Part I

Feynman Path Integral

Chapter 1

Feynman Path Integral

The version of quantum mechanics:

1. Schrödinger's wavefuction (operator form):

$$\hat{H}|\psi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle \tag{1.1}$$

2. Feynman's Path Integral (Common number form):

$$\mathrm{i}G(\text{Green's function}) \propto \int \mathcal{D}(x,t)\mathrm{e}^{\mathrm{i}\int \mathcal{L}\mathrm{d}t} \tag{1.2}$$

There are many advantages of Feynman Path Integral:

1. Make the double-slit experiment more understandable.
2. The classical limit "$\hbar \to 0$" is "tractable": quantum $\xrightarrow{\hbar\to 0}$ classical.
3. Provide a semi-classical picture. for. quantum mechanics.
4. "Quantum fluctuations" are more "understandable".
5. A natural route. to low energy effective theory of quantum many-body systems.
6. A natural language for describing topological properties of quantum many-body systems.

But the practical calculation in the path-integral representation of simple quantum mechanical problem many be notoriously difficult and lengthy.

1.1 Propagators

Consider a quantum partical confined in a one-dimensional space:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \tag{1.3}$$

and the canonical pair: $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = \mathrm{i}\hbar$

The Schrödinger's equation is:

$$\mathrm{i}\hbar \frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle \tag{1.4}$$

This first-order nature allows us to define a time evolution operator $\hat{U}(t, t_0)$ which propagates the state vector from an initial time t_0 to a final time t:

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \tag{1.5}$$

Assuming the Hamiltonian H is not explicitly depend on time, the formal solution of $\hat{U}$ is:

$$\hat{U} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_0)} \tag{1.6}$$

A crucial property of $\hat{U}$ is the "chain-like" rule, or composition property. For any intermediate time t' such that $t > t' > t_0$:

$$\hat{U}(t, t_0) = \hat{U}(t, t')\hat{U}(t', t_0) \tag{1.7}$$

This property is the key to the entire path integral derivation. And $\hat{U}(t, t_0)$ is unitary:

$$\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \hat{\mathbb{I}} \tag{1.8}$$

where $\hat{\mathbb{I}}$ is the identity operator.

In the position representation, we can obtain matrix elements:

$$\begin{aligned} U(x, t; x_0, t_0) &= \langle x|\hat{U}(t, t_0)|x_0\rangle \\ &= \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_0)}|x_0\rangle \end{aligned} \tag{1.9}$$

We can define a propagators (Green's function) of the quantum system by using the matrix elements:

$$\mathrm{i}G(x, t; x_0, t_0) = U(x, t; x_0, t_0) \tag{1.10}$$

Using the matrix elements, $\psi(x, t)$ can be reformulated as:

$$\begin{aligned} \psi(x, t) &= \langle x|\hat{U}(t, t_0)|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_0 \langle x|\hat{U}(t, t_0)|x_0\rangle\langle x_0|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_0 U(x, t; x_0, t_0)\psi(x_0, t_0) \end{aligned} \tag{1.11}$$

Also, the propagator also satisfies the Schrödinger's equation:

$$\boxed{\mathrm{i}\hbar\frac{\partial}{\partial t}G(x,t;x_0,t_0)=\hat{H}G(x,t;x_0,t_0)} \tag{1.12}$$

And the initial condition is:

$$G(x,t_0;x_0,t_0)=-\mathrm{i}\langle x|\hat{U}(t_0,t_0)|x_0\rangle=-\mathrm{i}\delta(x-x_0) \tag{1.13}$$

$\delta(x-x_0)$ is Dirac function.

Example 1.1.1. For free particle: $\hat{H}=\frac{1}{2m}\hat{p}^2$, in position representation:

$$\hat{p}\rightarrow -\mathrm{i}\hbar\frac{\partial}{\partial x} \tag{1.14}$$

So the PDE is:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}G(x,t;x_0,t_0)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x,t;x_0,t_0) \tag{1.15}$$

Solve the PDE:
Use Fourier Transform: (we use $G(x,t)$ instead of $G(x,t;x_0,t_0)$). We solve the free-particle Green's function by transforming to momentum space:

$$\begin{cases} G(x,t)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int \mathrm{d}k\cdot\tilde{G}(k,t)\mathrm{e}^{\mathrm{i}kx} \\ \tilde{G}(k,t)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int \mathrm{d}x\ G(x,t)\mathrm{e}^{-\mathrm{i}kx} \end{cases} \tag{1.16}$$

With these conventions, spatial derivatives become algebraic in k-space while the time derivative remains unchanged:

$$\begin{cases} \mathcal{F}\left\{\mathrm{i}\hbar\dfrac{\partial G(x,t)}{\partial t}\right\}=\mathrm{i}\hbar\dfrac{\partial\tilde{G}(k,t)}{\partial t} \\ \mathcal{F}\left\{-\dfrac{\hbar^2}{2m}\dfrac{\partial^2 G}{\partial x^2}\right\}=-\dfrac{\hbar^2}{2m}[-k^2\tilde{G}(k,t)]=\dfrac{\hbar^2k^2}{2m}\tilde{G}(k,t) \end{cases} \tag{1.17}$$

Applying the transform to the PDE yields an ordinary differential equation in time for each k:

$$\begin{cases} \mathrm{i}\hbar\dfrac{\partial}{\partial t}\tilde{G}(k,t)=\dfrac{\hbar^2k^2}{2m}\tilde{G}(k,t) \\ \dfrac{\mathrm{d}\tilde{G}}{\tilde{G}}=-\mathrm{i}\dfrac{\hbar k^2}{2m}\mathrm{d}t \end{cases} \tag{1.18}$$

Integrating in time gives the logarithm of the solution up to a k-dependent constant:

$$\ln\tilde{G}=-\mathrm{i}\frac{\hbar k^2}{2m}t+C(k) \tag{1.19}$$

So we can get the solution:

$$\tilde{G}(k,t)=A(k)\mathrm{e}^{-\mathrm{i}\frac{\hbar k^2}{2m}t},\quad A(k)=\mathrm{e}^{C(k)} \tag{1.20}$$

To determine $A(k)$, impose the initial condition at time t_0 in position space:

$$\begin{aligned}\tilde{G}(k,t_0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x G(x,t_0)\mathrm{e}^{-\mathrm{i}kx} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x \delta(x-x_0)\mathrm{e}^{-\mathrm{i}kx}\end{aligned} \tag{1.21}$$

Using the Fourier transform of the Dirac delta, we find:

$$\tilde{G}(k,t_0) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0} \tag{1.22}$$

Matching at t_0 fixes the k-space amplitude:

$$A(k) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0} \cdot \mathrm{e}^{\mathrm{i}\frac{\hbar k^2}{2m}t_0}. \tag{1.23}$$

Therefore, for general time t we have:

$$\tilde{G}(k,t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0}\mathrm{e}^{-\mathrm{i}\frac{\hbar k^2}{2m}(t-t_0)} \tag{1.24}$$

Finally, inverse-transform back to position space to obtain the integral representation of the propagator:

$$\begin{aligned}G(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}k \cdot \tilde{G}(k,t)\mathrm{e}^{\mathrm{i}kx} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k(x-x_0)} \cdot \mathrm{e}^{-\mathrm{i}\frac{\hbar(t-t_0)}{2m}k^2}\end{aligned} \tag{1.25}$$

This is a standard Gaussian integral of the form:

$$\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-ak^2+bk} = \sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}}. \tag{1.26}$$

Let's identify the coefficients:

$$\begin{cases} a = \mathrm{i}\dfrac{\hbar(t-t_0)}{2m} \\ b = \mathrm{i}(x-x_0) \end{cases} \tag{1.27}$$

So we can get the solution:

$$\mathrm{i}G(x,t) = \left[\frac{m}{2\pi\hbar\mathrm{i}(t-t_0)}\right]^{\frac{1}{2}} \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\cdot\frac{m(x-x_0)^2}{2(t-t_0)}} \tag{1.28}$$

Also, we can solve this PDE via definition:

$$\begin{aligned}\mathrm{i}G &= \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}}|x_0\rangle \\ &= \int \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}}|p\rangle\langle p|x_0\rangle\, \mathrm{d}p \\ &= \int \mathrm{d}p\, \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)p^2}{2m\hbar}}\langle x|p\rangle\langle p|x_0\rangle \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\, \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)p^2}{2m\hbar}+\mathrm{i}\frac{(x-x_0)}{\hbar}p}.\end{aligned} \tag{1.29}$$

we use $P = \hbar k$ and can get the same equation as the Fourier Transform Method.

1.2 Path-Integral

When $t > t_1 > t_0$, and t_1 is an arbitrarily selected intermediate time, we can write:

$$\begin{aligned} \mathrm{i}G(x,t;x_0,t_0) &= \langle x|\hat{U}(t,t_0)|x_0\rangle \\ &= \langle x|\hat{U}(t,t_1)\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_1 \langle x|\hat{U}(t,t_1)|x_1\rangle\langle x_1|\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_1\, \mathrm{i}G(x,t;x_1,t_1)\cdot \mathrm{i}G(x_1,t_1;x_0,t_0) \end{aligned} \tag{1.30}$$

This intergral over x_1 means "superposition" of all possible "path" that connect x and x_0. Next, we try to "smooth" the path along time directly. We can insert more time slices between x and x_0. If we insert infinite time slices, the path become smooth.

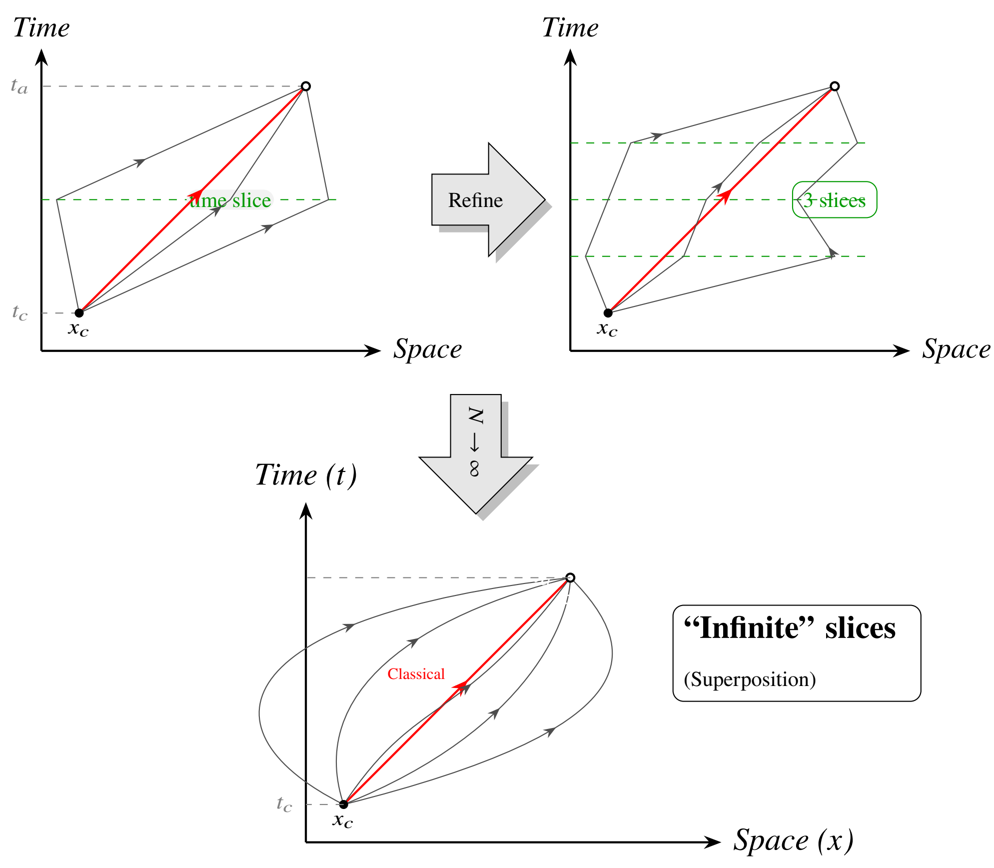


Figure 1: Evolution of the path integral formulation: (Top Left) A single time slice yields coarse, kinked paths. (Top Right) Adding more slices ($N = 3$) refines the grid. (Bottom) Taking the limit of infinite time slices ($N \to \infty$) recovers the smooth paths of standard quantum mechanics.

Firstly, let's discretize time. domain $[t_0, t]$ into N pieces of equal length $\Delta t = \frac{t-t_0}{N}$:

$$\begin{aligned} \mathrm{i}G(x,t;x_0,t_0) &= \langle x|\hat{U}(t,t_{N-1})\hat{U}(t_{N-1},t_{N-2})\cdots\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_{N-1}\cdots\mathrm{d}x_1 \prod_{l=1}^{N} \mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) \end{aligned} \tag{1.31}$$

let $\mathcal{D}_x = \prod_{l=1}^{N-1} \mathrm{d}x_l$. Consider $N \to \infty$, so $\Delta t = \frac{t-t_0}{N} \to 0$, which means $t_l - t_{l-1} = \Delta t$.

$$\mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) = \langle x_l|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\Delta t}|x_{l-1}\rangle \tag{1.32}$$

Because Δt is small, we can approximate the exponential function by its Taylor series:

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\Delta t} \approx \hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\hat{H}\Delta t = \hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\Delta t\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right] \tag{1.33}$$

Substitute (1.33) into (1.32):

$$\begin{aligned} \mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) &= \int \mathrm{d}p_l\langle x_l|p_l\rangle\langle p_l|\hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\Delta t\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]|x_{l-1}\rangle \\ &= \int \mathrm{d}p_l\langle x_l|p_l\rangle\langle p_l|x_{l-1}\rangle\left[1 - \frac{\mathrm{i}}{\hbar}\left(\frac{p_l^2}{2m} + V(x_{l-1})\right)\Delta t\right] \end{aligned} \tag{1.34}$$

With the approximations $V(x_l) \approx V(x_{l-1})$:

$$\left[1 - \frac{\mathrm{i}}{\hbar}\left(\frac{p_l^2}{2m} + V(x_{l-1})\right)\Delta t\right] \approx \left(1 - \frac{\mathrm{i}}{\hbar}H_l\Delta t\right) \approx \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_l\Delta t} \tag{1.35}$$

So $\mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1})$ can be written as:

$$\begin{aligned} \mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) &= \int \mathrm{d}p_l\frac{1}{2\pi\hbar}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p_l(x_l-x_{l-1})}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_{cl}\Delta t} \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p_l e^{\frac{\mathrm{i}}{\hbar}[p_l(x_l-x_{l-1})-H_l\Delta t]} \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p_l\mathrm{e}^{\frac{\mathrm{i}}{\hbar}[p_l(\frac{x_l-x_{l-1}}{\Delta t})-H_l]\Delta t} \end{aligned} \tag{1.36}$$

where, H_l is the classical Hamiltonion as a function of p_l and x_l.

When $\Delta t \to 0$:

$$\frac{x_l - x_{l-1}}{\Delta t} = \dot{x}_l \tag{1.37}$$

So we can get:

$$p_l\dot{x}_l - H_l = \mathcal{L}_l. \tag{1.38}$$

where, $\mathcal{L}_l$ is the classical Lagrangian.

So $\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1})$ can be written as the form with Lagrangian:

$$\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1}) = \int \mathrm{d}p_l \cdot \frac{1}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{L}_l \Delta t}. \tag{1.39}$$

Substitute $\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1})$ into the path integral:

$$\prod_{l=1}^{N} \mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1}) = \int \frac{\mathrm{d}p_N}{2\pi\hbar} \cdots \frac{\mathrm{d}p_1}{2\pi\hbar} \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N} \mathcal{L}_l \cdot \Delta t} \tag{1.40}$$

let $\mathcal{D}_p = \prod_{l=1}^{N} \frac{\mathrm{d}p_l}{2\pi\hbar}$, when $\Delta t \to 0$, which means:

$$\sum_{l=1}^{N} \mathcal{L}_l \Delta t = \int_{t_0}^{t} \mathrm{d}\tau \cdot \mathcal{L}[p(\tau), x(\tau)] \tag{1.41}$$

Finally, we can get the propagators by the path integral:

Theorem 1.2.1. *The propagators path integral:*

$$\mathrm{i}G(x, t; x_0, t_0) = \int \mathcal{D}_x \mathcal{D}_p \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau \cdot \mathcal{L}[p(\tau), x(\tau)]} \tag{1.42}$$

where, the pair of $p(t)$ and $\dot{x}(t)$ characterizes a path in the px phase space.

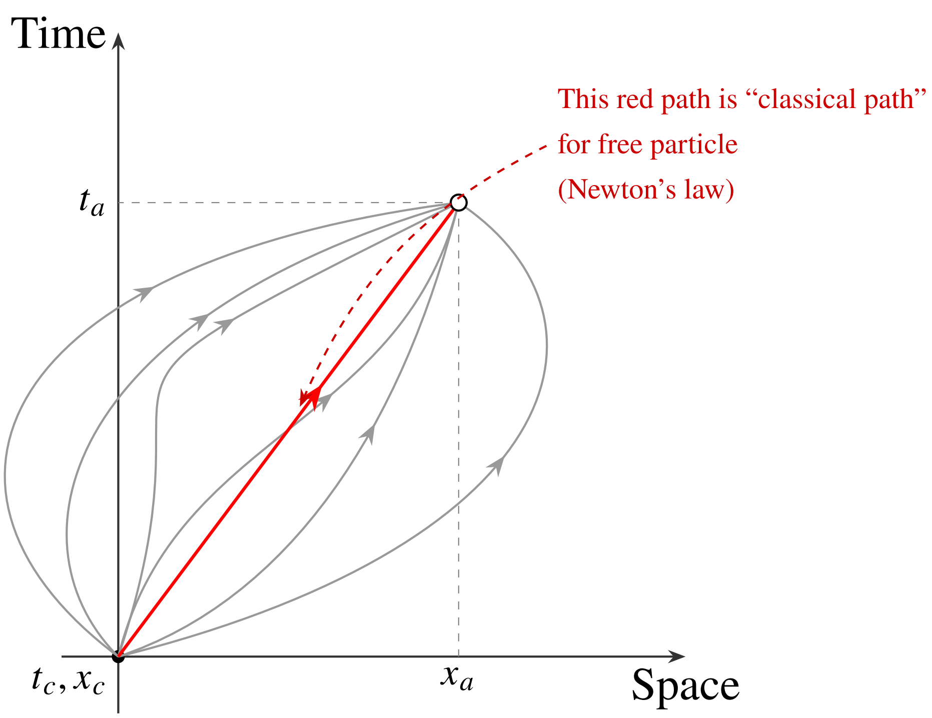


Figure 2: Schematic of Feynman Path Integral. The red path is the classical path (Newton's law) for a free particle, while the gray paths represent the quantum sum over histories.

1.3 Gaussian Integration

If the functional integration over p is Gaussian, we can exactly integrate out p. For example, $H = \frac{p^2}{2m} + V$, so $\mathcal{L} = p\dot{x} - H = p\dot{x} - \frac{p^2}{2m} - V(x)$,we can get:

$$\mathrm{i}G = \int \mathcal{D}p\mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_t \left(p_t\dot{x}_t - \frac{p_t^2}{2m} - V(x_t)\right)\Delta t\right] \tag{1.43}$$

Let:

$$\boldsymbol{p} = \begin{pmatrix} p_l \\ \vdots \\ p_1 \end{pmatrix}, \quad \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_l \\ \vdots \\ \dot{x}_1 \end{pmatrix} \tag{1.44}$$

So we can rewrite the integral as:

$$\mathrm{i}G = \int \mathcal{D}x \cdot \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}(-V(x_l))\Delta t\right] \cdot \int \mathcal{D}p \cdot \exp\left[\frac{\mathrm{i}}{2m\hbar}(-\boldsymbol{p}^T\boldsymbol{p} + 2m\boldsymbol{p}^T\dot{\boldsymbol{x}})\Delta t\right] \tag{1.45}$$

We have an useful formula for Gaussian integral (Proof in A):

$$\boxed{\int \prod_{n=1}^{N} \mathrm{d}x_n \exp\left[-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{y}\right] = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \exp\left[\frac{1}{2}\boldsymbol{y}^T A^{-1}\boldsymbol{y}\right]} \tag{1.46}$$

where, $\boldsymbol{x}, \boldsymbol{y}$ are real vectors and A is real symmetric matrix.

