger’s equation:

$$i\hbar \frac{\partial}{\partial t} G(x, t; x_0, t_0) = \hat{H} G(x, t; x_0, t_0) \quad (12)$$

And the initial condition is:

$$G(x, t_0; x_0, t_0) = -i \langle x | \hat{U}(t_0, t_0) | x_0 \rangle = -i \delta(x - x_0) \quad (13)$$

$\delta(x - x_0)$ is Dirac function.

Example 1.1. For free particle: $\hat{H} = \frac{1}{2m} \hat{p}^2$, in position representation:

$$\hat{p} \rightarrow -i\hbar \frac{\partial}{\partial x} \quad (14)$$

So the PDE is:

$$i\hbar \frac{\partial}{\partial t} G(x, t; x_0, t_0) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} G(x, t; x_0, t_0) \quad (15)$$

Solve the PDE:

Use Fourier Transform: (we use $G(x, t)$ instead of $G(x, t; x_0, t_0)$). We solve the free-particle Green's function by transforming to momentum space:

$$\begin{cases} G(x, t) = \frac{1}{\sqrt{2\pi}} \int dk \cdot \tilde{G}(k, t) e^{ikx} \\ \tilde{G}(k, t) = \frac{1}{\sqrt{2\pi}} \int dx G(x, t) e^{-ikx} \end{cases} \quad (16)$$

With these conventions, spatial derivatives become algebraic in k -space while the time derivative remains unchanged:

$$\begin{cases} \mathcal{F} \left\{ i\hbar \frac{\partial G(x, t)}{\partial t} \right\} = i\hbar \frac{\partial \tilde{G}(k, t)}{\partial t} \\ \mathcal{F} \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2 G}{\partial x^2} \right\} = -\frac{\hbar^2}{2m} [-k^2 \tilde{G}(k, t)] = \frac{\hbar^2 k^2}{2m} \tilde{G}(k, t) \end{cases} \quad (17)$$

Applying the transform to the PDE yields an ordinary differential equation in time for each k :

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \tilde{G}(k, t) = \frac{\hbar^2 k^2}{2m} \tilde{G}(k, t) \\ \frac{d\tilde{G}}{\tilde{G}} = -i \frac{\hbar k^2}{2m} dt \end{cases} \quad (18)$$

Integrating in time gives the logarithm of the solution up to a k -dependent constant:

$$\ln \tilde{G} = -i \frac{\hbar k^2}{2m} t + C(k) \quad (19)$$

So we can get the solution:

$$\tilde{G}(k, t) = A(k) e^{-i \frac{\hbar k^2}{2m} t}, \quad A(k) = e^{C(k)} \quad (20)$$

To determine $A(k)$, impose the initial condition at time t_0 in position space:

$$\begin{aligned} \tilde{G}(k, t_0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx G(x, t_0) e^{-ikx} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \delta(x - x_0) e^{-ikx} \end{aligned} \quad (21)$$

Using the Fourier transform of the Dirac delta, we find:

$$\tilde{G}(k, t_0) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0} \quad (22)$$

Matching at t_0 fixes the k -space amplitude:

$$A(k) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0} \cdot e^{i \frac{\hbar k^2}{2m} t_0}. \quad (23)$$

Therefore, for general time t we have:

$$\tilde{G}(k, t) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0} e^{-i\frac{\hbar k^2}{2m}(t-t_0)} \quad (24)$$

Finally, inverse-transform back to position space to obtain the integral representation of the propagator:

$$\begin{aligned} G(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \cdot \tilde{G}(k, t) e^{ikx} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x_0)} \cdot e^{-i\frac{\hbar(t-t_0)}{2m} k^2} \end{aligned} \quad (25)$$

This is a standard Gaussian integral of the form:

$$\int_{-\infty}^{\infty} dk e^{-ak^2+bk} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}. \quad (26)$$

Let's identify the coefficients:

$$\begin{cases} a = i\frac{\hbar(t-t_0)}{2m} \\ b = i(x-x_0) \end{cases} \quad (27)$$

So we can get the solution:

$$iG(x, t) = \left[\frac{m}{2\pi\hbar i(t-t_0)} \right]^{\frac{1}{2}} \cdot e^{i\frac{m(x-x_0)^2}{2\hbar(t-t_0)}} \quad (28)$$

Also, we can solve this PDE via definition:

$$\begin{aligned} iG &= \langle x | e^{-\frac{i}{\hbar}(t-t_0)\hat{H}} | x_0 \rangle \\ &= \int \langle x | e^{-\frac{i}{\hbar}(t-t_0)\hat{H}} | p \rangle \langle p | x_0 \rangle dp \\ &= \int dp e^{-\frac{i(t-t_0)p^2}{2m\hbar}} \langle x | p \rangle \langle p | x_0 \rangle \\ &= \frac{1}{2\pi\hbar} \int dp e^{-\frac{i(t-t_0)p^2}{2m\hbar} + i\frac{(x-x_0)}{\hbar}p}. \end{aligned} \quad (29)$$

we use $P = \hbar k$ and can get the same equation as the Fourier Transform Method.

1.2 Path-Integral

When $t > t_1 > t_0$, and t_1 is an arbitrarily selected intermediate time, we can write:

$$\begin{aligned}
 iG(x, t; x_0, t_0) &= \langle x | \hat{U}(t, t_0) | x_0 \rangle \\
 &= \langle x | \hat{U}(t, t_1) \hat{U}(t_1, t_0) | x_0 \rangle \\
 &= \int dx_1 \langle x | \hat{U}(t, t_1) | x_1 \rangle \langle x_1 | \hat{U}(t_1, t_0) | x_0 \rangle \\
 &= \int dx_1 iG(x, t; x_1, t_1) \cdot iG(x_1, t_1; x_0, t_0)
 \end{aligned} \tag{30}$$

This integral over x_1 means “superposition” of all possible “path” that connect x and x_0 . Next, we try to “smooth” the path along time directly. We can insert more time slices between x and x_0 . If we insert infinite time slices, the path become smooth.

