2. Quantum spin, path integral and topology

1. Quantum spin

We begin by considering the Hilbert space $\mathcal{H}$ for a single quantum ~~space~~ spin-$\frac{1}{2}$ particle. The orthonormal basis is given by: $|\uparrow\rangle$ and $|\downarrow\rangle$, which:

$$|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \hat{\mathbb{I}}.$$

But, this ~~sp~~ simple, discrete basis have drawbacks for advanced thoeries:

1. This discrete parametrized complete-set is not convenient for constructing the path-integral formalism of quantum spins.
2. SU(2) spin ~~sys~~symmetry is broken either explicitly or not manifest.

let's introduce a new complete set of spin states.

$$|\vec{n}\rangle = e^{i\delta} \cdot \begin{bmatrix} e^{-\frac{i\varphi}{2}} \cos\frac{\theta}{2} \\ e^{\frac{i\varphi}{2}} \sin\frac{\theta}{2} \end{bmatrix}$$

$$= e^{i\delta} e^{-\frac{i\varphi}{2}} \cos\frac{\theta}{2} |\uparrow\rangle + e^{i\delta} \cdot e^{\frac{i\varphi}{2}} \sin\frac{\theta}{2} |\downarrow\rangle$$

$$= z_1 |\uparrow\rangle + z_2 |\downarrow\rangle$$

with :

$$\begin{cases} z_1 = e^{i\delta} e^{-\frac{i\varphi}{2}} \cos\frac{\theta}{2} \\ z_2 = e^{i\delta} e^{\frac{i\varphi}{2}} \sin\frac{\theta}{2}. \end{cases}$$

The normalization condition $\langle\psi|\psi\rangle = 1$ imposes a constraint on these coefficients:

$$|z_1|^2 + |z_2|^2 = 1$$

This leaves $4-1=3$ real, physical degrees of freedom. And $e^{i\delta}$ is an overall phase, corresponding to a gauge choice.

For seeing the physical interpretation of $|\theta, \varphi\rangle$, we compute the expectation value of spin operator $\hat{\vec{S}}$ in the state $|\vec{n}\rangle$, where I choose $e^{i\delta} = 1$ & and set $\hbar = 1$).

$$\hat{\vec{S}} = \frac{\hbar}{2}(\hat{\sigma}_x + \hat{\sigma}_y + \hat{\sigma}_z), \quad |\vec{n}\rangle = \begin{pmatrix} e^{-\frac{i\varphi}{2}} \cos\frac{\theta}{2} \\ e^{\frac{i\varphi}{2}} \sin\frac{\theta}{2} \end{pmatrix}$$

So we can calulate the expectation value of $\hat{S}_z$:

$$\langle \hat{S}_z \rangle = \langle \vec{n} | \tfrac{1}{2} \hat{\sigma}_z | \vec{n} \rangle = \tfrac{1}{2} \cos\theta.$$

expectation value of $\hat{S}_x$:

$$\langle \hat{S}_x \rangle = \langle \vec{n} | \tfrac{1}{2} \hat{\sigma}_x | \vec{n} \rangle = \tfrac{1}{2} \sin\theta \cos\varphi$$

expectation value of $\hat{S}_y$

$$\langle \hat{S}_y \rangle = \langle \vec{n} | \frac{1}{2} \hat{\sigma}_y | \vec{n} \rangle = \frac{1}{2} \sin\theta \sin\varphi$$

So the expectation value of the spin state $|\vec{n}\rangle$ is:

$$\langle \hat{S} \rangle = \frac{1}{2} \cdot \vec{n}$$

where: $\vec{n} = (\sin\theta \cos\varphi, \cos\theta \sin\varphi, \sin\theta)$

Thus, the state $|\vec{n}\rangle$ is the quantum state that "points" in the classical direction $\vec{n}$. (θ, φ) is a pair of Euler angler. This direction vector lives on the surface of a unit sphere, known as the Bloch sphere.

