If we look at: $\vec{n}\times(\vec{n}\times\nabla^2\vec{n})$:

$$\vec{n}\times(\vec{n}\times\nabla^2\vec{n}) = \vec{n}\cdot(\vec{n}\cdot\nabla^2\vec{n}) - \nabla^2\vec{n}\,(\vec{n}\cdot\vec{n})$$
$$= \vec{n}\,(\vec{n}\cdot\nabla^2\vec{n}) - \nabla^2\vec{n}.$$

So:

$$\frac{S}{a_0^d}(\vec{n}\times\partial_t\vec{n}) = \frac{|J|S^2}{a_0^{d-2}}\left[\vec{n}\times(\vec{n}\times\nabla^2\vec{n})\right]$$

Simplified to:

~~We co~~

$$\vec{n}\times\cancel{\partial_t\vec{n}}\,\partial_t\vec{n} = \vec{n}\times\left[\cancel{\frac{|J|S^2}{a_0}}\,|J|S^2 a_0^2(\nabla^2\vec{n}))\right]$$

We can get the Landau-Lifshitz equation:

$$\cancel{\partial_t\vec{n} =}\ \frac{\partial\vec{n}}{\partial t} = |J|S^2 a_0^2(\vec{n}\times\nabla^2\vec{n})$$

~~From this equation, we can know that the Landau-Lifshitz equations can be solved in the linear regime.~~

From this equation, we can know that the spins move in a precessional fashion with an angular velocity $\vec{\Omega}$ given by:

$$\vec{\Omega} = -|J|Sa_0^2\nabla^2\vec{n}.$$

The landau-Lifshitz equations can be solved in the linear regime. Let's parametrize $\vec{n}$ by the components:

$$\vec{n} = \begin{pmatrix} \sigma \\ \vec{\pi} \end{pmatrix} \quad \text{and} \quad \vec{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$$

where σ and π_i $(i=1,2)$ satify the constraint:

$$\sigma^2 + \vec{\pi}^2 = 1$$

If the system is ordered, $\sigma \simeq 1$, $|\vec{\pi}|$ is very small, which means that. $\vec{\pi}$ is ~~fluctuation~~ small fluctuation. around the direction of σ.

In this ~~con~~situation, we can treat σ as a constat:

$$\begin{cases} \sigma \approx 1 \\ \nabla^2 \sigma \approx 0 \end{cases}$$

Therefore, ~~#~~ we can linearize the Landau-Lifshitz equation:

$$\vec{n} \times \nabla^2 \vec{n} = \begin{pmatrix} \pi_1 \nabla^2 \pi_2 - \pi_2 \nabla^2 \pi_1 \\ -\nabla^2 \pi_2 \\ \nabla^2 \pi_1 \end{pmatrix}$$

and we remain the one order term:

$$\vec{n} \times \nabla^2 \vec{n} = \begin{pmatrix} 0 \\ -\nabla^2 \pi_2 \\ \nabla^2 \pi_1 \end{pmatrix}$$

So the linearizing Landau-Lifshitz equation:

$$\begin{cases} \partial_t \pi_1 = -|J| S a_0^2 \nabla^2 \pi_2 \\ \partial_t \pi_2 = +|J| S a_0^2 \nabla^2 \pi_1 \end{cases}$$

We look for plane wave solutions (spin wave) of the term:

$$\pi_j(\vec{x},t) = \tilde{\pi}_j e^{i(\vec{p}\cdot\vec{x}-\omega t)}$$

taking derivatives this ansart converts differential operators into algebraic variables:

$$\begin{cases} \partial_t \rightarrow -i\omega \\ \nabla^2 \rightarrow -|\vec{p}|^2 \end{cases}$$

