

If we look at: $\vec{n} \times (\vec{n} \times \nabla^2 \vec{n})$:

$$\begin{aligned}\vec{n} \times (\vec{n} \times \nabla^2 \vec{n}) &= \vec{n} \cdot (\vec{n} \cdot \nabla^2 \vec{n}) - \nabla^2 \vec{n} (\vec{n} \cdot \vec{n}) \\ &= \vec{n} (\vec{n} \cdot \nabla^2 \vec{n}) - \nabla^2 \vec{n}.\end{aligned}$$

So:

$$\frac{S}{a_0^d} (\vec{n} \times \partial_t \vec{n}) = \frac{|J| S^2}{a_0^{d+2}} [\vec{n} \times (\vec{n} \times \nabla^2 \vec{n})]$$

Simplified to:

$$\cancel{\vec{n} \times \frac{\partial \vec{n}}{\partial t}} = \vec{n} \times \left[\cancel{\frac{|J| S^2}{a_0^d}} |J| S^2 a_0^2 (\nabla^2 \vec{n}) \right]$$

We can get the Landau-Lifshitz equation:

$$\cancel{\frac{\partial \vec{n}}{\partial t}} = |J| S^2 a_0^2 (\vec{n} \times \nabla^2 \vec{n}).$$

From this equation, we can know that the Landau-Lifshitz equations can be solved in the linear regime.

From this equation, we can know that the spins move in a precessional fashion with an angular velocity $\vec{\omega}$ given by:

$$\vec{\omega} = -|J| S a_0^2 \nabla^2 \vec{n}.$$

The Landau-Lifshitz equations can be solved in the linear regime. Let's parametrize \vec{n} by the components:

$$\vec{n} = \begin{pmatrix} \sigma \\ \pi_1 \\ \pi_2 \end{pmatrix} \text{ and } \vec{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$$

where σ and $\pi_i (i=1, 2)$ satisfy the constraint:

$$\sigma^2 + \pi^2 = 1$$

If the system is ordered, $\sigma \approx 1$, $|\vec{\pi}|$ is very small, which means that $\vec{\pi}$ is a small fluctuation around the direction of σ .

In this situation, we can treat σ as a constat:

$$\begin{cases} \sigma \approx 1 \\ \nabla^2 \sigma \approx 0 \end{cases}$$

Therefore, we can linearize the Landau-Lifshitz equation:

$$\vec{n} \times \nabla^2 \vec{n} = \begin{pmatrix} \pi_1 \nabla^2 \pi_2 - \pi_2 \nabla^2 \pi_1 \\ -\nabla^2 \pi_2 \\ \nabla^2 \pi_1 \end{pmatrix}$$

and we remain the one order term:

$$\vec{n} \times \nabla^2 \vec{n} = \begin{pmatrix} 0 \\ -\nabla^2 \pi_2 \\ \nabla^2 \pi_1 \end{pmatrix}$$

So the linearizing Landau-Lifshitz equation:

$$\begin{cases} \partial_t \pi_1 = -|J| S a_s^2 \nabla^2 \pi_2 \\ \partial_t \pi_2 = +|J| S a_s^2 \nabla^2 \pi_1 \end{cases}$$

We look for plane wave solutions (spin wave) of the form:

$$\pi_j(\vec{x}, t) = \tilde{\pi}_j e^{i(\vec{p} \cdot \vec{x} - \omega t)}$$

taking derivatives this ansatz converts differential operators into algebraic variables:

$$\begin{cases} \partial_t \rightarrow -i\omega \\ \partial^2 \rightarrow -|\vec{p}|^2 \end{cases}$$

