Part I
Feynman Path Integral

Chapter 1
Feynman Path Integral

The version of quantum mechanics:

1. Schrödinger's wavefuction (operator form):

$$\hat{H}|\psi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle \tag{1.1}$$

2. Feynman's Path Integral (Common number form):

$$\mathrm{i}G(\text{Green's function}) \propto \int \mathcal{D}(x,t)\mathrm{e}^{\mathrm{i}\int \mathcal{L}\mathrm{d}t} \tag{1.2}$$

There are many advantages of Feynman Path Integral:

1. Make the double-slit experiment more understandable.
2. The classical limit "$\hbar \to 0$" is "tractable": quantum $\xrightarrow{\hbar\to 0}$ classical.
3. Provide a semi-classical picture. for. quantum mechanics.
4. "Quantum fluctuations" are more "understandable".
5. A natural route. to low energy effective theory of quantum many-body systems.
6. A natural language for describing topological properties of quantum many-body systems.

But the practical calculation in the path-integral representation of simple quantum mechanical problem many be notoriously difficult and lengthy.

1.1 Propagators

Consider a quantum partical confined in a one-dimensional space:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \tag{1.3}$$

and the canonical pair: $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = \mathrm{i}\hbar$

The Schrödinger's equation is:

$$\mathrm{i}\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle \tag{1.4}$$

This first-order nature allows us to define a time evolution operator $\hat{U}(t, t_0)$ which propagates the state vector from an initial time t_0 to a final time t:

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \tag{1.5}$$

Assuming the Hamiltonian H is not explicitly depend on time, the formal solution of $\hat{U}$ is:

$$\hat{U} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_0)} \tag{1.6}$$

A crucial property of $\hat{U}$ is the "chain-like" rule, or composition property. For any intermediate time t' such that $t > t' > t_0$:

$$\hat{U}(t, t_0) = \hat{U}(t, t')\hat{U}(t', t_0) \tag{1.7}$$

This property is the key to the entire path integral derivation. And $\hat{U}(t, t_0)$ is unitary:

$$\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \hat{\mathbb{I}} \tag{1.8}$$

where $\hat{\mathbb{I}}$ is the identity operator.

In the position representation, we can obtain matrix elements:

$$\begin{aligned} U(x, t; x_0, t_0) &= \langle x|\hat{U}(t, t_0)|x_0\rangle \\ &= \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_0)}|x_0\rangle \end{aligned} \tag{1.9}$$

We can define a propagators (Green's function) of the quantum system by using the matrix elements:

$$\mathrm{i}G(x, t; x_0, t_0) = U(x, t; x_0, t_0) \tag{1.10}$$

Using the matrix elements, $\psi(x, t)$ can be reformulated as:

$$\begin{aligned} \psi(x, t) &= \langle x|\hat{U}(t, t_0)|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_0 \langle x|\hat{U}(t, t_0)|x_0\rangle\langle x_0|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_0 U(x, t; x_0, t_0)\psi(x_0, t_0) \end{aligned} \tag{1.11}$$

Also, the propagator also satisfies the Schrödinger's equation:

$$\boxed{\mathrm{i}\hbar\frac{\partial}{\partial t}G(x,t;x_0,t_0) = \hat{H}G(x,t;x_0,t_0)} \tag{1.12}$$

And the initial condition is:

$$G(x,t_0;x_0,t_0) = -\mathrm{i}\langle x|\hat{U}(t_0,t_0)|x_0\rangle = -\mathrm{i}\delta(x-x_0) \tag{1.13}$$

$\delta(x-x_0)$ is Dirac function.

Example 1.1.1. For free particle: $\hat{H} = \frac{1}{2m}\hat{p}^2$, in position representation:

$$\hat{p} \longrightarrow -\mathrm{i}\hbar\frac{\partial}{\partial x} \tag{1.14}$$

So the PDE is:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}G(x,t;x_0,t_0) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x,t;x_0,t_0) \tag{1.15}$$

Solve the PDE:

Use Fourier Transform: (we use $G(x,t)$ instead of $G(x,t;x_0,t_0)$). We solve the free-particle Green's function by transforming to momentum space:

$$\begin{cases} G(x,t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int \mathrm{d}k\cdot\tilde{G}(k,t)\mathrm{e}^{\mathrm{i}kx} \\ \tilde{G}(k,t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int \mathrm{d}x\ G(x,t)\mathrm{e}^{-\mathrm{i}kx} \end{cases} \tag{1.16}$$

With these conventions, spatial derivatives become algebraic in k-space while the time derivative remains unchanged:

$$\begin{cases} \mathcal{F}\left\{\mathrm{i}\hbar\dfrac{\partial G(x,t)}{\partial t}\right\} = \mathrm{i}\hbar\dfrac{\partial\tilde{G}(k,t)}{\partial t} \\ \mathcal{F}\left\{-\dfrac{\hbar^2}{2m}\dfrac{\partial^2 G}{\partial x^2}\right\} = -\dfrac{\hbar^2}{2m}[-k^2\tilde{G}(k,t)] = \dfrac{\hbar^2k^2}{2m}\tilde{G}(k,t) \end{cases} \tag{1.17}$$

Applying the transform to the PDE yields an ordinary differential equation in time for each k:

$$\begin{cases} \mathrm{i}\hbar\dfrac{\partial}{\partial t}\tilde{G}(k,t) = \dfrac{\hbar^2k^2}{2m}\tilde{G}(k,t) \\ \dfrac{\mathrm{d}\tilde{G}}{\tilde{G}} = -\mathrm{i}\dfrac{\hbar k^2}{2m}\mathrm{d}t \end{cases} \tag{1.18}$$

Integrating in time gives the logarithm of the solution up to a k-dependent constant:

$$\ln\tilde{G} = -\mathrm{i}\frac{\hbar k^2}{2m}t + C(k) \tag{1.19}$$

So we can get the solution:

$$\tilde{G}(k,t) = A(k)\mathrm{e}^{-\mathrm{i}\frac{\hbar k^2}{2m}t}, \quad A(k) = \mathrm{e}^{C(k)} \tag{1.20}$$

To determine $A(k)$, impose the initial condition at time t_0 in position space:

$$\begin{aligned}\tilde{G}(k,t_0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x G(x,t_0)\mathrm{e}^{-\mathrm{i}kx} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x \delta(x-x_0)\mathrm{e}^{-\mathrm{i}kx}\end{aligned} \tag{1.21}$$

Using the Fourier transform of the Dirac delta, we find:

$$\tilde{G}(k,t_0) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0} \tag{1.22}$$

Matching at t_0 fixes the k-space amplitude:

$$A(k) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0} \cdot \mathrm{e}^{\mathrm{i}\frac{\hbar k^2}{2m}t_0}. \tag{1.23}$$

Therefore, for general time t we have:

$$\tilde{G}(k,t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0}\mathrm{e}^{-\mathrm{i}\frac{\hbar k^2}{2m}(t-t_0)} \tag{1.24}$$

Finally, inverse-transform back to position space to obtain the integral representation of the propagator:

$$\begin{aligned}G(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}k \cdot \tilde{G}(k,t)\mathrm{e}^{\mathrm{i}kx} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k(x-x_0)} \cdot \mathrm{e}^{-\mathrm{i}\frac{\hbar(t-t_0)}{2m}k^2}\end{aligned} \tag{1.25}$$

This is a standard Gaussian integral of the form:

$$\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-ak^2+bk} = \sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}}. \tag{1.26}$$

Let's identify the coefficients:

$$\begin{cases} a = \mathrm{i}\dfrac{\hbar(t-t_0)}{2m} \\ b = \mathrm{i}(x-x_0) \end{cases} \tag{1.27}$$

So we can get the solution:

$$\mathrm{i}G(x,t) = \left[\frac{m}{2\pi\hbar\mathrm{i}(t-t_0)}\right]^{\frac{1}{2}} \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\cdot\frac{m(x-x_0)^2}{2(t-t_0)}} \tag{1.28}$$

Also, we can solve this PDE via definition:

$$\begin{aligned}\mathrm{i}G &= \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}}|x_0\rangle \\ &= \int \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}}|p\rangle\langle p|x_0\rangle\, \mathrm{d}p \\ &= \int \mathrm{d}p\, \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)p^2}{2m\hbar}}\langle x|p\rangle\langle p|x_0\rangle \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\, \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)p^2}{2m\hbar}+\mathrm{i}\frac{(x-x_0)}{\hbar}p}.\end{aligned} \tag{1.29}$$

we use $P = \hbar k$ and can get the same equation as the Fourier Transform Method.

1.2 Path-Integral

When $t > t_1 > t_0$, and t_1 is an arbitrarily selected intermediate time, we can write:

$$\begin{aligned} \mathrm{i}G(x,t;x_0,t_0) &= \langle x|\hat{U}(t,t_0)|x_0\rangle \\ &= \langle x|\hat{U}(t,t_1)\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_1 \langle x|\hat{U}(t,t_1)|x_1\rangle\langle x_1|\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_1\, \mathrm{i}G(x,t;x_1,t_1)\cdot \mathrm{i}G(x_1,t_1;x_0,t_0) \end{aligned} \tag{1.30}$$

This intergral over x_1 means “superposition” of all possible “path” that connect x and x_0. Next, we try to “smooth” the path along time directly. We can insert more time slices between x and x_0. If we insert infinite time slices, the path become smooth.