Let:

$$\begin{aligned} I &= \int \mathcal{D}p \exp\left[\frac{\mathrm{i}}{2m\hbar}(-\boldsymbol{p}^T\boldsymbol{p} + 2m\boldsymbol{p}^T\dot{\boldsymbol{x}})\Delta t\right] \\ &= \left(\frac{1}{2\pi\hbar}\right)^N \int \prod_{n=1}^{N} \mathrm{d}p_n \cdot \exp\left[-\frac{1}{2}\boldsymbol{p}^T A\boldsymbol{p} - \boldsymbol{p}^T\dot{\boldsymbol{x}}'\right] \end{aligned} \tag{1.47}$$

where $A = \frac{\mathrm{i}\Delta t}{m\hbar}\mathbb{I}_{N\times N}$ and $\dot{\boldsymbol{x}}' = -\frac{\mathrm{i}\Delta t}{\hbar}\dot{\boldsymbol{x}}$, $\mathbb{I}_{N\times N}$ is the $N \times N$ identity matrix.

So we can get:

$$\begin{cases} (\det A)^{-\frac{1}{2}} = \left(\dfrac{\mathrm{i}\Delta t}{m\hbar}\right)^{-\frac{N}{2}} \\ A^{-1} = \dfrac{m\hbar}{\mathrm{i}\Delta t}\mathbb{I}_{N\times N} \end{cases} \tag{1.48}$$

The exponent term is:

$$\begin{aligned} \frac{1}{2}\dot{\boldsymbol{x}}'^T A^{-1}\dot{\boldsymbol{x}}' &= \frac{1}{2}\cdot\frac{m\hbar}{\mathrm{i}\Delta t}\cdot\left(-\frac{\mathrm{i}\Delta t}{\hbar}\right)^2 \sum_{l=1}^{N}\dot{x}_l^2 \\ &= \frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t \end{aligned} \tag{1.49}$$

So:

$$\begin{aligned} I &= \left(\frac{1}{2\pi\hbar}\right)^N \cdot (2\pi)^{\frac{N}{2}} \cdot \left(\frac{\mathrm{i}\Delta t}{m\hbar}\right)^{-\frac{N}{2}} \cdot \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right] \\ &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right] \end{aligned} \tag{1.50}$$

So we can get the integration without p:

$$\begin{aligned} \mathrm{i}G &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t - V(x_l)\Delta t\right] \\ &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau\, \mathcal{L}(x,\dot{x})\right] \end{aligned} \tag{1.51}$$

So the path-integral is proportional to :

$$\boxed{\mathrm{i}G \propto \int \mathcal{D}x \cdot e^{\frac{\mathrm{i}}{\hbar}S[x(t)]}} \tag{1.52}$$

where, $S[x(t)] = \int_{t_0}^{t} \mathrm{d}t\, \mathcal{L}(x,\dot{x})$ is action and $\mathcal{L}(x,\dot{x}) = \frac{m\dot{x}^2}{2} - V(x)$ is Lagrangean.

From the Path-integral in real space-time, we can get some information about Physics Picture:

(1) Each path is weighted with a $U(1)$ phase factor $e^{\frac{\mathrm{i}}{\hbar}S}$. The Quantum interference effect between different paths.

(2) Since $\hbar \sim 10^{-34}\,\mathrm{J}\cdot\mathrm{s}$, any "small change" in S (we change S to $S+\delta S$), will drastically lead to quantum destructive interference. So only the paths that satisfy $\delta S = 0$ make dominant contributions to the path-integral.

(3) Remarkably, $\delta S = 0$ is exactly Hamilton's Principle in classical mechanics. So the classical paths ($\delta S = 0$) dominate the path integral in the limit $\hbar \to 0$. In other words, in classical mechanics, as $\hbar \to 0$, it neglects the contribution of the integral over all other paths near the path with $\delta S = 0$. So we can get the conclusion:

$$\text{Quantum system} \xrightarrow{\hbar\to 0} \text{classical system}$$

Example 1.3.1. Free particles' Hamiltonian is $H = \frac{p^2}{2m}$.
Using this Hamiltonian, we can get the path-integral:

$$\mathrm{i}G = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right]. \tag{1.53}$$

we let:

$$I = \int \mathcal{D}x \exp\left[\frac{\mathrm{i}m}{2\hbar\Delta t}\sum_{l=1}^{N}(x_l^2 - 2x_l x_{l-1} + x_{l-1}^2)\right] \tag{1.54}$$

In order to use Gaussian integral:

$$\boxed{\int \prod_{n=1}^{N} \mathrm{d}x_n \, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{y}} = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}} e^{\frac{1}{2}\boldsymbol{y}^T A^{-1} \boldsymbol{y}}} \tag{1.55}$$

we should rewrite the form of $\exp[\frac{im}{2\hbar\Delta t}\sum(x_l^2 - 2x_l x_{l-1} + x_{l-1}^2)]$, so we let:

$$A = \frac{-\mathrm{i}m}{\hbar\Delta t} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(N-1)\times(N-1)} \tag{1.56}$$

and:

$$\boldsymbol{x} = \begin{bmatrix} x_{N-1} \\ \vdots \\ x_1 \end{bmatrix}, \quad \boldsymbol{y} = \frac{-\mathrm{i}m}{\hbar\Delta t} \begin{bmatrix} -x_N \\ 0 \\ \vdots \\ \text{all zero} \\ \vdots \\ 0 \\ x_0 \end{bmatrix} \tag{1.57}$$

We get the new form of the exponent term:

$$\exp\left[\frac{\mathrm{i}m}{2\hbar\Delta t}\sum_{l=1}^{N-1}(x_l^2 - 2x_l x_{l-1} + x_{l-1}^2)\right] = \exp(-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{y}) e^{\frac{\mathrm{i}m}{2\hbar\Delta t}(x_N^2 + x_0^2)} \tag{1.58}$$

Because of $\mathcal{D}x = \prod_{l=1}^{N-1} \mathrm{d}x_l$ without x_N and x_0, so:

$$\mathrm{i}G = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \cdot e^{\frac{\mathrm{i}m}{2\hbar\Delta t}(x_N^2 + x_0^2)} \int \mathcal{D}x \, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{y}} \tag{1.59}$$

So we can compute $(\det A) = N \cdot \left(\frac{-\mathrm{i}m}{\hbar\Delta t}\right)^N$ (Proof in B) and get:

$$\mathrm{i}G = \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \cdot e^{\frac{\mathrm{i}m}{2\hbar\Delta t}(x_N^2 + x_0^2)} (2\pi)^{\frac{N}{2}} N^{-1/2} \left(\frac{-\mathrm{i}m}{\hbar\Delta t}\right)^{-\frac{N}{2}} \cdot e^{\frac{1}{2}\boldsymbol{y}^T A^{-1} \boldsymbol{y}} \tag{1.60}$$

Although A^{-1} is difficult to compute, we notice that $\boldsymbol{y} = \begin{bmatrix} -x_N \\ 0 \\ \vdots \\ 0 \\ x_0 \end{bmatrix}$ only have two non-zero elements, which locate in the first row and the last row respectively. So we only need to calculate the first and last columns of matrix A^{-1}, denoted $\boldsymbol{A}_1^{-1}$ and $\boldsymbol{A}_{N-1}^{-1}$, respectively (Proof in B):

$$\boldsymbol{A}_1^{-1} = \frac{\mathrm{i}\hbar\Delta t}{mN}\begin{bmatrix} N-1 \\ N-2 \\ \vdots \\ 1 \end{bmatrix}, \quad \boldsymbol{A}_{N-1}^{-1} = \frac{\mathrm{i}\hbar\Delta t}{mN}\begin{bmatrix} 1 \\ 2 \\ \vdots \\ N-1 \end{bmatrix} \tag{1.61}$$

The last term of the propagator(1.60) is:

$$\begin{aligned} -\frac{1}{2}\boldsymbol{y}^T A^{-1}\boldsymbol{y} &= \frac{1}{2}\boldsymbol{y}^T[x_N\boldsymbol{A}_1^{-1} + x_0\boldsymbol{A}_{N-1}^{-1}] \\ &= \frac{\mathrm{i}m}{2\hbar\Delta t}\left[-(x_N^2 + x_0^2) + \frac{(x_N - x_0)^2}{N}\right] \end{aligned} \tag{1.62}$$

So we can get the complete integral :

$$\begin{aligned} \mathrm{i}G &= \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} (2\pi)^{\frac{N}{2}} N^{-\frac{1}{2}} \left(\frac{-\mathrm{i}m}{\hbar\Delta t}\right)^{-\frac{N-1}{2}} e^{\frac{\mathrm{i}}{\hbar}\frac{m(x_N-x_0)^2}{2N\Delta t}} \\ &= \left(\frac{m}{i2\pi\hbar\Delta t N}\right)^{\frac{1}{2}} e^{\frac{\mathrm{i}}{\hbar}\frac{m(x_N-x_0)^2}{2N\Delta t}} \end{aligned} \tag{1.63}$$

where $t - t_0 = N\Delta t$, $x = x_N$. So the free particle's propagator is:

$$\boxed{\mathrm{i}G = \left[\frac{m}{i2\pi\hbar(t-t_0)}\right]^{\frac{1}{2}} \cdot e^{\frac{\mathrm{i}}{\hbar}\cdot\frac{m(x-x_0)^2}{2(t-t_0)}}} \tag{1.64}$$

Chapter 2

Stationary Phase Approximation (Semiclassical Approximation)

We have got the propagator $\mathrm{i}G$ for a partide to travel from an spacetime point (x_0, t_0) to a final spacetime point (x_f, t_f), with which is given by the Feynman path integral:

$$\boxed{\mathrm{i}G = K(x_f, t_f; x_0, t_0) = \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}[x(t)]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[x(t)]}} \tag{2.1}$$

where:

- $x(t)$ is the passition position of the particle at time t, respresenting a possible path.
- $\mathcal{D}[x(t)]$ is the functional measure for integrating over all paths that satisfy the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$.
- $S[x(t)]$ is the action for a path $x(t)$;

This integral is infinite-dimensional and generally very difficult to calculate directly. So we need a effective method to approximate it. The stationary phase approximation provides a method for approximating it.

2.1 One-dimensional integral of stationary phase approximation

Consider a integral:

$$I = \int \mathrm{e}^{\mathrm{i}f(x)/a}\mathrm{d}x \tag{2.2}$$

where a is a small parameter and $f(x)$ is a real-valued and regular function.

Our objective is to understant the physical picture of the stationary phase approximation for the propagator path integral through this one dimensional integral stationary phase approximation.

Let's further define a new notation $\Theta(x)$ by:

$$\Theta(x) = \frac{1}{a}f(x) \tag{2.3}$$

$\Theta(x)$ is a phase angle, so:

$$I = \int \mathrm{e}^{\mathrm{i}\Theta(x)}\mathrm{d}x \tag{2.4}$$

This integral can be physically regarded as an interference experiment.

Because each source at x contributes a phase factor $\mathrm{e}^{\mathrm{i}\Theta(x)}$, so the total integral I is the result of adding up all these infinite tiny vectors.

In order to compute the integral I, what we really need to do is to find the "dominant contribution" to I.

Firstly, let's pick up a point x_0 and evaluate the integral near x_0. The vicinity of x_0 is given by $x \in (x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2})$. Within this small domain, we may linearize $\Theta(x)$ by a Taylor series:

$$\begin{aligned} \Theta(x) &\approx \Theta(x_0) + \left.\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right|_{x=x_0} (x - x_0) \\ &= \frac{f(x_0)}{a} + \frac{f'(x_0)}{a}(x - x_0) \end{aligned} \tag{2.5}$$

The contributions to I in the small domain near x_0 are given by:

$$\begin{aligned} I_{x_0}^{\Delta x}(x_0) &= \int_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}} \mathrm{d}x\ \mathrm{e}^{\mathrm{i}\Theta(x_0)} \cdot \mathrm{e}^{\frac{\mathrm{i}f'(x_0)}{a}(x-x_0)} \\ &= \mathrm{e}^{\mathrm{i}\Theta(x_0)} \int_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}} \mathrm{d}x\ \mathrm{e}^{\frac{\mathrm{i}f'(x_0)}{a}(x-x_0)} \end{aligned} \tag{2.6}$$

we let $\Theta(x_0) = \Theta_0$ and $f'(x_0) = f_0'$, so:

$$\begin{aligned} I_{(x_0)}^{\Delta x} &\approx \mathrm{e}^{\mathrm{i}\Theta_0} \frac{a}{\mathrm{i}f_0'} \cdot \left. \mathrm{e}^{\frac{\mathrm{i}f_0'}{a}(x-x_0)} \right|_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}} \\ &= \mathrm{e}^{\mathrm{i}\Theta_0} \frac{2a}{\mathrm{i}f_0'} \sin \frac{f_0'\Delta x}{2a}. \end{aligned} \tag{2.7}$$

Let $\alpha = \frac{f_0'\Delta x}{2a}$, so:

$$I^{\Delta x}(x_0) \approx \mathrm{e}^{\mathrm{i}\Theta_0} \Delta x \cdot \frac{\sin\alpha}{\alpha} \tag{2.8}$$

Because a is a very small but nonzero parameter (in quantum mechanics, it corresponds to $\hbar$ being a very small but nonzero number), if $f_0' = 0$, α is strictly equal to 0 and it is not infinitely large near the zero point. So if we consider the case $f'(x_0) = 0$, we get:

$$I^{\Delta x}(x_0) = \mathrm{e}^{\mathrm{i}\Theta(x_0)} \cdot \Delta x \tag{2.9}$$

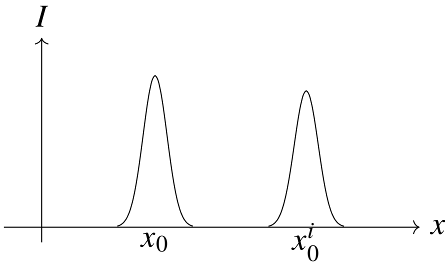
Physically, the result means that all U(1) phase in the vicinity of x_0 are completely the same. It's a perfect phase-coherence. The superposition of constant phases leads to linearly enhanced amplitude "Δx".

Next, let us consider $f'(x_0) \neq 0$. Because a is a very small parameter, $\alpha = \frac{f_0'\Delta x}{2a}$ is very large. So:

$$\frac{\sin\alpha}{\alpha} \to 0 \implies I^{\Delta x}(x_0) \approx 0 \tag{2.10}$$

Physically, the summation of all U(1) phases in the vicinity of x_0 leads to destructive interference. The parameter a is smaller, the destructive interference is more severe.

In conclusion, if we consider small enough but nozero parameter a, it is computationlly economic to merely focus on the integral contributions from the vicinity of these special point (denoted by a set $\{x_0^i\}$) that satisfy $f'(x_0^i) = 0$.



We consider the quadratic approximation in the vicinity of x_0:

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \\ &= f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 \end{aligned} \tag{2.11}$$

As a result, the original integral I can be evaluated by:

$$\begin{aligned} I &\approx \sum_{\{x_0^i\}} I^{\Delta x}(x_0^i) \\ &= \sum_{\{x_0^i\}} \mathrm{e}^{\frac{\mathrm{i}}{a} f(x_0^i)} \int_{x_0^i - \frac{\Delta x}{2}}^{x_0^i + \frac{\Delta x}{2}} \mathrm{d}x \; \mathrm{e}^{\frac{\mathrm{i}}{2a} f''(x_0^i)(x - x_0^i)^2} \end{aligned} \tag{2.12}$$

Because parameter a is a very small number, far away the stationary point x_0^i, the contributions of $\mathrm{e}^{\frac{\mathrm{i}}{2a} f''(x_0^i)(x - x_0^i)^2}$ to the integral cancel each other out through destructive interference.

So in the above Gaussian integral near x_0, we can extend the integral bounds to infinity:

$$\boxed{\begin{aligned} I &\approx \sum_{\{x_0^i\}} \mathrm{e}^{\frac{\mathrm{i} f(x_0^i)}{a}} \int_{-\infty}^{\infty} \mathrm{d}x \cdot \mathrm{e}^{\frac{\mathrm{i}}{2a} f''(x_0^i)(x - x_0^i)^2} \\ &= \sum_{\{x_0^i\}} \mathrm{e}^{\frac{\mathrm{i} f(x_0^i)}{a}} \sqrt{\frac{2\pi a \mathrm{i}}{f''(x_0^i)}} \end{aligned}} \tag{2.13}$$

In the above integral, we use the Fresnel integral (Proof see Appendix C):

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x = \sqrt{\frac{\pi}{2}}(1 + \mathrm{i}) = \sqrt{\pi} \mathrm{e}^{\mathrm{i}\pi/4}} \tag{2.14}$$

2.2 Semiclassical approximation of Feynman path integrals

We have discussed that if $\hbar$ tends to zero, then the quantum system will transitions to the classical system. We only need to treat $f(x)$ as $S[x(t)]$ and the parameter a as $\hbar$, then we can see the reason based on the discussion in the previous section. If $\hbar$ is zero, $I^{\Delta x}(x_0)$ equal to zero strictly for $x \neq x_0$ and is nonzero only at $x = x_0$. So we only need to consider the classical path with $\delta S = 0$ and not need to consider the quantum fluctuation near the classical path. But if $\hbar$ is a very small but nonzero number, we need to consider the quantum fluctuation near the classical path. In other word, in classical mechanics, $\hbar = 0$, $I^{\Delta x} \propto \frac{\sin\alpha}{\alpha}$ equal to zero in $(x_0 - \frac{\Delta x}{2}, x_0) \cup (x_0, x_0 + \frac{\Delta x}{2})$, because $\alpha = \frac{f_0'\Delta x}{2a}$ tends to infinity no matter how small the radius of this deleted neighbourhood is. But in quantum mechanics, $\hbar \sim 10^{-34}\mathrm{J}\cdot\mathrm{s}$, I is not equal to zero in the neighbourhood whose radius length matches to the order of magnitude of $\hbar$. Therefore, we cannot ignore the impact generated by I in this neighbourhood.