Substituting these into the linearized equations:

$$\begin{cases} -i\omega \tilde{\pi}_1 = (|J|S_{\alpha_0}^2 |\vec{p}|^2) \tilde{\pi}_2 \\ i\omega \tilde{\pi}_2 = -(|J|S_{\alpha_0}^2 |\vec{p}|^2) \tilde{\pi}_1 \end{cases}$$

let $A = |J|S_{\alpha_0}^2 P^2$, the system become:

$$\begin{cases} -i\omega \tilde{\pi}_1 - A \tilde{\pi}_2 = 0 \\ A \tilde{\pi}_1 - i\omega \tilde{\pi}_2 = 0 \end{cases}$$

for a non-trivial solution ($\tilde{\pi}_1, \tilde{\pi}_2 \neq 0$), the determinant of the coefficient matrix must be ~~not~~ zero:

$$\begin{vmatrix} -i\omega & -A \\ A & -i\omega \end{vmatrix} = 0$$

we can get:

$$\omega = \pm A.$$

So we obtain the dispersion relation for ferromagnetic spin waves:

$$|\omega| \approx J |S a^2 |\vec{p}|^2$$

We find that the frequency of the low-energy excitations of a quantum ferromagnetic scales as the square of the momentum.

3.3 Quantum Antiferromagnet of one-dimension.

Consider a spin chain with an even number of sites N ; we can write down the action of it ($J=|J|$):

$$S_m[\vec{n}] = S \sum_i^N S_{\text{xx}}[\vec{n}] - \int dt \cdot \sum_i^N JS^2 \vec{n}(i,t) \cdot \vec{n}(i+1,t)$$

where we have assumed periodic boundary conditions:

Since we expect to be close to a Néel state, we will stagger the configuration:

$$\vec{n}(j) \rightarrow (-1)^j \vec{n}(j)$$

On a bipartite lattice, the substitution of $|{\text{ref}}\}$ into $|{\text{ref}}\}$ will change the sign of the exchange term of the action to a ferromagnetic:

$$S_m[\vec{n}] = S \sum_i^N (-1)^i S_{WZ}[\vec{n}(i)] - \frac{|J|S^2}{2} \int dt \cdot \sum_{i=1}^N (-1)^i \vec{n}(i, t) \cdot (-1)^{i+1} \vec{n}(i+1, t)$$

so the ~~anti~~-energy term become to:

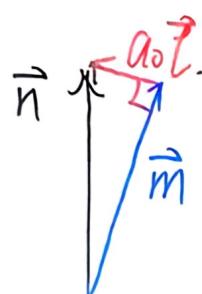
$$S_e = \frac{|J|S^2}{2} \cdot \int dt \sum_{i=1}^N \vec{n}(i, t) \vec{n}(i+1, t)$$

which is similar to ferromagnetic.

But the Wess-Zumino terms are odd ~~term~~ under the replacement of $|{\text{ref}}\}$ and thus become staggered. Thus, it is the Wess-Zumino term, a purely quantum-mechanical effect, which will distinguish ferromagnets from antiferromagnets.

As ferromagnetic, we can split \vec{n} into a slowly varying part \vec{m} and a small rapidly varying part $a_0 \vec{t}(i)$, so:

$$\vec{n}(i) = \vec{m}(i) + (-1)^i a_0 \vec{t}(i)$$



The constraint $|\vec{m}|^2 = 1$ and requirement that \vec{m} & $\vec{\ell}$ should obey the same constraint, and demand that \vec{m} and $\vec{\ell}$ be orthogonal vector:

$$\begin{cases} |\vec{m}|^2 = 1 \\ \vec{m} \cdot \vec{\ell} = 0 \end{cases}$$

The Wess-Zumino terms are rewritten as:

$$S \sum_{j=1}^N (-1)^j S_{WZ}[\vec{n}(j)] = S \sum_{r=1}^{\frac{N}{2}} \{ S_{WZ}[\vec{n}(2r)] - S_{WZ}[\vec{n}(2r-1)] \}$$

by making use of the approximation:

