2. Stationary Phase Approximation / Semiclassical Approximation

We have got the ~~prog~~ propagator iG for a particle to travel from an spacetime point (x_0, t_0) to a final spacetime point (x_f, t_f), ~~with~~ which is given by the Feynman path integral:

$$iG = K(x_f, t_f; x_0, t_0) \propto \int D[x(t)] e^{\frac{i}{\hbar} \cdot S[x(t)]}$$

where:

$x(t)$ is the ~~possition~~ position of the particle at time t, respresenting a possible path.

$D[x(t)]$ is the functional measure for integrating over all paths that statisfy the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$

$S[x(t)]$ is the action for a path $x(t)$;

This integral is infinite-dimensional and generally very difficult to calculate directly. So we need a effective method to approximate it. The stationary phase approximation priovides a method for approximating it.

2.1 one dimentional integral of stationary phase approximation

Consider a ~~intergr~~ integral.

$$I = \int e^{\frac{i f(x)}{a}} \cdot dx$$

where a is a small parameter. and the function $f(x)$ is real-valued and regular.

Our objective is to understant the physical picture of the stationary phase approximation for the propagator ~~ph~~ path integral through this one-dimensional integral stationary phase approximation.

let's further define a new notation: ~~$\theta(x)=$~~

$$\theta(x) = \frac{1}{a} \cdot f(x).$$

$\theta(x)$ is a phase angle, so

$$I = \int e^{i\theta(x)} dx$$

This integral can be " physically regarded as an interference experiment. Because each source at x_i contributes a phase factor. $e^{i\theta(x)}$, ~~so~~ the total integral. I is the result of adding up all these infinite tiny vectors.

In order to compute the integral I, what we really need to do is to find " dominant contribution" to I

Firstly, let's pick up a point x_0 and evaluate the integral near x_0. The vicinity of x_0 is given by $x \in (x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2})$. Within ~~thi~~ this "small" domain, we may linearize $\theta(x)$.

$$\theta(x) \approx \theta(x_0) + \left.\frac{d\theta}{dx}\right|_{x=x_0} (x - x_0)$$
$$= \frac{f(x_0)}{a} + \frac{f'(x_0)}{a} (x - x_0)$$

The contributions to I in the small domain near x_0 are given by:

$$I_{\cancel{x_0}}^{\Delta x}(x_0) = \int_{x_0 - \frac{\Delta x}{2}}^{x_0 + \frac{\Delta x}{2}} dx \; e^{i\theta(x_0)} \cdot e^{i\frac{f'(x_0)}{a}\cdot(x - x_0)} \, dx$$

$$= e^{i\theta(x_0)} \cdot \int_{x_0 - \frac{\Delta x}{2}}^{x_0 + \frac{\Delta x}{2}} dx \; e^{i\frac{f'(x_0)}{a}\cdot(x - x_0)}$$

we let $\theta(x_0) = \theta_0$ and $f'(x_0) = f_0'$, so:

$$I^{\Delta x}_{(x_0)} \approx e^{i\theta_0} \frac{a}{if_0'} \cdot e^{\frac{if_0'}{a}(x - x_0)} \Bigg|_{x_0 - \frac{\Delta x}{2}}^{x_0 + \frac{\Delta x}{2}}$$

$$= e^{i\theta_0} \cdot \frac{2a}{if_0'} \sin \frac{f' \Delta x}{2a}.$$

let $\alpha = \frac{f' \Delta x}{2a}$, so:

$$I^{\Delta x}_{(x_0)} \approx e^{i\theta_0} \cdot \Delta x \cdot \frac{\sin \alpha}{\alpha}$$