Firstly, let’s discretize time. domain $[t_0, t]$ into N pieces of equal length $\Delta t = \frac{t-t_0}{N}$:

$$\begin{aligned} \mathrm{i}G(x,t;x_0,t_0) &= \langle x|\hat{U}(t,t_{N-1})\hat{U}(t_{N-1},t_{N-2})\cdots\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_{N-1}\cdots\mathrm{d}x_1 \prod_{l=1}^{N} \mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) \end{aligned} \tag{1.31}$$

let $\mathcal{D}_x = \prod_{l=1}^{N-1}\mathrm{d}x_l$. Consider $N \to \infty$, so $\Delta t = \frac{t-t_0}{N} \to 0$, which means $t_l - t_{l-1} = \Delta t$.

$$\mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) = \langle x_l|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\Delta t}|x_{l-1}\rangle \tag{1.32}$$

Because Δt is small, we can approximate the exponential function by its Taylor series:

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\Delta t} \approx \hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\hat{H}\Delta t = \hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\Delta t\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right] \tag{1.33}$$

Substitute (1.33) into (1.32):

$$\begin{aligned} \mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) &= \int \mathrm{d}p_l \langle x_l|p_l\rangle\langle p_l|\hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\Delta t\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]|x_{l-1}\rangle \\ &= \int \mathrm{d}p_l \langle x_l|p_l\rangle\langle p_l|x_{l-1}\rangle\left[1 - \frac{\mathrm{i}}{\hbar}\left(\frac{p_l^2}{2m} + V(x_{l-1})\right)\Delta t\right] \end{aligned} \tag{1.34}$$

With the approximations $V(x_l) \approx V(x_{l-1})$:

$$\left[1 - \frac{\mathrm{i}}{\hbar}\left(\frac{p_l^2}{2m} + V(x_{l-1})\right)\Delta t\right] \approx \left(1 - \frac{\mathrm{i}}{\hbar}H_l\Delta t\right) \approx \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_l\Delta t} \tag{1.35}$$

So $\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1})$ can be written as:

$$\begin{aligned}\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1}) &= \int \mathrm{d}p_l \frac{1}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p_l (x_l - x_{l-1})} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H_{cl} \Delta t} \\ &= \frac{1}{2\pi\hbar} \int \mathrm{d}p_l e^{\frac{\mathrm{i}}{\hbar}[p_l(x_l - x_{l-1}) - H_l \Delta t]} \\ &= \frac{1}{2\pi\hbar} \int \mathrm{d}p_l \mathrm{e}^{\frac{\mathrm{i}}{\hbar}[p_l(\frac{x_l - x_{l-1}}{\Delta t}) - H_l]\Delta t}\end{aligned} \tag{1.36}$$

where, H_l is the classical Hamiltonion as a function of p_l and x_l.

When $\Delta t \to 0$:

$$\frac{x_l - x_{l-1}}{\Delta t} = \dot{x}_l \tag{1.37}$$

So we can get:

$$p_l \dot{x}_l - H_l = \mathcal{L}_l. \tag{1.38}$$

where, $\mathcal{L}_l$ is the classical Lagrangian.

So $\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1})$ can be written as the form with Lagrangian:

$$\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1}) = \int \mathrm{d}p_l \cdot \frac{1}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \mathcal{L}_l \Delta t}. \tag{1.39}$$

Substitute $\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1})$ into the path integral:

$$\prod_{l=1}^{N} \mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1}) = \int \frac{\mathrm{d}p_N}{2\pi\hbar} \cdots \frac{\mathrm{d}p_1}{2\pi\hbar} \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \sum_{l=1}^{N} \mathcal{L}_l \cdot \Delta t} \tag{1.40}$$

let $\mathcal{D}_p = \prod_{l=1}^{N} \frac{\mathrm{d}p_l}{2\pi\hbar}$, when $\Delta t \to 0$, which means:

$$\sum_{l=1}^{N} \mathcal{L}_l \Delta t = \int_{t_0}^{t} \mathrm{d}\tau \cdot \mathcal{L}[p(\tau), x(\tau)] \tag{1.41}$$

Finally, we can get the propagators by the path integral:

Theorem 1.2.1. *The propagators path integral:*

$$\mathrm{i}G(x, t; x_0, t_0) = \int \mathcal{D}_x \mathcal{D}_p \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} \mathrm{d}\tau \cdot \mathcal{L}[p(\tau), x(\tau)]} \tag{1.42}$$

where, the pair of $p(t)$ and $\dot{x}(t)$ characterizes a path in the px phase space.

1.3 Gaussian Integration

If the functional integration over p is Gaussian, we can exactly integrate out p. For example, $H = \frac{p^2}{2m} + V$, so $\mathcal{L} = p\dot{x} - H = p\dot{x} - \frac{p^2}{2m} - V(x)$,we can get:

$$\mathrm{i}G = \int \mathcal{D}p\mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_t \left(p_t\dot{x}_t - \frac{p_t^2}{2m} - V(x_t)\right)\Delta t\right] \tag{1.43}$$

Let:

$$\boldsymbol{p} = \begin{pmatrix} p_l \\ \vdots \\ p_1 \end{pmatrix}, \quad \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_l \\ \vdots \\ \dot{x}_1 \end{pmatrix} \tag{1.44}$$

So we can rewrite the integral as:

$$\mathrm{i}G = \int \mathcal{D}x \cdot \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}(-V(x_l))\Delta t\right] \cdot \int \mathcal{D}p \cdot \exp\left[\frac{\mathrm{i}}{2m\hbar}(-\boldsymbol{p}^T\boldsymbol{p} + 2m\boldsymbol{p}^T\dot{\boldsymbol{x}})\Delta t\right] \tag{1.45}$$

We have an useful formula for Gaussian integral (Proof in A):

$$\boxed{\int \prod_{n=1}^{N} \mathrm{d}x_n \exp\left[-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{y}\right] = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \exp\left[\frac{1}{2}\boldsymbol{y}^T A^{-1}\boldsymbol{y}\right]} \tag{1.46}$$

where, $\boldsymbol{x}, \boldsymbol{y}$ are real vectors and A is real symmetric matrix.

Let:

$$\begin{aligned} I &= \int \mathcal{D}p \exp\left[\frac{\mathrm{i}}{2m\hbar}(-\boldsymbol{p}^T\boldsymbol{p} + 2m\boldsymbol{p}^T\dot{\boldsymbol{x}})\Delta t\right] \\ &= \left(\frac{1}{2\pi\hbar}\right)^N \int \prod_{n=1}^{N} \mathrm{d}p_n \cdot \exp\left[-\frac{1}{2}\boldsymbol{p}^T A\boldsymbol{p} - \boldsymbol{p}^T\dot{\boldsymbol{x}}'\right] \end{aligned} \tag{1.47}$$

where $A = \frac{\mathrm{i}\Delta t}{m\hbar}\mathbb{I}_{N\times N}$ and $\dot{\boldsymbol{x}}' = -\frac{\mathrm{i}\Delta t}{\hbar}\dot{\boldsymbol{x}}$, $\mathbb{I}_{N\times N}$ is the $N \times N$ identity matrix.

So we can get:

$$\begin{cases} (\det A)^{-\frac{1}{2}} = \left(\dfrac{\mathrm{i}\Delta t}{m\hbar}\right)^{-\frac{N}{2}} \\ A^{-1} = \dfrac{m\hbar}{\mathrm{i}\Delta t}\mathbb{I}_{N\times N} \end{cases} \tag{1.48}$$

The exponent term is:

$$\begin{aligned} \frac{1}{2}\dot{\boldsymbol{x}}'^T A^{-1}\dot{\boldsymbol{x}}' &= \frac{1}{2} \cdot \frac{m\hbar}{\mathrm{i}\Delta t} \cdot \left(-\frac{\mathrm{i}\Delta t}{\hbar}\right)^2 \sum_{l=1}^{N} \dot{x}_l^2 \\ &= \frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t \end{aligned} \tag{1.49}$$

So:

$$\begin{aligned} I &= \left(\frac{1}{2\pi\hbar}\right)^N \cdot (2\pi)^{\frac{N}{2}} \cdot \left(\frac{\mathrm{i}\Delta t}{m\hbar}\right)^{-\frac{N}{2}} \cdot \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right] \\ &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right] \end{aligned} \tag{1.50}$$

So we can get the integration without p:

$$\begin{aligned}\mathrm{i}G &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t - V(x_l)\Delta t\right] \\ &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau\, \mathcal{L}(x,\dot{x})\right]\end{aligned} \tag{1.51}$$

So the path-integral is proportional to :

$$\boxed{\mathrm{i}G \propto \int \mathcal{D}x \cdot e^{\frac{\mathrm{i}}{\hbar}S[x(t)]}} \tag{1.52}$$

where, $S[x(t)] = \int_{t_0}^{t} \mathrm{d}t\, \mathcal{L}(x,\dot{x})$ is action and $\mathcal{L}(x,\dot{x}) = \frac{m\dot{x}^2}{2} - V(x)$ is Lagrangean.

From the Path-integral in real space-time, we can get some information about Physics Picture:

(1) Each path is weighted with a $U(1)$ phase factor $e^{\frac{\mathrm{i}}{\hbar}S}$. The Quantum interference effect between different paths.

(2) Since $\hbar \sim 10^{-34}\,\mathrm{J}\cdot\mathrm{s}$, any "small change" in S (we change S to $S+\delta S$), will drastically lead to quantum destructive interference. So only the paths that satisfy $\delta S = 0$ make dominant contributions to the path-integral.

(3) Remarkably, $\delta S = 0$ is exactly Hamilton's Principle in classical mechanics. So the classical paths ($\delta S = 0$) dominate the path integral in the limit $\hbar \to 0$. In other words, in classical mechanics, as $\hbar \to 0$, it neglects the contribution of the integral over all other paths near the path with $\delta S = 0$. So we can get the conclusion:

$$\text{Quantum system} \xrightarrow{\hbar\to 0} \text{classical system}$$

Example 1.3.1. Free particles' Hamiltonian is $H = \frac{p^2}{2m}$.
Using this Hamiltonian, we can get the path-integral:

$$\mathrm{i}G = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right]. \tag{1.53}$$

we let:

$$I = \int \mathcal{D}x \exp\left[\frac{im}{2\hbar\Delta t}\sum_{l=1}^{N}(x_l^2 - 2x_l x_{l-1} + x_{l-1}^2)\right] \tag{1.54}$$

In order to use Gaussian integral:

$$\boxed{\int \prod_{n=1}^{N} \mathrm{d}x_n\, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{y}} = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} e^{\frac{1}{2}\boldsymbol{y}^T A^{-1}\boldsymbol{y}}} \tag{1.55}$$

we should rewrite the form of $\exp[\frac{im}{2\hbar\Delta t}\sum(x_l^2 - 2x_l x_{l-1} + x_{l-1}^2)]$, so we let:

$$A = \frac{-\mathrm{i}m}{\hbar\Delta t}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(N-1)\times(N-1)} \tag{1.56}$$

and:

$$\boldsymbol{x} = \begin{bmatrix} x_{N-1} \\ \vdots \\ x_1 \end{bmatrix}, \quad \boldsymbol{y} = \frac{-\mathrm{i}m}{\hbar\Delta t}\begin{bmatrix} -x_N \\ 0 \\ \vdots \\ \text{all zero} \\ \vdots \\ 0 \\ x_0 \end{bmatrix} \tag{1.57}$$