Substituting these into the linearized equations:

$$\begin{cases} -i\omega\tilde{\pi}_1 = (|J|Sa_0^2|\vec{p}|^2)\tilde{\pi}_2 \\ -i\omega\tilde{\pi}_2 = -(|J|Sa_0^2|\vec{p}|^2)\tilde{\pi}_1 \end{cases}$$

let $A = |J|Sa_0^2 p^2$, the system become:

$$\begin{cases} -i\omega\tilde{\pi}_1 - A\tilde{\pi}_2 = 0 \\ A\tilde{\pi}_1 - i\omega\tilde{\pi}_2 = 0 \end{cases}$$

for a non-trivial solution $(\tilde{\pi}_1, \tilde{\pi}_2 \neq 0)$, the determinant of the coefficient matrix must be ~~zer~~ zero:

$$\begin{vmatrix} -i\omega & -A \\ A & -i\omega \end{vmatrix} = 0$$

we can get:

$$\omega = \pm A.$$

So we obtain the ~~dispers~~ dispersion relation for ~~fre ferromagn~~ ferromagnetic spin waves:

$$|\omega| \approx |J| S a_0^2 |\vec{p}|^2$$

We find that the frequency of the low-energy excitations of a quantum ferromagnetic scales as the square of the momentum.

3.3 Quantum Antiferromagnetic of one-dimension.

Consider a spin chain with an even number of sites N; we can write down the action of it ($J = |J|$):

$$S_m[\vec{n}] = S \sum_{i}^{N} S_{wz}[\vec{n}] - \int dt \cdot \sum_{i}^{N} J S^2 \vec{n}(i,t) \cdot \vec{n}(i+1,t)$$

where we have assumed periodic boundary conditions:

Since we expect to be close to a Néel state, we will stagger the configuration:

$$\vec{n}(j) \rightarrow (-1)^j \vec{n}(j)$$

On a bipartite lattice, the substitution of \ref{ } into \ref{ } will change the sign of the exchange term of the action to a ferromagnetic:

$$S_m[\vec{n}] = S\sum_{i}^{N}(-1)^i S_{WZ}[\vec{n}(i)] - \frac{|J|S^2}{2}\int dt \cdot \sum_{i=1}^{N}(-1)^i\vec{n}(i,t)\cdot(-1)^{i+1}\vec{n}(i+1,t)$$

so the ~~en~~ energy term become to:

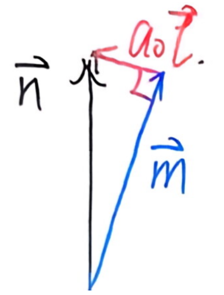
$$S_e = \frac{|J|S^2}{2}\cdot\int dt\sum_{i=1}^{N}\vec{n}(i,t)\vec{n}(i+1,t)$$

which is similar to ferromagnetic.

But the Wess-Zumino terms are odd ~~term~~ under the replacement of \ref{ } and thus become staggered. Thus, it is the Wess-Zumino term, a purely quantum-mechanical effect, which will distingush ferromagnets from antiferromagnets.

As ferromagnetic, we can split $\vec{n}$ into a slowly varying pice ~~m(j)~~ $\vec{m}$ a small rapidly varying part $a_0\vec{l}(i)$, so:

$$\vec{n}(i) = \vec{m}(i) + (-1)^i a_0\vec{l}(i)$$



The constraint $|\vec{n}|^2=1$ and requirement that $\vec{m}$ should obey the same constraint, ~~and~~ demand that $\vec{m}$ and $\vec{l}$ be orthogonal vector:

$$\begin{cases} |\vec{m}|^2=1 \\ \vec{m}\cdot\vec{l}=0 \end{cases}$$

The Wess-Zumino terms are rewritten as:

$$S\sum_{i}^{N}(-1)^{j}S_{WZ}[\vec{n}(j)]=S\sum_{r=1}^{\frac{N}{2}}\{S_{WZ}[\vec{n}(2r)]-S_{WZ}[\vec{n}(2r-1)]\}$$

by making use of the approximation:

$$\vec{n}(2r)-\vec{n}(2r-1)=\vec{m}(2r)-\vec{m}(2r-1)+a_0[\vec{l}(2r)+\vec{l}(2r-1)]$$

$$\approx a_0\left[\frac{\partial\vec{m}(2r)}{\partial x}+2\vec{l}(2r)\right]+O(a_0^2)$$

So becomes (use the smilar method of area variation)