Firstly, let's discretize time. domain $[t_0, t]$ into N pieces of equal length $\Delta t = \frac{t-t_0}{N}$:

$$\begin{aligned}
 iG(x, t; x_0, t_0) &= \langle x | \hat{U}(t, t_{N-1}) \hat{U}(t_{N-1}, t_{N-2}) \cdots \hat{U}(t_1, t_0) | x_0 \rangle \\
 &= \int dx_{N-1} \cdots dx_1 \prod_{l=1}^N iG(x_l, t_l; x_{l-1}, t_{l-1})
 \end{aligned} \tag{31}$$

let $\mathcal{D}_x = \prod_{l=1}^{N-1} dx_l$. Consider $N \rightarrow \infty$, so $\Delta t = \frac{t-t_0}{N} \rightarrow 0$, which means $t_l - t_{l-1} = \Delta t$.

$$iG(x_l, t_l; x_{l-1}, t_{l-1}) = \langle x_l | e^{-\frac{i}{\hbar} \hat{H} \Delta t} | x_{l-1} \rangle \tag{32}$$

Because Δt is small, we can approximate the exponential function by its Taylor series:

$$e^{-\frac{i}{\hbar} \hat{H} \Delta t} \approx \hat{\mathbb{I}} - \frac{i}{\hbar} \hat{H} \Delta t = \hat{\mathbb{I}} - \frac{i}{\hbar} \Delta t \left[\frac{\hat{p}^2}{2m} + V(\hat{x}) \right] \tag{33}$$

Substitute (33) into (32):

$$\begin{aligned}
 iG(x_l, t_l; x_{l-1}, t_{l-1}) &= \int dp_l \langle x_l | p_l \rangle \langle p_l | \hat{\mathbb{I}} - \frac{i}{\hbar} \Delta t \left[\frac{\hat{p}^2}{2m} + V(\hat{x}) \right] | x_{l-1} \rangle \\
 &= \int dp_l \langle x_l | p_l \rangle \langle p_l | x_{l-1} \rangle \left[1 - \frac{i}{\hbar} \left(\frac{p_l^2}{2m} + V(x_{l-1}) \right) \Delta t \right]
 \end{aligned} \tag{34}$$

With the approximations $V(x_l) \approx V(x_{l-1})$:

$$\left[1 - \frac{i}{\hbar} \left(\frac{p_l^2}{2m} + V(x_{l-1}) \right) \Delta t \right] \approx \left(1 - \frac{i}{\hbar} H_l \Delta t \right) \approx e^{-\frac{i}{\hbar} H_l \Delta t} \tag{35}$$

So $iG(x_l, t_l; x_{l-1}, t_{l-1})$ can be written as:

$$\begin{aligned}
 iG(x_l, t_l; x_{l-1}, t_{l-1}) &= \int dp_l \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar} p_l (x_l - x_{l-1})} e^{-\frac{i}{\hbar} H_l \Delta t} \\
 &= \frac{1}{2\pi\hbar} \int dp_l e^{\frac{i}{\hbar} [p_l (x_l - x_{l-1}) - H_l \Delta t]} \\
 &= \frac{1}{2\pi\hbar} \int dp_l e^{\frac{i}{\hbar} [p_l (\frac{x_l - x_{l-1}}{\Delta t}) - H_l] \Delta t}
 \end{aligned} \tag{36}$$

where, H_l is the classical Hamiltonian as a function of p_l and x_l .

When $\Delta t \rightarrow 0$:

$$\frac{x_l - x_{l-1}}{\Delta t} = \dot{x}_l \quad (37)$$

So we can get:

$$p_l \dot{x}_l - H_l = \mathcal{L}_l. \quad (38)$$

where, \mathcal{L}_l is the classical Lagrangian.

So $iG(x_l, t_l; x_{l-1}, t_{l-1})$ can be written as the form with Lagrangian:

$$iG(x_l, t_l; x_{l-1}, t_{l-1}) = \int dp_l \cdot \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar} \mathcal{L}_l \Delta t}. \quad (39)$$

Substitute $iG(x_l, t_l; x_{l-1}, t_{l-1})$ into the path integral:

$$\prod_{l=1}^N iG(x_l, t_l; x_{l-1}, t_{l-1}) = \int \frac{dp_N}{2\pi\hbar} \dots \frac{dp_1}{2\pi\hbar} \cdot e^{\frac{i}{\hbar} \sum_{l=1}^N \mathcal{L}_l \Delta t} \quad (40)$$

let $\mathcal{D}_p = \prod_{l=1}^N \frac{dp_l}{2\pi\hbar}$, when $\Delta t \rightarrow 0$, which means:

$$\sum_{l=1}^N \mathcal{L}_l \Delta t = \int_{t_0}^t d\tau \cdot \mathcal{L}[p(\tau), x(\tau)] \quad (41)$$

Finally, we can get the propagators by the path integral:

Theorem 1.1. *The propagators path integral:*

$$iG(x, t; x_0, t_0) = \int \mathcal{D}_x \mathcal{D}_p \cdot e^{\frac{i}{\hbar} \int_{t_0}^t d\tau \cdot \mathcal{L}[p(\tau), x(\tau)]} \quad (42)$$

where, the pair of $p(t)$ and $\dot{x}(t)$ characterizes a path in the px phase space.

2 Quantum Spin

We begin by considering the Hilbert space \mathcal{H} for a single quantum spin-1/2 particle. This is a two-dimensional complex vector space.

The conventional approach is to use an orthonormal basis formed by the eigenvectors of the spin operator along a chosen axis, typically the z -axis, denoted \hat{S}_z .

$$\hat{S}_z |\uparrow\rangle = +\frac{\hbar}{2} |\uparrow\rangle \quad \text{and} \quad \hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle \quad (43)$$

Here, $|\uparrow\rangle$ represents the "spin up" state and $|\downarrow\rangle$ represents the "spin down" state. These two states form a complete orthonormal basis, satisfying:

- **Orthogonality:** $\langle\uparrow|\downarrow\rangle = 0$
- **Normalization:** $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$
- **Completeness:** $|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \hat{\mathbb{I}}$

where $\hat{\mathbb{I}}$ is the identity operator in \mathcal{H} .

2.1 Spin Coherent States

A general, normalized state $|\psi\rangle$ in the spin-1/2 Hilbert space can be written as a complex linear combination of the basis states:

$$|\psi\rangle = z_1 |\uparrow\rangle + z_2 |\downarrow\rangle \quad (44)$$

where $z_1, z_2 \in \mathbb{C}$ are complex coefficients.