We get the new form of the exponent term:

$$\exp\left[\frac{\mathrm{i}m}{2\hbar\Delta t}\sum_{l=1}^{N-1}(x_l^2 - 2x_l x_{l-1} + x_{l-1}^2)\right] = \exp(-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{y})e^{\frac{\mathrm{i}m}{2\hbar\Delta t}(x_N^2 + x_0^2)} \tag{1.58}$$

Because of $\mathcal{D}x = \prod_{l=1}^{N-1}\mathrm{d}x_l$ without x_N and x_0, so:

$$\mathrm{i}G = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}}\cdot e^{\frac{im}{2\hbar\Delta t}(x_N^2 + x_0^2)}\int \mathcal{D}x\, e^{-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{y}} \tag{1.59}$$

So we can compute $(\det A) = N\cdot\left(\frac{-\mathrm{i}m}{\hbar\Delta t}\right)^N$ (Proof in B) and get:

$$\mathrm{i}G = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}}\cdot e^{\frac{\mathrm{i}m}{2\hbar\Delta t}(x_N^2 + x_0^2)}(2\pi)^{\frac{N}{2}}N^{-1/2}\left(\frac{-\mathrm{i}m}{\hbar\Delta t}\right)^{-\frac{N}{2}}\cdot e^{\frac{1}{2}\boldsymbol{y}^T A^{-1}\boldsymbol{y}} \tag{1.60}$$

Although A^{-1} is difficult to compute, we notice that $\boldsymbol{y} = \begin{bmatrix} -x_N \\ 0 \\ \vdots \\ 0 \\ x_0 \end{bmatrix}$ only have two non-zero elements, which locate in the first row and the last row respectively. So we only need to calculate the first and last columns of matrix A^{-1}, denoted $\boldsymbol{A}_1^{-1}$ and $\boldsymbol{A}_{N-1}^{-1}$, respectively (Proof in B):

$$\boldsymbol{A}_1^{-1} = \frac{\mathrm{i}\hbar\Delta t}{mN}\begin{bmatrix} N-1 \\ N-2 \\ \vdots \\ 1 \end{bmatrix}, \quad \boldsymbol{A}_{N-1}^{-1} = \frac{\mathrm{i}\hbar\Delta t}{mN}\begin{bmatrix} 1 \\ 2 \\ \vdots \\ N-1 \end{bmatrix} \tag{1.61}$$

The last term of the propagator(1.60) is:

$$-\frac{1}{2}\boldsymbol{y}^T A^{-1}\boldsymbol{y} = \frac{1}{2}\boldsymbol{y}^T[x_N \boldsymbol{A}_1^{-1} + x_0 \boldsymbol{A}_{N-1}^{-1}] \\ = \frac{\mathrm{i}m}{2\hbar\Delta t}\left[-(x_N^2 + x_0^2) + \frac{(x_N - x_0)^2}{N}\right] \tag{1.62}$$

So we can get the complete integral :

$$\mathrm{i}G = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} (2\pi)^{\frac{N}{2}} N^{-\frac{1}{2}} \left(\frac{-\mathrm{i}m}{\hbar\Delta t}\right)^{-\frac{N-1}{2}} e^{\frac{\mathrm{i}}{\hbar}\frac{m(x_N - x_0)^2}{2N\Delta t}} \\ = \left(\frac{m}{i2\pi\hbar\Delta t N}\right)^{\frac{1}{2}} e^{\frac{\mathrm{i}}{\hbar}\frac{m(x_N - x_0)^2}{2N\Delta t}} \tag{1.63}$$

where $t - t_0 = N\Delta t$, $x = x_N$. So the free particle's propagator is:

$$\boxed{\mathrm{i}G = \left[\frac{m}{i2\pi\hbar(t - t_0)}\right]^{\frac{1}{2}} \cdot e^{\frac{\mathrm{i}}{\hbar}\cdot\frac{m(x - x_0)^2}{2(t - t_0)}}} \tag{1.64}$$

Chapter 2

Stationary Phase Approximation (Semiclassical Approximation)

We have got the propagator iG for a partide to travel from an spacetime point (x_0, t_0) to a final spacetime point (x_f, t_f), with which is given by the Feynman path integral:

$$\mathrm{i}G = K(x_f, t_f; x_0, t_0) = \int \mathcal{D}[x(t)] \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S[x(t)]} \tag{2.1}$$

where:

- $x(t)$ is the passition position of the particle at time t, respresenting a possible path.
- $\mathcal{D}[x(t)]$ is the functional measure for integrating over all paths that satisfy the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$.
- $S[x(t)]$ is the action for a path $x(t)$;

This integral is infinite-dimensional and generally very difficult to calculate directly. So we need a effective method to approximate it. The stationary phase approximation provides a method for approximating it.

2.1 One-dimensional integral of stationary phase approximation

Consider a integral:

$$I = \int \mathrm{e}^{\mathrm{i}f(x)/a} \mathrm{d}x \tag{2.2}$$

where a is a small parameter. and the function $f(x)$ is real-valued and regular.

Our objective is to understant the physical picture of the startiomong phuse approximation for the propagutor ph path integral through this one dimensional integral stortionary phase approximation.

let's further define a new notation $\Theta(x)$ by:

$$\Theta(x) = \frac{1}{a} f(x) \tag{2.3}$$

$\Theta(x)$ is a phase angle, so

$$I = \int \mathrm{e}^{\mathrm{i}\Theta(x)} \mathrm{d}x \tag{2.4}$$

This integral can be " physically regarded as an interference experiment. Because each source at x contributes a phase factor $\mathrm{e}^{\mathrm{i}\Theta(x)}$, so the total integral I is the result of adding up all these infinite tiny vectors.

In order to compute the integral I, what we really need to do is to find "dominant contribution" to I.

Firstly, let's pick up a point x_0 and evaluate the integral near x_0. The vicinity of x_0 is given by $x \in (x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2})$. Within this small domain, we may linearize $\Theta(x)$ by:

$$\begin{aligned} \Theta(x) &\approx \Theta(x_0) + \left.\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right|_{x=x_0} (x - x_0) \\ &= \frac{f(x_0)}{a} + \frac{f'(x_0)}{a}(x - x_0) \end{aligned} \tag{2.5}$$

The contributions to I in the small domain near x_0 are given by:

$$\begin{aligned} I_{x_0}^{\Delta x}(x_0) &= \int_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}} \mathrm{d}x \, \mathrm{e}^{\mathrm{i}\Theta(x_0)} \cdot \mathrm{e}^{\frac{\mathrm{i}f'(x_0)}{a}(x-x_0)} \\ &= \mathrm{e}^{\mathrm{i}\Theta(x_0)} \int_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}} \mathrm{d}x \, \mathrm{e}^{\frac{\mathrm{i}f'(x_0)}{a}(x-x_0)} \end{aligned} \tag{2.6}$$

we let $\Theta(x_0) = \Theta_0$ and $f'(x_0) = f_0'$, so:

$$\begin{aligned} I_{(x_0)}^{\Delta x} &\sim \mathrm{e}^{\mathrm{i}\Theta_0} \frac{a}{\mathrm{i}f_0'} \cdot \left. \mathrm{e}^{\frac{\mathrm{i}f_0'}{a}(x-x_0)} \right|_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}} \\ &= \mathrm{e}^{\mathrm{i}\Theta_0} \frac{2a}{\mathrm{i}f_0'} \sin \frac{f_0' \Delta x}{2a}. \end{aligned} \tag{2.7}$$

let $\alpha = \frac{f_0' \Delta x}{2a}$, so:

$$I^{\Delta x}(x_0) \approx \mathrm{e}^{\mathrm{i}\Theta_0} \Delta x \cdot \frac{\sin \alpha}{\alpha} \tag{2.8}$$

Because a is a small parameter very small but nonzero parameter (in quantum mechanics, it corresponds to $\hbar$ being a very small but nonzero number), if $f_0' = 0$, α is strictly equal to 0 and it is not infinitely large near the zero point. So if we consider the case $f'(x_0) = 0$, we get:

$$I^{\Delta x}(x_0) = \mathrm{e}^{\mathrm{i}\Theta(x_0)} \cdot \Delta x \tag{2.9}$$

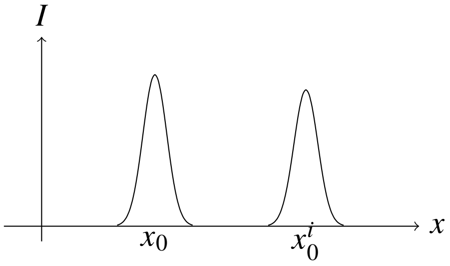
Physically, the result means that all U(1) phase in the vicinity of x_0 are completely the same. It's a perfect phase-coherence. The superposition of constant phases leads to linearly enhanced amplitude "Δx".

Next, let us consider $f'(x_0) \neq 0$. Because a is a very small parameter, $\alpha = \frac{f_0' \Delta x}{2a}$ is very large. So:

$$\frac{\sin \alpha}{\alpha} \to 0, \implies I^{\Delta x}(x_0) \approx 0 \tag{2.10}$$

Physically, the summation of all U(1) phases in the vicinity of x_0 leads to destructive interference. The parameter a is smaller, the destructive interference is more severe.

In conclusion, if we consider small enough but nozero parameter a, it is computationlly economic to merely focus on the integral contributions from the vicinity of these special point (denoted by a set $\{x_0^i\}$) that satisfy $f'(x_0^i) = 0$.



We consider the quadratic approximation in the vicinity of x_0:

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \\ &= f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \end{aligned} \tag{2.11}$$

As a result, the original integral I can be evaluated by:

$$\begin{aligned} I &\approx \sum_{\{x_0^i\}} I^{\Delta x}(x_0^i) \\ &= \sum_{\{x_0^i\}} \mathrm{e}^{\frac{\mathrm{i}}{a} f(x_0^i)} \int_{x_0^i - \frac{\Delta x}{2}}^{x_0^i + \frac{\Delta x}{2}} \mathrm{d}x \; \mathrm{e}^{\frac{\mathrm{i}}{2a} f''(x_0^i)(x - x_0^i)^2} \end{aligned} \tag{2.12}$$

Because parameter a is a very small number, far away the stationary point x_0^i, the contributions of $\mathrm{e}^{\frac{\mathrm{i}}{2a} f''(x_0^i)(x - x_0^i)^2}$ to the integral cancel each other out through destructive interference.