The domains of (θ, φ) can be defined as usual:

$$\begin{cases} \theta \in [0, \pi) \\ \varphi \in [0, 2\pi) \end{cases}$$

In order to restore $|\uparrow\rangle$ unambiguously at the North Pole when θ tends to zero; where choose $\delta = \frac{\varphi}{2}$, such that:

$$\begin{cases} z_1 = \cos\frac{\theta}{2} \\ z_2 = e^{i\varphi} \cdot \sin\frac{\theta}{2} \end{cases}$$

when $\theta = 0$:

$$\begin{cases} z_1 = 1 \\ z_2 = 0 \end{cases}$$

Nevertheless, in this choice, the South Pole becomes irregular. when $\theta = \pi$:

$$\begin{cases} z_1 = 0 \\ z_2 = e^{i\varphi} \end{cases}$$

~~there~~ Therefore, when we choose $\delta = \frac{\varphi}{2}$, the North pole is regular, but the south pole is a singularity

Now let's consider the eigenvector equation about $|\vec{n}\rangle$:

$$\cancel{\hat{\vec{S}} \cdot \vec{n} = 1} \quad (\hat{\vec{S}} \cdot \vec{n})|\vec{n}\rangle = \frac{1}{2}|\vec{n}\rangle$$

Proof: Construct the operator $\hat{\vec{S}} \cdot \vec{n}$ in matrix form:

$$\hat{\vec{S}} \cdot \vec{n} = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

~~So~~ So we apply this operator to the coherent state $|\vec{n}\rangle$:

$$(\hat{\vec{S}} \cdot \vec{n})|\vec{n}\rangle = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

So the Top component:

$$\cancel{\#}(\cos\theta\cos\tfrac{\theta}{2} + \sin\theta\sin\tfrac{\theta}{2}) = \cos^3\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2} + 2\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$= \cos\tfrac{\theta}{2}(\cos^2\tfrac{\theta}{2} + \sin^2\tfrac{\theta}{2})$$

$$= \cos\tfrac{\theta}{2}.$$

the Bottom component:

$$\left(\cos\frac{\theta}{2}\sin\theta-\cos\theta\sin\frac{\theta}{2}\right)e^{i\varphi}=\left(2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}-\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}+\sin^3\frac{\theta}{2}\right)e^{i\varphi}$$
$$=\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)\sin\frac{\theta}{2}\,e^{i\varphi}$$
$$=\sin\frac{\theta}{2}\,e^{i\varphi}$$

$$\therefore\ (\vec{S}\cdot\vec{n})|\vec{n}\rangle=\frac{1}{2}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix}=\frac{1}{2}|\vec{n}\rangle$$

with above preparation. we may prove the completeness of $\{|\vec{n}\rangle\}$:

$$\int\frac{d^2\vec{n}}{2\pi}|\vec{n}\rangle\langle\vec{n}|=\hat{I}.$$

Proof:

$$|\vec{n}\rangle\langle\vec{n}|=\begin{pmatrix}\cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi}\\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} & \sin^2\frac{\theta}{2}\end{pmatrix}$$

The off-diagonal terms depend on $e^{\pm i\varphi}$, so:

$$\int_0^{2\pi}d\varphi\, e^{\pm i\varphi}=0.$$

The. diagonal terms

$$\begin{cases} \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \cos^2\frac{\theta}{2} = \int_0^{\pi} d\theta \, \frac{1}{2}\sin\theta(\cos\theta+1) = 1 \\ \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \sin^2\frac{\theta}{2} = \int_0^{\pi} d\theta \, \frac{1}{2}\sin\theta(1-\cos\theta) = 1 \end{cases}$$

So the completeness 可 $\{|\vec{n}\rangle\}$:

$$\int \frac{d^2\vec{n}}{2\pi} |\vec{n}\rangle\langle\vec{n}| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{\mathbb{I}}$$

We can know that $\{|\vec{n}\rangle\}$ forms an over-complete basis for spin-$\frac{1}{2}$ Hilbert space. And all basis states are continuously parametrized by Euler's angles.

Last, let's see the ~~orthoga~~ orthogonality of antipodal states $|-\vec{n}\rangle$, ~~其~~ which is the antipodal point $-\vec{n}$ corresponding to the angle $(\theta', \varphi') = (\pi-\theta, \varphi+\pi)$, so.

$$|-\vec{n}\rangle = \begin{pmatrix} \cos(\frac{\pi-\theta}{2}) \\ \sin(\frac{\pi-\theta}{2}) e^{i(\varphi+\pi)} \end{pmatrix} = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \cdot e^{i\varphi} \end{pmatrix} \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\varphi} \end{pmatrix}$$

So:

$$\langle -\vec{n}|\vec{n}\rangle = \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$$

This confirms that antipodal states are orthogonal, as expected. For example: $|\vec{n}\rangle = |\uparrow\rangle$ and $|-\vec{n}\rangle = |\downarrow\rangle$ where: $\vec{n} = (0,0,1)$ and $-\vec{n} = (0,0,-1)$.