2.1.1 Degrees of Freedom and Normalization

The normalization condition $\langle\psi|\psi\rangle = 1$ imposes a constraint on these coefficients:

$$\langle\psi|\psi\rangle = (|z_1|^2 + |z_2|^2) = 1 \quad (45)$$

A complex number $z = x + iy$ has two real parameters. Therefore, the pair (z_1, z_2) is defined by four real parameters. The normalization condition $|z_1|^2 + |z_2|^2 = 1$ removes one degree of freedom, leaving three.

Furthermore, in quantum mechanics, the overall phase of a state vector is unphysical. The states $|\psi\rangle$ and $e^{i\gamma}|\psi\rangle$ (for any real γ) represent the same physical state (i.e., they belong to the same ray in Hilbert space). This "gauge freedom" removes one more degree of freedom.

This leaves $4 - 1 - 1 = 2$ real, physical degrees of freedom. This is a crucial observation: the state space of a spin-1/2 particle is topologically equivalent to the surface of a 2D sphere, which is also parameterized by two angles (like latitude and longitude).

2.1.2 Parametrization

We can explicitly parameterize z_1 and z_2 using two angles, θ and ϕ , which will map directly to the surface of a sphere. A standard (but not unique) parametrization for the spin coherent state, labeled by a unit vector \mathbf{n} , is:

$$|\mathbf{n}\rangle \equiv |\theta, \phi\rangle = \cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} |\uparrow\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi/2} |\downarrow\rangle \quad (46)$$

Here, the spherical coordinate angles have the domains $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. We can easily verify that this state is normalized:

$$\langle \mathbf{n} | \mathbf{n} \rangle = \left| \cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} \right|^2 + \left| \sin\left(\frac{\theta}{2}\right) e^{i\phi/2} \right|^2 = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1 \quad (47)$$

This set of states $\{|\mathbf{n}\rangle\}$ is continuously parameterized by the angles (θ, ϕ) , addressing the first drawback of the discrete basis.

2.2 Physical Interpretation: The Bloch Sphere

To understand the physical meaning of θ and ϕ , we compute the expectation value of the vector spin operator $\hat{\mathbf{S}}$ in the state $|\mathbf{n}\rangle$. We will set $\hbar = 1$ from here on for simplicity. The spin operator is $\hat{\mathbf{S}} = \frac{1}{2} \hat{\boldsymbol{\sigma}}$, where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (48)$$

In the $|\uparrow\rangle, |\downarrow\rangle$ basis, $|\mathbf{n}\rangle$ is represented by the column vector:

$$|\mathbf{n}\rangle = \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix} \implies \langle \mathbf{n} | = \begin{pmatrix} \cos(\theta/2) e^{i\phi/2} & \sin(\theta/2) e^{-i\phi/2} \end{pmatrix} \quad (49)$$

Expectation value of \hat{S}_z :

$$\begin{aligned} \langle \hat{S}_z \rangle &= \langle \mathbf{n} | \left(\frac{1}{2} \hat{\sigma}_z \right) | \mathbf{n} \rangle = \frac{1}{2} \begin{pmatrix} \cos(\theta/2) e^{i\phi/2} & \sin(\theta/2) e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix} \\ &= \frac{1}{2} \left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \right) = \frac{1}{2} \cos(\theta) \end{aligned} \quad (50)$$

Expectation value of \hat{S}_x :

$$\begin{aligned} \langle \hat{S}_x \rangle &= \langle \mathbf{n} | \left(\frac{1}{2} \hat{\sigma}_x \right) | \mathbf{n} \rangle = \frac{1}{2} \begin{pmatrix} \dots \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dots \end{pmatrix} \\ &= \frac{1}{2} \left(\cos(\theta/2) \sin(\theta/2) e^{i\phi/2} e^{i\phi/2} + \sin(\theta/2) \cos(\theta/2) e^{-i\phi/2} e^{-i\phi/2} \right) \\ &= \frac{1}{2} \cos(\theta/2) \sin(\theta/2) \left(e^{i\phi} + e^{-i\phi} \right) = \left(\frac{1}{2} \sin \theta \right) \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right) = \frac{1}{2} \sin \theta \cos \phi \end{aligned} \quad (51)$$

Expectation value of \hat{S}_y :

$$\begin{aligned}
 \langle \hat{S}_y \rangle &= \langle \mathbf{n} | \left(\frac{1}{2} \hat{\sigma}_y \right) | \mathbf{n} \rangle = \frac{1}{2} \begin{pmatrix} \dots \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \dots \end{pmatrix} \\
 &= \frac{1}{2} \left(\cos(\theta/2)(-i) \sin(\theta/2) e^{i\phi/2} e^{i\phi/2} + \sin(\theta/2)(i) \cos(\theta/2) e^{-i\phi/2} e^{-i\phi/2} \right) \quad (52) \\
 &= \frac{1}{2} \cos(\theta/2) \sin(\theta/2) \left(-ie^{i\phi} + ie^{-i\phi} \right) = \left(\frac{1}{2} \sin \theta \right) \left(\frac{e^{i\phi} - e^{-i\phi}}{2i} \right) = \frac{1}{2} \sin \theta \sin \phi
 \end{aligned}$$

2.2.1 Conclusion: The Classical Spin Vector

Combining these results, the expectation value of the spin vector is:

$$\langle \hat{\mathbf{S}} \rangle = \frac{1}{2} (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (53)$$

This is a vector of length $S = 1/2$ pointing in the direction specified by the unit vector \mathbf{n} :

$$\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (54)$$

Thus, the state $|\mathbf{n}\rangle$ is the quantum state that "points" in the classical direction \mathbf{n} . This direction vector lives on the surface of a unit sphere, known as the **Bloch Sphere**.

This formalism treats all directions \mathbf{n} on an equal footing, making the SU(2) rotational symmetry manifest. This addresses the second drawback of the discrete basis.