So in the above Gaussian integral near x_0, we can extend the integral bounds to infinity:

$$\begin{aligned} I &\approx \sum_{\{x_0^i\}} \mathrm{e}^{\frac{\mathrm{i} f(x_0^i)}{a}} \int_{-\infty}^{\infty} \mathrm{d}x \cdot \mathrm{e}^{\frac{\mathrm{i}}{2a} f''(x_0^i)(x - x_0^i)^2} \\ &= \sum_{\{x_0^i\}} \mathrm{e}^{\frac{\mathrm{i} f(x_0^i)}{a}} \sqrt{\frac{2\pi a \mathrm{i}}{f''(x_0^i)}} \end{aligned} \tag{2.13}$$

2.2 Semiclassical approximation of Feynman path integrals

We have discussed that if $\hbar$ tends to zero, then the quantum system will transitions to the classical system. We only need to treat $f(x)$ as $S[x(t)]$ and the parameter a as $\hbar$, then we can see the reason based on the discussion in the previous section. If $\hbar$ is zero, $I^{\Delta x}(x_0)$ equal to zero strictly for $x \neq x_0$ and is nonzero only at $x = x_0$. So we only need to consider the classical. path.

with $\delta S = 0$ and not need to consider the quantum fluctuation near the classical path. But if $\hbar$ is a very small but nonzero number, we need to consider the quantum fluctuation near the classical path. In other word, in classical mechanics, $\hbar = 0$, $I^{\Delta x} \propto \frac{\sin\alpha}{\alpha}$ is equal to zero in $(x_0 - \frac{\Delta x}{2}, x_0) \cup (x_0, x_0 + \frac{\Delta x}{2})$, because $\alpha = \frac{f_0' \Delta x}{2a}$ tends to infinity no matter how small the radius of this deleted neighbourhood is. But in quantum mechanics, $\hbar \sim 10^{-34}\mathrm{J}\cdot\mathrm{s}$, I is not equal to zero in the neighbourhood whose radius length matches to the order of magnitude of $\hbar$. Therefore, we cannot ignore the impact generated by I in this neighbourhood.

The Feynman path integral:

$$K(x_f, t_f; x_0, t_0) \propto \int \mathcal{D}[x(t)] \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S[x(t)]} \tag{2.14}$$

it can be regarded as path integral version of $\int \mathrm{d}x \mathrm{e}^{\frac{\mathrm{i}}{a} f(x)}$. $f(x)$ is replaced by Classical action S_c when $\delta S_c = 0$. Liking the previous section, we consider. the quadratic approximation in the vicinity of classical. path:

$$\begin{aligned} S &= S_c + \delta S + \frac{1}{2}\delta^2 S \\ &= S_c + \frac{1}{2}\delta^2 S. \end{aligned} \tag{2.15}$$

Now, we consider can decompose the path near classical path into the classical path $x_c(t)$ plus a quantum fluctuation $y(t)$ around it:

$$x(t) = x_c(t) + y(t) \tag{2.16}$$

$x(t)$ must satisfy the same boundary conditions, so:

$$\begin{cases} y(t_0) = 0 \\ y(t_f) = 0 \end{cases} \tag{2.17}$$

let's perform a functional Taylor expansion of the action $S[x(t)]$ around the classical path $x_c(t)$:

$$\begin{aligned} S[x(t)] = S[x_c + y] = S[x_c] &+ \int_{t_0}^{t_f} \mathrm{d}t \left.\frac{\delta S}{\delta x(t)}\right|_{x=x_c} y(t) \\ &+ \frac{1}{2}\int_{t_0}^{t_f} \mathrm{d}t_1 \int_{t_0}^{t_f} \mathrm{d}t_2 \left.\frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\right|_{x=x_c} y(t_1)y(t_2) + O(y^3) \end{aligned} \tag{2.18}$$

In semiclassical approximation, we assume the fluctuations y are small, so we neglect terms of $O(y^3)$ and higher. At the same time, $\delta S = 0$, so the action $S[x(t)]$:

$$S[x(t)] \approx S_c + \frac{1}{2}\int\int \mathrm{d}t_1 \mathrm{d}t_2 \; y(t_1) \cdot \left.\frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\right|_{x=x_c} \cdot y(t_2) \tag{2.19}$$

Now, let's solve the fluctuation term $\delta^2 S$:

$$\delta^2 S = \iint \mathrm{d}t_1 \mathrm{d}t_2 \cdot y(t_1) \cdot \left.\frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\right|_{x=x_c} \cdot y(t_2) \tag{2.20}$$

Consider a standard Lagrangian $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$, so the action is:

$$S = \int \mathrm{d}t \left[\frac{1}{2}m\dot{x}^2 - V(x)\right] \tag{2.21}$$

it's second variation is:

$$\delta^2 S = \int \mathrm{d}t \cdot \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}V''(x_c) \cdot y^2\right) \tag{2.22}$$

We substitute the approximated action back into the path integral. expression:

$$\begin{aligned} K &\approx \int \mathcal{D}[x(t)] \exp\left\{\frac{\mathrm{i}}{\hbar}\left[S_c + \frac{1}{2}\delta^2 S\right]\right\} \\ &= \int \mathcal{D}[x(t)] \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c} \exp\left[\int \mathrm{d}t \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}V''(x_c)y^2\right)\right] \end{aligned} \tag{2.23}$$

Now, we need to change the integration variable from an integral over all paths $x(t)$ to an integral over all fluctuations $y(t)$. Since $x_c(t)$ is a fixed. path, the path measure:

$$\mathcal{D}[x(t)] = \mathcal{D}[x_c(t) + y(t)] = \mathcal{D}[y(t)] \tag{2.24}$$

and S_c is a constant, it can be factored out of integral:

$$K \approx \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c} \cdot \int \mathcal{D}[y(t)] \cdot \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\int \mathrm{d}t\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}V''(x_c)y^2\right)} \tag{2.25}$$

$$= F(t_f, t_0) \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c}. \tag{2.26}$$

where:

1. $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c}$ is the Classical Phase Factor. It tells us that in the semiclassical approximation, the evolution of the system's quantum phase is dominated by the classical action. This is a bridge connecting classical and quantum mechanics.
2. $F(t_f, t_0) = \int \mathcal{D}[y(t)]\mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\delta^2 S}$ is the Quantum Fluctuation Prefactor. It describes the collective contribution of all the small quantum fluctuations around the classical path.

Now let's see a example: one dimensional free particle.

Example 2.2.1. For free particle:

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2. \tag{2.27}$$

First, we need to calculate the classical action. It is given by the Euler-Lagrange equertion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \tag{2.28}$$

For free particle:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \\ \frac{\partial \mathcal{L}}{\partial x} = 0 \end{cases} \tag{2.29}$$

so the equation of motion is:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = m\ddot{x} = 0 \tag{2.30}$$

we can get the general solution:

$$x_c(t) = at + b \tag{2.31}$$

we impose the boundary conditions:

$$\begin{cases} x_c(t_0) = x_0 \\ x_c(t_f) = x_f \end{cases} \tag{2.32}$$

to determin a and b:

$$\begin{cases} a = \frac{x_f - x_0}{t_f - t_0} \\ b = x_i - t_i\left(\frac{x_f - x_0}{t_f - t_0}\right) \end{cases} \tag{2.33}$$

the classical action is:

$$S_c = \int_{t_i}^{t_f} \mathrm{d}t \cdot \frac{1}{2}m(\dot{x}_c)^2 = \int_{t_i}^{t_f} \mathrm{d}t \cdot \frac{1}{2}ma^2 = \frac{m(x_f - x_0)^2}{2(t_f - t_0)} \tag{2.34}$$

So the classical phase factor is:

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c} = \mathrm{e}^{\frac{\mathrm{i}m(x_f - x_0)^2}{2\hbar(t_f - t_i)}} \tag{2.35}$$

now let's calculate the quantum fluctuation factor:

$$\begin{aligned} F(t_f, t_0) &= \int \mathcal{D}[y(t)] \exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_{t_0}^{t_f}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \mathrm{d}t\right\} \\ &= \int \prod_{i=1}^{N-1} \mathrm{d}y_i \exp\left[\frac{\mathrm{i}m}{2\hbar}\sum_{i=1}^{N}\frac{(y_i - y_{i-1})^2}{\Delta t^2}\Delta t\right] \end{aligned} \tag{2.36}$$

because $y_N = y_0 = 0$, we can rewrite the sum as:

$$\frac{\mathrm{i}m}{2\hbar}\sum_{i=1}^{N}\frac{(y_i - y_{i-1})^2}{\Delta t} = -\frac{1}{2}\boldsymbol{y}^T M \boldsymbol{y} \tag{2.37}$$

where

$$M = \frac{m}{\mathrm{i}\hbar\Delta t}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{(N-1)\times(N-1)}, \quad \boldsymbol{y} = \begin{pmatrix} y_{N-1} \\ \vdots \\ y_1 \end{pmatrix} \tag{2.38}$$

and use the Gaussion integral:

$$
\begin{aligned}
F(t_f, t_0) &= (2\pi)^{(N-1)/2}(\det M)^{-1/2} \\
&= (2\pi)^{(N-1)/2}\left(\left(\frac{\mathrm{i}\hbar\Delta t}{m}\right)^{N-1} N\right)^{-1/2}
\end{aligned} \tag{2.39}
$$

So we can get the complete propagator of free partide:

$$
\begin{aligned}
K &= J \cdot \left(\frac{2\pi\mathrm{i}\hbar\Delta t}{m}\right)^{N/2} \cdot \left(\frac{m}{2\pi\mathrm{i}N\Delta t}\right)^{1/2} \cdot \mathrm{e}^{\frac{\mathrm{i}S_c}{\hbar}} \\
&= \left(\frac{m}{2\pi\mathrm{i}\hbar\Delta t}\right)^{N/2} \cdot \left(\frac{2\pi\mathrm{i}\hbar\Delta t}{m}\right)^{N/2} \cdot \left(\frac{m}{2\pi\mathrm{i}(t_f - t_0)}\right)^{1/2} \cdot \mathrm{e}^{\frac{\mathrm{i}S_c}{\hbar}} \\
&= \left(\frac{m}{2\pi\mathrm{i}(t_f - t_0)}\right)^{1/2} \cdot \mathrm{e}^{\frac{\mathrm{i}S_c}{\hbar}}
\end{aligned} \tag{2.40}
$$

where, $S_c = \frac{m}{2}\frac{(x_f - x_0)^2}{t_f - t_0}$.