The Feynman path integral:

$$K(x_f, t_f; x_0, t_0) \propto \int \mathcal{D}[x(t)] \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[x(t)]} \tag{2.15}$$

it can be regarded as path integral version of $\int \mathrm{d}x \mathrm{e}^{\frac{\mathrm{i}}{a}f(x)}$. $f(x)$ is replaced by Classical action S_c when $\delta S_c = 0$. Liking the previous section, we consider the quadratic approximation in the vicinity of classical path:

$$\begin{aligned} S &= S_c + \delta S + \frac{1}{2}\delta^2 S \\ &= S_c + \frac{1}{2}\delta^2 S. \end{aligned} \tag{2.16}$$

Now, we can decompose the path near classical path into the classical path $x_c(t)$ plus a quantum fluctuation $y(t)$ around it:

$$x(t) = x_c(t) + y(t) \tag{2.17}$$

$x(t)$ must satisfy the same boundary conditions, so:

$$\begin{cases} y(t_0) = 0 \\ y(t_f) = 0 \end{cases} \tag{2.18}$$

let's perform a functional Taylor expansion of the action $S[x(t)]$ around the classical path $x_c(t)$:

$$\begin{aligned} S[x(t)] = S[x_c + y] = S[x_c] + \int_{t_0}^{t_f} \mathrm{d}t \left.\frac{\delta S}{\delta x(t)}\right|_{x=x_c} y(t) \\ + \frac{1}{2}\int_{t_0}^{t_f} \mathrm{d}t_1 \int_{t_0}^{t_f} \mathrm{d}t_2 \left.\frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\right|_{x=x_c} y(t_1)y(t_2) + O(y^3) \end{aligned} \tag{2.19}$$

In semiclassical approximation, we assume the fluctuations y are small, so we neglect terms of $O(y^3)$ and higher. At the same time, $\delta S = 0$, so the action $S[x(t)]$:

$$S[x(t)] \approx S_c + \frac{1}{2}\int\int \mathrm{d}t_1\mathrm{d}t_2\, y(t_1) \cdot \left.\frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\right|_{x=x_c} \cdot y(t_2) \tag{2.20}$$

Now, let's solve the fluctuation term $\delta^2 S$:

$$\delta^2 S = \iint \mathrm{d}t_1 \mathrm{d}t_2 \cdot y(t_1) \cdot \left.\frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}\right|_{x=x_c} \cdot y(t_2) \tag{2.21}$$

Consider a standard Lagrangian $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$, so the action is:

$$S = \int \mathrm{d}t \left[\frac{1}{2}m\dot{x}^2 - V(x)\right] \tag{2.22}$$

it's second variation is:

$$\delta^2 S = \int \mathrm{d}t \cdot \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}V''(x_c) \cdot y^2\right) \tag{2.23}$$

We substitute the approximated action back into the path integral expression:

$$\begin{aligned} K &\propto \int \mathcal{D}[x(t)] \exp\left\{\frac{\mathrm{i}}{\hbar}\left[S_c + \frac{1}{2}\delta^2 S\right]\right\} \\ &= \int \mathcal{D}[x(t)] \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c} \exp\left[\int \mathrm{d}t \left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}V''(x_c)y^2\right)\right] \end{aligned} \tag{2.24}$$

Now, we need to change the integration variable from an integral over all paths $x(t)$ to an integral over all fluctuations $y(t)$. Since $x_c(t)$ is a fixed classical path, the path measure is:

$$\mathcal{D}[x(t)] = \mathcal{D}[x_c(t) + y(t)] = \mathcal{D}[y(t)] \tag{2.25}$$

and S_c is a constant, it can be factored out of integral:

$$\boxed{\begin{aligned} K &\propto \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c} \cdot \int \mathcal{D}[y(t)] \cdot \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\int \mathrm{d}t\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}V''(x_c)y^2\right)} \\ &= F(t_f, t_0) \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c}. \end{aligned}} \tag{2.26}$$

where:

1. $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c}$ is the Classical Phase Factor. It tells us that in the semiclassical approximation, the evolution of the system's quantum phase is dominated by the classical action. This is a bridge connecting classical and quantum mechanics.
2. $F(t_f, t_0) = \int \mathcal{D}[y(t)] \mathrm{e}^{\frac{\mathrm{i}}{2\hbar}\delta^2 S}$ is the Quantum Fluctuation Prefactor. It describes the collective contribution of all the small quantum fluctuations around the classical path.

Now let's see a example: one dimensional free particle.

Example 2.2.1. For free particle:

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2. \tag{2.27}$$

First, we need to calculate the classical action. It is given by the Euler-Lagrange equertion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0 \tag{2.28}$$

For free particle:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \\ \frac{\partial \mathcal{L}}{\partial x} = 0 \end{cases} \tag{2.29}$$

so the equation of motion is:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = m\ddot{x} = 0 \tag{2.30}$$

we can get the general solution:

$$x_c(t) = at + b \tag{2.31}$$

we impose the boundary conditions:

$$\begin{cases} x_c(t_0) = x_0 \\ x_c(t_f) = x_f \end{cases} \tag{2.32}$$

to determine a and b:

$$\begin{cases} a = \frac{x_f - x_0}{t_f - t_0} \\ b = x_i - t_i\left(\frac{x_f - x_0}{t_f - t_0}\right) \end{cases} \tag{2.33}$$

the classical action is:

$$S_c = \int_{t_i}^{t_f} \mathrm{d}t \cdot \frac{1}{2}m(\dot{x}_c)^2 = \int_{t_i}^{t_f} \mathrm{d}t \cdot \frac{1}{2}ma^2 = \frac{m(x_f - x_0)^2}{2(t_f - t_0)} \tag{2.34}$$

So the classical phase factor is:

$$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_c} = \mathrm{e}^{\frac{\mathrm{i}m(x_f - x_0)^2}{2\hbar(t_f - t_i)}} \tag{2.35}$$

now let's calculate the quantum fluctuation factor:

$$\begin{aligned} F(t_f, t_0) &= \int \mathcal{D}[y(t)] \exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_{t_0}^{t_f}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \mathrm{d}t\right\} \\ &= \int \prod_{i=1}^{N-1} \mathrm{d}y_i \exp\left[\frac{\mathrm{i}m}{2\hbar}\sum_{i=1}^{N}\frac{(y_i - y_{i-1})^2}{\Delta t^2}\Delta t\right] \end{aligned} \tag{2.36}$$

because $y_N = y_0 = 0$, we can rewrite the sum as:

$$\frac{\mathrm{i}m}{2\hbar}\sum_{i=1}^{N}\frac{(y_i - y_{i-1})^2}{\Delta t} = -\frac{1}{2}\boldsymbol{y}^T M \boldsymbol{y} \tag{2.37}$$

where

$$M = \frac{m}{\mathrm{i}\hbar\Delta t}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{(N-1)\times(N-1)}, \quad \boldsymbol{y} = \begin{pmatrix} y_{N-1} \\ \vdots \\ y_1 \end{pmatrix} \tag{2.38}$$

and use the Gaussion integral:

$$\begin{aligned} F(t_f, t_0) &= (2\pi)^{(N-1)/2}(\det M)^{-1/2} \\ &= (2\pi)^{(N-1)/2}\left(\left(\frac{\mathrm{i}\hbar\Delta t}{m}\right)^{N-1} N\right)^{-1/2} \end{aligned} \tag{2.39}$$

So we can get the complete propagator of free partide:

$$\begin{aligned} K &= J \cdot \left(\frac{2\pi\mathrm{i}\hbar\Delta t}{m}\right)^{N/2} \cdot \left(\frac{m}{2\pi\mathrm{i}\hbar N\Delta t}\right)^{1/2} \cdot \mathrm{e}^{\frac{\mathrm{i}S_c}{\hbar}} \\ &= \left(\frac{m}{2\pi\mathrm{i}\hbar\Delta t}\right)^{N/2} \cdot \left(\frac{2\pi\mathrm{i}\hbar\Delta t}{m}\right)^{N/2} \cdot \left(\frac{m}{2\pi\mathrm{i}(t_f - t_0)}\right)^{1/2} \cdot \mathrm{e}^{\frac{\mathrm{i}S_c}{\hbar}} \\ &= \left(\frac{m}{2\pi\mathrm{i}\hbar(t_f - t_0)}\right)^{1/2} \cdot \mathrm{e}^{\frac{\mathrm{i}S_c}{\hbar}} \end{aligned} \tag{2.40}$$

where, $S_c = \frac{m}{2}\frac{(x_f - x_0)^2}{t_f - t_0}$ and $J = \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}}$ is the coefficient in front of the path integral 2.1.

Part II

Quantum Spins, Coherent-state Path Integral, and Topological Terms

Chapter 3

Quantum Spin

We begin by considering the Hilbert space $\mathcal{H}$ for a single quantum spin-1/2 particle. This is a two-dimensional complex vector space.

The conventional approach is to use an orthonormal basis formed by the eigenvectors of the spin operator along a chosen axis, typically the z-axis, denoted $\hat{S}_z$.

$$\hat{S}_z|\uparrow\rangle = +\frac{\hbar}{2}|\uparrow\rangle \quad \text{and} \quad \hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle \tag{3.1}$$

Here, $|\uparrow\rangle$ represents the ”spin up” state and $|\downarrow\rangle$ represents the ”spin down” state. These two states form a complete orthonormal basis, satisfying:

- **Orthogonality:** $\langle\uparrow|\downarrow\rangle = 0$
- **Normalization:** $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$
- **Completeness:** $|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \hat{\mathbb{I}}$

where $\hat{\mathbb{I}}$ is the identity operator in $\mathcal{H}$.

But this simple, discrete basis has two drawbacks:

- This discrete parametrized complete-set is not convenient for constructing the path integral formalism of quantum spins.
- $SU(2)$ spin symmetry is broken either explicitly or not manifest.

3.1 Spin Coherent States

A general, normalized state $|\psi\rangle$ in the spin-1/2 Hilbert space can be written as a complex linear combination of the basis states:

$$|\psi\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle \tag{3.2}$$

where $z_1, z_2 \in \mathbb{C}$ are complex coefficients.

The normalization condition $\langle\psi|\psi\rangle = 1$ imposes a constraint on these coefficients:

$$\langle\psi|\psi\rangle = (|z_1|^2 + |z_2|^2) = 1 \tag{3.3}$$

A complex number $z = x + \mathrm{i}y$ has two real parameters. Therefore, the pair (z_1, z_2) is defined by four real parameters. The normalization condition $|z_1|^2 + |z_2|^2 = 1$ removes one degree of freedom, leaving three.

Furthermore, in quantum mechanics, the overall phase of a state vector is unphysical. The states $|\psi\rangle$ and $\mathrm{e}^{\mathrm{i}\gamma}|\psi\rangle$ (for any real γ) represent the same physical state (i.e., they belong to the same ray in Hilbert space). This "gauge freedom" removes one more degree of freedom.

This leaves $4 - 1 - 1 = 2$ real, physical degrees of freedom. This is a crucial observation: the state space of a spin-1/2 particle is topologically equivalent to the surface of a 2D sphere, which is also parameterized by two angles (like latitude and longitude).

We can explicitly parameterize z_1 and z_2 using two angles, θ and ϕ, which will map directly to the surface of a sphere. A standard (but not unique) parametrization for the spin coherent state, labeled by a unit vector $\boldsymbol{n}$, is:

$$|\boldsymbol{n}\rangle \equiv |\theta, \phi\rangle = \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.4}$$

Here, the spherical coordinate angles have the domains $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. We can easily verify that this state is normalized:

$$\langle\boldsymbol{n}|\boldsymbol{n}\rangle = \left|\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\right|^2 + \left|\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\right|^2 = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1 \tag{3.5}$$

This set of states $\{|\boldsymbol{n}\rangle\}$ is continuously parameterized by the angles (θ, ϕ), addressing the first drawback of the discrete basis.

3.1.1 Physical Interpretation: The Bloch Sphere

To understand the physical meaning of θ and ϕ, we compute the expectation value of the vector spin operator $\hat{\boldsymbol{S}}$ in the state $|\boldsymbol{n}\rangle$. We will set $\hbar = 1$ from here on for simplicity. The spin operator is $\hat{\boldsymbol{S}} = \frac{1}{2}\hat{\boldsymbol{\sigma}}$, where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.6}$$

In the $|\uparrow\rangle, |\downarrow\rangle$ basis, $|\boldsymbol{n}\rangle$ is represented by the column vector:

$$|\boldsymbol{n}\rangle = \begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \quad \Longrightarrow \quad \langle\boldsymbol{n}| = \begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} & \sin(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix} \tag{3.7}$$

Expectation value of $\hat{S}_z$:

$$\begin{aligned} \langle\hat{S}_z\rangle &= \langle\boldsymbol{n}|\left(\frac{1}{2}\hat{\sigma}_z\right)|\boldsymbol{n}\rangle = \frac{1}{2}\begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} & \sin(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \\ &= \frac{1}{2}\left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}\cos(\theta) \end{aligned} \tag{3.8}$$

Expectation value of $\hat{S}_x$:

$$\begin{aligned} \langle\hat{S}_x\rangle &= \langle\boldsymbol{n}|\left(\frac{1}{2}\hat{\sigma}_x\right)|\boldsymbol{n}\rangle = \frac{1}{2}\left(\dots\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(\dots\right) \\ &= \frac{1}{2}\left(\cos(\theta/2)\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} + \sin(\theta/2)\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2}\right) \\ &= \frac{1}{2}\cos(\theta/2)\sin(\theta/2)\left(\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}\right) = \left(\frac{1}{2}\sin\theta\right)\left(\frac{\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}}{2}\right) = \frac{1}{2}\sin\theta\cos\phi \end{aligned} \tag{3.9}$$

Expectation value of $\hat{S}_y$:

$$\begin{aligned} \langle\hat{S}_y\rangle &= \langle\boldsymbol{n}|\left(\frac{1}{2}\hat{\sigma}_y\right)|\boldsymbol{n}\rangle = \frac{1}{2}\left(\dots\right)\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\left(\dots\right) \\ &= \frac{1}{2}\left(\cos(\theta/2)(-\mathrm{i})\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} + \sin(\theta/2)(\mathrm{i})\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2}\right) \\ &= \frac{1}{2}\cos(\theta/2)\sin(\theta/2)\left(-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\right) = \left(\frac{1}{2}\sin\theta\right)\left(\frac{\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi}}{2\mathrm{i}}\right) = \frac{1}{2}\sin\theta\sin\phi \end{aligned} \tag{3.10}$$

Combining these results, the expectation value of the spin vector is:

$$\langle\hat{\boldsymbol{S}}\rangle = \frac{1}{2}\left(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\right) \tag{3.11}$$

This is a vector of length $S = 1/2$ pointing in the direction specified by the unit vector $\boldsymbol{n}$:

$$\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{3.12}$$

Thus, the state $|\boldsymbol{n}\rangle$ is the quantum state that "points" in the classical direction $\boldsymbol{n}$. This direction vector lives on the surface of a unit sphere, known as the **Bloch Sphere**.

This formalism treats all directions $\boldsymbol{n}$ on an equal footing, making the SU(2) rotational symmetry manifest. This addresses the second drawback of the discrete basis.

The eigenvector equation:

$$(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}|\boldsymbol{n}\rangle \tag{3.13}$$

This equation signifies that the coherent state $|\boldsymbol{n}\rangle$ is, by definition, the "spin up" eigenvector of the spin operator projected along its own pointing direction $\boldsymbol{n}$, with the eigenvalue $+1/2$ (with $\hbar = 1$).

Proof. We first construct the operator $\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$ in matrix form:

$$\begin{aligned}
\hat{\boldsymbol{S}} \cdot \boldsymbol{n} &= \hat{S}_x n_x + \hat{S}_y n_y + \hat{S}_z n_z \\
&= \frac{1}{2}(\hat{\sigma}_x n_x + \hat{\sigma}_y n_y + \hat{\sigma}_z n_z) \\
&= \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\theta\cos\phi + \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\theta\right] \\
&= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - \mathrm{i}\sin\phi) \\ \sin\theta(\cos\phi + \mathrm{i}\sin\phi) & -\cos\theta \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, \mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\, \mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix}
\end{aligned} \tag{3.14}$$

Now, we apply this operator to the coherent state vector $|\boldsymbol{n}\rangle$:

$$(\hat{\boldsymbol{S}} \cdot \boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, \mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\, \mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos(\theta/2)\, \mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\, \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \tag{3.15}$$

We compute the top and bottom components of the resulting vector separately.
Top component:

$$\begin{aligned}
&\frac{1}{2}\left[\cos\theta\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} + \sin\theta\mathrm{e}^{-\mathrm{i}\phi}\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\right] \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\phi/2}\left[\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right] \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\phi/2}\left[\cos(\theta - \theta/2)\right] \quad \text{(using } \cos(A-B) \text{ identity)} \\
&= \frac{1}{2}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}
\end{aligned} \tag{3.16}$$

This is precisely $\frac{1}{2}$ times the top component of $|\boldsymbol{n}\rangle$.
Bottom component:

$$\begin{aligned}
&\frac{1}{2}\left[\sin\theta\mathrm{e}^{\mathrm{i}\phi}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} - \cos\theta\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\right] \\
&= \frac{1}{2}\mathrm{e}^{\mathrm{i}\phi/2}\left[\sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2)\right] \\
&= \frac{1}{2}\mathrm{e}^{\mathrm{i}\phi/2}\left[\sin(\theta - \theta/2)\right] \quad \text{(using } \sin(A-B) \text{ identity)} \\
&= \frac{1}{2}\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}
\end{aligned} \tag{3.17}$$

This is precisely $\frac{1}{2}$ times the bottom component of $|\boldsymbol{n}\rangle$.

Combining both components, we have shown:

$$(\hat{\boldsymbol{S}} \cdot \boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}\begin{pmatrix} \cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} = \frac{1}{2}|\boldsymbol{n}\rangle \tag{3.18}$$

This completes the proof. □

3.1.2 Gauge Choice and Topological Singularities

The parametrization in Eq. (3.4) is not unique, and it hides a subtle topological problem.

- **At the North Pole** ($\theta = 0$): The direction $\boldsymbol{n}$ is $(0, 0, 1)$. The angle ϕ is ill-defined. Our formula gives $|\theta = 0\rangle = \cos(0)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle + \sin(0)\cdots = \mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle$. The state vector itself depends on the meaningless angle ϕ. This is a **singularity**.
- **At the South Pole** ($\theta = \pi$): The direction $\boldsymbol{n}$ is $(0, 0, -1)$. Our formula gives $|\theta = \pi\rangle = \cos(\pi/2)\cdots + \sin(\pi/2)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle = \mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle$. This is also singular.

This is analogous to the problem of creating a flat map of the Earth: you cannot do so without singularities (e.g., at the poles) or cuts.

We can "fix" the singularity at one pole by making a ϕ-dependent gauge choice (i.e., multiplying by an overall phase $\mathrm{e}^{\mathrm{i}\gamma(\phi)}$).

Choice 1: Regular at North Pole. Let's choose an overall phase $\gamma = \phi/2$. The new state, $|\boldsymbol{n}\rangle_N$, is:

$$|\boldsymbol{n}\rangle_N = \mathrm{e}^{\mathrm{i}\phi/2}|\boldsymbol{n}\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi}|\downarrow\rangle \tag{3.19}$$

- At the North Pole ($\theta = 0$): $|\boldsymbol{n}\rangle_N = \cos(0)|\uparrow\rangle + \sin(0)\cdots = |\uparrow\rangle$. This is now regular and well-defined.
- At the South Pole ($\theta = \pi$): $|\boldsymbol{n}\rangle_N = \cos(\pi/2)|\uparrow\rangle + \sin(\pi/2)\mathrm{e}^{\mathrm{i}\phi}|\downarrow\rangle = \mathrm{e}^{\mathrm{i}\phi}|\downarrow\rangle$. The singularity has been "pushed" to the South Pole.

Choice 2: Regular at South Pole. Let's choose $\gamma = -\phi/2$. The new state, $|\boldsymbol{n}\rangle_S$, is:

$$|\boldsymbol{n}\rangle_S = \mathrm{e}^{-\mathrm{i}\phi/2}|\boldsymbol{n}\rangle = \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle \tag{3.20}$$

This state is regular at the South Pole ($|\boldsymbol{n}\rangle_S = |\downarrow\rangle$) but singular at the North Pole.

This unavoidable singularity is topological in nature and is the origin of the **Berry Phase**, or the "topological term," in the coherent-state path integral.

3.1.3 Over-Completeness and Orthogonality

The set of all coherent states $\{|\boldsymbol{n}\rangle\}$ for all $\boldsymbol{n}$ on the sphere is an **over-complete** basis. The Hilbert space is only 2-dimensional, but we have an infinite, continuous set of states. This means the states are not, in general, orthogonal.

$$\langle \boldsymbol{n}'|\boldsymbol{n}\rangle \neq 0 \quad \text{for } \boldsymbol{n}' \neq \boldsymbol{n} \text{ and } \boldsymbol{n}' \neq -\boldsymbol{n} \tag{3.21}$$

A special exception, as noted in the text, is for antipodal states.

