

### 3. The path integral for many-spin system

#### 3.1 General theory

Consider a spin system in a  $d$ -dimensional lattice which is called Heisenberg Model.

$$\hat{H} = J \sum_{\langle i,j \rangle} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j$$

where  $J$  is the exchange coupling and  $\hat{\vec{S}}$  are quantum spin operators. The  $\langle i,j \rangle$  denotes a summation over nearest-neighbor pairs.

The partition function of the spin system is:

$$\begin{aligned} Z &= \text{Tr } e^{-\beta \hat{H}} \\ &= \int d\vec{n}_0 \langle \vec{n}_0 | e^{-\beta \hat{H}} | \vec{n}_0 \rangle \end{aligned}$$

where

$|\vec{n}_0\rangle = \bigotimes_{i=1}^N |\vec{n}_0^i\rangle$  is the tensor product of all the states on the lattice.

In the previous chapters, we have seen that we can change real time  $t$  to imaginary time  $I$ :

$$\langle n_f | e^{-i\hat{H}(t_f - t_i)} | n_i \rangle \xrightarrow{i(t_f - t_i) \rightarrow I} \langle n_f | e^{-\hat{H}I} | n_i \rangle$$

In this version, ~~K~~on propagator  $K$  can be

written as (for closed path),

$$k = \int D[\vec{n}(t)] e^{S_E}$$

where,  $S_E = -iSS_{WZ} = i \cdot S \cdot A_r$ . for  $\vec{H} = 0$  and,

$$S_E = -iS \cdot S_{WZ} + \int_{T_i}^{T_f} \langle \vec{n}(t) | \vec{H} | \vec{n}(t) \rangle dt. \text{ for } \vec{H} \neq 0.$$

Because, the Hamiltonian diagonal element of partition function and propagator have the same mathematical form, so we can use the same method to solve the diagonal elements.

let's make  $\Delta T$  equal to  $\frac{\beta}{N}$  ( $N$  is the slice number) whose mean is that  $\beta$  play a same role in the diagonal linking  $T$  in the propagator.  $(T_i \rightarrow t_f \rightarrow 0 \rightarrow \beta)$

But in many-spin system, the identity operator for the entire system is the tensor product of the identities for each site:

$$\hat{I}_{\text{total}} = \bigotimes_{i=1}^N \hat{I}_i = \int (\prod_{i=1}^N d^2 \vec{n}_i) \vec{n}_1 \vec{n}_2 \cdots \vec{n}_N \otimes \langle \vec{n}_1 \rangle \langle \vec{n}_2 \rangle \cdots \langle \vec{n}_N \rangle$$

So the diagonal can be written as a mathematical form like propagator:

$$\langle \vec{n}_0 | e^{-\beta \vec{H}} | \vec{n}_0 \rangle = \int D(\vec{n}) \cdot e^{-S_E}$$

where  $D(\vec{n}) = \prod_i \prod_j d^2 \vec{n}_i j$ . Because diagonal likes the propagator with close path ( $\langle \vec{n}_0 | e^{-\beta \vec{H}} | \vec{n}_0 \rangle$  and not  $\langle \vec{n}_i | e^{-\beta \vec{H}} | \vec{n}_j \rangle$  for  $i \neq j$ ),  $S_E$  can be written as:

$$S_E = \int_0^\beta dI \langle \vec{n} | \frac{\partial}{\partial I} \cdot | \vec{n} \rangle + \int_0^\beta dI \cdot \langle \vec{n} | \vec{H} | \vec{n} \rangle$$

For the energy term:

$$\begin{aligned} S_E &= \int_0^\beta dI \cdot \langle \vec{n} | \vec{H} | \vec{n} \rangle \\ &= \int_0^\beta dI \int \sum_{\langle i,j \rangle} (\otimes | \vec{n} \rangle) \vec{S}_i \cdot \vec{S}_j (\otimes | \vec{n} \rangle). \end{aligned}$$

Use the formula:

$$\langle \vec{n} | \vec{S} | \vec{n} \rangle = S_{\text{spin}} \cdot \vec{n}$$

we can get:

$$S_E = \int_0^\beta dI \int \sum_{\langle i,j \rangle} S_{\text{spin}}^2 \vec{n}_i(I) \vec{n}_j(I).$$