2. Coherent-state path integral for spin

Consider the quantum amplitude ($\hbar = 1$):

$$\langle n_f | e^{-i\hat{H}(t_f - t_i)} | n_i \rangle \xrightarrow{i(t_f - t_i) \to T} \langle n_f | e^{-\hat{H}T} | n_i \rangle$$

First, consider $\hat{H} \cancel{=0;}$ that is an isolated spin

we divide the total time interval T into N small slices of duration $\Delta t = \frac{T}{N}$. In the limit $N \to \infty$ and $\Delta t \to 0$, the time evolution operator can be written as a product of infinitesimal operators:

$$e^{-i\hat{H}T} = \left(e^{-i\hat{H}\Delta t}\right)^N$$

The amplitude is the:

$$\langle n_f | e^{-i\hat{H}\Delta t} \cdots e^{-i\hat{H}\Delta t} | n_i \rangle$$

We now insert the resolution of identity operator: let's label our ~~intt~~ inital state as $|n_0\rangle = |n_i\rangle$ and final state as ~~x~~ $|n_N\rangle = |n_f\rangle$

$$K = \langle n_N | e^{-i\hat{H}\Delta t} \cdot \left(\int \frac{d\vec{n}_{N-1}}{2\pi} |n_{N-1}\rangle\langle n_{N-1}| \right) e^{-i\hat{H}\Delta t} \cdots e^{-i\hat{H}\Delta t} |n_0\rangle$$

$$= \int \left(\prod_{l=1}^{N-1} \frac{d\vec{n}_l}{2\pi} \right) \cdot \left(\prod_{l=1}^{N} \langle n_l | e^{-i\hat{H}\Delta t} | n_{l-1} \rangle \right)$$

Since Δt is small, we can expand the exponential to first order:

$$e^{-i\hat{H}\Delta t} \approx \hat{\mathbb{I}} - \cancel{\hbar} i\hat{H}\Delta t$$

The matrix element becomes:

$$\langle n_t | e^{-i\hat{H}\Delta t} | n_{t-1}\rangle = \langle n_t | \hat{\mathbb{I}} - i\hat{H}\Delta t | n_{t-1}\rangle$$

$$= \langle n_t | n_{t-1}\rangle - i\Delta t \langle n_t | \hat{H} | n_{t-1}\rangle$$

let's represent the state at time t_k as a small ~~deviation~~ deviation from the state at t_{t-1}

$$|n_t\rangle \approx |n_{t-1}\rangle + \delta |n_{t-1}\rangle = |n_{t-1}\rangle + \cancel{d|n_t\rangle} \left.\frac{d|n(t)\rangle}{dt}\right|_{t=t_{t-1}} \cdot \Delta t$$

The overlap is then:

$$\langle n_t | n_{t-1}\rangle \approx \langle n_{t-1} | n_{t-1}\rangle + \cancel{d|n(t)\rangle} \left(\left.\frac{d\langle n(t)|}{dt}\right|_{t=t_{t-1}} \right) |n_{t-1}\rangle \Delta t$$

$$= 1 + \left(\left.\frac{d\langle n|}{dt}\right|_{t=t_{t-1}} \right) |n_{t-1}\rangle \cdot \Delta t.$$

This can be written in more suggestive exponential form for small Δt:

$$\langle n_t | n_{t-1}\rangle \approx e^{\langle \dot{n}_{t-1} | n_{t-1}\rangle \cdot \Delta t}$$

where, $\langle \dot{n}_{t-1}| = \cancel{\frac{d\langle n_{t-1}|}{dt}} \left.\frac{d\langle n(t)|}{dt}\right|_{t=t_{t-1}}$