2.2.2 Coherent State as an Eigenvector

The formula presented in the original notes, $\langle \mathbf{n} | \hat{\mathbf{S}} \cdot \mathbf{n} | \mathbf{n} \rangle = \frac{1}{2} |\mathbf{n}\rangle$, appears to contain a typographical error, as an expectation value (a scalar) cannot be equal to a state vector.

The more fundamental property, which is likely intended, is the eigenvector equation:

$$(\hat{\mathbf{S}} \cdot \mathbf{n}) |\mathbf{n}\rangle = \frac{1}{2} |\mathbf{n}\rangle \quad (55)$$

This equation signifies that the coherent state $|\mathbf{n}\rangle$ is, by definition, the "spin up" eigenvector of the spin operator projected along its own pointing direction \mathbf{n} , with the eigenvalue $+1/2$ (with $\hbar = 1$).

Proof: We first construct the operator $\hat{\mathbf{S}} \cdot \mathbf{n}$ in matrix form:

$$\begin{aligned}
 \hat{\mathbf{S}} \cdot \mathbf{n} &= \hat{S}_x n_x + \hat{S}_y n_y + \hat{S}_z n_z \\
 &= \frac{1}{2} (\hat{\sigma}_x n_x + \hat{\sigma}_y n_y + \hat{\sigma}_z n_z) \\
 &= \frac{1}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin \theta \cos \phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin \theta \sin \phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos \theta \right] \\
 &= \frac{1}{2} \begin{pmatrix} \cos \theta & \sin \theta (\cos \phi - i \sin \phi) \\ \sin \theta (\cos \phi + i \sin \phi) & -\cos \theta \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}
 \end{aligned} \tag{56}$$

Now, we apply this operator to the coherent state vector $|\mathbf{n}\rangle$:

$$(\hat{\mathbf{S}} \cdot \mathbf{n})|\mathbf{n}\rangle = \frac{1}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix} \tag{57}$$

We compute the top and bottom components of the resulting vector separately.

Top component:

$$\begin{aligned}
 &\frac{1}{2} \left[\cos \theta \cos(\theta/2) e^{-i\phi/2} + \sin \theta e^{-i\phi} \sin(\theta/2) e^{i\phi/2} \right] \\
 &= \frac{1}{2} e^{-i\phi/2} [\cos \theta \cos(\theta/2) + \sin \theta \sin(\theta/2)] \\
 &= \frac{1}{2} e^{-i\phi/2} [\cos(\theta - \theta/2)] \quad (\text{using } \cos(A - B) \text{ identity}) \\
 &= \frac{1}{2} \cos(\theta/2) e^{-i\phi/2}
 \end{aligned} \tag{58}$$

This is precisely $\frac{1}{2}$ times the top component of $|\mathbf{n}\rangle$.

Bottom component:

$$\begin{aligned}
 &\frac{1}{2} \left[\sin \theta e^{i\phi} \cos(\theta/2) e^{-i\phi/2} - \cos \theta \sin(\theta/2) e^{i\phi/2} \right] \\
 &= \frac{1}{2} e^{i\phi/2} [\sin \theta \cos(\theta/2) - \cos \theta \sin(\theta/2)] \\
 &= \frac{1}{2} e^{i\phi/2} [\sin(\theta - \theta/2)] \quad (\text{using } \sin(A - B) \text{ identity}) \\
 &= \frac{1}{2} \sin(\theta/2) e^{i\phi/2}
 \end{aligned} \tag{59}$$

This is precisely $\frac{1}{2}$ times the bottom component of $|\mathbf{n}\rangle$.

Combining both components, we have shown:

$$(\hat{\mathbf{S}} \cdot \mathbf{n})|\mathbf{n}\rangle = \frac{1}{2} \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix} = \frac{1}{2} |\mathbf{n}\rangle \tag{60}$$

This completes the proof. The expectation value $\langle \hat{\mathbf{S}} \cdot \mathbf{n} \rangle = \langle \mathbf{n} | (\hat{\mathbf{S}} \cdot \mathbf{n}) | \mathbf{n} \rangle = \langle \mathbf{n} | (\frac{1}{2} |\mathbf{n}\rangle) = \frac{1}{2} \langle \mathbf{n} | \mathbf{n} \rangle = \frac{1}{2}$ follows directly.

2.3 Gauge Choice and Topological Singularities

The parametrization in Eq. (46) is not unique, and it hides a subtle topological problem.

- **At the North Pole** ($\theta = 0$): The direction \mathbf{n} is $(0, 0, 1)$. The angle ϕ is ill-defined. Our formula gives $|\theta = 0\rangle = \cos(0)e^{-i\phi/2}|\uparrow\rangle + \sin(0)\cdots = e^{-i\phi/2}|\uparrow\rangle$. The state vector itself depends on the meaningless angle ϕ . This is a **singularity**.
- **At the South Pole** ($\theta = \pi$): The direction \mathbf{n} is $(0, 0, -1)$. Our formula gives $|\theta = \pi\rangle = \cos(\pi/2)\cdots + \sin(\pi/2)e^{i\phi/2}|\downarrow\rangle = e^{i\phi/2}|\downarrow\rangle$. This is also singular.

This is analogous to the problem of creating a flat map of the Earth: you cannot do so without singularities (e.g., at the poles) or cuts.

We can "fix" the singularity at one pole by making a ϕ -dependent gauge choice (i.e., multiplying by an overall phase $e^{i\gamma(\phi)}$).

Choice 1: Regular at North Pole. Let's choose an overall phase $\gamma = \phi/2$. The new state, $|\mathbf{n}\rangle_N$, is:

$$|\mathbf{n}\rangle_N = e^{i\phi/2}|\mathbf{n}\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|\downarrow\rangle \quad (61)$$

- At the North Pole ($\theta = 0$): $|\mathbf{n}\rangle_N = \cos(0)|\uparrow\rangle + \sin(0)\cdots = |\uparrow\rangle$. This is now regular and well-defined.
- At the South Pole ($\theta = \pi$): $|\mathbf{n}\rangle_N = \cos(\pi/2)|\uparrow\rangle + \sin(\pi/2)e^{i\phi}|\downarrow\rangle = e^{i\phi}|\downarrow\rangle$. The singularity has been "pushed" to the South Pole.