For the ~~Hizze~~-geometric term:

$$S_g = \int_0^\beta d\vec{n} (\otimes |\vec{n}\rangle) \frac{\partial}{\partial \vec{n}} (\otimes |\vec{n}\rangle)$$

$$= \int_0^\beta (\otimes |\vec{n}\rangle) [(\otimes |\vec{n}\rangle + d|\vec{n}\rangle) - (\otimes |\vec{n}\rangle)]$$

let's retain the first order term:

$$S_g = \int_0^\beta (\otimes |\vec{n}\rangle) [(\otimes |\vec{n}\rangle + \sum_i (\otimes |\vec{n}_j\rangle d|\vec{n}_i\rangle) - (\otimes |\vec{n}\rangle)]$$

$$= \sum_i \int_0^\beta (\otimes |\vec{n}\rangle) (\otimes |\vec{n}_j\rangle) d|\vec{n}_i\rangle$$

because of  $\langle \vec{n}_i | \vec{n}_i \rangle = 1$ , so:

$$S_g = \sum_i \int_0^\beta \langle \vec{n}_i | d|\vec{n}_i\rangle$$

$$= \sum_i -i S \cdot S_{WZ} [\vec{n}_i]$$

$$= -i S \sum_i S_{WZ} [\vec{n}_i]$$

let's go back to real time ( $I \rightarrow it$ ):

$$-S_E \rightarrow \frac{i}{\hbar} S_M.$$

$$\Rightarrow S_M = \hbar S \sum_i S_{WZ} [\vec{n}_i] - S^2 \int_0^\beta dt \cdot \sum_i \vec{J} \vec{n}_i \cdot \vec{n}_j$$

$$\frac{\hbar S_M}{\hbar} = S \left\{ \sum_i S_{WZ} [\vec{n}_i] \right\} - \frac{S^2}{\hbar} \left\{ \int_0^\beta dt \sum_i \vec{J} \vec{n}_i \cdot \vec{n}_j \right\}$$

So  $\left\{ \sum_i S_{W2} [\vec{n}_i] \right\}$  and  $\left\{ \int_0^\beta dt \sum_{i,j} \vec{J} \cdot \vec{n}_i \cdot \vec{n}_j \right\}$ , can be regarded as the phase term of action ( $e^{\frac{i}{\hbar} \cdot \theta}$ ). Therefore,

$S$  and  $\frac{S^2}{\hbar}$  play the role of  $\hbar$  in simple action

Here, if  $S$  is very large,  ~~$S$~~  we called it Large- $S$  limit ( $\hbar \ll 1$ , so  $\frac{1}{\hbar} \approx S \gg 1$ ) This is a semi-classical limit.

### 3.2 Quantum Ferromagnetism.

let's consider a ferromagnetism system whose  $J$  equal to  $-|J|$ . So the action is:

$$S_M = S \sum_i S_{W2} [\vec{n}_i] + |J| S^2 \sum_{i,j} \int_0^\beta dt \vec{n}_i \cdot \vec{n}_j$$

because  $|\vec{n}|^2 = 1$ , so we can rewrite the term  $\vec{n}_i \cdot \vec{n}_j$  as:

~~$$\frac{1}{2} + \vec{n}_i \cdot \vec{n}_j = \frac{1}{2}(1)$$~~

$$\begin{aligned} -\vec{n}_i \cdot \vec{n}_j &= \frac{1}{2}(1 - 2\vec{n}_i \cdot \vec{n}_j + 1) - 1 \\ &= \frac{1}{2}(\vec{n}_i - \vec{n}_j)^2 - 1 \end{aligned}$$

So the energy term is

$$S_E = -\frac{|J| S^2}{2} \sum_{i,j} \int_0^\beta dt (\vec{n}_i - \vec{n}_j)^2 + \frac{|J| S^2}{2} \sum_{i,j} \int_0^\beta dt$$

The second term of  $S_e$  is the constant term (global phase), so we can drop it. and  $S_m$  can be written as:

$$S_m = S \sum_i S_{WZ}[\vec{n}_i] - \frac{|\vec{J}|S^2}{2} \sum_{\langle i,j \rangle} \int_0^\beta dt (\vec{n}_i - \vec{n}_j)^2$$

Next, we take the continuum limit where the lattice sums become integrals

$$\sum_i \rightarrow \int \frac{d^d x}{a_0^d} S \cdot S$$

where  $a_0$  is the lattice spacing and finite differences become gradients:

$$\begin{aligned} \sum_j (\vec{n}_i - \vec{n}_j)^2 &= \left( \sum_j \frac{\partial n_i}{\partial x_j} dx_j \right) \left( \sum_k \frac{\partial n_i}{\partial x_k} dx_k \right) a_0, \quad \sum_k \frac{\partial n_i}{\partial x_k} a_0, \dots \right)^2 \\ &= a_0^2 (\nabla \vec{n})^2. \end{aligned}$$

where  $\nabla \vec{n} = (\frac{\partial \vec{n}}{\partial x}, \dots)$  and not  $\nabla \vec{n} = (\frac{\partial n_x}{\partial x}, \dots)$

So the continuum action is:

$$\begin{aligned} S_m &= \int \frac{d^d x}{a_0^d} S \cdot S_{WZ}[\vec{n}] - \frac{|\vec{J}|S^2}{2} \int \frac{d^d x}{a_0^d} dt a_0^2 (\nabla \vec{n})^2 \\ &= \frac{S}{a_0^d} \int d^d x S_{WZ}[\vec{n}] - \frac{|\vec{J}|S^2}{2a_0^{d-2}} \int d^d x dt (\nabla \vec{n})^2 \end{aligned}$$