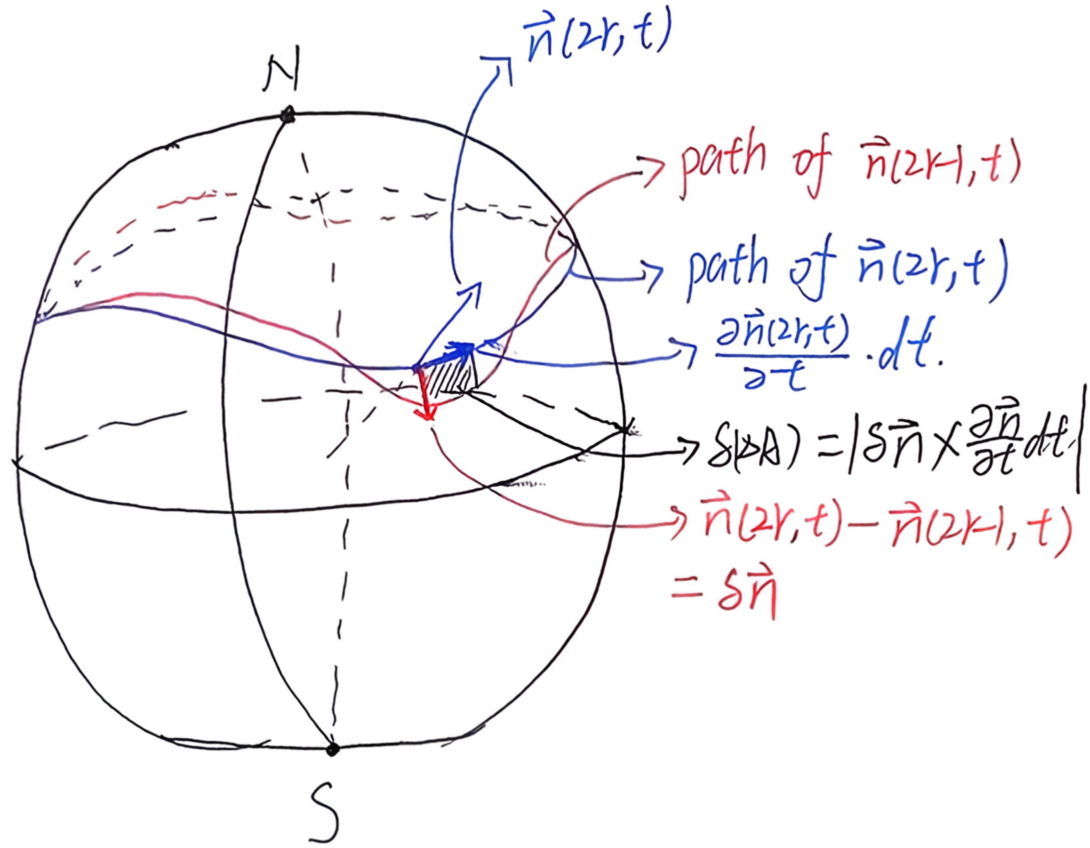
$$S\sum_{i}^{N}(-1)^{i}S_{WZ}[\vec{n}(i)]\approx S\sum_{i=1}^{\frac{N}{2}}(\vec{n}(2r)-\vec{n}(2r-1))\left[\vec{n}(2r)\times\frac{\partial\vec{n}(2r)}{\partial t}\right]$$

$$\approx S\sum_{i=1}^{\frac{N}{2}}a_0\int_0^{\beta}\left(\frac{\partial\vec{m}(2r)}{\partial x}+2\vec{l}(2r)\right)\left[\vec{m}(2r)\times\frac{\partial\vec{m}(2r)}{\partial t}\right]dt$$

Using continuum limit $(a_0\to 0)$:

$$\lim_{a_0\to 0}S_g\approx\frac{S}{2}\sum_{i=1}^{N}(\ldots)$$ (~~pre~~ periodic boundary conditions)

$$=\frac{S}{2}\int dx\,dt\cdot(\partial_x\vec{m}+2\vec{l})(\vec{m}\times\partial_t\vec{m})$$



~~So the~~ Thus, it can be simplified to:

$$S_g \simeq \frac{S}{2}\int dx dt \cdot \vec{m}(\partial_t \vec{m} \times \partial_x \vec{m}) + S\int dx dt\, \vec{l}(\vec{m} \times \partial_t \vec{m})$$

This term comes from alternating sum of WZ terms:

Similarly, the continuum limit of the energy terms can also be found to be given by:

$$\lim_{a_0 \to \infty} \frac{|J|S^2}{2} \cdot \int dt \sum_{i=1}^{N} \vec{n}(i,t)\vec{n}(i+1,t)$$

$$\xrightarrow{\text{drop global phase}} \lim_{a_0 \to 0} \frac{-|J|S^2}{2} \int dt \sum_{i=1}^{N} [\vec{n}(i+1,t) - \vec{n}(i,t)]^2$$

$$\cancel{\simeq \frac{-a_0|J|S^2}{2} \cdot \int dx dt [\vec{m}}$$

$$\approx \frac{-|J|S^2}{2}\int dt \sum_i^N \left[\vec{m}(i+1) - \vec{m}(i) + a_0\vec{l}(i+1) + a_0\vec{l}(i)\right]^2$$

$$\approx \frac{-|J|S^2}{2}\int dt \sum_i^N a_0^2\left(\frac{\Delta\vec{m}}{a_0} + 2\vec{l}\right)^2$$

$$= \frac{-|J|S^2}{2}\int dt\, a_0 \int dx\left[(\partial_x\vec{m})^2 + \underbrace{4\frac{\Delta\vec{m}}{a_0}\cdot\vec{l}}_{=0} + 4\vec{l}^2\right]$$

$$= \frac{-|J|S^2 a_0}{2}\int dt\,dx\left[(\partial_x\vec{m})^2 + 4\vec{l}^2\right]$$

So the total Lagrangian density involving both the orderparameter $\vec{m}$ and $\vec{l}$:

$$\mathcal{L}_m(\vec{m},\vec{l}) = -2a_0|J|S^2\vec{l}^2 + S\vec{l}(\vec{m}\times\partial_t\vec{m}) - \frac{a_0|J|S^2}{2}(\partial_x\vec{m})^2 + \frac{S}{2}\vec{m}\cdot(\partial_t\vec{m}\times\partial_x\vec{m})$$

where $\vec{m}$ is the order parameter field and $\vec{l}$ roughly represents the average spin ~~file~~ field.

The Lagrangian ~~have~~ has a quadratic term for $\vec{l}$:

$$\mathcal{L}_{\vec{l}} = -2a_0|J|S^2\vec{l}^2 + S(\vec{m}\times\partial_t\vec{m})\vec{l}.$$

To integrate ~~the~~ out $\vec{l}$, we minimize the action ~~we~~ with respect to $\vec{l}$

$$\frac{\partial \mathcal{L}_l}{\partial l} = -4a_0|J|S^2\vec{l} + S(\vec{m}\times\partial_t\vec{m}) = 0$$

so we solve for l:

$$\vec{l} = \frac{1}{4a_0|J|S}\cdot(\vec{m}\times\partial_t\vec{m})$$

Now we subsitute the expession for $\vec{l}$ found above back into $\mathcal{L}_l$:

$$\mathcal{L}_l^{eff} = \frac{1}{8a_0|J|}(\vec{m}\times\partial_t\vec{m})^2 \quad (\vec{m}\perp\partial_t\vec{m} \text{ because } \vec{m} \text{ is on the sphere})$$

$$= \frac{1}{8a_0|J|}(\partial_t\vec{m})^2$$

~~Final~~ we can combine the ~~all~~ new term with original term that ~~dio~~ do not depend on $\vec{l}$:

$$\mathcal{L}(\vec{m}) = \frac{1}{8a_0J}(\partial_t\vec{m})^2 - \frac{a_0JS^2}{2}(\partial_x\vec{m})^2 + \frac{S}{2}\vec{m}(\partial_t\vec{m}\times\partial_x\vec{m})$$

$$= \frac{1}{2g}\left[\frac{1}{v_s}(\partial_t\vec{m})^2 - v_s(\partial_x\vec{m})^2\right] + \frac{\theta}{8\pi}\varepsilon_{\mu\nu}\vec{m}(\partial_\mu\vec{m}\times\partial_\nu\vec{m})$$

where g and v_s are respectively the coupling constant and spin-wave velocity:

$$\begin{cases} g = \frac{2}{S} \\ v_s = 2a_0JS. \end{cases}$$