Because a is a ~~small parameter~~ very small but nonzero parameter (in quantum mechanics, it corresponds to $\hbar$ being a very small but nonzero number), if $f_0' = 0$, α is strictly equal to 0. and I_0 is not infinitely large near the zero point. ~~So~~ if we consider the case $f'(x_c) = 0$

$$I^{\Delta x}_{(x_c)} = e^{i\theta(x_c)} \cdot \Delta x$$

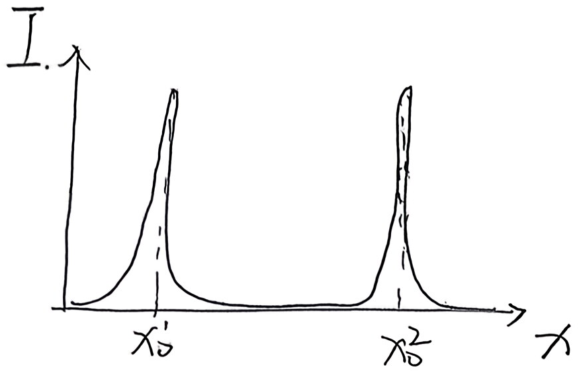
Physically, the result means that all U(1) phase in the vicinity of x_0 are completely the same. It's a perfect phase-coherence. The superposition of constant phases leads to linearly enhanced amplitude "Δx".

Next, let us consider $f'(x_0) \neq 0$. Because a is a very small parameter, $\alpha = \frac{f' \Delta x}{2a}$ is very large. So:

$$\frac{\sin\alpha}{\alpha} \rightarrow 0. \Rightarrow I^{\Delta x}_{(x_0)} \approx 0$$

Physically, the summation of all U(1) phases in the vicinity of x_0 leads to destructive interference. The parameter a is smaller, the destructive interference is more severe.

In conclusion, if we consider small enough but nozero parameter a, it is computationlly economic to merely focus on the integral contributions from the vicinity of these special point (denoted by a set $\{x_0^i\}$) that satisfy $f'(x_0^i)=0$.



We consider the quadratic approximation in the vicinity of x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$
$$= f(x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$

As a result, the original integral I can be evaluated by:

$$I \approx \sum_{\{x_0^i\}} I^{\Delta x}(x_0^i)$$
$$= \sum_{\{x_0^i\}} e^{i\frac{f(x_0^i)}{a}} \int_{x_0^i-\frac{\Delta x}{2}}^{x_0^i+\frac{\Delta x}{2}} dx \; e^{\frac{i}{2a} f''(x_0^i)(x-x_0^i)^2}$$

Because parameter a is a very small number, far away the stationary point x_0^i, the contributions of $e^{\frac{i}{2a} f''(x_0^i)(x-x_0)^2}$ to the integral cancel each other out through destructive interference

So in the above Gaussian integral near x_0, we can extended the integral bounds to infinity:

$$I \approx \sum_{\{x_0^i\}} e^{\frac{if(x_0^i)}{a}} \int_{-\infty}^{\infty} dx \cdot e^{\frac{i}{2a} \cdot f''(x_0^i)(x-x_0^i)^2}$$

$$= \sum_{\{x_0^i\}} e^{\frac{if(x_0^i)}{a}} \sqrt{\frac{2\pi a i}{f''(x_0^i)}}$$

2.2 Semiclassical approximation of Feynmen path integrals.

We have discussed that if $\hbar$ tends to zero, then the quantum system will transitions to the classical system. We only need to treat $f(x)$ as $S[x(t)]$ and ~~a as $\hbar$~~ parameter a as $\hbar$, then we can see the reason based on the discussion in the previous section. If $\hbar$ is zero, $I^{\Delta x}_{(x_0)}$ equal to zero strictly for $x \neq x_0$ and is nonzero only at $x \neq x_0$. So we only need to consider the classical path with $\delta S = 0$ and not need to consider the quantum fluctuation near the classical path. But if $\hbar$ is a very small ~~nu~~ but nonzero number, we need to consider the quantum fluctuation near the classical path. In other word.

in classical. mechanics, $\hbar=0$, $I^{\Delta x} \propto \frac{\sin \kappa}{\kappa}$. is equal. to zero

in $(x_0-\frac{\Delta x}{2}, x_0) \cup (x_0, x_0+\frac{\Delta x}{2})$ because. $\kappa=\frac{f'\Delta x}{2a}$ ~~zero~~ tends to infinity no matter how small the radius of this deleted neighbourhood ~~is~~. is. But in quantum mechanics, $\hbar \sim 10^{-34} J\cdot s$, I is not equal to zero in the neighbourhood whose radius length matches ~~$\hbar$~~ the order of magnitude of $\hbar$. Therefore, we cannot ignore the impact generated by I in this neighbourhood.

