If the functional integration over p is Gaussian, we can exactly integrate out P. For example, $H = \frac{p^2}{2m} + V$

so $\mathcal{L} = p\dot{x} - H = p\dot{x} - \frac{p^2}{2m} - V(x)$.

$$\cancel{iG = \int Dp\, Dx \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} \mathcal{L}\, d\tau\right]}$$

$$\cancel{= \int Dp\, Dx \exp\left[\frac{i}{\hbar}\int_{t_0}^{t}\right.}$$

$$iG = \int Dp\, Dx \exp\left[\frac{i}{\hbar}\sum_{l=1}^{N}\left(p_l \cdot \dot{x}_l - \frac{p_l^2}{2m} - V(x)\right)\Delta t\right]$$

let $\vec{p} = \begin{bmatrix} p_l \\ \vdots \\ p_1 \end{bmatrix}$, $\dot{x} = \begin{bmatrix} \dot{x}_l \\ \vdots \\ \dot{x}_1 \end{bmatrix}$

$$\therefore iG = \int Dx \cdot e^{\frac{i}{\hbar}\sum(-V(x))\Delta t} \cdot Dp \cdot \exp\left[\frac{i}{2m\hbar}\left(-p^T p + 2m p^T \cdot \dot{x}\right)\Delta t\right]$$

we have an useful formula: Gaussian integral:

$$\int \prod_{n=1}^{N} dx_n\, e^{-\frac{1}{2}x^T A x - x^T y} = (2\pi)^{\frac{N}{2}} (\operatorname{Det} A)^{-\frac{1}{2}} e^{\frac{1}{2} y^T A^{-1} y}$$

where, x, y are real vectors. A is real symmetric matrix.

let

$$I = \int Dp \exp\left[\frac{i}{2m\hbar}\left(-p^T p + 2m p^T \cdot \dot{x}\right)\Delta t\right]$$

$$= \left(\frac{1}{2\pi\hbar}\right)^{N} \int \prod_{l=1}^{N} dp_l \cdot \exp\left[-\frac{1}{2}p^T A p - p^T \cdot \dot{x}'\right]$$

where $A = \frac{i\Delta t}{m\hbar}\hat{I}_{N\times N}$, $\dot{x}' = -\frac{i\Delta t}{\hbar}\dot{x}$, $\hat{I}$ is a $N\times N$ identity matrix.

So we can get,

$$
\begin{cases}
(\mathrm{Det}\,A)^{-\frac{1}{2}} = \left(\frac{i\Delta t}{m\hbar}\right)^{-\frac{N}{2}}, \\
A^{-1} = \frac{m\hbar}{i\Delta t}\hat{I}_{N\times N}
\end{cases}
$$

So,

$$
\frac{1}{2}\dot{x}'^{T}A^{-1}\dot{x}' = \frac{1}{2}\cdot\frac{m\hbar}{i\Delta t}\cdot\left(-\frac{i\Delta t}{\hbar}\right)^{2}\sum_{l=1}^{N}\dot{x}_l^{2}
$$

$$
= \frac{i}{\hbar}\sum_{l=1}^{N}\frac{m}{2}\left(\frac{x_l - x_{l-1}}{\Delta t}\right)^{2}\Delta t
$$

So: $I = \left(\frac{1}{2\pi\hbar}\right)^{N}\cdot(2\pi)^{\frac{N}{2}}\cdot\left(\frac{i\Delta t}{m\hbar}\right)^{-\frac{N}{2}} e^{\frac{i}{\hbar}\sum_{l=1}^{N}\frac{m}{2}\left(\frac{x_l - x_{l-1}}{\Delta t}\right)^{2}\Delta t}$

$$
= \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} e^{\frac{i}{\hbar}\sum_{l=1}^{N}\frac{m}{2}\left(\frac{x_l - x_{l-1}}{\Delta t}\right)^{2}\Delta t}
$$

So we can get the integration without p.

$$
iG = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}}\int D_x \exp\left[\frac{i}{\hbar}\sum_{l=1}^{N}\left(\frac{m}{2}\left(\frac{x_l - x_{l-1}}{\Delta t}\right)^{2} - V(x)\right)\Delta t\right]
$$

$$
= \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}}\int D_x \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} d\tau\, L(x,\dot{x})\right]
$$

So $iG \propto \int \mathcal{D}x \cdot e^{\frac{i}{\hbar}S[x(t)]}$,

where, $S[x(t)] = \int_{t_0}^{t} d\tau \cdot \mathcal{L}(x, \dot{x})$ is action,

and $\mathcal{L}(x, \dot{x}) = \frac{m\dot{x}^2}{2} - V(x)$ is Lagrangean

From the Path-integral in real space-time, we can get some information about Physics Picture:

(1) Each path is weighted with a $U(1)$ phase factor $e^{\frac{i}{\hbar}S}$
The Quantum interference between different paths

(2). Since $\hbar \sim 10^{-34} J \cdot s$, any "small change" in S (we change S to $S + \delta S$), ~~w~~ will drastically lead to quantum destructive interference. So ~~the~~ only the paths that ~~st~~ satisfy $\delta S = 0$ make dominant contributions to the path-integral.