Part II

Quantum Spins, Coherent-state Path Integral, and Topological Terms

Chapter 3

Quantum Spin

We begin by considering the Hilbert space $\mathcal{H}$ for a single quantum spin-1/2 particle. This is a two-dimensional complex vector space.

The conventional approach is to use an orthonormal basis formed by the eigenvectors of the spin operator along a chosen axis, typically the z-axis, denoted $\hat{S}_z$.

$$\hat{S}_z|\uparrow\rangle = +\frac{\hbar}{2}|\uparrow\rangle \quad \text{and} \quad \hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle \tag{3.1}$$

Here, $|\uparrow\rangle$ represents the ”spin up” state and $|\downarrow\rangle$ represents the ”spin down” state. These two states form a complete orthonormal basis, satisfying:

- **Orthogonality:** $\langle\uparrow|\downarrow\rangle = 0$
- **Normalization:** $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$
- **Completeness:** $|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \hat{\mathbb{I}}$

where $\hat{\mathbb{I}}$ is the identity operator in $\mathcal{H}$.

But this simple, discrete basis has two drawbacks:

- This discrete parametrized complete-set is not convenient for constructing the path integral formalism of quantum spins.
- $SU(2)$ spin symmetry is broken either explicitly or not manifest.

3.1 Spin Coherent States

A general, normalized state $|\psi\rangle$ in the spin-1/2 Hilbert space can be written as a complex linear combination of the basis states:

$$|\psi\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle \tag{3.2}$$

where $z_1, z_2 \in \mathbb{C}$ are complex coefficients.

The normalization condition $\langle\psi|\psi\rangle = 1$ imposes a constraint on these coefficients:

$$\langle\psi|\psi\rangle = (|z_1|^2 + |z_2|^2) = 1 \tag{3.3}$$

A complex number $z = x + \mathrm{i}y$ has two real parameters. Therefore, the pair (z_1, z_2) is defined by four real parameters. The normalization condition $|z_1|^2 + |z_2|^2 = 1$ removes one degree of freedom, leaving three.

Furthermore, in quantum mechanics, the overall phase of a state vector is unphysical. The states $|\psi\rangle$ and $\mathrm{e}^{\mathrm{i}\gamma}|\psi\rangle$ (for any real γ) represent the same physical state (i.e., they belong to the same ray in Hilbert space). This "gauge freedom" removes one more degree of freedom.

This leaves $4 - 1 - 1 = 2$ real, physical degrees of freedom. This is a crucial observation: the state space of a spin-1/2 particle is topologically equivalent to the surface of a 2D sphere, which is also parameterized by two angles (like latitude and longitude).

We can explicitly parameterize z_1 and z_2 using two angles, θ and ϕ, which will map directly to the surface of a sphere. A standard (but not unique) parametrization for the spin coherent state, labeled by a unit vector $\boldsymbol{n}$, is:

$$|\boldsymbol{n}\rangle \equiv |\theta, \phi\rangle = \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.4}$$

Here, the spherical coordinate angles have the domains $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. We can easily verify that this state is normalized:

$$\langle\boldsymbol{n}|\boldsymbol{n}\rangle = \left|\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\right|^2 + \left|\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\right|^2 = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1 \tag{3.5}$$

This set of states $\{|\boldsymbol{n}\rangle\}$ is continuously parameterized by the angles (θ, ϕ), addressing the first drawback of the discrete basis.

3.2 Physical Interpretation: The Bloch Sphere

To understand the physical meaning of θ and ϕ, we compute the expectation value of the vector spin operator $\hat{\boldsymbol{S}}$ in the state $|\boldsymbol{n}\rangle$. We will set $\hbar = 1$ from here on for simplicity. The spin operator is $\hat{\boldsymbol{S}} = \frac{1}{2}\hat{\boldsymbol{\sigma}}$, where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.6}$$

In the $|\uparrow\rangle, |\downarrow\rangle$ basis, $|\boldsymbol{n}\rangle$ is represented by the column vector:

$$|\boldsymbol{n}\rangle = \begin{pmatrix} \cos(\theta/2)\, \mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\, \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \quad \Longrightarrow \quad \langle \boldsymbol{n}| = \begin{pmatrix} \cos(\theta/2)\, \mathrm{e}^{\mathrm{i}\phi/2} & \sin(\theta/2)\, \mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix} \tag{3.7}$$

Expectation value of $\hat{S}_z$:

$$\begin{aligned} \langle \hat{S}_z \rangle &= \langle \boldsymbol{n}| \left(\frac{1}{2}\hat{\sigma}_z\right) |\boldsymbol{n}\rangle = \frac{1}{2} \begin{pmatrix} \cos(\theta/2)\, \mathrm{e}^{\mathrm{i}\phi/2} & \sin(\theta/2)\, \mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta/2)\, \mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\, \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \\ &= \frac{1}{2}\left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}\cos(\theta) \end{aligned} \tag{3.8}$$

Expectation value of $\hat{S}_x$:

$$\begin{aligned} \langle \hat{S}_x \rangle &= \langle \boldsymbol{n}| \left(\frac{1}{2}\hat{\sigma}_x\right) |\boldsymbol{n}\rangle = \frac{1}{2} \left(\dots\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\dots\right) \\ &= \frac{1}{2}\left(\cos(\theta/2)\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} + \sin(\theta/2)\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2}\right) \\ &= \frac{1}{2}\cos(\theta/2)\sin(\theta/2)\left(\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}\right) = \left(\frac{1}{2}\sin\theta\right)\left(\frac{\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}}{2}\right) = \frac{1}{2}\sin\theta\cos\phi \end{aligned} \tag{3.9}$$

Expectation value of $\hat{S}_y$:

$$\begin{aligned} \langle \hat{S}_y \rangle &= \langle \boldsymbol{n}| \left(\frac{1}{2}\hat{\sigma}_y\right) |\boldsymbol{n}\rangle = \frac{1}{2} \left(\dots\right) \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \left(\dots\right) \\ &= \frac{1}{2}\left(\cos(\theta/2)(-\mathrm{i})\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} + \sin(\theta/2)(\mathrm{i})\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2}\right) \\ &= \frac{1}{2}\cos(\theta/2)\sin(\theta/2)\left(-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\right) = \left(\frac{1}{2}\sin\theta\right)\left(\frac{\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi}}{2\mathrm{i}}\right) = \frac{1}{2}\sin\theta\sin\phi \end{aligned} \tag{3.10}$$

Combining these results, the expectation value of the spin vector is:

$$\langle \hat{\boldsymbol{S}} \rangle = \frac{1}{2}\left(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\right) \tag{3.11}$$

This is a vector of length $S = 1/2$ pointing in the direction specified by the unit vector $\boldsymbol{n}$:

$$\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{3.12}$$

Thus, the state $|\boldsymbol{n}\rangle$ is the quantum state that "points" in the classical direction $\boldsymbol{n}$. This direction vector lives on the surface of a unit sphere, known as the **Bloch Sphere**.

This formalism treats all directions $\boldsymbol{n}$ on an equal footing, making the SU(2) rotational symmetry manifest. This addresses the second drawback of the discrete basis.

3.2.1 Coherent State as an Eigenvector

The eigenvector equation:

$$(\hat{\boldsymbol{S}} \cdot \boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}|\boldsymbol{n}\rangle \tag{3.13}$$

This equation signifies that the coherent state $|\boldsymbol{n}\rangle$ is, by definition, the "spin up" eigenvector of the spin operator projected along its own pointing direction $\boldsymbol{n}$, with the eigenvalue $+1/2$ (with $\hbar = 1$).

Proof. We first construct the operator $\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$ in matrix form:

$$\begin{aligned}
\hat{\boldsymbol{S}} \cdot \boldsymbol{n} &= \hat{S}_x n_x + \hat{S}_y n_y + \hat{S}_z n_z \\
&= \frac{1}{2}(\hat{\sigma}_x n_x + \hat{\sigma}_y n_y + \hat{\sigma}_z n_z) \\
&= \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\theta\cos\phi + \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\theta\right] \\
&= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - \mathrm{i}\sin\phi) \\ \sin\theta(\cos\phi + \mathrm{i}\sin\phi) & -\cos\theta \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix}
\end{aligned} \tag{3.14}$$

Now, we apply this operator to the coherent state vector $|\boldsymbol{n}\rangle$:

$$(\hat{\boldsymbol{S}} \cdot \boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \tag{3.15}$$

We compute the top and bottom components of the resulting vector separately.

Top component:

$$\begin{aligned}
&\frac{1}{2}\left[\cos\theta\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} + \sin\theta\mathrm{e}^{-\mathrm{i}\phi}\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\right] \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\phi/2}\left[\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right] \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\phi/2}\left[\cos(\theta - \theta/2)\right] \quad (\text{using } \cos(A-B) \text{ identity}) \\
&= \frac{1}{2}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}
\end{aligned} \tag{3.16}$$

This is precisely $\frac{1}{2}$ times the top component of $|\boldsymbol{n}\rangle$.

Bottom component:

$$\begin{aligned}
&\frac{1}{2}\left[\sin\theta \mathrm{e}^{\mathrm{i}\phi}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} - \cos\theta\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\right] \\
&= \frac{1}{2}\mathrm{e}^{\mathrm{i}\phi/2}\left[\sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2)\right] \\
&= \frac{1}{2}\mathrm{e}^{\mathrm{i}\phi/2}\left[\sin(\theta-\theta/2)\right] \quad \text{(using } \sin(A-B) \text{ identity)} \\
&= \frac{1}{2}\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}
\end{aligned} \tag{3.17}$$

This is precisely $\frac{1}{2}$ times the bottom component of $|\boldsymbol{n}\rangle$.
Combining both components, we have shown:

$$(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}\begin{pmatrix}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\end{pmatrix} = \frac{1}{2}|\boldsymbol{n}\rangle \tag{3.18}$$

This completes the proof. □

3.3 Gauge Choice and Topological Singularities

The parametrization in Eq. (3.4) is not unique, and it hides a subtle topological problem.