The ~~Fy~~ Feynmen ~~A~~ path integral:

$$K(x_f, t_f; x_0, t_0) \propto \int D[x(t)] \cdot e^{\frac{i}{\hbar}S[x(t)]}$$

it can be regarded as ~~ma~~ path integral version of $\int dx\, e^{\frac{if(x)}{a}}$. $f(x_0)$ is replaced by Classical action S_c when $\delta S_c=0$. Liking the previous section, we consider. the quadratic approximation in the vicinity of classical. path:

$$S = S_c + \delta S + \frac{1}{2}\delta^2 S$$
$$= S_c + \frac{1}{2}\delta^2 S.$$

Now, we ~~consider~~ can decompose the path near classical path into the classical path $x_c(t)$ plus a quantum fluctuation $y(t)$ around it:

$$x(t) = x_c(t) + y(t).$$

$x(t)$ must satisfy the same boundary conditions, so:

$$\begin{cases} y(t_0) = 0 \\ y(t_f) = 0 \end{cases}$$

let's perform a functional Taylor expansion of the action $S[x(t)]$ around the classical path $x_c(t)$:

$$S[x(t)] = S[x_c + y] = S[x_c] + \int_{t_0}^{t_f} dt \cdot \frac{\delta S}{\delta x}\Big|_{x=x_c} \cdot y \cancel{+ \frac{1}{2}\int_{t_0}^{t_f} dt_1 \int_{t_0}^{t_f} dt_2 \frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\Big|_{x=}}$$

$$+ \frac{1}{2}\int_{t_0}^{t_f} dt_1 \cdot \int_{t_0}^{t_f} dt_2 \cdot \frac{\delta^2 S}{\delta x(t_1)\,\delta x(t_2)}\Big|_{x=x_c} \cdot y^2 + O(y^3)$$

In semiclassical approximation, we assume the fluctuations y are small, so we neglect terms of $O(y^3)$ and higher. At the same time, $\delta S = 0$, so the action $S[x(t)]$:

$$S[x(t)] \approx S_c + \frac{1}{2}\iint dt_1 dt_2\, y(t_1) \cdot \frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\Big|_{x=x_c} \cdot y(t_2).$$

Now, let's solve the fluctuation term $\delta^2 S$:

$$\delta^2 S = \iint dt_1 dt_2 \cdot y(t_1) \cdot \frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\Big|_{x=x_c} \cdot y(t_2).$$

Consider a standard Lagrangian $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$, so the action is:

$$S = \int dt \left[\frac{1}{2}m\dot{x}^2 - V(x)\right]$$

its second variation is:

$$\delta^2 S = \int dt \cdot \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}V''(x_c)\cdot y^2\right).$$

We substitute the approximated action back into the path integral expression:

$$K \approx J \cdot \int \mathcal{D}[x(t)]\, \exp\left\{\frac{i}{\hbar}\left[S_c + \frac{1}{2}\delta^2 S\right]\right\}$$

$$= J \cdot \int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar}S_c} \cdot \exp\left[\int dt \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}V''(x_c)\, y^2\right)\right]$$

where, $J = \left(\frac{m}{2\pi\hbar\Delta t}\right)^{\frac{N}{2}}$.

Now, we need to change the integration variable from an integral over all paths $x(t)$ to an integral over all ~~flucti~~ fluctuations $y(t)$. Since $x_c(t)$ is a fixed. ~~fun~~ path, the path measure: $\cancel{\mathcal{D}[x(t)}$)

$$\mathcal{D}[x(t)] = \mathcal{D}[x_c(t) + y(t)] = \mathcal{D}[y(t)]$$

and S_c is a constant, it can be factored out of integral:

$$K \approx J e^{\frac{i}{\hbar} \cdot S_c} \cdot \int \mathcal{D}[y(t)] \cdot e^{\frac{i}{2\hbar} \int dt \cdot \left(\frac{1}{2} m \dot{y}^2 - \frac{1}{2} V''(x_c) y^2\right)}$$

$$= J \cdot F(t_f, t_0) \cdot e^{\frac{i}{\hbar} S_c}.$$

where:

1. $e^{\frac{i}{\hbar} S_c}$ is the Classical Phase Factor. It tells us that in the semiclassical approximation, the evolution of the system's quantum phase is dominated by the classical action. This is a bridge connecting classical and quantum mechanics.