3.1.4 Orthogonality of Antipodal States

Let us prove that $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle = 0$. The antipodal point $-\boldsymbol{n}$ corresponds to the angles $(\theta', \phi') = (\pi - \theta, \phi + \pi)$.

We write the state $|-\boldsymbol{n}\rangle$ using Eq. (3.4):

$$\begin{aligned} |-\boldsymbol{n}\rangle &= \cos\left(\frac{\pi-\theta}{2}\right) \mathrm{e}^{-\mathrm{i}(\phi+\pi)/2}|\uparrow\rangle + \sin\left(\frac{\pi-\theta}{2}\right) \mathrm{e}^{\mathrm{i}(\phi+\pi)/2}|\downarrow\rangle \\ &= \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\pi/2}|\uparrow\rangle + \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\pi/2}|\downarrow\rangle \end{aligned} \tag{3.22}$$

Using $\cos(\pi/2 - x) = \sin(x)$, $\sin(\pi/2 - x) = \cos(x)$, $\mathrm{e}^{-\mathrm{i}\pi/2} = -\mathrm{i}$, and $\mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}$:

$$|-\boldsymbol{n}\rangle = \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\phi/2}(-\mathrm{i})|\uparrow\rangle + \cos\left(\frac{\theta}{2}\right) \mathrm{e}^{\mathrm{i}\phi/2}(\mathrm{i})|\downarrow\rangle \tag{3.23}$$

Now we compute the inner product $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle$:

$$\begin{aligned} &\langle -\boldsymbol{n}|\boldsymbol{n}\rangle \\ &= \left(\mathrm{i}\sin\left(\frac{\theta}{2}\right) \mathrm{e}^{\mathrm{i}\phi/2}\langle\uparrow| + (-\mathrm{i})\cos\left(\frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\phi/2}\langle\downarrow|\right)\left(\cos\left(\frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle\right) \\ &= \mathrm{i}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2} + (-\mathrm{i})\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} \\ &= \mathrm{i}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \mathrm{i}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ &= 0 \end{aligned} \tag{3.24}$$

This confirms that antipodal states are orthogonal, as expected. For example, $|\boldsymbol{n} = \hat{z}\rangle = |\uparrow\rangle$ is orthogonal to $|\boldsymbol{n} = -\hat{z}\rangle = |\downarrow\rangle$.

Distinction Between $|-\boldsymbol{n}\rangle$ and $-|\boldsymbol{n}\rangle$

It is a common point of confusion to mistake the antipodal state $|-\boldsymbol{n}\rangle$ for the state $-|\boldsymbol{n}\rangle$. We must justify that, in general, $|-\boldsymbol{n}\rangle \neq -|\boldsymbol{n}\rangle$.

From our derivation in the previous section, the antipodal state is:

$$|-\boldsymbol{n}\rangle = \mathrm{i}\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle - \mathrm{i}\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.25}$$

In contrast, the state $-|\boldsymbol{n}\rangle$ is:

$$-|\boldsymbol{n}\rangle = -\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle - \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.26}$$

By simple inspection, these two state vectors are clearly not identical. They are, in fact, orthogonal to each other, as we just proved $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle = 0$. If $|-\boldsymbol{n}\rangle$ were equal to $-|\boldsymbol{n}\rangle$, then we would have $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle = \langle -\boldsymbol{n}|-(-\boldsymbol{n})\rangle = -1\cdot\langle -\boldsymbol{n}|-\boldsymbol{n}\rangle = -1$, which contradicts our result of 0 (unless the state is null, which is not the case).

The state $|-\boldsymbol{n}\rangle$ represents a spin pointing in the *opposite direction* (e.g., spin down), while $-|\boldsymbol{n}\rangle$ represents the *same physical state* as $|\boldsymbol{n}\rangle$ but with a phase shift of π (since $\mathrm{e}^{\mathrm{i}\pi} = -1$).

3.1.5 Completeness Relation

Despite being over-complete, the spin coherent states provide a resolution of the identity operator $\hat{\mathbb{I}}$.

Let's call the integral $J = \int \mathrm{d}\Omega\,|\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\,|\boldsymbol{n}\rangle\langle\boldsymbol{n}|$.

1. Integrate over ϕ: The off-diagonal terms depend on $\mathrm{e}^{\pm\mathrm{i}\phi}$.

$$\int_0^{2\pi}\mathrm{e}^{\pm\mathrm{i}\phi}\,\mathrm{d}\phi = 0 \tag{3.27}$$

The diagonal terms are independent of ϕ.

$$\int_0^{2\pi} 1\,\mathrm{d}\phi = 2\pi \tag{3.28}$$

After integrating over ϕ, the matrix J becomes diagonal:

$$J = \int_0^{\pi}\sin\theta\,\mathrm{d}\theta\begin{pmatrix}2\pi\cos^2(\theta/2) & 0\\ 0 & 2\pi\sin^2(\theta/2)\end{pmatrix} \tag{3.29}$$

2. Integrate over θ: Both integrals evaluate to 2π. Thus, the full integral is:

$$J = \int \mathrm{d}\Omega\,|\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \begin{pmatrix}2\pi & 0\\ 0 & 2\pi\end{pmatrix} = 2\pi\,\hat{\mathbb{I}} \tag{3.30}$$

Dividing by 2π, we arrive at the completeness relation:

$$\frac{1}{2\pi}\int \mathrm{d}\Omega\,|\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \hat{\mathbb{I}} \tag{3.31}$$

This relation is the foundation for the coherent-state path integral. It allows us to insert the identity operator at infinitesimally small time steps, t_j, as an integral over the Bloch sphere: $\hat{\mathbb{I}} = \int \frac{\mathrm{d}\Omega_j}{2\pi}|\boldsymbol{n}_j\rangle\langle\boldsymbol{n}_j|$. Summing over all paths becomes an integral over all $\boldsymbol{n}_j$ at all times t_j.

3.2 Coherent-state path integral for spin

Consider the quantum amplitude ($\hbar = 1$):

$$\langle n_f|\mathrm{e}^{-\mathrm{i}\hat{H}(t_f-t_i)}|n_i\rangle \xrightarrow{\mathrm{i}(t_f-t_i)\to T} \langle n_f|\mathrm{e}^{-\hat{H}T}|n_i\rangle \tag{3.32}$$

First, consider $\hat{H}$ that is an isolated spin we divide the total time interval T into N small slices of duration $\Delta t = \frac{T}{N}$. In the limit $N \to \infty$ and $\Delta t \to 0$, the time evolution operator can be written as a product of infinitesimal operators:

$$\mathrm{e}^{-\mathrm{i}\hat{H}T} = \left(\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t}\right)^N \tag{3.33}$$

The amplitude is the:

$$\langle n_f|\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t}\dots\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t}|n_i\rangle \tag{3.34}$$

We now insert the resolution of identity operator: let's label our initial state as $|n_0\rangle = |n_i\rangle$ and final state as $|n_N\rangle = |n_f\rangle$

$$\begin{aligned} K &= \langle n_N|\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t}\cdot\left(\int\frac{\mathrm{d}\boldsymbol{n}_{N-1}}{2\pi}|n_{N-1}\rangle\langle n_{N-1}|\right)\cdot\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t}\dots\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t}|n_0\rangle \\ &= \int\left(\prod_{l=1}^{N-1}\frac{\mathrm{d}\boldsymbol{n}_l}{2\pi}\right)\cdot\left(\prod_{l=1}^{N}\langle n_l|\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t}|n_{l-1}\rangle\right) \end{aligned} \tag{3.35}$$

Since Δt is small, we can expand the exponential to first order:

$$\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t} \approx \hat{\mathbb{I}} - \mathrm{i}\hat{H}\Delta t \tag{3.36}$$

The matrix element becomes:

$$\begin{aligned} \langle n_l|\mathrm{e}^{-\mathrm{i}\hat{H}\Delta t}|n_{l-1}\rangle &= \langle n_l|\hat{\mathbb{I}} - \mathrm{i}\hat{H}\Delta t|n_{l-1}\rangle \\ &= \langle n_l|n_{l-1}\rangle - \mathrm{i}\Delta t\langle n_l|\hat{H}|n_{l-1}\rangle \end{aligned} \tag{3.37}$$

let's represent the state at time t_l as a small deviation from the state at t_{l-1}

$$|n_l\rangle \approx |n_{l-1}\rangle + \delta|n_{l-1}\rangle = |n_{l-1}\rangle + \left.\frac{\mathrm{d}|n(t)\rangle}{\mathrm{d}t}\right|_{t=t_{l-1}}\cdot\Delta t \tag{3.38}$$

The overlap is then:

$$\begin{aligned} \langle n_l|n_{l-1}\rangle &\approx \langle n_{l-1}|n_{l-1}\rangle + \left(\left.\frac{\mathrm{d}\langle n(t)|}{\mathrm{d}t}\right|_{t=t_{l-1}}\right)|n_{l-1}\rangle\Delta t \\ &= 1 + \left(\left.\frac{\mathrm{d}\langle n_l|}{\mathrm{d}t}\right|_{t=t_{l-1}}\right)|n_{l-1}\rangle\cdot\Delta t \end{aligned} \tag{3.39}$$

This can be written in more suggestive exponential form for small Δt:

$$\langle n_l | n_{l-1} \rangle \approx \mathrm{e}^{\langle \dot{n}_{l-1} | n_{l-1} \rangle \cdot \Delta t} \tag{3.40}$$

where, $\langle \dot{n}_{l-1} | = \frac{\mathrm{d}\langle n(t)|}{\mathrm{d}t}\Big|_{t=t_{l-1}}$

By the way, from the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle n_l | n_l \rangle = \left(\frac{\mathrm{d}}{\mathrm{d}t}\langle n_l|\right)|n_l\rangle + \langle n_l|\left(\frac{\mathrm{d}}{\mathrm{d}t}|n_l\rangle\right) = 0 \tag{3.41}$$

we can get:

$$\begin{aligned} \langle \dot{n}_l | n_l \rangle &= -\langle n_l | \dot{n}_l \rangle \\ &= -\langle \dot{n}_l | n_l \rangle^* \end{aligned} \tag{3.42}$$

So $\langle \dot{n}_l | n_l \rangle$ is purely imaginary

Because we only retain first-order small quantities, so the second term of matrix element becomes:

$$\begin{aligned} -\mathrm{i}\Delta t \langle n_l | \hat{H} | n_{l-1} \rangle &\approx -\mathrm{i}\Delta t (\langle n_{l-1}| + \langle \dot{n}_{l-1}|\Delta t)\hat{H}|n_{l-1}\rangle \\ &= -\mathrm{i}\langle n_{l-1}|\hat{H}|n_{l-1}\rangle \Delta t + O(\Delta t^2) \\ &\approx -\mathrm{i}\langle n_{l-1}|\hat{H}|n_{l-1}\rangle \Delta t \end{aligned} \tag{3.43}$$

Putting everything back into the matrix elements:

$$\begin{aligned} \langle n_l | \mathrm{e}^{-\mathrm{i}\hat{H}\Delta t} | n_{l-1} \rangle &= 1 + (\langle \dot{n}_{l-1} | n_{l-1} \rangle - \mathrm{i}\langle n_{l-1}|\hat{H}|n_{l-1}\rangle)\Delta t \\ &\approx \exp\left[(\langle \dot{n}_{l-1} | n_{l-1} \rangle - \mathrm{i}\langle n_{l-1}|\hat{H}|n_{l-1}\rangle)\Delta t\right] \end{aligned} \tag{3.44}$$

So the amplitude K is:

$$K = \int \left(\prod_{l=1}^{N-1} \frac{\mathrm{d}\boldsymbol{n}_l}{2\pi}\right) \cdot \mathrm{e}^{\sum_{l=1}^{N}(\langle \dot{n}_{l-1}|n_{l-1}\rangle - \mathrm{i}\langle n_{l-1}|\hat{H}|n_{l-1}\rangle)\Delta t} \tag{3.45}$$

The product of integrals over all intermediate states becomes the formal path integral measure $D[n(t)]$:

$$\begin{aligned} K &= \int D[n(t)] \exp\left\{\mathrm{i}\int_{t_i}^{t_f} [-\mathrm{i}\langle \dot{n}(t)|n(t)\rangle - \langle n(t)|\hat{H}|n(t)\rangle]\mathrm{d}t\right\} \\ &= \int D[n(t)] \exp\{\mathrm{i}S[n(t)]\} \end{aligned} \tag{3.46}$$

Because of $-\langle \dot{n}(t)|n(t)\rangle = \langle n(t)|\dot{n}(t)\rangle$, $S[n(t)]$ can be written as:

$$S[n(t)] = \int_{t_i}^{t_f} \mathrm{d}t \left[\langle n(t)|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|n(t)\rangle - \langle n(t)|\hat{H}|n(t)\rangle\right] \tag{3.47}$$

where, $\hbar = 1$.

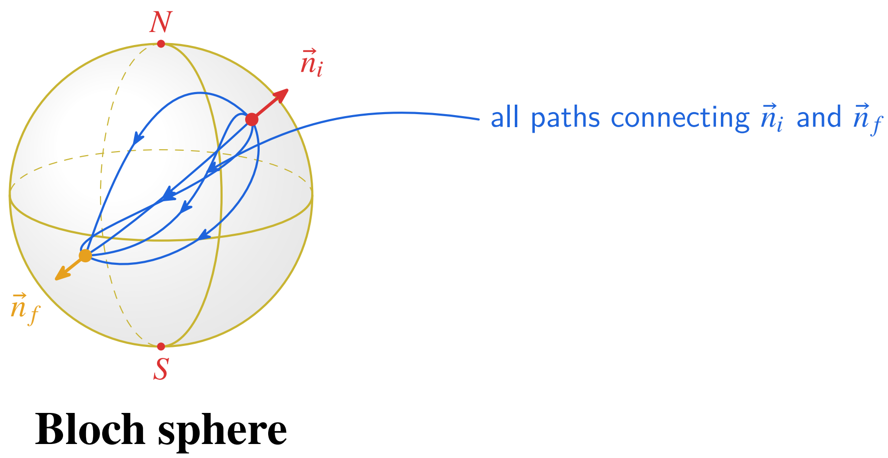


Figure 3: The paths connecting $\vec{n}_i$ and $\vec{n}_f$ in the Bloch sphere

3.2.1 Geometrical meaning of the Geometric term

We consider a special system whose Hamiltonian is zero; So the dynamical term vanishes:

$$\langle n(t)|\hat{H}|n(t)\rangle = 0 \tag{3.48}$$

and the action simplifies and contains only the geometric term:

$$S[n(t)] = \int_{t_i}^{t_f} \mathrm{d}t \cdot \mathrm{i}\langle n(t)|\frac{\mathrm{d}}{\mathrm{d}t}|n(t)\rangle \tag{3.49}$$

Again, we choose the gauge choice $\delta = \frac{\varphi}{2}$, such that the North Pole is regular:

$$|n(t)\rangle = \cos\frac{\theta(t)}{2}|\uparrow\rangle + \mathrm{e}^{\mathrm{i}\varphi(t)}\sin\frac{\theta(t)}{2}|\downarrow\rangle \tag{3.50}$$

So the geometric term is:

$$\begin{aligned}\langle n(t)|\frac{\mathrm{d}}{\mathrm{d}t}|n(t)\rangle &= \cos\frac{\theta}{2}(-\frac{1}{2}\sin\frac{\theta}{2}\cdot\dot{\theta}) + \mathrm{e}^{-\mathrm{i}\varphi}\sin\frac{\theta}{2}\cdot(\mathrm{i}\dot{\varphi}\mathrm{e}^{\mathrm{i}\varphi}\sin\frac{\theta}{2} + \mathrm{e}^{\mathrm{i}\varphi}\frac{1}{2}\cos\frac{\theta}{2}\cdot\dot{\theta}) \\ &= \mathrm{i}\dot{\varphi}\sin^2\frac{\theta}{2} \\ &= \mathrm{i}\dot{\varphi}\frac{(1-\cos\theta)}{2}\end{aligned} \tag{3.51}$$

So the action is:

$$\begin{aligned}S[n(t)] &= -\frac{1}{2}\int_{t_i}^{t_f}\mathrm{d}t\cdot\dot{\varphi}(1-\cos\theta) \\ &= -\frac{1}{2}\int_{\varphi(t_i)}^{\varphi(t_f)}(1-\cos\theta(\varphi))\mathrm{d}\varphi\end{aligned} \tag{3.52}$$

Because θ and φ are the functions depending on t. we can express θ as a function of φ

We can rewrite the function $1-\cos\theta(\varphi)$ as a integral:

$$1-\cos\theta(\varphi)=\int_0^{\theta(\varphi)}\mathrm{d}\theta\cdot\sin\theta \tag{3.53}$$

So the action can be written as a area integral:

$$\boxed{\begin{aligned}S[n(t)]&=-\frac{1}{2}\int_{\varphi_i}^{\varphi_f}\mathrm{d}\varphi\int_0^{\theta(\varphi)}\sin\theta\mathrm{d}\theta\\&=-\frac{1}{2}\iint_{\partial A_\gamma}\mathrm{d}\Omega\\&=-\frac{1}{2}A_\gamma\quad\text{(Because the radius is 1 and } A_\gamma=R^2\Omega)\end{aligned}} \tag{3.54}$$

where γ is the path from $|n(t_i)\rangle=|n_i\rangle$ to $|n(t_f)\rangle=|n_f\rangle$ and A_γ is area enclosed by three points on the sphere ($|\uparrow\rangle,|n_i\rangle,|n_f\rangle$). Obviously, A_γ depends on where the North pole is defined, which is kind of "gauge-dependence". If we multiply $|n(t)\rangle$ by a overall phase:

$$|n(t)\rangle\longrightarrow\mathrm{e}^{\mathrm{i}\delta}|n(t)\rangle \tag{3.55}$$

$S[n(t)]$ will be added an other term:

$$\begin{aligned}S'[n(t)]&=\mathrm{i}\int_{t_i}^{t_f}\mathrm{d}t\langle n(t)|(\mathrm{e}^{\mathrm{i}\delta}\frac{\mathrm{d}}{\mathrm{d}t}|n(t)\rangle+\mathrm{i}\dot{\delta}\mathrm{e}^{\mathrm{i}\delta}|n(t)\rangle)\\&=S[n(t)]+(-1)\int_{t_i}^{t_f}\mathrm{d}t\dot{\delta}\\&=S[n(t)]+(-1)(\delta(t_f)-\delta(t_i))\end{aligned} \tag{3.56}$$

To be more precise, δ is a function of θ and φ:

$$\delta(t_f)-\delta(t_i)=\delta(\theta_f,\varphi_f)-\delta(\theta_i,\varphi_i) \tag{3.57}$$

So if we choose different gauge, $\Delta\delta$ is different and $S[n(t)]$ is gauge-dependence.

