1. Feynman Path Integral.

The version of quantum mechanics.

- Schrödinger's wavefuction. (operator form)

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

- Feynman's Path Integral (Common number form)

$$iG \text{ (Green's function)} \propto \int D(x,t)\, e^{i\int \mathcal{L}\,dt}.$$

There are many ~~properties~~ advantages of Feynman Path Integral:

1. Make the double-slit experiment more understandable.
2. The classical limit "$\hbar \to 0$" is "tractable"

   quantum $\xrightarrow{\hbar\to 0}$ classical.

3. Provide a semi-classical picture for quantum mechanics
4. "Quantum fluctuations" are more "understandable"
5. A natural route to low energy effective theory of quantum many-body systems
6. A natural language for describing topological properties of quantum many-body systems

But the practical calculation in the path-integral representation of simple quantum ~~x~~mechanical problem many be notoriously difficult and lengthy

1.1. Propagators.

Consider a quantum partical confined in a one-dimensional space:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

Canonical pair: $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$

the Schrödinger's equations:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} \psi(t)$$

This first-order natura allows us to define a time evolution operator $\hat{U}(t; t_0)$ which propagates the state vector from an ~~initial~~ initial tim t_0 to a final time t:

$$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle$$

Assuming the Hamiltonian $\hat{H}$ is not explicitly depend on time, the formal solution of $\hat{U}$ is:

$$\hat{U} = e^{-\frac{i\hat{H}}{\hbar}(t-t_0)}$$

A crucial property of $\hat{U}$ is the "chain-like" rule, or composition property. For any intermediate time t', such that $t > t' > t_0$:

$$\hat{U}(t,t_0) = \hat{U}(t,t')\,\hat{U}(t',t_0)$$

This property is the key to the entire path integral derivation

And $\hat{U}(t,t_0)$ is unitary:

$$\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \hat{\mathbb{I}}.$$

where $\hat{\mathbb{I}}$ is the identity operator.

In the position respresentation, we can obtain matrix elements

$$\begin{aligned} U(x,t;x_0,t_0) &= \langle x|\hat{U}(t,t_0)|x_0\rangle \\ &= \langle x|e^{-\frac{i\hat{H}}{\hbar}(t-t_0)}|x_0\rangle \end{aligned}$$

we can define a propagators (Green's function) ~~f using this ma~~ of the quantum system by using the matrix elements:

$$iG(x,t;x_0,t_0) = U(x,t;x_0,t_0)$$

Using the matrix elements, $\psi(x,t)$ can be reformulated as:

$$\begin{aligned} \psi(x,t) &= \langle x|\hat{U}(t,t_0)|\psi(t_0)\rangle \\ &= \int dx_0\, \langle x|\hat{U}(t,t_0)|x_0\rangle\langle x_0|\psi(t_0)\rangle \\ &= \int dx_0\, \hat{U}(x,t;x_0,t_0)\,\psi(x_0,t_0) \end{aligned}$$

Also, the propagator. also satisfies the Schödinger's equation:

$$i\hbar \frac{\partial}{\partial t} G(x,t;x_0,t_0) = \hat{H} G(x,t;x_0,t_0)$$

initial condition is:

$$G(x,t_0;x_0,t_0) = -i \langle x|\hat{U}(t_0,t_0)|x_0\rangle = -i\,\delta(x-x_0)$$

$\delta(x-x_0)$ is Dirac function

For free partical; $\hat{H} = \frac{1}{2m}\hat{p}^2$, In position representation:

$$\hat{p} \to -i\hbar \frac{\partial}{\partial x}$$

$$\therefore i\hbar \frac{\partial G}{\partial t}(x,t;x_0,t_0) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} G(x,t;x_0,t_0)$$

Solve:

Use Fouries Transform: (we use $G(x,t)$ to instead ~~instede~~ $G(x,t;x_0,t_0)$)

$$G(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{G}(k,t) e^{ikx}$$

$$\tilde{G}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, G(x,t) e^{-ikx}$$

$$\therefore \begin{cases} \mathcal{F}\left(i\hbar \frac{\partial G(x,t)}{\partial t}\right) = i\hbar \frac{\partial}{\partial t} \tilde{G}(k,t) \\ \mathcal{F}\left[-\frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial x^2}\right] = -\frac{\hbar^2}{2m}\left[-k^2 \tilde{G}(k,t)\right] = \frac{\hbar^2 k^2}{2m}\tilde{G}(k,t) \end{cases}$$

$$\therefore i\hbar \frac{\partial}{\partial t}\tilde{G}(k,t) = \frac{\hbar^2 k^2}{2m}\tilde{G}(k,t)$$

$$\therefore \frac{d\tilde{G}}{\tilde{G}} = -i\frac{\hbar k^2}{2m}\, dt.$$

$$\therefore \ln \tilde{G} = -i\frac{\hbar k^2}{2m}t + C(k)$$

$$\therefore \tilde{G}(k,t) = A(k)\, e^{-\frac{i\hbar k^2}{2m}\cdot t}, \quad A(k) = e^{C(k)}$$

$$\tilde{G}(k,k_0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, G(x,x_0)\, e^{-ikx}$$