By the way, from the equation:

$$\frac{d}{dt}\langle n_t|n_t\rangle = \left(\frac{d}{dt}\langle n_t|\right)|n_t\rangle + \langle n_t|\left(\frac{d}{dt}|n_t\rangle\right) = 0$$

we can get:

$$\langle \dot{n}_t|n_t\rangle = -\langle n_t|\dot{n}_t\rangle$$
$$= -\langle \dot{n}_t|n_t\rangle^*$$

So $\langle \dot{n}_t|n_t\rangle$ is ~~purely~~ purly imaginary

Because we only retain first-order small quantities, so the second term of matrix element becomes:

$$-i\Delta t\langle n_t|\hat{H}|n_{t-1}\rangle \approx -i\Delta t\left(\langle n_{t-1}| + \langle \dot{n}_{t-1}|\Delta t\right)\hat{H}|n_{t-1}\rangle$$
$$= -i\langle n_{t-1}|\hat{H}|n_{t-1}\rangle\Delta t + O(\Delta t^2)$$
$$\approx -i\langle n_{t-1}|\hat{H}|n_{t-1}\rangle\Delta t$$

Putting everything back into the matrix elements:

$$\langle n_t|e^{-i\hat{H}\Delta t}|n_{t-1}\rangle = 1 + \left(\langle \dot{n}_{t-1}|n_{t-1}\rangle - i\langle n_{t-1}|\hat{H}|n_{t-1}\rangle\right)\Delta t$$
$$\approx \exp\left[\left(\langle \dot{n}_{t-1}|n_{t-1}\rangle - i\langle n_{t-1}|\hat{H}|n_{t-1}\rangle\right)\Delta t\right]$$

So the amplitude K is:

$$K = \int\left(\prod_{t=1}^{N-1}\frac{d^2\vec{n}_t}{2\pi}\right)\cdot e^{\sum_{t=1}^{N}\left(\langle \dot{n}_{t-1}|n_{t-1}\rangle - i\langle n_{t-1}|\hat{H}|n_{t-1}\rangle\right)\Delta t}$$

The product of integrals over all intermediate states becomes the formal path integral measure $D[n(t)]$:

$$K = \int D[n(t)] \exp\left\{ i \int_{t_i}^{t_f} [-i\langle \dot{n}(t)|n(t)\rangle - \langle n(t)|\hat{H}|n(t)\rangle] dt \right\}$$

$$= \int D[n(t)] \exp\{ iS[n(t)] \}$$

Because of $-\langle \dot{n}(t)|n(t)\rangle = \langle n(t)|\dot{n}(t)\rangle$, $S[n(t)]$ can be written as:
~~where. $-\langle\dot{n}(t)$~~

$$S[n(t)] = \int_{t_i}^{t_f} dt \, [i\langle n(t)|\frac{d}{dt}|n(t)\rangle - \langle n(t)|\hat{H}|n(t)\rangle]$$

where, $\hbar = 1$.

2.1 Geometrical meaning of the ~~dynamical~~ ~~non-Hamiltonian~~ Geometric term

We consider a special system whose Hamiltonian is zero, So the dynamical term vanishes:

$$\langle n(t)|\hat{H}|n(t)\rangle = 0$$

and the action simplifies and ~~conta~~ contains only the geometric term:

$$S[n(t)] = \int_{t_i}^{t_f} dt \cdot i\langle n(t)|\frac{d}{dt}|n(t)\rangle$$

Again, we choose the gauge choice $\delta = \frac{\varphi}{2}$, such that the North Pole is regular:

$$|n(t)\rangle = \cos\frac{\theta(t)}{2}|\uparrow\rangle + e^{i\varphi(t)} \cdot \sin\frac{\theta(t)}{2}\cdot|\downarrow\rangle$$

So the ~~ove term~~ geometric term is:

$$\langle n(t)|\frac{d}{dt}|n(t)\rangle = \cos\frac{\theta}{2}\left(-\frac{1}{2}\sin\frac{\theta}{2}\cdot\dot{\theta}\right) +$$
$$\left(e^{-i\varphi}\sin\frac{\theta}{2}.\right)\left(+i\dot{\varphi}e^{i\varphi}\sin\frac{\theta}{2} + e^{i\varphi}\frac{1}{2}\cos\frac{\theta}{2}\cdot\dot{\theta}\right)$$

$$= i\dot{\varphi}\sin^2\frac{\theta}{2}$$
$$= \frac{i\dot{\varphi}(1-\cos\theta)}{2}.$$