Choice 2: Regular at South Pole. Let's choose $\gamma = -\phi/2$. The new state, $|\mathbf{n}\rangle_S$, is:

$$|\mathbf{n}\rangle_S = e^{-i\phi/2}|\mathbf{n}\rangle = \cos\left(\frac{\theta}{2}\right)e^{-i\phi}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle \quad (62)$$

This state is regular at the South Pole ($|\mathbf{n}\rangle_S = |\downarrow\rangle$) but singular at the North Pole.

This unavoidable singularity is topological in nature and is the origin of the **Berry Phase**, or the "topological term," in the coherent-state path integral.

2.4 Over-Completeness and Orthogonality

The set of all coherent states $\{|\mathbf{n}\rangle\}$ for all \mathbf{n} on the sphere is an **over-complete** basis. The Hilbert space is only 2-dimensional, but we have an infinite, continuous set of states. This means the states are not, in general, orthogonal.

$$\langle \mathbf{n}' | \mathbf{n} \rangle \neq 0 \quad \text{for } \mathbf{n}' \neq \mathbf{n} \text{ and } \mathbf{n}' \neq -\mathbf{n} \quad (63)$$

A special exception, as noted in the text, is for antipodal states.

2.4.1 Orthogonality of Antipodal States

Let us prove that $\langle -\mathbf{n} | \mathbf{n} \rangle = 0$. The antipodal point $-\mathbf{n}$ corresponds to the angles $(\theta', \phi') = (\pi - \theta, \phi + \pi)$.

We write the state $| - \mathbf{n} \rangle$ using Eq. (46):

$$\begin{aligned} | - \mathbf{n} \rangle &= \cos\left(\frac{\pi - \theta}{2}\right) e^{-i(\phi + \pi)/2} | \uparrow \rangle + \sin\left(\frac{\pi - \theta}{2}\right) e^{i(\phi + \pi)/2} | \downarrow \rangle \\ &= \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) e^{-i\phi/2} e^{-i\pi/2} | \uparrow \rangle + \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) e^{i\phi/2} e^{i\pi/2} | \downarrow \rangle \end{aligned} \quad (64)$$

Using $\cos(\pi/2 - x) = \sin(x)$, $\sin(\pi/2 - x) = \cos(x)$, $e^{-i\pi/2} = -i$, and $e^{i\pi/2} = i$:

$$| - \mathbf{n} \rangle = \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} (-i) | \uparrow \rangle + \cos\left(\frac{\theta}{2}\right) e^{i\phi/2} (i) | \downarrow \rangle \quad (65)$$

Now we compute the inner product $\langle -\mathbf{n} | \mathbf{n} \rangle$:

$$\begin{aligned} \langle -\mathbf{n} | \mathbf{n} \rangle &= \left(i \sin\left(\frac{\theta}{2}\right) e^{i\phi/2} \langle \uparrow | + (-i) \cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} \langle \downarrow | \right) \left(\cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} | \uparrow \rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi/2} | \downarrow \rangle \right) \\ &= i \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) e^{i\phi/2} e^{-i\phi/2} + (-i) \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} e^{i\phi/2} \\ &= i \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) - i \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \\ &= 0 \end{aligned} \quad (66)$$

This confirms that antipodal states are orthogonal, as expected. For example, $|\mathbf{n} = \hat{z}\rangle = | \uparrow \rangle$ is orthogonal to $|\mathbf{n} = -\hat{z}\rangle = | \downarrow \rangle$.

2.4.2 Distinction Between $| - \mathbf{n} \rangle$ and $-|\mathbf{n}\rangle$

It is a common point of confusion to mistake the antipodal state $| - \mathbf{n} \rangle$ for the state $-|\mathbf{n}\rangle$. We must justify that, in general, $| - \mathbf{n} \rangle \neq -|\mathbf{n}\rangle$.

From our derivation in the previous section, the antipodal state is:

$$| - \mathbf{n} \rangle = i \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} | \uparrow \rangle - i \cos\left(\frac{\theta}{2}\right) e^{i\phi/2} | \downarrow \rangle \quad (67)$$

In contrast, the state $-|\mathbf{n}\rangle$ is:

$$-|\mathbf{n}\rangle = -\cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} | \uparrow \rangle - \sin\left(\frac{\theta}{2}\right) e^{i\phi/2} | \downarrow \rangle \quad (68)$$

By simple inspection, these two state vectors are clearly not identical. They are, in fact, orthogonal to each other, as we just proved $\langle -\mathbf{n} | \mathbf{n} \rangle = 0$. If $| - \mathbf{n} \rangle$ were equal to $-|\mathbf{n}\rangle$, then we would

have $\langle -\mathbf{n} | \mathbf{n} \rangle = \langle -\mathbf{n} | -(-\mathbf{n}) \rangle = -1 \cdot \langle -\mathbf{n} | -\mathbf{n} \rangle = -1$, which contradicts our result of 0 (unless the state is null, which is not the case).

The state $|- \mathbf{n} \rangle$ represents a spin pointing in the *opposite direction* (e.g., spin down), while $-\mathbf{n} \rangle$ represents the *same physical state* as $|\mathbf{n} \rangle$ but with a phase shift of π (since $e^{i\pi} = -1$).

2.5 Completeness Relation

Despite being over-complete, the spin coherent states provide a resolution of the identity operator $\hat{\mathbb{I}}$.

Let's call the integral $J = \int d\Omega |\mathbf{n} \rangle \langle \mathbf{n}| = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta |\mathbf{n} \rangle \langle \mathbf{n}|$.