2. $F(t_f, t_0) = \int \mathcal{D}[y(t)] e^{\frac{i}{\hbar} \cdot \frac{1}{2} \delta^2 S}$ is the Quantum Fluctuation Prefactor. It describes the collective contribution of all the small quantum fluctuations around the classical path.

Now et's see a example: one dimensional free particle.

1 ~ free particle:

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} m \dot{x}^2.$$

t, we need to calulate the classical action. It is given by the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)-\frac{\partial \mathcal{L}}{\partial x}=0$$

For free particle:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \\ \frac{\partial \mathcal{L}}{\partial x} = 0. \end{cases}$$

So the equation of motion is:

$$\frac{d}{dt}(m\dot{x}) = m\ddot{x} = 0$$

we can get the general solution:

$$x_c(t) = at + b$$

we impose the boundary conditions:

$$\begin{cases} x(t_0) = x_0 \\ x(t_f) = x_f \end{cases}$$

to determin a and b:

$$\begin{cases} a = \frac{x_f - \cancel{x_0} x_0}{t_f - t_0}. \\ b = x_i - t_i\left(\frac{x_f - x_0}{t_f - t_0}\right) \end{cases}$$

the classical action is:

$$S_c = \int_{t_i}^{t_f} dt \cdot \frac{1}{2}m(\dot{x}_c)^2 = \int_{t_i}^{t_f} dt \cdot \frac{1}{2}ma^2 = \frac{m(x_f - x_0)^2}{2(t_f - t_0)}$$

So the classical phase factor is:

$$e^{\frac{i}{\hbar}S_c} = e^{\frac{im(x_f - x_0)^2}{2\hbar(t_f - t_i)}}$$

now let's ~~calu~~ calculate the quantum fluctuation factor:

$$F(t_f, t_0) = \int D[y(t)] \exp\left\{\frac{im}{2\hbar}\int_{t_0}^{t_f} dt\, \dot{y}^2(t)\right\}.$$

$$= \int \prod_{l=1}^{N-1} dy_l \cdot \exp\left[\frac{im}{2\hbar} \cdot \sum_{l=1}^{N} \left(\frac{y_l - y_{l-1}}{\Delta t}\right)^2 \cdot \Delta t\right]$$

because $y_N = y_0 = 0$, we can rewrite the sum as:

$$\frac{im}{2\hbar}\sum_{l=1}^{N}\left(\frac{y_l - y_{l-1}}{\Delta t}\right)^2 \Delta t = -\frac{1}{2} y^T M y$$

where

$$M = \frac{m}{i\Delta t\hbar}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & \cdots & -1 & \\ & & \cdots & & \\ & & & -1 & 2 \end{bmatrix}_{(N-1)\times(N-1)}, \quad y = \begin{bmatrix} y_{N-1} \\ \vdots \\ y_1 \end{bmatrix}$$

and use the Gaussion integral:

$$F(t_f, t_0) = (2\pi)^{\frac{N-1}{2}} \cdot (\mathrm{Det}\, M)^{-\frac{1}{2}}.$$

$$= (2\pi)^{\frac{N-1}{2}} \cdot \left(\frac{i\Delta t\hbar}{m}\right)^{\frac{N-1}{2}} \cdot N^{-\frac{1}{2}}$$

So we can get the complete ~~prog~~ propagator of free particle:

$$K = J \cdot \left(\frac{2\pi i \Delta t \hbar}{m}\right)^{\frac{N}{2}} \cdot \left(\frac{m}{2\pi i N \Delta t}\right) \cdot e^{\frac{iS_c}{\hbar}}$$

$$= \left(\frac{m}{2\pi i \hbar \Delta t}\right)^{\frac{N}{2}} \cdot \left(\frac{2\pi i \Delta t \hbar}{m}\right)^{\frac{N}{2}} \cdot \left(\frac{m}{2\pi i (t_f - t_0)}\right) \cdot e^{\frac{iS_c}{\hbar}}$$

$$= \left(\frac{m}{2\pi i (t_f - t_0)}\right) \cdot e^{\frac{iS_c}{\hbar}}$$

where, $S_c = \frac{m}{2} \frac{(x_f - x_0)^2}{t_f - t_0}$.