But if weconsider a close path ($n_f=n_i$), the $S[n(t)]$ is gauge-independent:

$$\begin{aligned}\Delta\delta&=\delta(\theta_f,\varphi_f)-\delta(\theta_i,\varphi_i)\\&=2\pi\cdot N\quad(N\in\mathbb{Z})\end{aligned} \tag{3.58}$$

The reason as follow: To ensure that the transformed basis is single-valued, which

$$\mathrm{e}^{\mathrm{i}\delta_i}|n_i\rangle\xrightarrow{\text{close path}}\mathrm{e}^{\mathrm{i}\delta_f}|n_f\rangle=\mathrm{e}^{\mathrm{i}\delta_i}|n_i\rangle \tag{3.59}$$

and

$$\mathrm{e}^{\mathrm{i}\delta_f}|n_i\rangle=\mathrm{e}^{\mathrm{i}\delta_i}|n_i\rangle \tag{3.60}$$

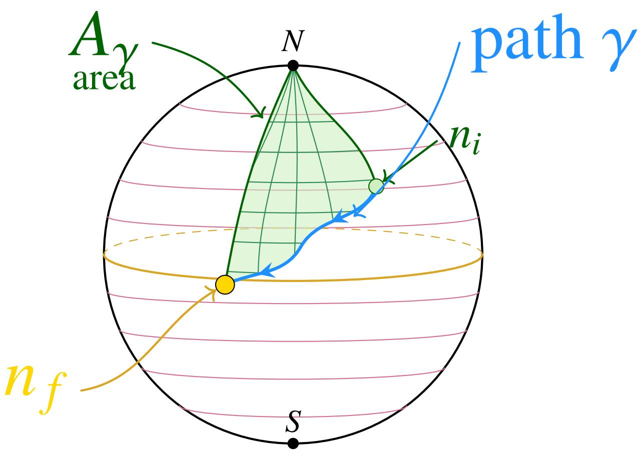


Figure 4: The integral path γ in the Bloch sphere

This implies that the value of δ can change by integer multiples of 2π after completing a full cycle:

$$\delta_f - \delta_i = 2\pi \cdot N \quad (N \in \mathbb{Z}). \tag{3.61}$$

So for a close path, $S[n(t)]$, to be more precise, the geometric phase factor:

$$\mathrm{e}^{\mathrm{i}S'} = \mathrm{e}^{\mathrm{i}S+\mathrm{i}\Delta\delta} = \mathrm{e}^{\mathrm{i}S+2\pi N\cdot \mathrm{i}} = \mathrm{e}^{\mathrm{i}S} \tag{3.62}$$

remains unchanged, or that the geometric phase S is invariant under the model modulo 2π operation.

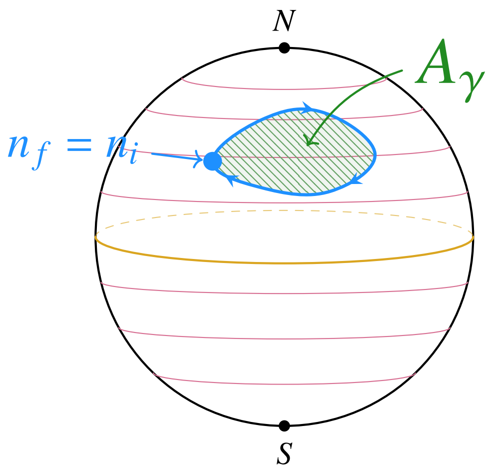


Figure 5: The integral close path γ in the Bloch sphere

3.2.2 Wess-Zumino term and Witten extension

The Wess-Zumino term is S_{WZ}, which satisfies:

$$S = -\frac{1}{2}\iint \mathrm{d}\Omega = \frac{1}{2}S_{\mathrm{WZ}} = S_{\mathrm{spin}} \cdot S_{\mathrm{WZ}}. \tag{3.63}$$

It means that:

$$S_{\mathrm{WZ}} = -\iint_{A_\gamma} \mathrm{d}\Omega = -\iint_{A_\gamma} \boldsymbol{n} \cdot \mathrm{d}\boldsymbol{S} \tag{3.64}$$

where $\boldsymbol{n}$ can regrad as the monopole's field strength whose charge is 1.

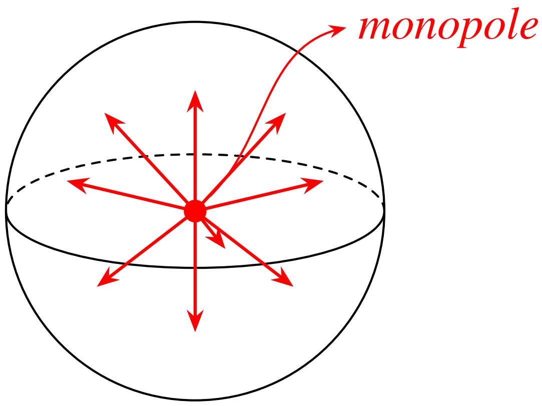


Figure 6: The monopole

Because $\boldsymbol{n}$ locates on the surface of the sphere, it can be the function of two parameter t and k, which form an orthonormal basis on the sphere. So $\mathrm{d}\boldsymbol{S}$ can be written as:

$$\begin{aligned}
\mathrm{d}\boldsymbol{S} &= \mathrm{d}\boldsymbol{n}(t+\mathrm{d}t, \kappa) \times \mathrm{d}\boldsymbol{n}(t, \kappa + \mathrm{d}\kappa) \\
&= (\frac{\partial \boldsymbol{n}}{\partial t} \cdot \mathrm{d}t) \times (\frac{\partial \boldsymbol{n}}{\partial \kappa} \cdot \mathrm{d}\kappa) \\
&= (\frac{\partial \boldsymbol{n}}{\partial t} \times \frac{\partial \boldsymbol{n}}{\partial \kappa}) \mathrm{d}t \cdot \mathrm{d}\kappa.
\end{aligned} \tag{3.65}$$

So the Wess-Zumion term have another expression that is broadly used:

Theorem 3.2.1 (Wess-Zumino term)**.**

$$S_{\mathrm{WZ}} = -\int_0^1 \mathrm{d}\kappa \int_{t_i}^{t_f} \mathrm{d}t \cdot \boldsymbol{n}(t, \kappa) \cdot \left[\frac{\partial \boldsymbol{n}}{\partial t} \times \frac{\partial \boldsymbol{n}}{\partial \kappa}\right] \tag{3.66}$$

where κ is an auxiliary parameter, in order to extend $\boldsymbol{n}(t)$ to $\boldsymbol{n}(t, \kappa)$ with:

$$\begin{cases}
\boldsymbol{n}(t, 0) = \boldsymbol{n}(t) \\
\boldsymbol{n}(t, 1) = \boldsymbol{n}_0 (\textit{North Pole}) \\
\boldsymbol{n}(t_i, \kappa) = \boldsymbol{n}(t_f, \kappa) \ (\textit{Closed Path})
\end{cases} \tag{3.67}$$

3.2.3 Emergence of U(1) gauge degrees of freedom

$$S = \mathrm{i} \int_\gamma \mathrm{d}t \langle \boldsymbol{n}(t) | \partial_t | \boldsymbol{n}(t) \rangle = \mathrm{i} \int_\gamma \mathrm{d}t \, \boldsymbol{z}^\dagger \cdot \partial_t \boldsymbol{z} \tag{3.68}$$

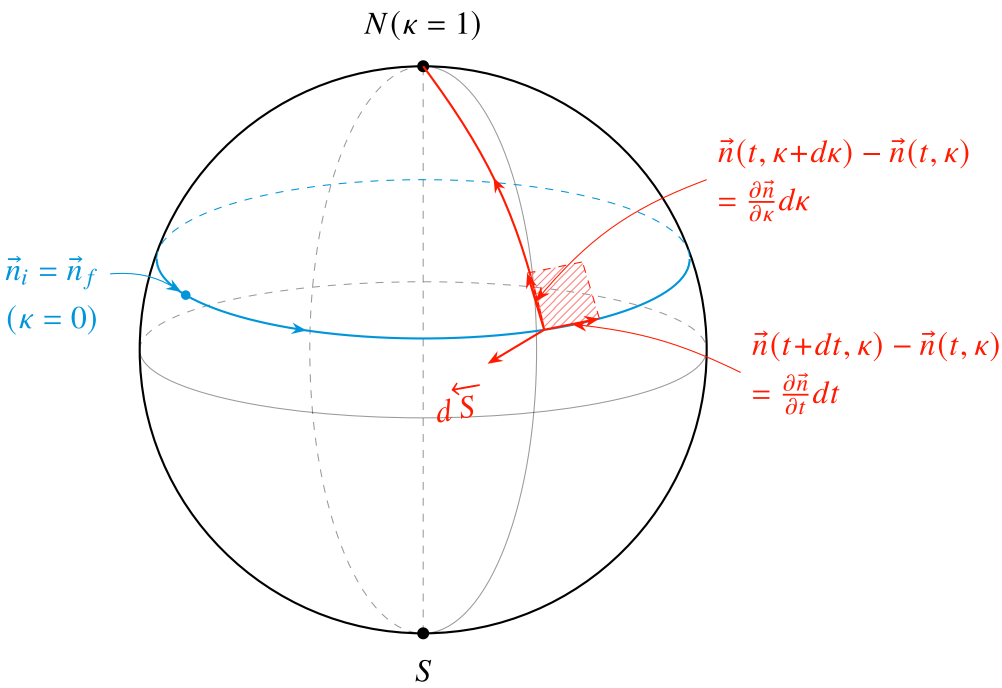


Figure 7: The differential of the sphere with κ and t.

where, $\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = |\boldsymbol{n}\rangle$. Using the chain rule where $\boldsymbol{x} = (x_1, x_2)$ are coordinates on Bloch Sphere:

$$S = \mathrm{i} \int_\gamma \boldsymbol{z}^\dagger \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \mathrm{d}t \tag{3.69}$$

Define the Gauge Potential $\mathcal{A}$ as:

$$\boxed{\mathcal{A} = \mathrm{i}\boldsymbol{z}^\dagger \cdot \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}}} \tag{3.70}$$

Therefore, S can be written as:

$$S = \int_\gamma \mathcal{A} \cdot \mathrm{d}\boldsymbol{l} \tag{3.71}$$

where $\mathrm{d}\boldsymbol{l} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \cdot \mathrm{d}t = \boldsymbol{v} \cdot \mathrm{d}t$ is the differential of path γ.

We apply a local gauge transformation where the state z picks up a position-dependent phase $\phi(\boldsymbol{x})$:

$$z \longrightarrow z \cdot \mathrm{e}^{\mathrm{i}\phi} \tag{3.72}$$

So the Gauge Potential change as:

$$\boxed{\begin{aligned} \mathcal{A}' &= \mathrm{i}\boldsymbol{z}^\dagger \mathrm{e}^{-\mathrm{i}\phi}(\mathrm{e}^{\mathrm{i}\phi}\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}} + \boldsymbol{z}\mathrm{e}^{\mathrm{i}\phi}\nabla\phi) \\ &= \mathcal{A} + \nabla\phi \end{aligned}} \tag{3.73}$$

So the potential $\mathcal{A}$ is not gauge-invariant.

From what we have discussed above, we can know that: A single spin $\xLeftrightarrow{\text{equivalent to}}$ A spherical Landau problem with a monopole.

Because of : (for closed path)

$$S = \oint_{\gamma} \boldsymbol{\mathcal{A}} \cdot \mathrm{d}\boldsymbol{l} \tag{3.74}$$

where $\boldsymbol{\mathcal{A}}$ can be regard as a magnetic potential that is responsible for a "magnetic monopole" at the center of Bloch Ball. Therefore, S_{WZ} can be regarded as a quantum mechanical problem: a charge particle with sufficiently small mass is confined on the unit sphere, and feels electromagnetic field generated by the "magnetic monopole" at the center of the ball.

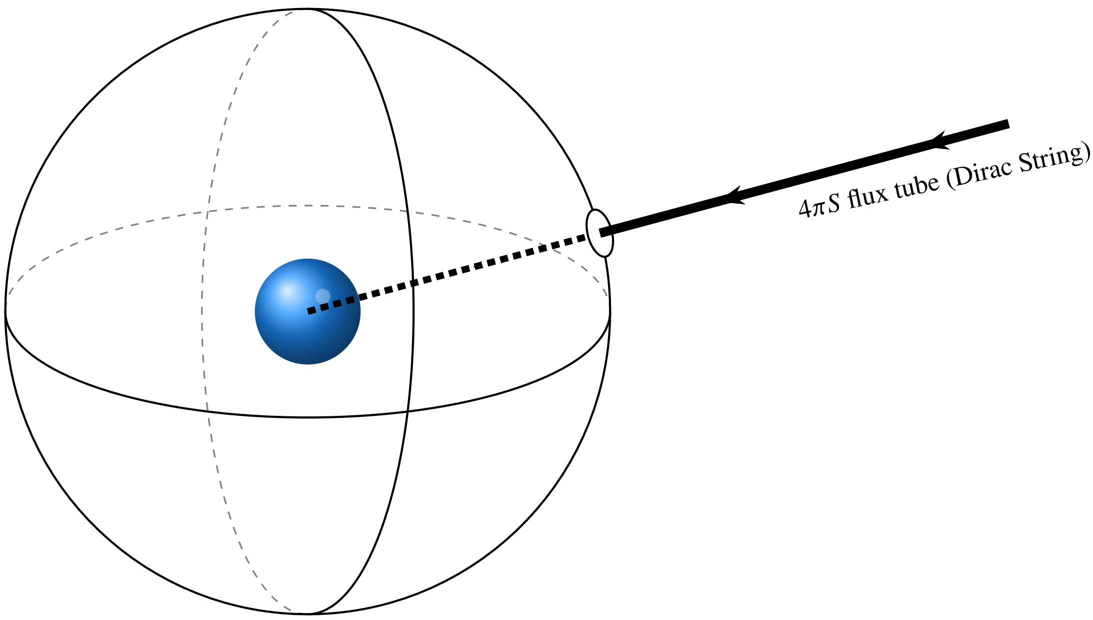


Figure 8: The Dirac String

3.2.4 Conclusion.

In the previous discussion, we have obtained path integral formalism of a single spin-$\frac{1}{2}$ ($S = \frac{1}{2}$).

Theorem 3.2.2 (Action for a generic spin-S)**.** *For a generic spin-S ($S = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$), the action is given by:*

$$\begin{cases} S[\boldsymbol{n}(t)] = S_{\text{spin}} \cdot S_{\mathrm{WZ}}[\boldsymbol{n}(t)] \quad (\hat{H} = 0) \\ \langle \boldsymbol{n} | \boldsymbol{S} | \boldsymbol{n} \rangle = S_{\text{spin}} \cdot \boldsymbol{n} \end{cases} \tag{3.75}$$

Chapter 4

The path integral for many-spin system

4.1 General theory

Consider a spin system in a d-dimensional lattice which is called Heisenberg Model:

$$\hat{H} = J \cdot \sum_{\langle i,j \rangle} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j \tag{4.1}$$

where J is the exchange coupling and $\hat{\boldsymbol{S}}$ is quantum spin operators. The $\langle i, j \rangle$ denotes a summation over nearest-neighbor pairs.

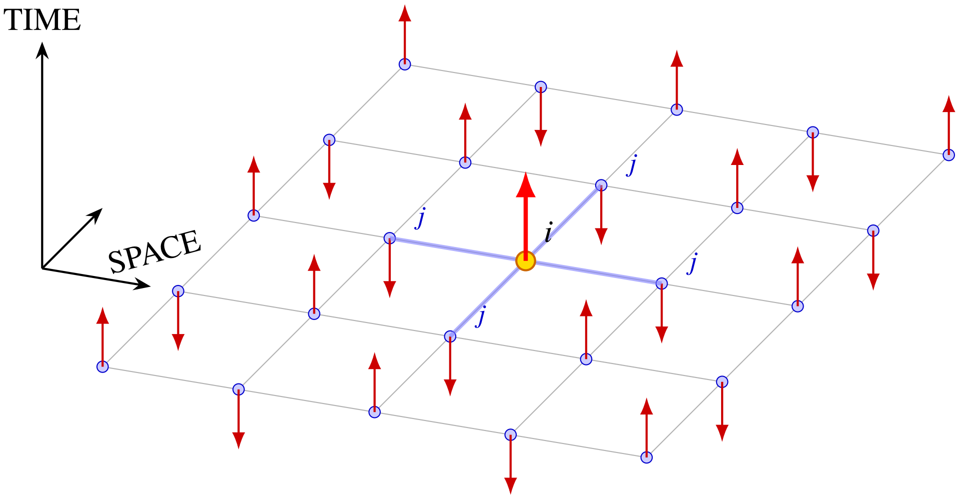


Figure 9: The Heisenberg Model on a 2D square lattice

The partition function of the spin system is:

$$\begin{aligned} Z &= \mathrm{Tr} e^{-\beta \hat{H}} \\ &= \int \mathrm{d}\boldsymbol{n}_0 \cdot \langle \boldsymbol{n}_0 | e^{-\beta \hat{H}} | \boldsymbol{n}_0 \rangle \end{aligned} \tag{4.2}$$

where $|\boldsymbol{n}_0\rangle = \bigotimes_{i=1} |\boldsymbol{n}_0^i\rangle$ is the tensor product of all the state on the lattice.

In the previous chapters, we have seen that we can change real time t to imaginary time τ:

$$\langle n_f | e^{-i\hat{H}(t_f - t_i)} | n_i \rangle \xrightarrow{i(t_f - t_i) \to \tau} \langle n_f | e^{-\hat{H}\tau} | n_i \rangle \tag{4.3}$$

In this version, propagator K can be written as (for closed path):

$$K = \int D[\boldsymbol{n}(t)] e^{-S_E} \tag{4.4}$$

where, $S_E = -\mathrm{i}SS_{WZ} = \mathrm{i}\cdot S\cdot A_\gamma$ for $\hat{H}=0$ and $S_E = -\mathrm{i}S\cdot S_{WZ} + \int_{\tau_i}^{\tau_f}\langle \boldsymbol{n}(\tau)|\hat{H}|\boldsymbol{n}(\tau)\rangle \mathrm{d}\tau$ for $\hat{H}\neq 0$.

Because the diagonal element of partition function and propagator have the same mathematical form, so we can use the same method to solve the diagonal elements.

let's make $\Delta\tau$ equal to $\frac{\beta}{N}$ (N is the slice number) whose mean is that β play a same role in the diagonal liking τ in the propagator ($\tau_i \to \tau_f \Longrightarrow 0 \to \beta$).

But in many-spin system, the identity operator for the entire system is the tensor product of the identities for each site:

$$\hat{\mathbb{I}}_{total} = \bigotimes_{i=1} \hat{\mathbb{I}}_i = \int (\prod_{i=1} \mathrm{d}^2\boldsymbol{n}_i)|\vec{n}_1\ldots\vec{n}_N\rangle\langle\vec{n}_1\ldots\vec{n}_N| \tag{4.5}$$

So the diagonal can be written as a mathematical form like propagator:

$$\langle\boldsymbol{n}_0|\mathrm{e}^{-\beta\hat{H}}|\boldsymbol{n}_0\rangle = \int D(\boldsymbol{n})\cdot \mathrm{e}^{-S_E} \tag{4.6}$$

where $D(\boldsymbol{n}) = \prod_i\prod_l \mathrm{d}^2\boldsymbol{n}_i^l$.