1. Integrate over ϕ : The off-diagonal terms depend on $e^{\pm i\phi}$.

$$\int_0^{2\pi} e^{\pm i\phi} d\phi = 0 \quad (69)$$

The diagonal terms are independent of ϕ .

$$\int_0^{2\pi} 1 d\phi = 2\pi \quad (70)$$

After integrating over ϕ , the matrix J becomes diagonal:

$$J = \int_0^\pi \sin \theta d\theta \begin{pmatrix} 2\pi \cos^2(\theta/2) & 0 \\ 0 & 2\pi \sin^2(\theta/2) \end{pmatrix} \quad (71)$$

2. Integrate over θ : Both integrals evaluate to 2π . Thus, the full integral is:

$$J = \int d\Omega |\mathbf{n} \rangle \langle \mathbf{n}| = \begin{pmatrix} 2\pi & 0 \\ 0 & 2\pi \end{pmatrix} = 2\pi \hat{\mathbb{I}} \quad (72)$$

Dividing by 2π , we arrive at the completeness relation:

$$\frac{1}{2\pi} \int d\Omega |\mathbf{n} \rangle \langle \mathbf{n}| = \hat{\mathbb{I}} \quad (73)$$

This relation is the foundation for the coherent-state path integral. It allows us to insert the identity operator at infinitesimally small time steps, t_j , as an integral over the Bloch sphere: $\hat{\mathbb{I}} = \int \frac{d\Omega_j}{2\pi} |\mathbf{n}_j \rangle \langle \mathbf{n}_j|$. Summing over all paths becomes an integral over all \mathbf{n}_j at all times t_j .

Appendix

A Appendix A: Evaluation of the Fresnel Integral

The value of the **Fresnel Integral** is:

$$\int_{-\infty}^{\infty} e^{ix^2} dx = \sqrt{\frac{\pi}{2}}(1 + i) = \sqrt{\pi}e^{i\pi/4} \quad (\text{A.1})$$

A.1 Derivation (Using Contour Integration)

This derivation is somewhat advanced and requires a basic understanding of complex analysis.

A.1.1 Step 1: Define the Contour

We consider the complex function $f(z) = e^{iz^2}$, where z is a complex variable. We construct a closed path (contour) C in the complex plane. This path is a sector of a circle, composed of three parts:

1. **Path C_1 :** A line segment along the real axis from 0 to R .
2. **Path C_2 :** A circular arc of radius R , centered at the origin, running counter-clockwise from R to $Re^{i\pi/4}$.
3. **Path C_3 :** A line segment from $Re^{i\pi/4}$ back to the origin 0.

We will eventually let $R \rightarrow \infty$.

A.1.2 Step 2: Apply Cauchy's Integral Theorem

The function $f(z) = e^{iz^2}$ is analytic over the entire complex plane (it is an entire function) as it has no singularities. According to **Cauchy's Integral Theorem**, its integral over any closed path C is zero:

$$\oint_C e^{iz^2} dz = 0 \quad (\text{A.2})$$

This closed-loop integral can be split into the sum of integrals over the three paths:

$$\int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz + \int_{C_3} e^{iz^2} dz = 0 \quad (\text{A.3})$$

A.1.3 Step 3: Evaluate the Integral on Each Path

1. Integral along Path C_1 (The part we want to find) On path C_1 , we have $z = x$ (a real number) and $dz = dx$. Therefore:

$$\lim_{R \rightarrow \infty} \int_{C_1} e^{iz^2} dz = \int_0^{\infty} e^{ix^2} dx \quad (\text{A.4})$$

This is exactly half of the integral we wish to compute, since the integrand e^{ix^2} is an even function.

2. Integral along Path C_2 (Show it vanishes as $R \rightarrow \infty$) On path C_2 , we parameterize $z = Re^{i\theta}$, where θ varies from 0 to $\pi/4$. Then $dz = iRe^{i\theta}d\theta$ and $z^2 = R^2e^{i2\theta}$. The integral becomes:

$$\begin{aligned} \int_{C_2} e^{iz^2} dz &= \int_0^{\pi/4} \exp(i(R^2e^{i2\theta})) \cdot iRe^{i\theta} d\theta \\ &= \int_0^{\pi/4} \exp(iR^2(\cos(2\theta) + i\sin(2\theta))) \cdot iRe^{i\theta} d\theta \\ &= \int_0^{\pi/4} \exp(-R^2\sin(2\theta)) \exp(i(R^2\cos(2\theta) + \theta)) \cdot iRd\theta \end{aligned} \quad (\text{A.5})$$

Taking the magnitude:

$$\left| \int_{C_2} e^{iz^2} dz \right| \leq \int_0^{\pi/4} |\exp(-R^2\sin(2\theta))| \cdot Rd\theta = \int_0^{\pi/4} R \exp(-R^2\sin(2\theta)) d\theta \quad (\text{A.6})$$

On the interval $[0, \pi/4]$, 2θ is in $[0, \pi/2]$. We can use Jordan's inequality, which states $\sin(x) \geq \frac{2x}{\pi}$ for $x \in [0, \pi/2]$. Thus, $\sin(2\theta) \geq \frac{4\theta}{\pi}$.

$$\begin{aligned} \left| \int_{C_2} e^{iz^2} dz \right| &\leq \int_0^{\pi/4} R \exp(-R^2(4\theta/\pi)) d\theta \\ &= R \left[\frac{-\pi}{4R^2} \exp(-4R^2\theta/\pi) \right]_0^{\pi/4} \\ &= \frac{\pi}{4R} (1 - \exp(-R^2)) \end{aligned} \quad (\text{A.7})$$

As $R \rightarrow \infty$, this expression approaches 0. Therefore:

$$\lim_{R \rightarrow \infty} \int_{C_2} e^{iz^2} dz = 0 \quad (\text{A.8})$$

3. Integral along Path C_3 (Connection to the Gaussian Integral) On path C_3 , we parameterize $z = re^{i\pi/4}$, where r varies from R to 0. Then $dz = e^{i\pi/4}dr$ and $z^2 = (re^{i\pi/4})^2 = r^2e^{i\pi/2} = r^2i$. The integral becomes:

$$\int_{C_3} e^{iz^2} dz = \int_R^0 \exp(i(r^2i)) e^{i\pi/4} dr = \int_R^0 \exp(-r^2) e^{i\pi/4} dr \quad (\text{A.9})$$

Reversing the limits of integration and factoring out the constant:

$$= -e^{i\pi/4} \int_0^R \exp(-r^2) dr \quad (\text{A.10})$$

As $R \rightarrow \infty$, we get the well-known Gaussian integral:

$$\lim_{R \rightarrow \infty} \int_{C_3} e^{iz^2} dz = -e^{i\pi/4} \int_0^\infty \exp(-r^2) dr \quad (\text{A.11})$$