So the action is:

$$S[n(t)] = \frac{1}{2}\int_{t_i}^{t_f} dt\cdot\dot{\varphi}(1-\cos\theta)$$
$$\cancel{= i\int_{t_i}^{\varphi(t_f)}(1-\cos\theta)\,d\varphi}$$
$$= \frac{1}{2}\int_{\varphi(t_i)}^{\varphi(t_f)}[1-\cos\theta(\varphi)]\,d\varphi.$$

Because θ and φ ~~is the~~ are function dependenting t.

we can ~~change~~ express θ ~~to~~ as a function of φ

We can rewrite the function $1-\cos\theta(\varphi)$ as a integral:

$$1-\cos\theta(\varphi) = \int_0^{\theta(\varphi)} d\theta . \sin\theta$$

So ~~th~~ the action can be written as a area integral:

$$S[n(t)] = -\frac{1}{2}\int_{\varphi_i}^{\varphi_f} \cancel{d\varphi} \int_0^{\theta(\varphi)} \sin\theta \, d\theta \, d\varphi$$

$$= -\frac{1}{2}\iint_{\partial A_\gamma} d\Omega .$$

$$= -\frac{1}{2} A_\gamma \quad (\text{Because the radius is } 1 \text{ and } A_\gamma = R^2\Omega)$$

where γ is the path $|n(t)\rangle$ from $|n(t_i)\rangle = |n_i\rangle$ to $|n(t_f)\rangle = |n_f\rangle$ and A_γ is area enclosed by three points on the sphere. $(|\uparrow\rangle, |n_i\rangle, |n_f\rangle)$. Obviously, A_γ depends on where the North pole is defined, which is kind of "gauge-dependence"

If we multiply $|n(t)\rangle$ by a overall ~~pa~~ phase:

$$|n(t)\rangle \longrightarrow e^{i\delta}|n(t)\rangle .$$

$S[n(t)]$ will be added an other term:

$$S'[n(t)] = i\int_{t_i}^{t_f} dt \, \langle n(t)| (e^{i\delta}\frac{d}{dt}|n(t)\rangle + i\dot{\delta} e^{i\delta}|n(t)\rangle$$

$$= S[n(t)] + (-1)\int_{t_i}^{t_f} dt \, \dot{\delta}$$

$= S[n(t)] + (-1)(\delta(t_f) - \delta(t_i))$

To be more precise, δ is a function of θ and φ

$$\delta(t_f) - \delta(t_i) = \delta(\theta_f, \varphi_f) - \delta(\theta_i, \varphi_i)$$

So if we choose different gauge. $\Delta\delta$ is different and $S[n(t)]$ is ~~gage~~ gauge-dependence. But if we ~~cho~~ consider a close path ($n_f = n_i$), the $S[n(t)]$ is ~~gauge-indenpe~~ gauge-independent:

$$\Delta\delta = \delta(\theta_f, \varphi_f) - \delta(\theta_i, \varphi_i)$$
$$= 2\pi N \quad (N \in \mathbb{Z})$$

The reason as follow: To ensure that the transformed basis is single-valued, which.

$$e^{i\delta_i}|n_i\rangle \xrightarrow{\text{close path}} e^{i\delta_f}|n_f\rangle = e^{i\delta_i}|n_i\rangle$$

and

$$e^{i\delta_f}|n_i\rangle = e^{i\delta_i}|n_i\rangle$$

This implies that the value of δ can change by integer multiples of ~~2π~~ 2π after completing a full cycle:

$$S_f - S_i = 2\pi \cdot N \quad (N \in \mathbb{Z}).$$

So for a close path, $\dot{S}[n(t)]$, to be more precise, the geometric phase factor: ~~$e^{iS'} = e^{iS + i\Delta S} = e^{iS + 2\pi N i} = e$~~

$$e^{iS'} = e^{iS + i\Delta S} = e^{iS + 2\pi N \cdot i} = e^{iS}$$

remains unchanged, or that the geometric phase S is invariant under the ~~model~~ modulo 2π operation.

2.7 Wess-Zumino term and Witten extention