- **At the North Pole** ($\theta = 0$): The direction $\boldsymbol{n}$ is $(0,0,1)$. The angle ϕ is ill-defined. Our formula gives $|\theta = 0\rangle = \cos(0)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle + \sin(0)\cdots = \mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle$. The state vector itself depends on the meaningless angle ϕ. This is a **singularity**.
- **At the South Pole** ($\theta = \pi$): The direction $\boldsymbol{n}$ is $(0,0,-1)$. Our formula gives $|\theta = \pi\rangle = \cos(\pi/2)\cdots + \sin(\pi/2)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle = \mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle$. This is also singular.

This is analogous to the problem of creating a flat map of the Earth: you cannot do so without singularities (e.g., at the poles) or cuts.

We can "fix" the singularity at one pole by making a ϕ-dependent gauge choice (i.e., multiplying by an overall phase $\mathrm{e}^{\mathrm{i}\gamma(\phi)}$).

Choice 1: Regular at North Pole. Let's choose an overall phase $\gamma = \phi/2$. The new state, $|\boldsymbol{n}\rangle_N$, is:

$$|\boldsymbol{n}\rangle_N = \mathrm{e}^{\mathrm{i}\phi/2}|\boldsymbol{n}\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi}|\downarrow\rangle \tag{3.19}$$

- At the North Pole ($\theta = 0$): $|\boldsymbol{n}\rangle_N = \cos(0)|\uparrow\rangle + \sin(0)\cdots = |\uparrow\rangle$. This is now regular and well-defined.
- At the South Pole ($\theta = \pi$): $|\boldsymbol{n}\rangle_N = \cos(\pi/2)|\uparrow\rangle + \sin(\pi/2)\mathrm{e}^{\mathrm{i}\phi}|\downarrow\rangle = \mathrm{e}^{\mathrm{i}\phi}|\downarrow\rangle$. The singularity has been "pushed" to the South Pole.

Choice 2: Regular at South Pole. Let's choose $\gamma = -\phi/2$. The new state, $|\boldsymbol{n}\rangle_S$, is:

$$|\boldsymbol{n}\rangle_S = \mathrm{e}^{-\mathrm{i}\phi/2}|\boldsymbol{n}\rangle = \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle \tag{3.20}$$

This state is regular at the South Pole ($|\boldsymbol{n}\rangle_S = |\downarrow\rangle$) but singular at the North Pole.

This unavoidable singularity is topological in nature and is the origin of the **Berry Phase**, or the "topological term," in the coherent-state path integral.

3.4 Over-Completeness and Orthogonality

The set of all coherent states $\{|\boldsymbol{n}\rangle\}$ for all $\boldsymbol{n}$ on the sphere is an **over-complete** basis. The Hilbert space is only 2-dimensional, but we have an infinite, continuous set of states. This means the states are not, in general, orthogonal.

$$\langle \boldsymbol{n}'|\boldsymbol{n}\rangle \neq 0 \quad \text{for } \boldsymbol{n}' \neq \boldsymbol{n} \text{ and } \boldsymbol{n}' \neq -\boldsymbol{n} \tag{3.21}$$

A special exception, as noted in the text, is for antipodal states.

3.4.1 Orthogonality of Antipodal States

Let us prove that $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle = 0$. The antipodal point $-\boldsymbol{n}$ corresponds to the angles $(\theta', \phi') = (\pi - \theta, \phi + \pi)$.

We write the state $|-\boldsymbol{n}\rangle$ using Eq. (3.4):

$$\begin{aligned} |-\boldsymbol{n}\rangle &= \cos\left(\frac{\pi-\theta}{2}\right)\mathrm{e}^{-\mathrm{i}(\phi+\pi)/2}|\uparrow\rangle + \sin\left(\frac{\pi-\theta}{2}\right)\mathrm{e}^{\mathrm{i}(\phi+\pi)/2}|\downarrow\rangle \\ &= \cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\pi/2}|\uparrow\rangle + \sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\pi/2}|\downarrow\rangle \end{aligned} \tag{3.22}$$

Using $\cos(\pi/2 - x) = \sin(x)$, $\sin(\pi/2 - x) = \cos(x)$, $\mathrm{e}^{-\mathrm{i}\pi/2} = -\mathrm{i}$, and $\mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}$:

$$|-\boldsymbol{n}\rangle = \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}(-\mathrm{i})|\uparrow\rangle + \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}(\mathrm{i})|\downarrow\rangle \tag{3.23}$$

Now we compute the inner product $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle$:

$$\begin{aligned} &\langle -\boldsymbol{n}|\boldsymbol{n}\rangle \\ &= \left(\mathrm{i}\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\langle\uparrow| + (-\mathrm{i})\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\langle\downarrow|\right)\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle\right) \\ &= \mathrm{i}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2} + (-\mathrm{i})\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} \\ &= \mathrm{i}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \mathrm{i}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ &= 0 \end{aligned} \tag{3.24}$$

This confirms that antipodal states are orthogonal, as expected. For example, $|\boldsymbol{n} = \hat{z}\rangle = |\uparrow\rangle$ is orthogonal to $|\boldsymbol{n} = -\hat{z}\rangle = |\downarrow\rangle$.

Distinction Between $|-\boldsymbol{n}\rangle$ and $-|\boldsymbol{n}\rangle$

It is a common point of confusion to mistake the antipodal state $|-\boldsymbol{n}\rangle$ for the state $-|\boldsymbol{n}\rangle$. We must justify that, in general, $|-\boldsymbol{n}\rangle \neq -|\boldsymbol{n}\rangle$.

From our derivation in the previous section, the antipodal state is:

$$|-\boldsymbol{n}\rangle = \mathrm{i}\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle - \mathrm{i}\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.25}$$

In contrast, the state $-|\boldsymbol{n}\rangle$ is:

$$-|\boldsymbol{n}\rangle = -\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle - \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.26}$$

By simple inspection, these two state vectors are clearly not identical. They are, in fact, orthogonal to each other, as we just proved $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle = 0$. If $|-\boldsymbol{n}\rangle$ were equal to $-|\boldsymbol{n}\rangle$, then we would have $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle = \langle -\boldsymbol{n}| - (-\boldsymbol{n})\rangle = -1 \cdot \langle -\boldsymbol{n}| - \boldsymbol{n}\rangle = -1$, which contradicts our result of 0 (unless the state is null, which is not the case).

The state $|-\boldsymbol{n}\rangle$ represents a spin pointing in the *opposite direction* (e.g., spin down), while $-|\boldsymbol{n}\rangle$ represents the *same physical state* as $|\boldsymbol{n}\rangle$ but with a phase shift of π (since $\mathrm{e}^{\mathrm{i}\pi} = -1$).

3.5 Completeness Relation

Despite being over-complete, the spin coherent states provide a resolution of the identity operator $\hat{\mathbb{I}}$.

Let's call the integral $J = \int \mathrm{d}\Omega\, |\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta\, |\boldsymbol{n}\rangle\langle\boldsymbol{n}|$.

1. Integrate over ϕ: The off-diagonal terms depend on $\mathrm{e}^{\pm\mathrm{i}\phi}$.

$$\int_0^{2\pi} \mathrm{e}^{\pm\mathrm{i}\phi}\, \mathrm{d}\phi = 0 \tag{3.27}$$

The diagonal terms are independent of ϕ.

$$\int_0^{2\pi} 1\, \mathrm{d}\phi = 2\pi \tag{3.28}$$

After integrating over ϕ, the matrix J becomes diagonal:

$$J = \int_0^{\pi} \sin\theta\, \mathrm{d}\theta \begin{pmatrix} 2\pi\cos^2(\theta/2) & 0 \\ 0 & 2\pi\sin^2(\theta/2) \end{pmatrix} \tag{3.29}$$

2. Integrate over θ: Both integrals evaluate to 2π. Thus, the full integral is:

$$J = \int \mathrm{d}\Omega \, |\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \begin{pmatrix} 2\pi & 0 \\ 0 & 2\pi \end{pmatrix} = 2\pi \, \hat{\mathbb{I}} \tag{3.30}$$

Dividing by 2π, we arrive at the completeness relation:

$$\frac{1}{2\pi} \int \mathrm{d}\Omega \, |\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \hat{\mathbb{I}} \tag{3.31}$$

This relation is the foundation for the coherent-state path integral. It allows us to insert the identity operator at infinitesimally small time steps, t_j, as an integral over the Bloch sphere: $\hat{\mathbb{I}} = \int \frac{\mathrm{d}\Omega_j}{2\pi} |\boldsymbol{n}_j\rangle\langle\boldsymbol{n}_j|$. Summing over all paths becomes an integral over all $\boldsymbol{n}_j$ at all times t_j.

Part III

Appendix

Appendix A

Multivariate Gaussian Integral

The multivariate Gaussian integral:

$$\boxed{I = \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left(-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y}\right) = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right)} \tag{A.1}$$

where:

- $\boldsymbol{x}$ and $\boldsymbol{y}$ are N-dimensional column vectors.
- A is an $N \times N$ real, symmetric, and positive-definite matrix.
- The notation $\int \prod_{n=1}^{N} \mathrm{d}x_n$ denotes integration over all components of $\boldsymbol{x}$ from $-\infty$ to $+\infty$.

A.1 Proof of the Multivariate Gaussian Integral

The proof relies on the assumptions that A is symmetric ($A^\top = A$) and positive-definite (all eigenvalues are positive), which ensures the integral converges. The proof proceeds in several key steps.

Step 1: Completing the Square

The primary technique is to complete the square for the quadratic form in the exponent. We want to rewrite the argument of the exponential, $-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y}$, into the form $-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^\top A(\boldsymbol{x}-\boldsymbol{x}_0) + C$, where $\boldsymbol{x}_0$ and C are constants with respect to $\boldsymbol{x}$.