Because diagonal likes the propagator with close path ($\langle\boldsymbol{n}_0|\mathrm{e}^{-\beta\hat{H}}|\boldsymbol{n}_0\rangle$ and not $\langle\boldsymbol{n}_i|\mathrm{e}^{-\beta\hat{H}}|\boldsymbol{n}_j\rangle$ for $i\neq j$), S_E can be written as:

$$S_E = \int_0^\beta \mathrm{d}\tau\langle\boldsymbol{n}|\frac{\partial}{\partial\tau}|\boldsymbol{n}\rangle + \int_0^\beta \mathrm{d}\tau\langle\boldsymbol{n}|\hat{H}|\boldsymbol{n}\rangle \tag{4.7}$$

For the energy term:

$$\begin{aligned} S_e &= \int_0^\beta \mathrm{d}\tau\langle\boldsymbol{n}|\hat{H}|\boldsymbol{n}\rangle \\ &= \int_0^\beta \mathrm{d}\tau J\sum_{\langle i,j\rangle}(\otimes\langle\boldsymbol{n}|)\hat{\boldsymbol{S}}_i\cdot\hat{\boldsymbol{S}}_j(\otimes|\boldsymbol{n}\rangle) \end{aligned} \tag{4.8}$$

Use the formula:

$$\langle\boldsymbol{n}|\boldsymbol{S}|\boldsymbol{n}\rangle = S_{spin}\boldsymbol{n} \tag{4.9}$$

we can get:

$$S_e = \int_0^\beta J\sum_{\langle i,j\rangle} S_{spin}^2\boldsymbol{n}_i(\tau)\boldsymbol{n}_j(\tau) \tag{4.10}$$

For the geometric term:

$$\begin{aligned} S_g &= \int_0^\beta \mathrm{d}\tau(\otimes\langle\boldsymbol{n}|)\frac{\partial}{\partial\tau}(\otimes|\boldsymbol{n}\rangle) \\ &= \int_0^\beta (\otimes\langle\boldsymbol{n}|)[\otimes(|\boldsymbol{n}\rangle + \mathrm{d}|\boldsymbol{n}\rangle) - \otimes|\boldsymbol{n}\rangle] \end{aligned} \tag{4.11}$$

Let's retain the first order term:

$$
\begin{aligned}
S_g &= \int_0^\beta \ldots (\otimes\langle \boldsymbol{n}|)[\otimes|\boldsymbol{n}\rangle + \sum_i (\bigotimes_{j\neq i} |\boldsymbol{n}_j\rangle)\mathrm{d}|\boldsymbol{n}_i\rangle) - \otimes|\boldsymbol{n}\rangle] \\
&= \sum_i \int_0^\beta (\otimes\langle \boldsymbol{n}|)(\bigotimes_{j\neq i} |\boldsymbol{n}_j\rangle|\mathrm{d}\boldsymbol{n}_i\rangle)
\end{aligned}
\tag{4.12}
$$

because of $\langle \boldsymbol{n}_i|\boldsymbol{n}_i\rangle = 1$, so:

$$
\begin{aligned}
S_g &= \sum_i \int_0^\beta \langle \boldsymbol{n}_i|\mathrm{d}\boldsymbol{n}_i\rangle \\
&= \sum_i -\mathrm{i}S \cdot S_{WZ}[\boldsymbol{n}_i] \\
&= -\mathrm{i}S \sum_i S_{WZ}[\boldsymbol{n}_i]
\end{aligned}
\tag{4.13}
$$

let's go back to real time ($\tau \longrightarrow \mathrm{i}t$):

$$
\begin{aligned}
-S_E &\longrightarrow \frac{\mathrm{i}}{\hbar} S_M \\
&\Rightarrow S_M = \hbar S \cdot \sum_i S_{WZ}[\boldsymbol{n}_i] - S^2 \int_0^\beta \mathrm{d}t \cdot \sum_{\langle \boldsymbol{n}_i, \boldsymbol{n}_j\rangle} J\boldsymbol{n}_i \cdot \boldsymbol{n}_j \\
&\quad \frac{\mathrm{i}S_M}{\hbar} = S\{\sum_i S_{WZ}[\boldsymbol{n}_i]\} - \frac{S^2}{\hbar}\{\int_0^\beta \mathrm{d}t \sum_{\langle \boldsymbol{n}_i, \boldsymbol{n}_j\rangle} J \cdot \boldsymbol{n}_i \cdot \boldsymbol{n}_j\}
\end{aligned}
\tag{4.14}
$$

So $\{\sum_i S_{WZ}[\boldsymbol{n}_i]\}$ and $\{\int_0^\beta \mathrm{d}t \sum_{\langle i,j\rangle} J \cdot \boldsymbol{n}_i \cdot \boldsymbol{n}_j\}$ can be regard as the phase term of action ($\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\theta}$). Therefore, S and $\frac{S^2}{\hbar}$ play the role of $\hbar$ in simple action. Here, if S is very large, we called it Large-S limit ($\hbar \ll 1$, so $\frac{1}{\hbar} \approx S \gg 1$). This is a semi-classical limit.

4.2 Quantum Ferromagnetism

let's consider a ferromagnetism system whose J equal to $-|J|$. So the action is:

$$
S_M = S \sum_i S_{WZ}[\boldsymbol{n}_i] + |J|S^2 \sum_{\langle i,j\rangle} \int_0^\beta \mathrm{d}t \boldsymbol{n}_i \cdot \boldsymbol{n}_j \tag{4.15}
$$

Because $|\boldsymbol{n}|^2 = 1$, so we can rewrite the term $\boldsymbol{n}_i \cdot \boldsymbol{n}_j$ as:

$$
\begin{aligned}
-\boldsymbol{n}_i \cdot \boldsymbol{n}_j &= \frac{1}{2}(|\boldsymbol{n}_i|^2 - 2\boldsymbol{n}_i \cdot \boldsymbol{n}_j + |\boldsymbol{n}_j|^2) - 1 \\
&= \frac{1}{2}(\boldsymbol{n}_i - \boldsymbol{n}_j)^2 - 1
\end{aligned}
\tag{4.16}
$$

So the energy term is:

$$
S_e = -\frac{|J|S^2}{2} \sum_{\langle i,j\rangle} \int_0^\beta \mathrm{d}t(\boldsymbol{n}_i - \boldsymbol{n}_j)^2 + |J|S^2 \sum_{\langle i,j\rangle} \int_0^\beta \mathrm{d}t \tag{4.17}
$$

The second term of S_e is the constant term (global phase), so we can drop it and S_M can be written as:

$$S_M = S\sum_i S_{WZ}[\boldsymbol{n}_i] - \frac{|J|S^2}{2}\sum_{\langle i,j\rangle}\int_0^\beta \mathrm{d}t(\boldsymbol{n}_i - \boldsymbol{n}_j)^2 \tag{4.18}$$

Next, we take the continuum limit where the lattice sums become integrals

$$\sum_i \to \int \frac{\mathrm{d}^d x}{a_0^d} \tag{4.19}$$

where a_0 is the lattice spacing and finite differences become gradients:

$$\begin{aligned}\sum_j(\boldsymbol{n}_i - \boldsymbol{n}_j)^2 &= \dots \\ &= a_0^2(\nabla\boldsymbol{n})^2\end{aligned} \tag{4.20}$$

where $\nabla\boldsymbol{n} = (\frac{\partial\boldsymbol{n}}{\partial x}, \dots)$ and not $\nabla\boldsymbol{n} = (\frac{\partial n_x}{\partial x},\quad)$

So the continuum action is:

$$\begin{aligned}S_M &= \int \frac{\mathrm{d}^d x}{a_0^d} S\cdot S_{WZ}[\boldsymbol{n}] - \frac{|J|S^2}{2}\int\frac{\mathrm{d}^d x}{a_0^d}\mathrm{d}t\, a_0^2(\nabla\boldsymbol{n})^2 \\ &= \frac{S}{a_0^d}\int \mathrm{d}^d x S_{WZ}[\boldsymbol{n}] - \frac{|J|S^2}{2a_0^{d-2}}\int \mathrm{d}^d x\mathrm{d}t(\nabla\boldsymbol{n})^2\end{aligned} \tag{4.21}$$

We want to find the classical path that minimizes the action. However, simply varying S_m is incorrect because the variable $\boldsymbol{n}$ is not free; it is constrained to the surface of a sphere:

$$\boldsymbol{n}^2 - 1 = 0 \tag{4.22}$$

To handle this, we introduce a Lagrange multiplier field λ. So the total action becomes:

$$S_{\text{tot}} = S_m + \int \mathrm{d}^d x\mathrm{d}t\cdot\frac{\lambda}{2}(\boldsymbol{n}^2 - 1) \tag{4.23}$$

Now, we can safely solve the variation equation below:

$$\delta S_{\text{tot}} = 0 \tag{4.24}$$

We first need to variation of the Wess-Zumion term δS_{WZ}. Geometrically, the S_{WZ} is proportional to the area swept out by the spin vector on the unit sphere.

From the picture, we can see that the variation of this area element $\delta(\Delta A)$ with respect to a small change in path $\delta\boldsymbol{n}$ is given by:

$$\boldsymbol{n}\cdot\delta(\Delta A) = \delta\boldsymbol{n}\times(\frac{\partial\boldsymbol{n}}{\partial t})\mathrm{d}t \tag{4.25}$$

we product $\boldsymbol{n}$ on the both sides of the equation :

$$\begin{aligned}\delta(\Delta A) &= \boldsymbol{n}(\delta\boldsymbol{n}\times\frac{\partial\boldsymbol{n}}{\partial t})\mathrm{d}t \\ &= -\delta\boldsymbol{n}(\boldsymbol{n}\times\frac{\partial\boldsymbol{n}}{\partial t})\mathrm{d}t.\end{aligned} \tag{4.26}$$

Therefore, the variation of the Wess-Zumion part of the action is:

$$\delta(\int \mathrm{d}^d x S_{\mathrm{WZ}}(\boldsymbol{n})) = \frac{S}{a_0^d}\int \mathrm{d}^d x[\delta\boldsymbol{n}\cdot(\boldsymbol{n}\times\frac{\partial\boldsymbol{n}}{\partial t})] \tag{4.27}$$

This implies the functional derivative is:

$$\frac{\delta S_g}{\delta\boldsymbol{n}} = \frac{S}{a_0^d}\cdot(\boldsymbol{n}\times\frac{\partial\boldsymbol{n}}{\partial t}) \tag{4.28}$$

Next, we calculate the variation of energy term:

$$\begin{aligned}\delta[\int \mathrm{d}^d x\mathrm{d}t(\nabla\boldsymbol{n})^2] &= \int \mathrm{d}^d x\mathrm{d}t\frac{1}{2}\delta(\frac{\partial\boldsymbol{n}}{\partial x_i})^2 \\ &= \int \mathrm{d}^d x\mathrm{d}t\cdot\sum_i\frac{1}{2}\cdot 2(\frac{\partial\boldsymbol{n}}{\partial x_i})\cdot\delta(\frac{\partial\boldsymbol{n}}{\partial x_i})\end{aligned} \tag{4.29}$$

and we use the partial integration method:

$$\begin{aligned}\int d^d x\sum_i^d 2\left(\frac{\partial\vec{n}}{\partial x_i}\right)\cdot\delta\left(\frac{\partial\vec{n}}{\partial x_i}\right) &\overset{\frac{\partial\vec{n}}{\partial x_i}=\partial_i\vec{n}}{=\!=} 2\int d^d x\sum_i^d(\partial_i\vec{n})\cdot\partial_i(\delta\vec{n}) \\ &= 2\left\{\left[\sum_i^d(\partial_i\vec{n})\cdot\delta\vec{n}\right]\Bigg| - \int d^d x\sum_i^d\delta\vec{n}\cdot\partial_i^2\vec{n}\right\} \\ &= -2\int d^d x(\nabla^2\vec{n})\cdot\delta\vec{n}\end{aligned} \tag{4.30}$$

So the variation of energy term is

$$\frac{\delta S_e}{\delta\boldsymbol{n}} = \frac{|J|S^2}{a_0^{d-2}}(\nabla^2\boldsymbol{n}) \tag{4.31}$$

Finally, we calculate the variation of λ term:

$$\delta\int \mathrm{d}^d x\mathrm{d}t\frac{\lambda}{2}(\boldsymbol{n}^2-1) = \int \mathrm{d}^d x\mathrm{d}t\cdot\lambda\boldsymbol{n}\delta\boldsymbol{n} \tag{4.32}$$

So the variation of λ term is:

$$\frac{\delta S_\lambda}{\delta\boldsymbol{n}} = \lambda\boldsymbol{n} \tag{4.33}$$

Now we can get the total action's variation:

$$\delta S_{\mathrm{tot}} = 0 \implies \frac{S}{a_0^d}(\boldsymbol{n}\times\frac{\partial\boldsymbol{n}}{\partial t}) + \lambda\boldsymbol{n} = -\frac{|J|S^2}{a_0^{d-2}}\nabla^2\boldsymbol{n}. \tag{4.34}$$

To find the value of the Lagrange multiplier, take the dot product of the above equation with $\boldsymbol{n}$:

$$\underbrace{\frac{S}{a_0^d}\vec{n}\cdot\left(\vec{n}\times\frac{\partial\vec{n}}{\partial t}\right)}_{\substack{=\frac{S}{a_0^d}\frac{\partial\vec{n}}{\partial t}(\vec{n}\times\vec{n})\\=0}}+\lambda\underbrace{\vec{n}\cdot\vec{n}}_{=1}=-\frac{|J|S^2}{a_0^{d-2}}(\vec{n}\cdot\nabla^2\vec{n}) \tag{4.35}$$

Thus:

$$\lambda=-\frac{|J|S^2}{a_0^{d-2}}(\boldsymbol{n}\cdot\nabla^2\boldsymbol{n}) \tag{4.36}$$

Now, substitute the λ back into equation:

$$\frac{S}{a_0^d}\cdot(\boldsymbol{n}\times\frac{\partial\boldsymbol{n}}{\partial t})-[\frac{|J|S^2}{a_0^{d-2}}(\boldsymbol{n}\cdot\nabla^2\boldsymbol{n})]\boldsymbol{n}=-\frac{|J|S^2}{a_0^{d-2}}\nabla^2\boldsymbol{n} \tag{4.37}$$

Rearrange to group the derivative terms on the right:

$$\frac{S}{a_0^d}(\boldsymbol{n}\times\frac{\partial\boldsymbol{n}}{\partial t})=-\frac{|J|S^2}{a_0^{d-2}}[\nabla^2\boldsymbol{n}-(\boldsymbol{n}\cdot\nabla^2\boldsymbol{n})\cdot\boldsymbol{n}] \tag{4.38}$$

If we look at: $\boldsymbol{n}\times(\boldsymbol{n}\times\nabla^2\boldsymbol{n})$:

$$\begin{aligned}\boldsymbol{n}\times(\boldsymbol{n}\times\nabla^2\boldsymbol{n})&=\boldsymbol{n}\cdot(\boldsymbol{n}\cdot\nabla^2\boldsymbol{n})-\nabla^2\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{n})\\&=\boldsymbol{n}(\boldsymbol{n}\cdot\nabla^2\boldsymbol{n})-\nabla^2\boldsymbol{n}.\end{aligned} \tag{4.39}$$

So,

$$\frac{S}{a_0^d}(\boldsymbol{n}\times\partial_t\boldsymbol{n})=\frac{|J|S^2}{a_0^{d-2}}[\boldsymbol{n}\times(\boldsymbol{n}\times\nabla^2\boldsymbol{n})] \tag{4.40}$$

Simplified to:

$$\boldsymbol{n}\times\frac{\partial\boldsymbol{n}}{\partial t}=\boldsymbol{n}\times[|J|Sa_0^2(\nabla^2\boldsymbol{n})] \tag{4.41}$$

We can get the Landau-Lifshitz equation:

$$\frac{\partial\boldsymbol{n}}{\partial t}=|J|Sa_0^2(\boldsymbol{n}\times\nabla^2\boldsymbol{n}). \tag{4.42}$$

From this equation, we can know that the spins move in a precessional fashion with an angular velocity $\boldsymbol{\Omega}$ given by:

$$\boldsymbol{\Omega}=-|J|Sa_0^2\nabla^2\boldsymbol{n}. \tag{4.43}$$

The Landau-Lifshitz equations can be solved in the linear regime. Let's parametrize $\boldsymbol{n}$ by the components:

$$\boldsymbol{n}=\begin{pmatrix}\sigma\\\boldsymbol{\pi}\end{pmatrix}\text{ and }\boldsymbol{\pi}=\begin{pmatrix}\pi_1\\\pi_2\end{pmatrix} \tag{4.44}$$

where σ and $\pi_i (i = 1, 2)$ satisfy the constraint:

$$\sigma^2 + \boldsymbol{\pi}^2 = 1 \tag{4.45}$$

If the system is ordered, $\sigma \sim 1$, $|\boldsymbol{\pi}|$ is very small, which means that $\boldsymbol{\pi}$ is small fluctuation around the direction of σ. In this situation, we can treat σ as a constant:

$$\begin{cases} \sigma \approx 1 \\ \nabla^2 \sigma \approx 0 \end{cases} \tag{4.46}$$

Therefore, we can linearize the Landau-Lifshitz equation:

$$\boldsymbol{n} \times \nabla^2 \boldsymbol{n} = \begin{pmatrix} \pi_1 \nabla^2 \pi_2 - \pi_2 \nabla^2 \pi_1 \\ -\sigma \nabla^2 \pi_2 \\ \sigma \nabla^2 \pi_1 \end{pmatrix} \tag{4.47}$$

and we remain the one order term :

$$\boldsymbol{n} \times \nabla^2 \boldsymbol{n} = \begin{pmatrix} 0 \\ -\nabla^2 \pi_2 \\ \nabla^2 \pi_1 \end{pmatrix} \tag{4.48}$$

So the linearizing Landau-Lifshitz equation:

$$\begin{cases} \partial_t \pi_1 = -|J| S a_0^2 \nabla^2 \pi_2 \\ \partial_t \pi_2 = +|J| S a_0^2 \nabla^2 \pi_1 \end{cases} \tag{4.49}$$

We look for plane wave solutions (spin wave) of the term:

$$\pi_j(\boldsymbol{x}, t) = \tilde{\pi}_j e^{\mathrm{i}(\boldsymbol{p} \cdot \boldsymbol{x} - \omega t)} \tag{4.50}$$

taking derivatives this ansatz converts differential operators into algebraic variables:

$$\begin{cases} \partial_t \to -\mathrm{i}\omega \\ \nabla^2 \to -|\boldsymbol{p}|^2 \end{cases} \tag{4.51}$$

Substituting these into the linearized equations:

$$\begin{cases} -\mathrm{i}\omega \tilde{\pi}_1 = (|J| S a_0^2 |\boldsymbol{p}|^2) \tilde{\pi}_2 \\ -\mathrm{i}\omega \tilde{\pi}_2 = -(|J| S a_0^2 |\boldsymbol{p}|^2) \tilde{\pi}_1 \end{cases} \tag{4.52}$$

let $A = |J| S a_0^2 p^2$, the system become:

$$\begin{cases} -\mathrm{i}\omega \tilde{\pi}_1 - A \tilde{\pi}_2 = 0 \\ A \tilde{\pi}_1 - \mathrm{i}\omega \tilde{\pi}_2 = 0 \end{cases} \tag{4.53}$$

for a non-trivial solution ($\tilde{\pi}_1, \tilde{\pi}_2 \neq 0$), the determinant of the coefficient matrix must be zero:

$$\begin{vmatrix} -\mathrm{i}\omega & -A \\ A & -\mathrm{i}\omega \end{vmatrix} = 0 \tag{4.54}$$

we can get:

$$\omega = \pm A. \tag{4.55}$$

So we obtain the dispersion relation for ferromagnetic spin waves:

$$|\omega| \approx |J| S a_0^2 |\boldsymbol{p}|^2 \tag{4.56}$$

We find that the frequency of the low-energy excitations of a quantum ferromagnetic scales as the square of the momentum.

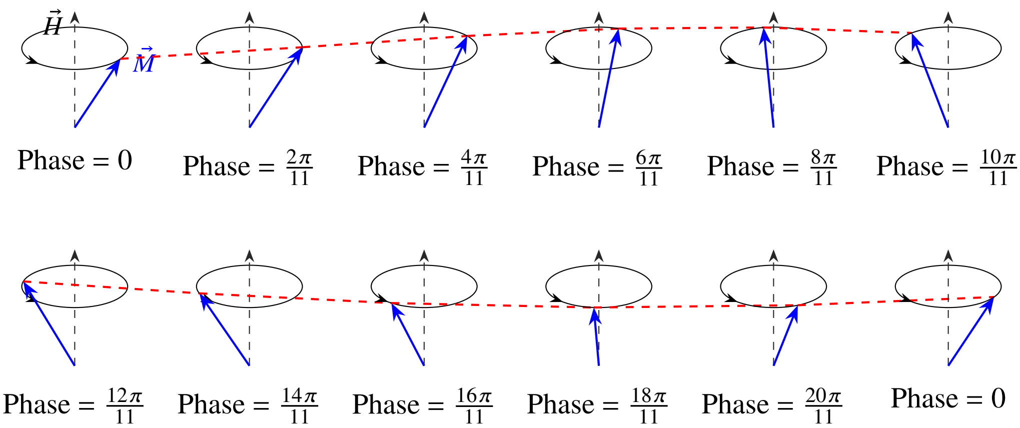


Figure 10: Schematic diagram of spin wave phase variation

4.3 Quantum Antiferromagnetic of one-dimension

Consider a spin chain with an even number of sites N; we can write down the action of it ($J = |J|$):

$$S_m[\boldsymbol{n}] = S \sum_i^N S_{WZ}[\boldsymbol{n}] - \int \mathrm{d}t \sum_i^N J S^2 \boldsymbol{n}(i,t) \cdot \boldsymbol{n}(i+1,t) \tag{4.57}$$

where we have assumed periodic boundary conditions.