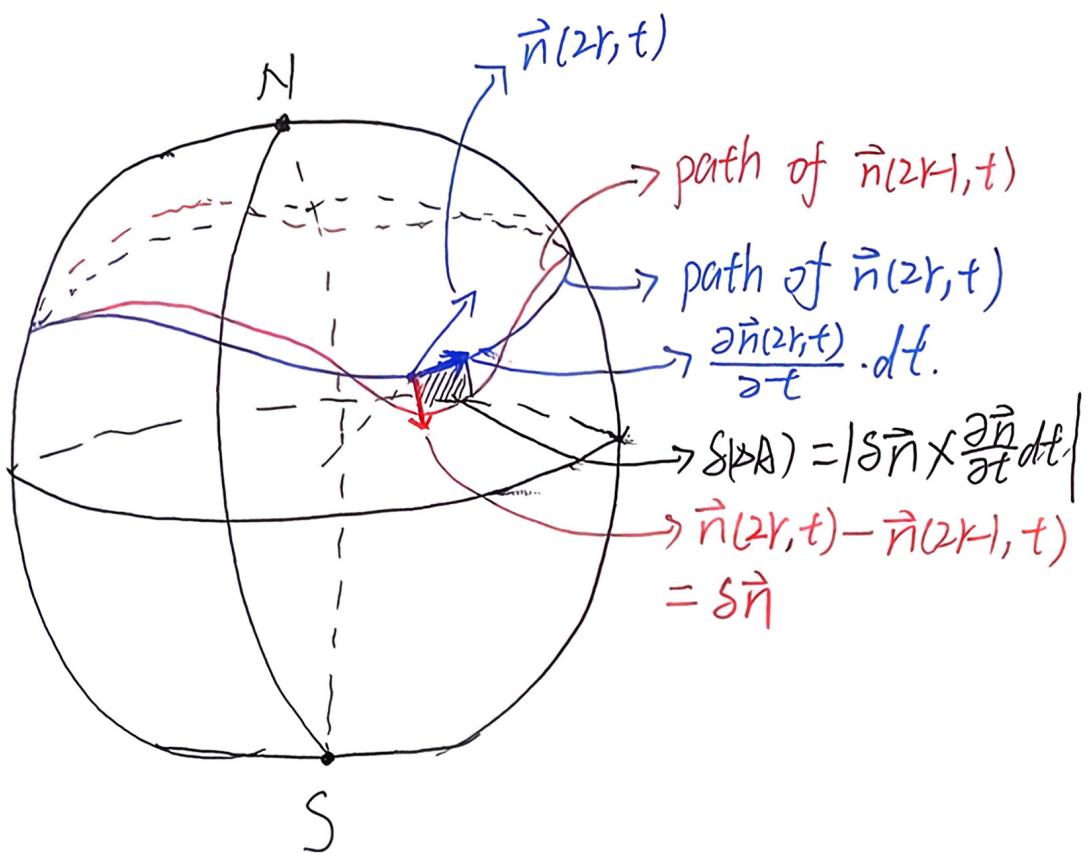
$$\begin{aligned} \vec{n}(2r) - \vec{n}(2r-1) &= \vec{m}(2r) - \vec{m}(2r-1) + a_0 [\vec{\ell}(2r) + \vec{\ell}(2r-1)] \\ &\approx a_0 \left[\frac{\partial \vec{m}(2r)}{\partial x} + 2\vec{\ell}(2r) \right] + \mathcal{O}(a_0^2) \end{aligned}$$

So becomes [use the similar method of area variation,

$$\begin{aligned} S \sum_{i=1}^N (-1)^i S_{WZ}[\vec{n}(i)] &\approx S \sum_{i=1}^{\frac{N}{2}} (\vec{n}(2i) - \vec{n}(2i-1)) \left[\vec{n}(2i) \times \frac{\partial \vec{n}(2i)}{\partial t} \right] \\ &\approx S \sum_{i=1}^{\frac{N}{2}} a_0 \int_0^\beta \left[\frac{\partial \vec{m}(2i)}{\partial x} + 2\vec{\ell}(2i) \right] \left[\vec{m}(2i) \times \frac{\partial \vec{m}(2i)}{\partial t} \right] dt \end{aligned}$$

Using continuum limit ($a_0 \rightarrow 0$):

$$\begin{aligned} \lim_{a_0 \rightarrow 0} Sg &\approx \frac{S}{2} \sum_{i=1}^N (\dots) \quad (\text{periodic boundary conditions}) \\ &= \frac{S}{2} \int dx dt \cdot (\partial x \vec{m} + 2\vec{\ell}) (\vec{m} \times \partial t \vec{m}) \end{aligned}$$



~~So the~~ Thus, it can be simplified to:

$$Sg \approx \frac{S}{2} \int d\vec{m} dt \cdot \vec{m} (\partial \vec{m} \times \partial \vec{m}) + S \int d\vec{x} dt \vec{B} (\vec{m} \times \partial \vec{m})$$

This term comes from alternating sum of WZ terms.