Expanding this target form, we get:

$$\begin{aligned} -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^\top A(\boldsymbol{x}-\boldsymbol{x}_0) &= -\frac{1}{2}(\boldsymbol{x}^\top - \boldsymbol{x}_0^\top)A(\boldsymbol{x}-\boldsymbol{x}_0) \\ &= -\frac{1}{2}(\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top A\boldsymbol{x}_0 - \boldsymbol{x}_0^\top A\boldsymbol{x} + \boldsymbol{x}_0^\top A\boldsymbol{x}_0) \end{aligned} \tag{A.2}$$

Since A is symmetric ($A = A^\top$), the scalar term $\boldsymbol{x}_0^\top A\boldsymbol{x}$ is equal to its own transpose: $(\boldsymbol{x}_0^\top A\boldsymbol{x})^\top = \boldsymbol{x}^\top A^\top\boldsymbol{x}_0 = \boldsymbol{x}^\top A\boldsymbol{x}_0$. Thus, the two cross-terms are equal.

$$-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^\top A(\boldsymbol{x}-\boldsymbol{x}_0) = -\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} + \boldsymbol{x}^\top A\boldsymbol{x}_0 - \frac{1}{2}\boldsymbol{x}_0^\top A\boldsymbol{x}_0 \tag{A.3}$$

Comparing this to the original exponent, $-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y}$, we can equate the terms linear in $\boldsymbol{x}$:

$$-\boldsymbol{x}^\top\boldsymbol{y} = \boldsymbol{x}^\top A\boldsymbol{x}_0 \implies A\boldsymbol{x}_0 = -\boldsymbol{y} \tag{A.4}$$

Since A is positive-definite, it is invertible. We can solve for $\boldsymbol{x}_0$:

$$\boldsymbol{x}_0 = -A^{-1}\boldsymbol{y} \tag{A.5}$$

With this definition of $\boldsymbol{x}_0$, the original exponent can be written as:

$$-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y} = -\frac{1}{2}(\boldsymbol{x}+A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x}+A^{-1}\boldsymbol{y}) + \frac{1}{2}(A^{-1}\boldsymbol{y})^\top A(A^{-1}\boldsymbol{y}) \tag{A.6}$$

Let's simplify the constant term (the term not involving $\boldsymbol{x}$):

$$\begin{aligned}\frac{1}{2}(A^{-1}\boldsymbol{y})^\top A(A^{-1}\boldsymbol{y}) &= \frac{1}{2}\boldsymbol{y}^\top (A^{-1})^\top AA^{-1}\boldsymbol{y} \\ &= \frac{1}{2}\boldsymbol{y}^\top A^{-1}AA^{-1}\boldsymbol{y} \quad (\text{since } (A^{-1})^\top = (A^\top)^{-1} = A^{-1}) \\ &= \frac{1}{2}\boldsymbol{y}^\top IA^{-1}\boldsymbol{y} = \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\end{aligned} \tag{A.7}$$

So, the exponent is:

$$-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y} = -\frac{1}{2}(\boldsymbol{x}+A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x}+A^{-1}\boldsymbol{y}) + \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y} \tag{A.8}$$

Step 2: Change of Variables (Translation)

Substituting the completed square back into the integral:

$$I = \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left[-\frac{1}{2}(\boldsymbol{x}+A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x}+A^{-1}\boldsymbol{y}) + \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right] \tag{A.9}$$

The term $\exp(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y})$ is constant with respect to $\boldsymbol{x}$ and can be factored out of the integral:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left[-\frac{1}{2}(\boldsymbol{x}+A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x}+A^{-1}\boldsymbol{y})\right] \tag{A.10}$$

Now, we perform a change of variables. Let $\boldsymbol{z} = \boldsymbol{x} + A^{-1}\boldsymbol{y}$. This is a simple translation of the coordinate system. The differential element $\prod \mathrm{d}x_n$ transforms as $\prod \mathrm{d}z_n$, as the Jacobian of this transformation is 1. The limits of integration remain from $-\infty$ to $+\infty$. The integral becomes:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \int \prod_{n=1}^{N} \mathrm{d}z_n \, \exp\left(-\frac{1}{2}\boldsymbol{z}^\top A\boldsymbol{z}\right) \tag{A.11}$$

The problem is now reduced to evaluating the simpler, centered Gaussian integral:

$$I_0 = \int \prod \mathrm{d}z_n \, \exp(-\frac{1}{2}\boldsymbol{z}^\top A\boldsymbol{z}) \tag{A.12}$$

Step 3: Diagonalization

To compute I_0, we diagonalize the matrix A. Since A is a real symmetric matrix, it is orthogonally diagonalizable:

$$A = PDP^\top \tag{A.13}$$

where P is an orthogonal matrix ($PP^\top = P^\top P = I$) whose columns are the orthonormal eigenvectors of A, and D is a diagonal matrix whose entries are the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. Substituting this into the quadratic form $\boldsymbol{z}^\top A\boldsymbol{z}$:

$$\boldsymbol{z}^\top A\boldsymbol{z} = \boldsymbol{z}^\top (PDP^\top)\boldsymbol{z} = (\boldsymbol{z}^\top P)D(P^\top \boldsymbol{z}) = (P^\top \boldsymbol{z})^\top D(P^\top \boldsymbol{z}) \tag{A.14}$$

We perform another change of variables. Let $\boldsymbol{w} = P^\top \boldsymbol{z}$. This transformation corresponds to a rotation of the coordinate system. The Jacobian determinant is $|\det(P^\top)| = 1$, so the volume element is unchanged: $\prod \mathrm{d}z_n = \prod \mathrm{d}w_n$. The quadratic form simplifies to:

$$\boldsymbol{w}^\top D\boldsymbol{w} = \sum_{i=1}^{N} \lambda_i w_i^2 \tag{A.15}$$

This is because D is a diagonal matrix.

Step 4: Computing the Decoupled Integral

The integral I_0 now becomes:

$$I_0 = \int \prod_{n=1}^{N} \mathrm{d}w_n \, \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \lambda_i w_i^2\right) \tag{A.16}$$

The exponential of a sum is the product of exponentials, which allows us to separate the multi-dimensional integral into a product of N one-dimensional integrals:

$$I_0 = \int \prod_{n=1}^{N} \mathrm{d}w_n \prod_{i=1}^{N} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right) = \prod_{i=1}^{N}\left(\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right) \mathrm{d}w_i\right) \tag{A.17}$$

We use the standard formula for a 1D Gaussian integral: $\int_{-\infty}^{\infty} \exp(-au^2)\mathrm{d}u = \sqrt{\pi/a}$. In our case, $a = \lambda_i/2$.

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right) \mathrm{d}w_i = \sqrt{\frac{\pi}{\lambda_i/2}} = \sqrt{\frac{2\pi}{\lambda_i}} \tag{A.18}$$

Multiplying these N results together:

$$I_0 = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} = (2\pi)^{\frac{N}{2}} \prod_{i=1}^{N} (\lambda_i)^{-\frac{1}{2}} = (2\pi)^{\frac{N}{2}} \left(\prod_{i=1}^{N} \lambda_i\right)^{-\frac{1}{2}} \tag{A.19}$$

The determinant of a matrix is equal to the product of its eigenvalues. Thus, $\det A = \det D = \prod_{i=1}^{N} \lambda_i$.

$$I_0 = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}} \tag{A.20}$$

Step 5: Combining the Results

Finally, we substitute the value of I_0 back into our expression for I from Step 2:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \cdot I_0 = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right)(2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \tag{A.21}$$

Rearranging the terms yields the final result:

$$I = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \tag{A.22}$$

This completes the proof.

Appendix B

Calculation of the Determinant and Inverse of the a Special Matrix

Let the given $n \times n$ matrix be denoted by A_n.

$$A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{n\times n} \tag{B.1}$$

B.1 Calculation of the Determinant

We will find the determinant by establishing a recurrence relation. Let $D_n = \det(A_n)$ be the determinant of the $n \times n$ version of this matrix.

Determinants for Small Sizes

We compute the determinant for small values of n to identify a pattern.

- For $n = 1$:

$$A_1 = \begin{bmatrix} 2 \end{bmatrix} \implies D_1 = \det(A_1) = 2 \tag{B.2}$$

- For $n = 2$:

$$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \implies D_2 = \det(A_2) = (2)(2) - (-1)(-1) = 3 \tag{B.3}$$

- For $n = 3$:

$$\begin{aligned} A_3 &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ D_3 &= 2 \cdot \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - (-1) \cdot \det\begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix} \\ &= 2D_2 + ((-1)(2) - (-1)(0)) = 2(3) - 2 = 4 \end{aligned} \tag{B.4}$$

The sequence of determinants $D_1 = 2, D_2 = 3, D_3 = 4$ suggests the pattern $D_n = n + 1$.

Recurrence Relation

We use cofactor expansion along the first row of A_n to derive a general recurrence relation for $D_n = \det(A_n)$.

$$D_n = 2 \cdot \det \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}_{(n-1)\times(n-1)} - (-1) \cdot \det \begin{pmatrix} -1 & -1 & 0 & \dots \\ 0 & 2 & -1 & \\ 0 & -1 & 2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}_{(n-1)\times(n-1)} \tag{B.5}$$

The first sub-determinant is simply D_{n-1}. For the second sub-determinant, we perform a cofactor expansion along its first column, which yields $-1 \cdot D_{n-2}$.

$$\begin{aligned} D_n &= 2D_{n-1} - (-1)(-1 \cdot D_{n-2}) \\ D_n &= 2D_{n-1} - D_{n-2} \end{aligned} \tag{B.6}$$

This recurrence is valid for $n \geq 3$. We check if our hypothesized formula $D_n = n + 1$ satisfies this recurrence.

$$2D_{n-1} - D_{n-2} = 2((n-1)+1) - ((n-2)+1) = 2n - (n-1) = n + 1 = D_n \tag{B.7}$$

The formula holds for the base cases and satisfies the recurrence, so it is correct by induction.

Final Result

The given matrix A has size $n = N - 1$. Therefore, its determinant is:

$$\det(A) = D_{N-1} = (N-1) + 1 = N \tag{B.8}$$

B.2 Calculation of the Inverse of the Matrix

Let $B = A^{-1}$. The j-th column of B, denoted by the vector $\boldsymbol{b}_j$, is the solution to the system $A\boldsymbol{b}_j = \boldsymbol{e}_j$, where $\boldsymbol{e}_j$ is the j-th standard basis vector. Writing this out for the i-th component (where b_{ij} is the (i, j)-th element of B) gives the system of equations:

$$-b_{i-1,j} + 2b_{i,j} - b_{i+1,j} = \delta_{ij} \quad \text{for } i, j = 1, \dots, N-1 \tag{B.9}$$

with boundary conditions $b_{0,j} = 0$ and $b_{N,j} = 0$.