We know that the Gaussian integral $\int_0^\infty \exp(-x^2) dx = \frac{\sqrt{\pi}}{2}$. So:

$$\lim_{R \rightarrow \infty} \int_{C_3} e^{iz^2} dz = -e^{i\pi/4} \frac{\sqrt{\pi}}{2} \quad (\text{A.12})$$

A.1.4 Step 4: Combine the Results

Returning to the equation from Step 2 and taking the limit as $R \rightarrow \infty$:

$$\left(\int_0^\infty e^{ix^2} dx \right) + (0) + \left(-e^{i\pi/4} \frac{\sqrt{\pi}}{2} \right) = 0 \quad (\text{A.13})$$

Rearranging the terms, we find:

$$\int_0^\infty e^{ix^2} dx = e^{i\pi/4} \frac{\sqrt{\pi}}{2} \quad (\text{A.14})$$

A.1.5 Step 5: Calculate the Final Integral

The integral we want to evaluate is from $-\infty$ to ∞ . Since the integrand $e^{ix^2} = \cos(x^2) + i \sin(x^2)$ is an even function (i.e., $f(-x) = f(x)$), we have:

$$\begin{aligned} \int_{-\infty}^\infty e^{ix^2} dx &= 2 \int_0^\infty e^{ix^2} dx \\ &= 2 \cdot \left(e^{i\pi/4} \frac{\sqrt{\pi}}{2} \right) = \sqrt{\pi} e^{i\pi/4} \end{aligned} \quad (\text{A.15})$$

Finally, we can use Euler's formula, $e^{i\theta} = \cos \theta + i \sin \theta$, to expand $e^{i\pi/4}$:

$$e^{i\pi/4} = \cos(\pi/4) + i \sin(\pi/4) = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} = \frac{1+i}{\sqrt{2}} \quad (\text{A.16})$$

Substituting this into our result gives:

$$\int_{-\infty}^\infty e^{ix^2} dx = \sqrt{\pi} \cdot \frac{1+i}{\sqrt{2}} = \sqrt{\frac{\pi}{2}} (1+i) \quad (\text{A.17})$$

A.2 Conclusion

The result of the integral is a complex number. Its real and imaginary parts correspond to two other important integrals:

$$\int_{-\infty}^\infty \cos(x^2) dx = \sqrt{\frac{\pi}{2}} \quad (\text{A.18})$$

$$\int_{-\infty}^\infty \sin(x^2) dx = \sqrt{\frac{\pi}{2}} \quad (\text{A.19})$$

B Appendix B: Proof of the Multivariate Gaussian Integral

B.1 Statement of the Formula

The formula to be proven is the multivariate Gaussian integral:

$$I = \int \prod_{n=1}^N dx_n \exp\left(-\frac{1}{2}\mathbf{x}^\top A \mathbf{x} - \mathbf{x}^\top \mathbf{y}\right) = (2\pi)^{N/2} (\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\mathbf{y}^\top A^{-1} \mathbf{y}\right) \quad (\text{B.1})$$

where:

- \mathbf{x} and \mathbf{y} are N -dimensional column vectors.
- A is an $N \times N$ real, symmetric, and positive-definite matrix.
- The notation $\int \prod_{n=1}^N dx_n$ denotes integration over all components of \mathbf{x} from $-\infty$ to $+\infty$.

The proof relies on the assumptions that A is symmetric ($A^\top = A$) and positive-definite (all eigenvalues are positive), which ensures the integral converges. The proof proceeds in several key steps.

B.2 Step 1: Completing the Square

The primary technique is to complete the square for the quadratic form in the exponent. We want to rewrite the argument of the exponential, $-\frac{1}{2}\mathbf{x}^\top A \mathbf{x} - \mathbf{x}^\top \mathbf{y}$, into the form $-\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^\top A (\mathbf{x} - \mathbf{x}_0) + C$, where \mathbf{x}_0 and C are constants with respect to \mathbf{x} .

Expanding this target form, we get:

$$\begin{aligned} -\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^\top A (\mathbf{x} - \mathbf{x}_0) &= -\frac{1}{2}(\mathbf{x}^\top - \mathbf{x}_0^\top) A (\mathbf{x} - \mathbf{x}_0) \\ &= -\frac{1}{2}(\mathbf{x}^\top A \mathbf{x} - \mathbf{x}^\top A \mathbf{x}_0 - \mathbf{x}_0^\top A \mathbf{x} + \mathbf{x}_0^\top A \mathbf{x}_0) \end{aligned} \quad (\text{B.2})$$

Since A is symmetric ($A = A^\top$), the scalar term $\mathbf{x}_0^\top A \mathbf{x}$ is equal to its own transpose: $(\mathbf{x}_0^\top A \mathbf{x})^\top = \mathbf{x}^\top A^\top \mathbf{x}_0 = \mathbf{x}^\top A \mathbf{x}_0$. Thus, the two cross-terms are equal.

$$-\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^\top A (\mathbf{x} - \mathbf{x}_0) = -\frac{1}{2}\mathbf{x}^\top A \mathbf{x} + \mathbf{x}^\top A \mathbf{x}_0 - \frac{1}{2}\mathbf{x}_0^\top A \mathbf{x}_0 \quad (\text{B.3})$$

Comparing this to the original exponent, $-\frac{1}{2}\mathbf{x}^\top A \mathbf{x} - \mathbf{x}^\top \mathbf{y}$, we can equate the terms linear in \mathbf{x} :

$$-\mathbf{x}^\top \mathbf{y} = \mathbf{x}^\top A \mathbf{x}_0 \implies A \mathbf{x}_0 = -\mathbf{y} \quad (\text{B.4})$$

Since A is positive-definite, it is invertible. We can solve for \mathbf{x}_0 :

$$\mathbf{x}_0 = -A^{-1} \mathbf{y} \quad (\text{B.5})$$

With this definition of \mathbf{x}_0 , the original exponent can be written as:

$$-\frac{1}{2}\mathbf{x}^\top A\mathbf{x} - \mathbf{x}^\top \mathbf{y} = -\frac{1}{2}(\mathbf{x} + A^{-1}\mathbf{y})^\top A(\mathbf{x} + A^{-1}\mathbf{y}) + \frac{1}{2}(A^{-1}\mathbf{y})^\top A(A^{-1}\mathbf{y}) \quad (\text{B.6})$$