Similarly, the continuum limit of the energy terms can also be found to be given by:

$$\lim_{n \rightarrow \infty} \frac{|J|S^2}{2} \cdot \int dt \sum_{i=1}^N \vec{n}(i, t) \vec{n}(i+1, t)$$

~~drop global phase~~

$$\lim_{n \rightarrow \infty} \frac{|J|S^3}{2} \int dt \sum_{i=1}^N [\vec{n}(i, t) - \vec{n}(i+1, t)]^2$$

~~$$\frac{-\alpha_0 |J| S^2}{2} \int dx dt \vec{B} \vec{m}$$~~

$$\begin{aligned}
&\approx -\frac{1J|S^2}{2} \int dt \sum_i^N \left[\vec{m}(i+1) - \vec{m}(i) + D\omega \vec{\ell}(i+1) + a_0 \vec{\ell}(i) \right]^2 \\
&\approx -\frac{1J|S^2}{2} \int dt \sum_i^N a_0^2 \left(\frac{\Delta \vec{m}}{a_0} + 2 \vec{\ell} \right)^2 \\
&= -\frac{1J\beta^2}{2} \int dt a_0 \int dx \left[(\partial_x \vec{m})^2 + \underbrace{4 \frac{\Delta \vec{m}}{a_0} \cdot \vec{\ell}}_{=0} + 4 \vec{\ell}^2 \right] \\
&= -\frac{1J|S^2 a_0}{2} \cdot \int dt dx \left[(\partial_x \vec{m})^2 + 4 \vec{\ell}^2 \right]
\end{aligned}$$

So the total Lagrangian density involving both the orderparameter \vec{m} and $\vec{\ell}$:

$$\begin{aligned}
\mathcal{L}_M(\vec{m}, \vec{\ell}) = &-2a_0 J|S^2 \vec{\ell}^2 + S \vec{\ell} (\vec{m} \times \partial_t \vec{m}) \\
&- \frac{a_0 J \beta^2}{2} (\partial_x \vec{m})^2 + \frac{S}{2} \vec{m} \cdot (\partial_t \vec{m} \times \partial_x \vec{m})
\end{aligned}$$

where \vec{m} is the order parameter field and $\vec{\ell}$ roughly represents the average spin field.

The Lagrangian have has a quadratic term for $\vec{\ell}$:

$$\mathcal{L}_{\vec{\ell}} = -2a_0 J|S^2 \vec{\ell}^2 + S (\vec{m} \times \partial_t \vec{m}) \vec{\ell}.$$

To integrate the out $\vec{\ell}$, we minimize the action we with respect to $\vec{\ell}$

$$\frac{\partial \mathcal{L}}{\partial t} = -4\alpha_0 J S^2 \vec{\ell} + S(\vec{m} \times \partial_t \vec{m}) = 0$$

so we solve for $\vec{\ell}$:

$$\vec{\ell} = \frac{1}{4\alpha_0 J S} \cdot (\vec{m} \times \partial_t \vec{m})$$

Now we substitute the expression for $\vec{\ell}$ found above back into \mathcal{L}_t :

$$\begin{aligned}\mathcal{L}_t^{\text{eff}} &= \frac{1}{8\alpha_0 J} (\vec{m} \times \partial_t \vec{m})^2 \quad (\vec{m} \perp \partial_t \vec{m} \text{ because } \vec{m} \text{ is on the sphere}) \\ &= \frac{1}{8\alpha_0 J} (\partial_t \vec{m})^2\end{aligned}$$