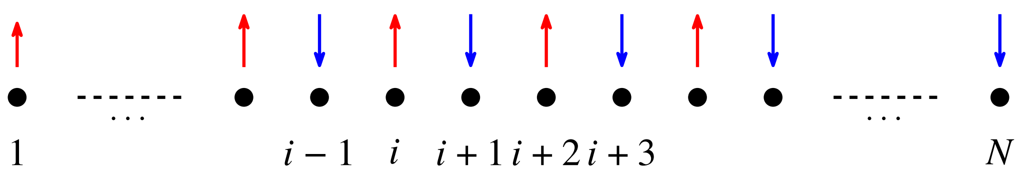


Figure 11: Illustration of a 1D spin chain with an even number of sites, showing alternating spins around site i.

Since we expect to be close to a Néel state, we will stagger the configuration:

$$\boldsymbol{n}(j) \to (-1)^j \boldsymbol{n}(j) \tag{4.58}$$

On a bipartite lattice, the substitution of {ref} into {ref} will change the sign of the exchange term of the action to a ferromagnetic:

$$S_m[\boldsymbol{n}] = S\sum_{i}^{N}(-1)^i S_{WZ}[\boldsymbol{n}(i)] - \frac{|J|S^2}{2}\int \mathrm{d}t \sum_{i=1}^{N}(-1)^i \boldsymbol{n}(i,t)\cdot(-1)^{i+1}\boldsymbol{n}(i+1,t) \tag{4.59}$$

so the energy term become to:

$$S_e = \frac{|J|S^2}{2}\int \mathrm{d}t \sum_{i=1}^{N}\boldsymbol{n}(i,t)\boldsymbol{n}(i+1,t) \tag{4.60}$$

which is similar to ferromagnetic. But the Wess-Zumino terms are odd under the replacement of {ref} and thus become staggered. Thus, it is the Wess-Zumino term, a purely quantum-mechanical effect, which will distinguish ferromagnets from antiferromagnets.

As ferromagnetic, we can split $\boldsymbol{n}$ into a slowly varying piece $\boldsymbol{m}$ plus a small rapidly varying part $a_0\boldsymbol{l}$, so:

$$\boldsymbol{n}(i) = \boldsymbol{m}(i) + (-1)^i a_0 \boldsymbol{l}(i) \tag{4.61}$$

The constraint $|\boldsymbol{n}|^2 = 1$ and requirement that $\boldsymbol{m}$ should obey the same constraint, and demand that $\boldsymbol{m}$ and $\boldsymbol{l}$ be orthogonal vector:

$$\begin{cases} |\boldsymbol{m}|^2 = 1 \\ \boldsymbol{m}\cdot\boldsymbol{l} = 0 \end{cases} \tag{4.62}$$

The Wess-Zumino terms are rewritten as:

$$S\sum_{i}^{N}(-1)^i S_{WZ}[\boldsymbol{n}(i)] = S\sum_{r=1}^{N/2}\{S_{WZ}[\boldsymbol{n}(2r)] - S_{WZ}[\boldsymbol{n}(2r-1)]\} \tag{4.63}$$

Figure 12: Schematic of 1D Antiferromagnetic Spin Chain

By making use of the approximation:

$$\begin{aligned} \boldsymbol{n}(2r) - \boldsymbol{n}(2r-1) &= \boldsymbol{m}(2r) - \boldsymbol{m}(2r-1) + a_0[\boldsymbol{l}(2r) + \boldsymbol{l}(2r-1)] \\ &\approx a_0\left[\frac{\partial \boldsymbol{m}(2r)}{\partial x} + 2\boldsymbol{l}(2r)\right] + O(a_0^2) \end{aligned} \tag{4.64}$$

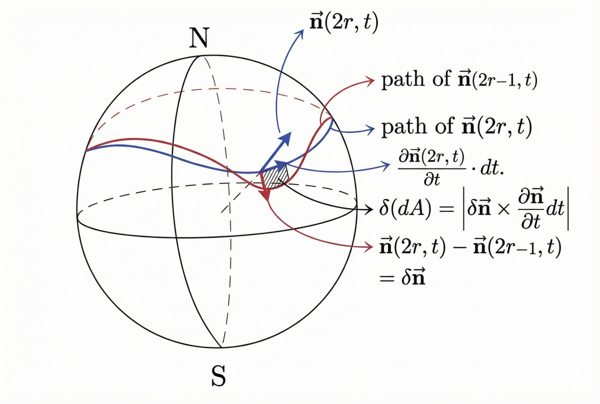


Figure 13: Antiferromagnetic Variation Diagram

So becomes (use the similar method of area variation, see figure 13)

$$\begin{aligned} S\sum_{i}^{N}(-1)^i S_{WZ}[\boldsymbol{n}(i)] &\approx S\sum_{i=1}^{N/2}(\boldsymbol{n}(2r)-\boldsymbol{n}(2r-1))[\boldsymbol{n}(2r)\times\frac{\partial \boldsymbol{n}(2r)}{\partial t}] \\ &\approx S\sum_{i=1}^{N/2} a_0\left[\frac{\partial \boldsymbol{m}(2r)}{\partial x}+2\boldsymbol{l}(2r)\right]\left[\boldsymbol{m}(2r)\times\frac{\partial \boldsymbol{m}(2r)}{\partial t}\right] \end{aligned} \tag{4.65}$$

Using continuum limit ($a_0 \to 0$):

$$\begin{aligned} \lim_{a_0\to 0} S_g &\approx \frac{S}{2}\sum_{i=1}^{N}(\dots) \quad \text{(pre periodic boundary conditions)} \\ &= \frac{S}{2}\int \mathrm{d}x\mathrm{d}t(\partial_x \boldsymbol{m}+2\boldsymbol{l})\cdot(\boldsymbol{m}\times\partial_t\boldsymbol{m}) \end{aligned} \tag{4.66}$$

Thus, it can be simplified to:

$$S_g \approx \frac{S}{2}\int \mathrm{d}x\mathrm{d}t\cdot\boldsymbol{m}(\partial_t\boldsymbol{m}\times\partial_x\boldsymbol{m}) + S\int \mathrm{d}x\mathrm{d}t\boldsymbol{l}(\boldsymbol{m}\times\partial_t\boldsymbol{m}) \tag{4.67}$$

This term comes from alternating sum of WZ terms: Similarly, the continuum limit of the energy terms can also be found to be given by:

$$
\begin{aligned}
&\lim_{a_0\to 0}\frac{|J|S^2}{2}\int \mathrm{d}t\sum_{i=1}^{N}\boldsymbol{n}(i,t)\boldsymbol{n}(i+1,t)\\
\xrightarrow{\text{drop global phase}}&\lim_{a_0\to 0}\frac{-|J|S^2}{2}\int \mathrm{d}t\sum_{i=1}^{N}[\boldsymbol{n}(i+1,t)-\boldsymbol{n}(i,t)]^2\\
&\approx\frac{-|J|S^2}{2}\int \mathrm{d}t\sum_{i}[(\boldsymbol{m}(i+1)-\boldsymbol{m}(i))+a_0\boldsymbol{l}(i+1)+a_0\boldsymbol{l}(i)]^2\\
&\approx\frac{-|J|S^2}{2}\int \mathrm{d}t\sum_{i}a_0^2(\frac{\Delta\boldsymbol{m}}{a_0}+2\boldsymbol{l})^2\\
&=\frac{-|J|S^2}{2}\int \mathrm{d}ta_0\int \mathrm{d}x[(\partial_x\boldsymbol{m})^2+\underbrace{4\frac{\Delta\boldsymbol{m}}{a_0}\cdot\boldsymbol{l}}_{=0}+4\boldsymbol{l}^2]\\
&=\frac{-|J|S^2a_0}{2}\int \mathrm{d}t\mathrm{d}x[(\partial_x\boldsymbol{m})^2+4\boldsymbol{l}^2]
\end{aligned}
\tag{4.68}
$$

So the total Lagrangian density involving both the order parameter $\boldsymbol{m}$ and $\boldsymbol{l}$:

$$
\mathcal{L}_m(\boldsymbol{m},\boldsymbol{l})=-2a_0|J|S^2\boldsymbol{l}^2+S\boldsymbol{l}(\boldsymbol{m}\times\partial_t\boldsymbol{m})-\frac{a_0|J|S^2}{2}(\partial_x\boldsymbol{m})^2+\frac{S}{2}\boldsymbol{m}\cdot(\partial_t\boldsymbol{m}\times\partial_x\boldsymbol{m}) \tag{4.69}
$$

where $\boldsymbol{m}$ is the order parameter field and $\boldsymbol{l}$ roughly represents the average spin field.

The Lagrangian has a quadratic term for $\boldsymbol{l}$:

$$
\mathcal{L}_l=-2a_0|J|S^2\boldsymbol{l}^2+S(\boldsymbol{m}\times\partial_t\boldsymbol{m})\boldsymbol{l}. \tag{4.70}
$$

To integrate out $\boldsymbol{l}$, we minimize the action with respect to $\boldsymbol{l}$

$$
\frac{\partial\mathcal{L}_l}{\partial\boldsymbol{l}}=-4a_0|J|S^2\boldsymbol{l}+S(\boldsymbol{m}\times\partial_t\boldsymbol{m})=0 \tag{4.71}
$$

so we solve for $\boldsymbol{l}$:

$$
\boldsymbol{l}=\frac{1}{4a_0|J|S}(\boldsymbol{m}\times\partial_t\boldsymbol{m}) \tag{4.72}
$$

Now we subsitute the expression for $\boldsymbol{l}$ found above back into $\mathcal{L}_l$:

$$
\begin{aligned}
\mathcal{L}_l^{\text{eff}}&=\frac{1}{8a_0|J|}(\boldsymbol{m}\times\partial_t\boldsymbol{m})^2\quad(\boldsymbol{m}\perp\partial_t\boldsymbol{m}\text{ because }\boldsymbol{m}\text{ is on the sphere})\\
&=\frac{1}{8a_0|J|}(\partial_t\boldsymbol{m})^2
\end{aligned}
\tag{4.73}
$$

Final we can combine the new term with original term that do not depend on $\boldsymbol{l}$:

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{m})&=\frac{1}{8a_0J}(\partial_t\boldsymbol{m})^2-\frac{a_0JS^2}{2}(\partial_x\boldsymbol{m})^2+\frac{S}{2}\boldsymbol{m}(\partial_t\boldsymbol{m}\times\partial_x\boldsymbol{m})\\
&=\frac{1}{2g}\left[\frac{1}{v_s}(\partial_t\boldsymbol{m})^2-v_s(\partial_x\boldsymbol{m})^2\right]+\underbrace{\frac{\theta}{8\pi}\varepsilon_{\mu\nu}\boldsymbol{m}(\partial_\mu\boldsymbol{m}\times\partial_\nu\boldsymbol{m})}_{\text{Einstein summation}}
\end{aligned}
\tag{4.74}
$$

where g and v_s are respectively the coupling constant and spin-wave velocity:

$$\begin{cases} g = \frac{2}{S} \\ v_s = 2a_0|J|S \end{cases} \tag{4.75}$$

The last term in {ref} has topological significance. We have chosen the normalization so that the coupling constant θ is given by:

$$\theta = 2\pi S \tag{4.76}$$

The tensor $\varepsilon_{\mu\nu}$ is the usual Levi-Civita antisymmetric tensor in two dimensions ($\mu, \nu = t, x$. $(t \to 0, x \to 1)$)

$$\varepsilon_{\mu\nu} = \begin{cases} 1 & (\mu,\nu) = (0,1) \\ -1 & (\mu,\nu) = (1,0) \\ 0 & \text{otherwise.} \end{cases} \tag{4.77}$$

The result is the Lagrangian density of non-linear sigma model.

4.4 The role of topological term.

Go back to Euclidean spacetime ($\mathrm{i}S_M \to -S_E$):

$$\mathrm{i}S_M = \mathrm{i}\int \mathrm{d}t\mathrm{d}x\mathcal{L}_M[\boldsymbol{m}(t,x)] = -\int \mathrm{d}\tau\mathrm{d}x\mathcal{L}_E[\boldsymbol{m}(\tau,x)] \tag{4.78}$$

So we can get the Lagrangian density $\mathcal{L}_E$ of Euclidean sector ($x_2 = \mathrm{i}t = \mathrm{i}x_0$)

$$\mathcal{L}_E = \frac{1}{2g}[(\partial_1\boldsymbol{m})^2 + (\partial_2\boldsymbol{m})^2] + \frac{\mathrm{i}\theta}{8\pi}\epsilon_{ij}\boldsymbol{m}\cdot(\partial_i\boldsymbol{m}\times\partial_j\boldsymbol{m}) \tag{4.79}$$

where $i, j = 1, 2$.

We now define the Pontryagin index or topological charge (or winding number) Q of the Euclidean space spin configuration $\{\boldsymbol{m}(\boldsymbol{x})\}$ by the expression:

$$\begin{aligned} Q &= \frac{1}{8\pi}\int \mathrm{d}^2x\, \epsilon_{ij}\boldsymbol{m}(\partial_i\boldsymbol{m}\times\partial_j\boldsymbol{m}) \\ &= \frac{1}{4\pi}\int \mathrm{d}^2x\, \boldsymbol{m}(\partial_1\boldsymbol{m}\times\partial_2\boldsymbol{m}) \end{aligned} \tag{4.80}$$

We impose the boundary condition that the Euclidean action be finite. This is equivalent to the requirement that asymptotically $\boldsymbol{m}$ becomes a constant vector $\boldsymbol{m}_0$ at spatial-time infinity:

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{m}_0 \tag{4.81}$$

Thus, the 2D Euclidean space-time is isomorphic to a sphere S_2 since the fields are identified with $\boldsymbol{m}_0$ at the point of infinity. However, the order parameter manifold (”target space”) is also isomorphic to a sphere S_2, since the constraint $\boldsymbol{m}^2 = 1$ has to be satisfied everywhere.

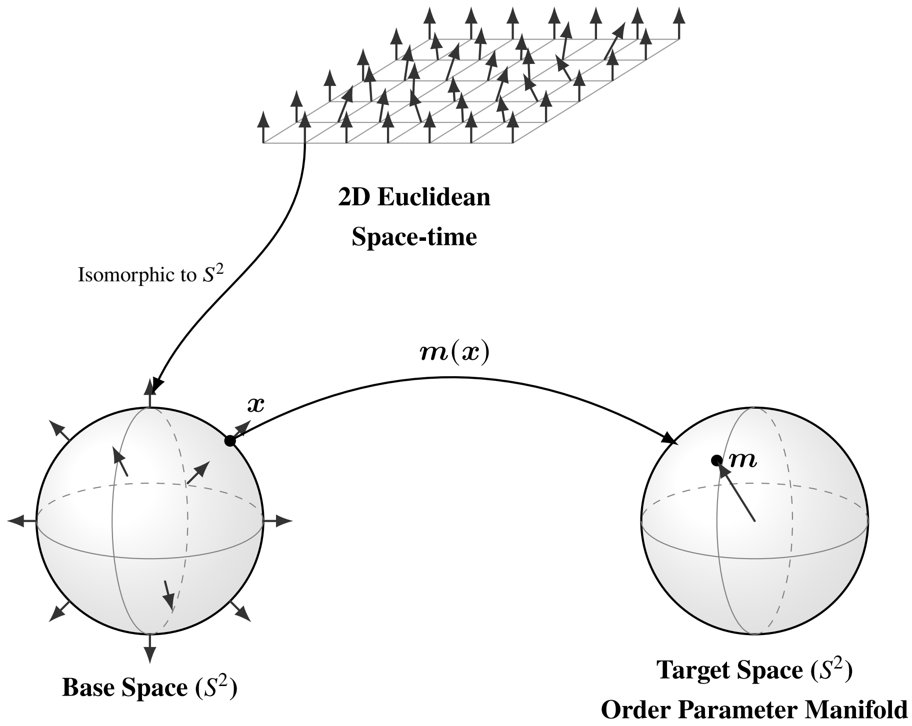


Figure 14: Topological mapping from 2D Euclidean space-time to the base space S^2 and the target space S^2.

The topological charge Q is in the sense that it counts how many times the spin configuration $\boldsymbol{m}$ has wrapped around the sphere S_2.

So the topological term of action is:

$$S_{\text{top}} = \mathrm{e}^{\mathrm{i}\cdot 2\pi SQ} = (-1)^{2SQ} \tag{4.82}$$

Thus:

$$\mathrm{e}^{\mathrm{i}\cdot 2\pi SQ} = \begin{cases} 1 & \text{for } S = 1, 2 \dots \text{ (integer spin)} \\ (-1)^Q & \text{for } S = \frac{1}{2}, \frac{3}{2}, \dots \text{ (half-odd-integer)} \end{cases} \tag{4.83}$$

We can see that:

(1) S is an integer: the spin chain is described at low energies by the standard non-linear sigma model, without a topological term.

(2) S is half integral S: each topological class contributes to the weight of the integral with a sign that is positive (negative) if the winding number Q is even (odd)

The integer- and half-integer-spin chains fall in different universality classes. We will now see that this property implies a very important result, known as Haldane's conjecture, that states that the integer-spin chains are massive (have an energy gap), whereas the half-integer-spin chains are massless as in the spin one-half case.

Part III

Appendix

Appendix A

Multivariate Gaussian Integral

The multivariate Gaussian integral:

$$\boxed{I = \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left(-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{y}\right) = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right)} \tag{A.1}$$

where:

- $\boldsymbol{x}$ and $\boldsymbol{y}$ are N-dimensional column vectors.
- A is an $N \times N$ real, symmetric, and positive-definite matrix.
- The notation $\int \prod_{n=1}^{N} \mathrm{d}x_n$ denotes integration over all components of $\boldsymbol{x}$ from $-\infty$ to $+\infty$.

A.1 Proof of the Multivariate Gaussian Integral

The proof relies on the assumptions that A is symmetric ($A^\top = A$) and positive-definite (all eigenvalues are positive), which ensures the integral converges. The proof proceeds in several key steps.

Step 1: Completing the Square

The primary technique is to complete the square for the quadratic form in the exponent. We want to rewrite the argument of the exponential, $-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{y}$, into the form $-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^\top A(\boldsymbol{x} - \boldsymbol{x}_0) + C$, where $\boldsymbol{x}_0$ and C are constants with respect to $\boldsymbol{x}$.