Let's simplify the constant term (the term not involving \mathbf{x}):

$$\begin{aligned} \frac{1}{2}(A^{-1}\mathbf{y})^\top A(A^{-1}\mathbf{y}) &= \frac{1}{2}\mathbf{y}^\top (A^{-1})^\top A A^{-1}\mathbf{y} \\ &= \frac{1}{2}\mathbf{y}^\top A^{-1} A A^{-1}\mathbf{y} \quad (\text{since } (A^{-1})^\top = (A^\top)^{-1} = A^{-1}) \\ &= \frac{1}{2}\mathbf{y}^\top I A^{-1}\mathbf{y} = \frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y} \end{aligned} \quad (\text{B.7})$$

So, the exponent is:

$$-\frac{1}{2}\mathbf{x}^\top A\mathbf{x} - \mathbf{x}^\top \mathbf{y} = -\frac{1}{2}(\mathbf{x} + A^{-1}\mathbf{y})^\top A(\mathbf{x} + A^{-1}\mathbf{y}) + \frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y} \quad (\text{B.8})$$

B.3 Step 2: Change of Variables (Translation)

Substituting the completed square back into the integral:

$$I = \int \prod_{n=1}^N dx_n \exp \left[-\frac{1}{2}(\mathbf{x} + A^{-1}\mathbf{y})^\top A(\mathbf{x} + A^{-1}\mathbf{y}) + \frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y} \right] \quad (\text{B.9})$$

The term $\exp(\frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y})$ is constant with respect to \mathbf{x} and can be factored out of the integral:

$$I = \exp \left(\frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y} \right) \int \prod_{n=1}^N dx_n \exp \left[-\frac{1}{2}(\mathbf{x} + A^{-1}\mathbf{y})^\top A(\mathbf{x} + A^{-1}\mathbf{y}) \right] \quad (\text{B.10})$$

Now, we perform a change of variables. Let $\mathbf{z} = \mathbf{x} + A^{-1}\mathbf{y}$. This is a simple translation of the coordinate system. The differential element $\prod dx_n$ transforms as $\prod dz_n$, as the Jacobian of this transformation is 1. The limits of integration remain from $-\infty$ to $+\infty$. The integral becomes:

$$I = \exp \left(\frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y} \right) \int \prod_{n=1}^N dz_n \exp \left(-\frac{1}{2}\mathbf{z}^\top A\mathbf{z} \right) \quad (\text{B.11})$$

The problem is now reduced to evaluating the simpler, centered Gaussian integral:

$$I_0 = \int \prod dz_n \exp \left(-\frac{1}{2}\mathbf{z}^\top A\mathbf{z} \right) \quad (\text{B.12})$$

B.4 Step 3: Diagonalization

To compute I_0 , we diagonalize the matrix A . Since A is a real symmetric matrix, it is orthogonally diagonalizable:

$$A = P D P^\top \quad (\text{B.13})$$

where P is an orthogonal matrix ($PP^\top = P^\top P = I$) whose columns are the orthonormal eigenvectors of A , and D is a diagonal matrix whose entries are the corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$. Substituting this into the quadratic form $\mathbf{z}^\top A \mathbf{z}$:

$$\mathbf{z}^\top A \mathbf{z} = \mathbf{z}^\top (PDP^\top) \mathbf{z} = (\mathbf{z}^\top P) D (P^\top \mathbf{z}) = (P^\top \mathbf{z})^\top D (P^\top \mathbf{z}) \quad (\text{B.14})$$

We perform another change of variables. Let $\mathbf{w} = P^\top \mathbf{z}$. This transformation corresponds to a rotation of the coordinate system. The Jacobian determinant is $|\det(P^\top)| = 1$, so the volume element is unchanged: $\prod dz_n = \prod dw_n$. The quadratic form simplifies to:

$$\mathbf{w}^\top D \mathbf{w} = \sum_{i=1}^N \lambda_i w_i^2 \quad (\text{B.15})$$

This is because D is a diagonal matrix.

B.5 Step 4: Computing the Decoupled Integral

The integral I_0 now becomes:

$$I_0 = \int \prod_{n=1}^N dw_n \exp\left(-\frac{1}{2} \sum_{i=1}^N \lambda_i w_i^2\right) \quad (\text{B.16})$$

The exponential of a sum is the product of exponentials, which allows us to separate the multi-dimensional integral into a product of N one-dimensional integrals:

$$I_0 = \int \prod_{n=1}^N dw_n \prod_{i=1}^N \exp\left(-\frac{1}{2} \lambda_i w_i^2\right) = \prod_{i=1}^N \left(\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \lambda_i w_i^2\right) dw_i \right) \quad (\text{B.17})$$

We use the standard formula for a 1D Gaussian integral: $\int_{-\infty}^{\infty} \exp(-au^2) du = \sqrt{\pi/a}$. In our case, $a = \lambda_i/2$.

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \lambda_i w_i^2\right) dw_i = \sqrt{\frac{\pi}{\lambda_i/2}} = \sqrt{\frac{2\pi}{\lambda_i}} \quad (\text{B.18})$$

Multiplying these N results together:

$$I_0 = \prod_{i=1}^N \sqrt{\frac{2\pi}{\lambda_i}} = (2\pi)^{N/2} \prod_{i=1}^N (\lambda_i)^{-1/2} = (2\pi)^{N/2} \left(\prod_{i=1}^N \lambda_i \right)^{-1/2} \quad (\text{B.19})$$

The determinant of a matrix is equal to the product of its eigenvalues. Thus, $\det A = \det D = \prod_{i=1}^N \lambda_i$.

$$I_0 = (2\pi)^{N/2} (\det A)^{-1/2} \quad (\text{B.20})$$

B.6 Step 5: Combining the Results

Finally, we substitute the value of I_0 back into our expression for I from Step 2:

$$I = \exp\left(\frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y}\right) \cdot I_0 = \exp\left(\frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y}\right) (2\pi)^{N/2} (\det A)^{-1/2} \quad (\text{B.21})$$

Rearranging the terms yields the final result:

$$I = (2\pi)^{N/2} (\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\mathbf{y}^\top A^{-1}\mathbf{y}\right) \quad (\text{B.22})$$

This completes the proof.