$$= \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, \delta(x-x_0)\, e^{-ikx}$$

$$\therefore \tilde{G}(k,k_0) = \frac{-i}{\sqrt{2\pi}} e^{-ikx_0}.$$

$$\tilde{G}(k,t_0) = \tilde{G}(k,k_0) = \frac{-i}{\sqrt{2\pi}} e^{-ikx_0}.$$

$$\therefore A(k) = \frac{-i}{\sqrt{2\pi}} e^{-ikx_0} \cdot e^{\frac{i\hbar k^2}{2m}t_0}$$

$$\cancel{= \frac{-i}{\sqrt{2\pi}}}$$

$$\therefore \tilde{G}(k,t) = \frac{-i}{\sqrt{2\pi}} e^{-ikx_0}\, e^{-\frac{i\hbar k^2}{2m}(t-t_0)}$$

$$\therefore G(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \cdot \tilde{G}(k,t)\, e^{ikx}$$

$$= \frac{-i}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x_0)} \cdot e^{-\frac{i\hbar(t-t_0)}{2m}k^2}$$

This is a standard Gaussian integral of the form:

$$\int_{-\infty}^{\infty} dk\, e^{-ak^2+bk} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}.$$

1.3

Let's identify our coefficients:

$$\begin{cases} a = i\dfrac{\hbar(t-t_0)}{2m} \\ b = i(x-x_0) \end{cases}$$

$$\therefore iG(x,t) = \left[\frac{m}{2\pi\hbar i(t-t_0)}\right]^{\frac{1}{2}} \cdot e^{\frac{i}{\hbar}\cdot\frac{m(x-x_0)^2}{2(t-t_0)}}$$

Also, we can solve this PDE via definition:

$$\begin{aligned} iG &= \langle x| e^{-\frac{i}{\hbar}(t-t_0)\hat{H}} |x_0\rangle \\ &= \int \langle x| e^{-\frac{i}{\hbar}(t-t_0)\hat{H}} |p\rangle\langle p|x_0\rangle \, dp \\ &= \int dp \; e^{\frac{-i(t-t_0)p^2}{2m\hbar}} \langle x|p\rangle\langle p|x_0\rangle \\ &= \frac{1}{2\pi\hbar}\int dp \; e^{\frac{-i(t-t_0)p^2}{2m\hbar} + \frac{i(x-x_0)}{\hbar}p} \end{aligned}$$

we use $p = \hbar k$. and can get the same equation as the Fourier Transform Method.

1.2 Path-Integral

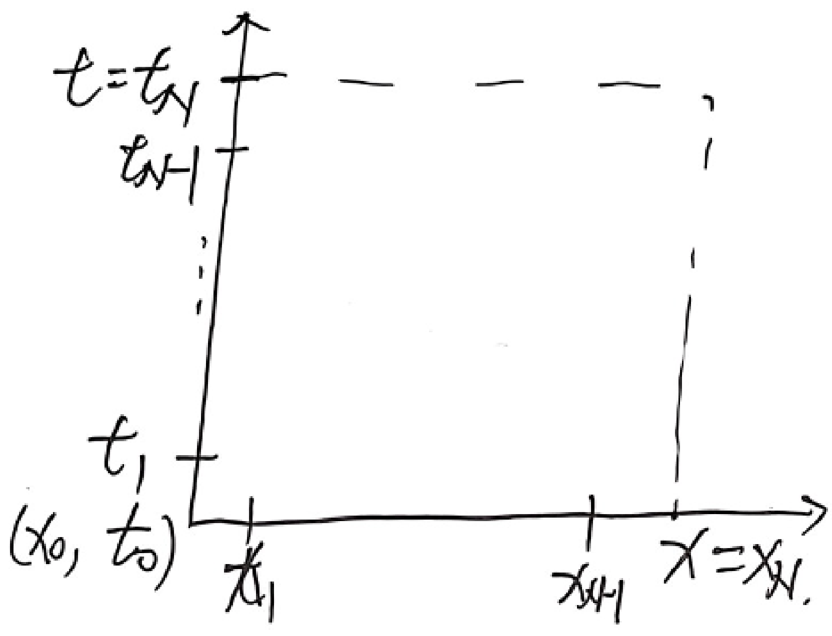
$$iG(x,t;x_0,t_0) = \cancel{\langle x | iG} \ \langle x | \hat{U}(t,t_0) | x_0 \rangle$$
$$= \langle x | \hat{U}(t,t_1)\hat{U}(t_1,t_0) | x_0 \rangle$$
$$= \int dx_1 \ \langle x | \hat{U}(t,t_1) | x_1 \rangle \langle x_1 | \hat{U}(t_1,t_0) | x_0 \rangle$$
$$= \int dx_1 \ iG(x,t;x_1,t_1) \cdot iG(x_1,t_1;x_0,t_0)$$

where $t > t_1 > t_0$, and t_1 is an arbitrarily selected inter-midiate time.

This intergral over ~~x_0~~ x_1 means "superposition" of all possible "path" that connect x and x_0

Next, we try to "smooth" the path along time directly. We can insert more time slices between x and x_0. If we insert infinite time slices, the path become smooth.