(3). Remarkably, $\delta S = 0$ is exactly Hamilton's Principle in classical mechanics. So the classical paths ($\delta S = 0$) dominate the path integral in the limit $\hbar \to 0$. In other words, in classical mechanics, ~~it~~ as $\hbar \to 0$, it neglects the contribution of the integral over all other paths near the path ~~w those~~ with $\delta S = 0$. So we can get the conclusion:

$$\text{Quantum system} \xrightarrow{\hbar \to 0} \text{classical system}.$$

Example: Computing the path-integral of free particles:

Free particles' Hamiltonian, $H = \frac{p^2}{2m}$.

So $iG = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int D_x \exp\left[\frac{i}{\hbar}\sum_{t=1}^{N}\frac{m}{2}\left(\frac{x_t - x_{t-1}}{\Delta t}\right)^2 \Delta t\right]$.

we let:

$$I = \int D_x \exp\left[\frac{im}{2\hbar\Delta t}\left[\sum_{t=1}^{N}(x_t^2 - 2x_t x_{t-1} + x_{t-1}^2)\right]\right]$$

In other to use Gaussian integral:

$$\int \prod_{n=1}^{N} dx_n\, e^{-\frac{1}{2}x^T A x - x^T y} = (2\pi)^{\frac{N}{2}} (\mathrm{Det} A)^{-\frac{1}{2}} e^{\frac{1}{2} y^T A^{-1} y}.$$

we should rewrite the form of $\exp\left[\frac{im}{2\hbar\Delta t}\sum(x_t^2 - 2x_t x_{t-1} + x_{t-1}^2)\right)$

So we let:

$$A = \frac{-im}{\hbar\Delta t}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & & \ddots \end{bmatrix}_{(N-1)\times(N-1)}, \qquad x = \begin{bmatrix} x_{N-1} \\ \vdots \\ x_1 \end{bmatrix}$$

$$y = \frac{im}{\hbar\Delta t}\begin{bmatrix} x_N \\ 0 \\ \text{all zero} \\ 0 \\ x_0 \end{bmatrix}$$

$$\exp\left[\frac{im}{2\hbar\Delta t}\sum_{t=1}^{N}(x_t^2 - 2x_t x_{t-1} + x_{t-1}^2)\right] = \exp\left(-\frac{1}{2}x^T A x - x^T y\right) e^{\frac{im}{2\hbar\Delta t}(x_N^2 + x_0^2)}$$

because of $Dx = \prod_{t=1}^{N-1} x_t$ without x_N and x_0,

So $iG = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \cdot e^{\frac{im}{2\hbar\Delta t}(x_N^2+x_0^2)} \int Dx \cdot e^{-\frac{1}{2}x^TAx - x^Ty}$

~~So~~we can compute $|\mathrm{Det}A| = N \cdot \left(\frac{-im}{\hbar\Delta t}\right)^{N}$

So we can get:

$$iG = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \cdot e^{\frac{im}{2\hbar\Delta t}(x_N^2+x_0^2)} \cdot (2\pi)^{\frac{N}{2}} \cdot N^{-\frac{1}{2}} \cdot \left(\frac{-im}{\hbar\Delta t}\right)^{-\frac{N}{2}} \cdot e^{\frac{1}{2}y^TA^{-1}y}.$$

Although A^{-1} is difficult to compute, we notice that $y = \begin{bmatrix} x_N \\ 0 \\ \vdots \\ 0 \\ x_0 \end{bmatrix}$ only have two non-~~zeor~~ zero elements, which locate in the fist row and the last row respectively.

So we only need to ~~m~~ calulate the first and last columns of matrix A^{-1}, denoted A_1^{-1} and A_{N-1}^{-1}, respectively

$$\cancel{A_1^{-1} = \frac{i\hbar\Delta t}{m}\begin{bmatrix} \frac{N-1}{N} \\ \frac{N-2}{N} \end{bmatrix}} \quad A_1^{-1} = \frac{i\hbar\Delta t}{mN}\begin{bmatrix} N-1 \\ N-2 \\ \vdots \\ 2 \\ 1 \end{bmatrix}, \quad A_{N-1}^{-1} = \frac{i\hbar\Delta t}{mN}\begin{bmatrix} 1 \\ 2 \\ \vdots \\ N-1 \end{bmatrix}$$

$$\therefore \frac{1}{2}y^TA^{-1}y = \frac{1}{2}y^T\left[x_N A_1^{-1} + x_0 A_{N-1}^{-1}\right]$$

$$= \frac{im}{2\hbar\Delta t}\left[-(x_N^2+x_0^2) + \frac{(x_N-x_0)^2}{N}\right]$$

So we can get the complete integral 式:

$$I \equiv \int iG = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} (2\pi)^{\frac{N-1}{2}} N^{-\frac{1}{2}} \left(\frac{-im}{\hbar\Delta t}\right)^{-\frac{(N-1)}{2}} e^{\frac{i}{\hbar}\cdot\frac{m(x_N-x_0)^2}{2N\Delta t}}.$$

$$= \left(\frac{m}{i2\pi\hbar\Delta t N}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\cdot\frac{(x_N-x_0)^2}{2N\Delta t}}.$$

where $t-t_0=N\Delta t$, $x=x_N$

So:

$$iG = \left[\frac{m}{i2\pi\hbar(t-t_0)}\right]^{\frac{1}{2}} \cdot e^{\frac{i}{\hbar}\cdot\frac{m(x-x_0)^2}{2(t-t_0)}}.$$