Homogeneous Solution

For $i \neq j$, the equation is homogeneous:

$$-b_{i-1,j} + 2b_{i,j} - b_{i+1,j} = 0 \tag{B.10}$$

This is a linear recurrence relation with characteristic equation $r^2 - 2r + 1 = 0$, or $(r-1)^2 = 0$. This has a repeated root $r = 1$, so the general solution is linear in i:

$$b_{i,j} = C_1 + C_2 i \tag{B.11}$$

We apply this solution to two regions.

Region 1: $1 \le i \le j$. The boundary condition $b_{0,j} = 0$ implies $C_1 + C_2(0) = 0$, so $C_1 = 0$. The solution has the form:

$$b_{i,j} = C \cdot i \tag{B.12}$$

Region 2: $j \le i \le N-1$. The boundary condition $b_{N,j} = 0$ implies $D_1 + D_2(N) = 0$, so $D_1 = -D_2 N$. The solution is $b_{i,j} = -D_2 N + D_2 i = D_2(i-N)$. Letting $D = -D_2$, the solution has the form:

$$b_{i,j} = D \cdot (N - i) \tag{B.13}$$

Stitching the Solutions

The full solution is given by:

$$b_{i,j} = \begin{cases} C \cdot i & \text{if } i \le j \\ D \cdot (N-i) & \text{if } i \ge j \end{cases} \tag{B.14}$$

The constants C and D are found by satisfying two conditions at $i = j$.

1. Continuity at $i = j$: The two forms must be equal.

$$C \cdot j = D \cdot (N-j) \implies D = C\frac{j}{N-j} \tag{B.15}$$

2. The inhomogeneous equation at $i = j$:

$$-b_{j-1,j} + 2b_{j,j} - b_{j+1,j} = 1 \tag{B.16}$$

Substituting the piecewise solutions into the inhomogeneous equation:

$$\begin{aligned} -C(j-1) + 2(Cj) - D(N-(j+1)) &= 1 \\ C\left[-(j-1) + 2j - \frac{j}{N-j}(N-j-1)\right] &= 1 \\ C\left[j + 1 - \frac{jN - j^2 - j}{N-j}\right] &= 1 \\ C\left[\frac{(j+1)(N-j) - (jN - j^2 - j)}{N-j}\right] &= 1 \\ C\left[\frac{jN - j^2 + N - j - jN + j^2 + j}{N-j}\right] &= 1 \\ C\left[\frac{N}{N-j}\right] &= 1 \end{aligned} \tag{B.17}$$

This gives the constants:

$$C = \frac{N-j}{N} \quad \text{and} \quad D = \left(\frac{N-j}{N}\right)\frac{j}{N-j} = \frac{j}{N} \tag{B.18}$$

Final Result

The element (i, j) of the inverse matrix A^{-1} is:

$$(A^{-1})_{ij} = \begin{cases} \frac{N-j}{N} \cdot i & \text{if } i \leq j \\ \frac{j}{N} \cdot (N-i) & \text{if } i \geq j \end{cases} \tag{B.19}$$

This can be written more compactly using min and max functions:

$$(A^{-1})_{ij} = \frac{\min(i, j) \cdot (N - \max(i, j))}{N} \tag{B.20}$$

Appendix C

Evaluation of the Fresnel Integral

The value of the **Fresnel Integral** is:

$$\boxed{\int_{-\infty}^{\infty} e^{ix^2} dx = \sqrt{\frac{\pi}{2}}(1+i) = \sqrt{\pi} e^{i\pi/4}} \tag{C.1}$$

C.1 Derivation (Using Contour Integration)

Step 1: Define the Contour

We consider the complex function $f(z) = e^{iz^2}$, where z is a complex variable. We construct a closed path (contour) C in the complex plane. This path is a sector of a circle, composed of three parts:

1. **Path** C_1: A line segment along the real axis from 0 to R.
2. **Path** C_2: A circular arc of radius R, centered at the origin, running counter-clockwise from R to $Re^{i\pi/4}$.
3. **Path** C_3: A line segment from $Re^{i\pi/4}$ back to the origin 0.

We will eventually let $R \to \infty$.

Step 2: Apply Cauchy's Integral Theorem

The function $f(z) = e^{iz^2}$ is analytic over the entire complex plane (it is an entire function) as it has no singularities. According to **Cauchy's Integral Theorem**, its integral over any closed path C is zero:

$$\oint_C e^{iz^2} dz = 0 \tag{C.2}$$

This closed-loop integral can be split into the sum of integrals over the three paths:

$$\int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz + \int_{C_3} e^{iz^2} dz = 0 \tag{C.3}$$

Step 3: Evaluate the Integral on Each Path

1. Integral along Path C_1 (The part we want to find)

On path C_1, we have $z = x$ (a real number) and $dz = dx$. Therefore:

$$\lim_{R\to\infty} \int_{C_1} e^{iz^2} dz = \int_0^{\infty} e^{ix^2} dx \tag{C.4}$$

This is exactly half of the integral we wish to compute, since the integrand e^{ix^2} is an even function.

2. Integral along Path C_2 (Show it vanishes as $R \to \infty$)

On path C_2, we parameterize $z = Re^{i\theta}$, where θ varies from 0 to $\pi/4$. Then $dz = iRe^{i\theta}d\theta$ and $z^2 = R^2e^{i2\theta}$. The integral becomes:

$$\begin{aligned}\int_{C_2} e^{iz^2}dz &= \int_0^{\pi/4} \exp(i(R^2e^{i2\theta})) \cdot iRe^{i\theta}d\theta \\ &= \int_0^{\pi/4} \exp(iR^2(\cos(2\theta) + i\sin(2\theta))) \cdot iRe^{i\theta}d\theta \\ &= \int_0^{\pi/4} \exp(-R^2\sin(2\theta))\exp(i(R^2\cos(2\theta) + \theta)) \cdot iRd\theta\end{aligned} \tag{C.5}$$

Taking the magnitude:

$$\left|\int_{C_2} e^{iz^2}dz\right| \leq \int_0^{\pi/4} \left|\exp(-R^2\sin(2\theta))\right| \cdot Rd\theta = \int_0^{\pi/4} R\exp(-R^2\sin(2\theta))d\theta \tag{C.6}$$

On the interval $[0, \pi/4]$, 2θ is in $[0, \pi/2]$. We can use Jordan's inequality, which states $\sin(x) \geq \frac{2x}{\pi}$ for $x \in [0, \pi/2]$. Thus, $\sin(2\theta) \geq \frac{4\theta}{\pi}$.

$$\begin{aligned}\left|\int_{C_2} e^{iz^2}dz\right| &\leq \int_0^{\pi/4} R\exp(-R^2(4\theta/\pi))d\theta \\ &= R\left[\frac{-\pi}{4R^2}\exp(-4R^2\theta/\pi)\right]\Big|_0^{\pi/4} \\ &= \frac{\pi}{4R}(1 - \exp(-R^2))\end{aligned} \tag{C.7}$$

As $R \to \infty$, this expression approaches 0. Therefore:

$$\lim_{R\to\infty} \int_{C_2} e^{iz^2}dz = 0 \tag{C.8}$$

3. Integral along Path C_3 (Connection to the Gaussian Integral)

On path C_3, we parameterize $z = re^{i\pi/4}$, where r varies from R to 0. Then $dz = e^{i\pi/4}dr$ and $z^2 = (re^{i\pi/4})^2 = r^2e^{i\pi/2} = r^2i$. The integral becomes:

$$\int_{C_3} e^{iz^2}dz = \int_R^0 \exp(i(r^2i))e^{i\pi/4}dr = \int_R^0 \exp(-r^2)e^{i\pi/4}dr \tag{C.9}$$

Reversing the limits of integration and factoring out the constant:

$$= -e^{i\pi/4}\int_0^R \exp(-r^2)dr \tag{C.10}$$

As $R \to \infty$, we get the well-known Gaussian integral:

$$\lim_{R\to\infty} \int_{C_3} \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = -\mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \exp(-r^2)\mathrm{d}r \tag{C.11}$$

We know that the Gaussian integral $\int_0^\infty \exp(-x^2)\mathrm{d}x = \frac{\sqrt{\pi}}{2}$. So:

$$\lim_{R\to\infty} \int_{C_3} \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = -\mathrm{e}^{\mathrm{i}\pi/4} \frac{\sqrt{\pi}}{2} \tag{C.12}$$

Step 4: Combine the Results

Returning to the equation from Step 2 and taking the limit as $R \to \infty$:

$$\left(\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x\right) + (0) + \left(-\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2}\right) = 0 \tag{C.13}$$

Rearranging the terms, we find:

$$\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x = \mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2} \tag{C.14}$$

Step 5: Calculate the Final Integral

The integral we want to evaluate is from $-\infty$ to ∞. Since the integrand $\mathrm{e}^{\mathrm{i}x^2} = \cos(x^2) + \mathrm{i}\sin(x^2)$ is an even function (i.e., $f(-x) = f(x)$), we have:

$$\begin{aligned} \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x &= 2\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x \\ &= 2\cdot\left(\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi/4} \end{aligned} \tag{C.15}$$

Finally, we can use Euler's formula, $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, to expand $\mathrm{e}^{\mathrm{i}\pi/4}$:

$$\mathrm{e}^{\mathrm{i}\pi/4} = \cos(\pi/4) + \mathrm{i}\sin(\pi/4) = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2} = \frac{1+\mathrm{i}}{\sqrt{2}} \tag{C.16}$$

Substituting this into our result gives:

$$\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x = \sqrt{\pi}\cdot\frac{1+\mathrm{i}}{\sqrt{2}} = \sqrt{\frac{\pi}{2}}(1+\mathrm{i}) \tag{C.17}$$

C.2 Conclusion

The result of the integral is a complex number. Its real and imaginary parts correspond to two other important integrals:

$$\begin{cases} \int_{-\infty}^\infty \cos(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{2}} \\ \int_{-\infty}^\infty \sin(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{2}} \end{cases} \tag{C.18}$$