Firstly, let's discretize time domain $[t_0, t]$ into N pieces of equal length $\Delta t = \frac{t-t_0}{N}$.



$\therefore iG(x,t;x_0,t_0)$

$$= \langle x | \hat{U}(t,t_{N-1})\hat{U}(t_{N-1},t_{N-2}) \cdots \hat{U}(t_1,t_0) | x_0 \rangle$$
$$= \int dx_{N-1} \cdots dx_1 \prod_{l=1}^{N} iG(x_l,t_l;x_{l-1},t_{l-1})$$

let $\mathcal{D}x = \prod_{l=1}^{N-1} dx_l$.

Consider $N \to \infty$, so $\Delta t = \frac{t-t_0}{N} \to 0$. $t_l - t_{l-1} = \Delta t$.

$$iG(x_l, t_l; x_{l-1}, t_{l-1}) = \langle x_l | e^{-\frac{i}{\hbar}\hat{H}\Delta t} | x_{l-1} \rangle$$

$$e^{-\frac{i}{\hbar}\hat{H}\Delta t} \approx \hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}\Delta t = \hat{\mathbb{1}} - \frac{i}{\hbar}\left[\frac{\hat{P}^2}{2m} + V(\hat{x})\right]$$

$$\therefore \cancel{iG(x_l, t_l; x_{l-1}, t_{l-1}) = \langle x_l | x_{l-1} \rangle - \frac{i}{\hbar}\Delta t \langle x_l | \hat{H} | x_{l-1} \rangle}$$

$\therefore iG(x_l, t_l; x_{l-1}, t_{l-1})$

$$= \int dP_l \cdot \langle x_l | P_l \rangle \langle P_l | \hat{\mathbb{1}} - \frac{i}{\hbar}\left[\frac{\hat{P}^2}{2m} + V(\hat{x})\right]\Delta t | x_{l-1} \rangle$$

$$= \int dP_l \langle x_l | P_l \rangle \left\{ \langle P_l | x_{l-1} \rangle - \frac{i}{2m\hbar} P_l^2 \langle P_l | x_{l-1} \rangle - \frac{i}{\hbar} V(x_{l-1}) \langle P_l | x_{l-1} \rangle \Delta t \right\}$$

$$= \int dP_l \langle x_l | P_l \rangle \langle P_l | x_{l-1} \rangle \left\{ 1 - \left[\frac{i}{\hbar}\left(\frac{P_l^2}{2m} + V(x_{l-1})\right)\right]\Delta t \right\}$$

$\because V(x_{l-1}) \approx V(x_l)$

$$\therefore \left[1 - \frac{i}{\hbar}\left(\frac{P_l^2}{2m} + V(x_{l-1})\right)\right]\Delta t \approx \left(1 - \frac{i}{\hbar} H_l\right)\Delta t \approx e^{-\frac{i}{\hbar} H_l \Delta t}.$$

$\therefore iG(x_l, t_l; x_{l-1}, t_{l-1})$

$$= \int dP_l \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar} P_l (x_l - x_{l-1}) - \frac{i}{\hbar} H_l \Delta t}.$$

$$e^{\frac{i}{\hbar}\left[P_l (x_l - x_{l-1}) - H_l \Delta t\right]} = e^{\frac{i}{\hbar}\left[P_l \frac{(x_l - x_{l-1})}{\Delta t} - H_l\right]\Delta t}$$

where, H_l is the classical Hamiltonian as a function of P_l and x_l.

when $\Delta t \to 0$,

$$\frac{x_l - x_{l-1}}{\Delta t} = \dot{x}_l$$

$$\therefore P_l \dot{x}_l - H_l = \mathcal{L}_l.$$

where, $\mathcal{L}_l$ is the classical Lagrangion.

$$\therefore iG(x_l, t_l; x_{l-1}, t_{l-1})$$

$$= \int dP_l \cdot \frac{1}{2\pi\hbar} \cdot e^{\frac{i}{\hbar}\mathcal{L}_l \Delta t}.$$

$$\therefore \prod_{l=1}^{N} iG(x_l; t_l; x_{l-1}, t_{l-1})$$

$$= \int \frac{dP_N}{2\pi\hbar} \cdots \frac{dP_1}{2\pi\hbar} \cdot e^{\frac{i}{\hbar}\sum_{l=1}^{N}\mathcal{L}_l \cdot \Delta t}$$

let $D_p = \prod_{l=1}^{N} \frac{dP_l}{2\pi\hbar}$, when $\Delta t \to 0$, $\sum_{l=1}^{N} \mathcal{L}_l \Delta t = \int_{t_0}^{t} d\tau \mathcal{L}[p(\tau), x(\tau)]$

$$\therefore iG(x, t; x_0, t_0) = \int D_x D_p \cdot e^{\frac{i}{\hbar} \int_{t_0}^{t} d\tau \cdot \mathcal{L}[p(\tau), x(\tau)]}$$

where, the pair of $p(\tau)$ and $\dot{x}(\tau)$ charaterizes a path in the ~~px~~ phase space.