Expanding this target form, we get:

$$\begin{aligned} -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^\top A(\boldsymbol{x} - \boldsymbol{x}_0) &= -\frac{1}{2}(\boldsymbol{x}^\top - \boldsymbol{x}_0^\top)A(\boldsymbol{x} - \boldsymbol{x}_0) \\ &= -\frac{1}{2}(\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top A\boldsymbol{x}_0 - \boldsymbol{x}_0^\top A\boldsymbol{x} + \boldsymbol{x}_0^\top A\boldsymbol{x}_0) \end{aligned} \tag{A.2}$$

Since A is symmetric ($A = A^\top$), the scalar term $\boldsymbol{x}_0^\top A\boldsymbol{x}$ is equal to its own transpose: $(\boldsymbol{x}_0^\top A\boldsymbol{x})^\top = \boldsymbol{x}^\top A^\top \boldsymbol{x}_0 = \boldsymbol{x}^\top A\boldsymbol{x}_0$. Thus, the two cross-terms are equal.

$$-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^\top A(\boldsymbol{x} - \boldsymbol{x}_0) = -\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} + \boldsymbol{x}^\top A\boldsymbol{x}_0 - \frac{1}{2}\boldsymbol{x}_0^\top A\boldsymbol{x}_0 \tag{A.3}$$

Comparing this to the original exponent, $-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{y}$, we can equate the terms linear in $\boldsymbol{x}$:

$$-\boldsymbol{x}^\top \boldsymbol{y} = \boldsymbol{x}^\top A\boldsymbol{x}_0 \implies A\boldsymbol{x}_0 = -\boldsymbol{y} \tag{A.4}$$

Since A is positive-definite, it is invertible. We can solve for $\boldsymbol{x}_0$:

$$\boldsymbol{x}_0 = -A^{-1}\boldsymbol{y} \tag{A.5}$$

With this definition of $\boldsymbol{x}_0$, the original exponent can be written as:

$$-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y} = -\frac{1}{2}(\boldsymbol{x} + A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x} + A^{-1}\boldsymbol{y}) + \frac{1}{2}(A^{-1}\boldsymbol{y})^\top A(A^{-1}\boldsymbol{y}) \tag{A.6}$$

Let's simplify the constant term (the term not involving $\boldsymbol{x}$):

$$\begin{aligned}\frac{1}{2}(A^{-1}\boldsymbol{y})^\top A(A^{-1}\boldsymbol{y}) &= \frac{1}{2}\boldsymbol{y}^\top (A^{-1})^\top AA^{-1}\boldsymbol{y} \\ &= \frac{1}{2}\boldsymbol{y}^\top A^{-1}AA^{-1}\boldsymbol{y} \quad (\text{since } (A^{-1})^\top = (A^\top)^{-1} = A^{-1}) \\ &= \frac{1}{2}\boldsymbol{y}^\top IA^{-1}\boldsymbol{y} = \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\end{aligned} \tag{A.7}$$

So, the exponent is:

$$-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y} = -\frac{1}{2}(\boldsymbol{x} + A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x} + A^{-1}\boldsymbol{y}) + \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y} \tag{A.8}$$

Step 2: Change of Variables (Translation)

Substituting the completed square back into the integral:

$$I = \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left[-\frac{1}{2}(\boldsymbol{x} + A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x} + A^{-1}\boldsymbol{y}) + \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right] \tag{A.9}$$

The term $\exp(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y})$ is constant with respect to $\boldsymbol{x}$ and can be factored out of the integral:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left[-\frac{1}{2}(\boldsymbol{x} + A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x} + A^{-1}\boldsymbol{y})\right] \tag{A.10}$$

Now, we perform a change of variables. Let $\boldsymbol{z} = \boldsymbol{x} + A^{-1}\boldsymbol{y}$. This is a simple translation of the coordinate system. The differential element $\prod \mathrm{d}x_n$ transforms as $\prod \mathrm{d}z_n$, as the Jacobian of this transformation is 1. The limits of integration remain from $-\infty$ to $+\infty$. The integral becomes:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \int \prod_{n=1}^{N} \mathrm{d}z_n \, \exp\left(-\frac{1}{2}\boldsymbol{z}^\top A\boldsymbol{z}\right) \tag{A.11}$$

The problem is now reduced to evaluating the simpler, centered Gaussian integral:

$$I_0 = \int \prod \mathrm{d}z_n \, \exp(-\frac{1}{2}\boldsymbol{z}^\top A\boldsymbol{z}) \tag{A.12}$$

Step 3: Diagonalization

To compute I_0, we diagonalize the matrix A. Since A is a real symmetric matrix, it is orthogonally diagonalizable:

$$A = PDP^\top \tag{A.13}$$

where P is an orthogonal matrix ($PP^\top = P^\top P = I$) whose columns are the orthonormal eigenvectors of A, and D is a diagonal matrix whose entries are the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. Substituting this into the quadratic form $\boldsymbol{z}^\top A\boldsymbol{z}$:

$$\boldsymbol{z}^\top A\boldsymbol{z} = \boldsymbol{z}^\top (PDP^\top)\boldsymbol{z} = (\boldsymbol{z}^\top P)D(P^\top \boldsymbol{z}) = (P^\top \boldsymbol{z})^\top D(P^\top \boldsymbol{z}) \tag{A.14}$$

We perform another change of variables. Let $\boldsymbol{w} = P^\top \boldsymbol{z}$. This transformation corresponds to a rotation of the coordinate system. The Jacobian determinant is $|\det(P^\top)| = 1$, so the volume element is unchanged: $\prod \mathrm{d}z_n = \prod \mathrm{d}w_n$. The quadratic form simplifies to:

$$\boldsymbol{w}^\top D\boldsymbol{w} = \sum_{i=1}^{N} \lambda_i w_i^2 \tag{A.15}$$

This is because D is a diagonal matrix.

Step 4: Computing the Decoupled Integral

The integral I_0 now becomes:

$$I_0 = \int \prod_{n=1}^{N} \mathrm{d}w_n \, \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \lambda_i w_i^2\right) \tag{A.16}$$

The exponential of a sum is the product of exponentials, which allows us to separate the multi-dimensional integral into a product of N one-dimensional integrals:

$$I_0 = \int \prod_{n=1}^{N} \mathrm{d}w_n \prod_{i=1}^{N} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right) = \prod_{i=1}^{N}\left(\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right)\mathrm{d}w_i\right) \tag{A.17}$$

We use the standard formula for a 1D Gaussian integral: $\int_{-\infty}^{\infty} \exp(-au^2)\mathrm{d}u = \sqrt{\pi/a}$. In our case, $a = \lambda_i/2$.

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right)\mathrm{d}w_i = \sqrt{\frac{\pi}{\lambda_i/2}} = \sqrt{\frac{2\pi}{\lambda_i}} \tag{A.18}$$

Multiplying these N results together:

$$I_0 = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} = (2\pi)^{\frac{N}{2}} \prod_{i=1}^{N} (\lambda_i)^{-\frac{1}{2}} = (2\pi)^{\frac{N}{2}} \left(\prod_{i=1}^{N} \lambda_i\right)^{-\frac{1}{2}} \tag{A.19}$$

The determinant of a matrix is equal to the product of its eigenvalues. Thus, $\det A = \det D = \prod_{i=1}^{N} \lambda_i$.

$$I_0 = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}} \tag{A.20}$$

Step 5: Combining the Results

Finally, we substitute the value of I_0 back into our expression for I from Step 2:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \cdot I_0 = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right)(2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \tag{A.21}$$

Rearranging the terms yields the final result:

$$I = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \tag{A.22}$$

This completes the proof.

Appendix B

Calculation of the Determinant and Inverse of the a Special Matrix

Let the given $n \times n$ matrix be denoted by A_n.

$$A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{n\times n} \tag{B.1}$$

B.1 Calculation of det(A)

Let D_n denote the determinant of an $n \times n$ matrix of the form described in the problem. The matrix A given in the problem has dimensions $(N-1) \times (N-1)$. For the sake of recurrence derivation, we analyze the matrix A_n of size $n \times n$:

$$A_n = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}_{n\times n} \tag{B.2}$$

Step 1: Find a recurrence relation

We compute the determinant using cofactor expansion along the first row. Let $D_n = \det(A_n)$.

$$D_n = 2 \cdot \det(M_{1,1}) - (-1) \cdot \det(M_{1,2}) \tag{B.3}$$

Here, $M_{i,j}$ represents the minor matrix obtained by removing row i and column j.

1. Removing row 1 and column 1 leaves a matrix of the same structure with size $(n-1) \times (n-1)$, so $\det(M_{1,1}) = D_{n-1}$.
2. Removing row 1 and column 2 leaves a matrix where the first column contains only -1 in the top position and zeros elsewhere. Expanding along this new first column yields -1

multiplied by the determinant of the remaining $(n-2)\times(n-2)$ matrix (which again has the original structure). Thus, $\det(M_{1,2}) = -1 \cdot D_{n-2}$.

Substituting these back into the expansion equation:

$$\begin{aligned} D_n &= 2(D_{n-1}) - (-1)(-1 \cdot D_{n-2}) \\ D_n &= 2D_{n-1} - D_{n-2} \end{aligned} \tag{B.4}$$

Step 2: Calculate base cases

$$\begin{aligned} &\text{For } n = 1: \quad A_1 = [2] \implies D_1 = 2 \\ &\text{For } n = 2: \quad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \implies D_2 = (2)(2) - (-1)(-1) = 3 \end{aligned} \tag{B.5}$$

Step 3: Solve the recurrence

We observe the pattern $D_1 = 2$, $D_2 = 3$, $D_3 = 4$. We hypothesize that $D_n = n + 1$. We prove this by induction.

1. The hypothesis holds for base cases $n = 1$ and $n = 2$.
2. Assume $D_k = k + 1$ for all $k < n$.
3. Apply the recurrence:

$$\begin{aligned} D_n &= 2D_{n-1} - D_{n-2} \\ &= 2((n-1)+1) - ((n-2)+1) \\ &= 2n - (n-1) \\ &= 2n - n + 1 \\ &= n + 1 \end{aligned} \tag{B.6}$$

Step 4: Final Calculation

The matrix A has size $(N-1)\times(N-1)$. Therefore, we substitute $n = N - 1$ into the formula:

$$\det(A) = D_{N-1} = (N-1) + 1 = N \tag{B.7}$$

B.2 Calculation of A^{-1}

Let $B = A^{-1}$. The elements of the inverse matrix are denoted as $b_{i,j}$. From the definition $AB = I$, for a fixed column j of the inverse matrix, the elements $x_i = b_{i,j}$ satisfy the linear system:

$$A\boldsymbol{x} = \boldsymbol{e}_j \tag{B.8}$$

where $\boldsymbol{e}_j$ is the standard basis vector. The indices i and j run from 1 to $N-1$.

Step 1: Set up the difference equations

The structure of A corresponds to a discrete second derivative. The equation for the i-th row is:

$$-x_{i-1} + 2x_i - x_{i+1} = \delta_{i,j} \tag{B.9}$$

The boundary conditions correspond to the nodes outside the matrix:

$$x_0 = 0 \quad \text{and} \quad x_N = 0 \tag{B.10}$$

Step 2: Solve the homogeneous regions

For $i \neq j$, the equation is $-x_{i-1} + 2x_i - x_{i+1} = 0$, which implies $x_{i+1} - x_i = x_i - x_{i-1}$. The solution is linear in i.

1. **Region 1 ($0 \leq i \leq j$):** Applying $x_0 = 0$, the solution is $x_i = c_1 i$.
2. **Region 2 ($j \leq i \leq N$):** Applying $x_N = 0$, the solution is $x_i = c_2(N-i)$.

Step 3: Match continuity at i = j

The solution must be continuous at $i = j$:

$$c_1 j = c_2(N-j) \implies c_2 = c_1 \frac{j}{N-j} \tag{B.11}$$

Step 4: Apply the jump condition at i = j

Substitute the neighbors into the non-homogeneous equation at $i = j$:

$$-x_{j-1} + 2x_j - x_{j+1} = 1 \tag{B.12}$$

Using the linear forms derived above (x_{j-1} from Region 1, x_{j+1} from Region 2):

$$\begin{aligned} -[c_1(j-1)] + 2[c_1 j] - [c_2(N-j-1)] &= 1 \\ c_1(j+1) - c_2(N-j-1) &= 1 \end{aligned} \tag{B.13}$$

Substitute $c_2 = c_1 \frac{j}{N-j}$:

$$
\begin{aligned}
c_1(j+1) - \left(c_1 \frac{j}{N-j}\right)(N-j-1) &= 1 \\
c_1 \left[(j+1) - \frac{j(N-j-1)}{N-j}\right] &= 1
\end{aligned}
\tag{B.14}
$$

Multiplying by $(N-j)$ to clear the denominator:

$$
\begin{aligned}
c_1\left[(j+1)(N-j) - j(N-j-1)\right] &= N-j \\
c_1\left[(jN - j^2 + N - j) - (jN - j^2 - j)\right] &= N-j \\
c_1\left[N\right] &= N-j
\end{aligned}
\tag{B.15}
$$

Thus, we find the coefficients:

$$
c_1 = \frac{N-j}{N}, \quad c_2 = \frac{j}{N}
\tag{B.16}
$$

Step 5: Final Matrix Elements

Substituting the constants back into the piecewise expressions: For $i \leq j$: $x_i = \frac{N-j}{N} i$. For $i \geq j$: $x_i = \frac{j}{N}(N-i)$.

The elements of the inverse matrix are given by:

$$
(A^{-1})_{i,j} = \frac{1}{N} \min(i,j)(N - \max(i,j))
\tag{B.17}
$$

Appendix C

Evaluation of the Fresnel Integral

The value of the **Fresnel Integral** is:

$$\boxed{\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x = \sqrt{\frac{\pi}{2}}(1+\mathrm{i}) = \sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi/4}} \tag{C.1}$$

C.1 Derivation (Using Contour Integration)

Step 1: Define the Contour

We consider the complex function $f(z) = \mathrm{e}^{\mathrm{i}z^2}$, where z is a complex variable. We construct a closed path (contour) C in the complex plane. This path is a sector of a circle, composed of three parts:

1. **Path** C_1: A line segment along the real axis from 0 to R.
2. **Path** C_2: A circular arc of radius R, centered at the origin, running counter-clockwise from R to $R\mathrm{e}^{\mathrm{i}\pi/4}$.
3. **Path** C_3: A line segment from $R\mathrm{e}^{\mathrm{i}\pi/4}$ back to the origin 0.

We will eventually let $R \to \infty$.

Step 2: Apply Cauchy's Integral Theorem

The function $f(z) = \mathrm{e}^{\mathrm{i}z^2}$ is analytic over the entire complex plane (it is an entire function) as it has no singularities. According to **Cauchy's Integral Theorem**, its integral over any closed path C is zero:

$$\oint_C \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = 0 \tag{C.2}$$

This closed-loop integral can be split into the sum of integrals over the three paths:

$$\int_{C_1} \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z + \int_{C_2} \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z + \int_{C_3} \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = 0 \tag{C.3}$$

Step 3: Evaluate the Integral on Each Path

1. Integral along Path C_1 (The part we want to find)

On path C_1, we have $z = x$ (a real number) and $\mathrm{d}z = \mathrm{d}x$. Therefore:

$$\lim_{R\to\infty} \int_{C_1} \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = \int_0^{\infty} \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x \tag{C.4}$$

This is exactly half of the integral we wish to compute, since the integrand $\mathrm{e}^{\mathrm{i}x^2}$ is an even function.

2. Integral along Path C_2 (Show it vanishes as $R \to \infty$)

On path C_2, we parameterize $z = R\mathrm{e}^{\mathrm{i}\theta}$, where θ varies from 0 to $\pi/4$. Then $\mathrm{d}z = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$ and $z^2 = R^2\mathrm{e}^{\mathrm{i}2\theta}$. The integral becomes:

$$\begin{aligned}\int_{C_2} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z &= \int_0^{\pi/4} \exp(\mathrm{i}(R^2\mathrm{e}^{\mathrm{i}2\theta})) \cdot \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta \\ &= \int_0^{\pi/4} \exp(\mathrm{i}R^2(\cos(2\theta) + \mathrm{i}\sin(2\theta))) \cdot \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta \\ &= \int_0^{\pi/4} \exp(-R^2\sin(2\theta))\exp(\mathrm{i}(R^2\cos(2\theta) + \theta)) \cdot \mathrm{i}R\mathrm{d}\theta\end{aligned} \tag{C.5}$$

Taking the magnitude:

$$\left|\int_{C_2} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z\right| \leq \int_0^{\pi/4} \left|\exp(-R^2\sin(2\theta))\right| \cdot R\mathrm{d}\theta = \int_0^{\pi/4} R\exp(-R^2\sin(2\theta))\mathrm{d}\theta \tag{C.6}$$

On the interval $[0, \pi/4]$, 2θ is in $[0, \pi/2]$. We can use Jordan's inequality, which states $\sin(x) \geq \frac{2x}{\pi}$ for $x \in [0, \pi/2]$. Thus, $\sin(2\theta) \geq \frac{4\theta}{\pi}$.

$$\begin{aligned}\left|\int_{C_2} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z\right| &\leq \int_0^{\pi/4} R\exp(-R^2(4\theta/\pi))\mathrm{d}\theta \\ &= R\left[\frac{-\pi}{4R^2}\exp(-4R^2\theta/\pi)\right]\Bigg|_0^{\pi/4} \\ &= \frac{\pi}{4R}(1 - \exp(-R^2))\end{aligned} \tag{C.7}$$

As $R \to \infty$, this expression approaches 0. Therefore:

$$\lim_{R\to\infty} \int_{C_2} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z = 0 \tag{C.8}$$

3. Integral along Path C_3 (Connection to the Gaussian Integral)

On path C_3, we parameterize $z = r\mathrm{e}^{\mathrm{i}\pi/4}$, where r varies from R to 0. Then $\mathrm{d}z = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{d}r$ and $z^2 = (r\mathrm{e}^{\mathrm{i}\pi/4})^2 = r^2\mathrm{e}^{\mathrm{i}\pi/2} = r^2\mathrm{i}$. The integral becomes:

$$\int_{C_3} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z = \int_R^0 \exp(\mathrm{i}(r^2\mathrm{i}))\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{d}r = \int_R^0 \exp(-r^2)\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{d}r \tag{C.9}$$

Reversing the limits of integration and factoring out the constant:

$$= -\mathrm{e}^{\mathrm{i}\pi/4}\int_0^R \exp(-r^2)\mathrm{d}r \tag{C.10}$$

As $R \to \infty$, we get the well-known Gaussian integral:

$$\lim_{R\to\infty}\int_{C_3} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z = -\mathrm{e}^{\mathrm{i}\pi/4}\int_0^\infty \exp(-r^2)\mathrm{d}r \tag{C.11}$$

We know that the Gaussian integral $\int_0^\infty \exp(-x^2)\mathrm{d}x = \frac{\sqrt{\pi}}{2}$. So:

$$\lim_{R\to\infty}\int_{C_3} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z = -\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2} \tag{C.12}$$

Step 4: Combine the Results

Returning to the equation from Step 2 and taking the limit as $R \to \infty$:

$$\left(\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x\right) + (0) + \left(-\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2}\right) = 0 \tag{C.13}$$

Rearranging the terms, we find:

$$\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x = \mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2} \tag{C.14}$$

Step 5: Calculate the Final Integral

The integral we want to evaluate is from $-\infty$ to ∞. Since the integrand $\mathrm{e}^{\mathrm{i}x^2} = \cos(x^2) + \mathrm{i}\sin(x^2)$ is an even function (i.e., $f(-x) = f(x)$), we have:

$$\begin{aligned}\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x &= 2\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x \\ &= 2\cdot\left(\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi/4}\end{aligned} \tag{C.15}$$

Finally, we can use Euler's formula, $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, to expand $\mathrm{e}^{\mathrm{i}\pi/4}$:

$$\mathrm{e}^{\mathrm{i}\pi/4} = \cos(\pi/4) + \mathrm{i}\sin(\pi/4) = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2} = \frac{1+\mathrm{i}}{\sqrt{2}} \tag{C.16}$$

Substituting this into our result gives:

$$\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x = \sqrt{\pi}\cdot\frac{1+\mathrm{i}}{\sqrt{2}} = \sqrt{\frac{\pi}{2}}(1+\mathrm{i}) \tag{C.17}$$

C.2 Conclusion

The result of the integral is a complex number. Its real and imaginary parts correspond to two other important integrals:

$$\begin{cases}\int_{-\infty}^\infty \cos(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{2}} \\ \int_{-\infty}^\infty \sin(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{2}}\end{cases} \tag{C.18}$$