~~Final~~ We can combine the new term with original term that do not depend on $\vec{\ell}$. ~~The result is the Lagrangian~~

$$\begin{aligned}\mathcal{L}(\vec{m}) &= \frac{1}{2\alpha_0 J} (\partial_t \vec{m})^2 - \frac{\alpha_0 JS^2}{2} (\partial_x \vec{m})^2 + \frac{S}{2} \vec{m} (\partial_t \vec{m} \times \partial_x \vec{m}) \\ &= \frac{1}{2g} \left[\frac{1}{V_3} (\partial_t \vec{m})^2 - V_S (\partial_x \vec{m})^2 \right] + \underbrace{\frac{S}{8\pi} \epsilon_{\mu\nu\lambda} \vec{m} (\partial_\mu \vec{m} \times \partial_\nu \vec{m})}_{\text{Einstein summation}}\end{aligned}$$

where g and V_S are respectively the coupling constant and spin-wave velocity:

$$\begin{cases} g = \frac{2}{S} \\ V_S = 2\alpha_0 JS \end{cases}$$

The last term in ref{} has ~~si~~ topological significance. We have chosen the normalization so that the coupling constant θ is given by:

$$\theta = 2\pi S$$

The tensor $\epsilon_{\mu\nu}$ is the usual Levi-Civita antisymmetric tensor in two dimensions (~~$\mu, \nu = t, x, t \rightarrow 0, x \rightarrow 1$~~).

$$\epsilon_{\mu\nu} = \begin{cases} 1 & (\mu, \nu) = (0, 1) \\ -1 & (\mu, \nu) = (1, 0) \\ 0 & \text{otherwise.} \end{cases}$$

The result is the Lagrangian density of non-linear sigma model.

3.4 The role of topological term.

Go back to Euclidean spacetime ($iS_m \rightarrow -S_E$):

$$iS_m \rightarrow S_E.$$

$$\begin{aligned} iS_m &= i \int dt dx \mathcal{L}_M [\vec{m}(t, x)] \\ &= - \int dT dx \mathcal{L}_E [\vec{m}(T, x)] \end{aligned}$$

So we can get the Lagrangian density \mathcal{L}_E of Euclidean sector ($x_2 = it = ix_0$)

$$\mathcal{L}_E = \frac{1}{2g} \left[\frac{\partial^2}{\partial T^2} (\partial_1 \vec{m})^2 + \frac{\partial^2}{\partial T^2} (\partial_2 \vec{m})^2 \right] + \frac{i\theta}{8\pi} \epsilon_{ij} \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m})$$

where $i, j = 1, 2$.

We now define the Pontryagin index or topological charge (or winding number) Q of the Euclidean-space spin configuration $\{\vec{m}(\vec{x})\}$ by the expression:

$$Q = \frac{1}{8\pi} \int d^2x \epsilon_{ij} \vec{m} (\partial_i \vec{m} \times \partial_j \vec{m}) \\ = \frac{1}{4\pi} \int d^2x \vec{m} (\partial_1 \vec{m} \times \partial_2 \vec{m})$$

We impose the boundary condition that the Euclidean action be finite. This is equivalent to the requirement that asymptotically \vec{m} becomes a constant vector \vec{m}_0 at spatial-time infinity:

$$\lim_{|\vec{x}| \rightarrow \infty} \vec{m}(\vec{x}) = \vec{m}_0$$

Thus, the 2D Euclidean space-time is isomorphic to a sphere S_2 since the fields are identified with \vec{m}_0 at the point of infinity. However, the order parameter manifold ("target space") is also isomorphic to a sphere S_2 , since the constraint $|\vec{m}|=1$ has to be satisfied everywhere.

The topological charge Q is in the sense that it counts how many times the spin configuration \vec{m} has wrapped around the sphere S_2 .

So the topological term of action is:

$$S_{\text{top}} = e^{i \cdot 2\pi \cdot S Q} = (-1)^{2SQ}.$$

Thus:

$$e^{i \cdot 2\pi \cdot SQ} = \begin{cases} 1 & \text{for } S=1, 2, \dots \text{ (integer spin)} \\ (-1)^Q & \text{for } S=\frac{1}{2}, \frac{3}{2}, \dots \text{ (half-odd-integer)} \end{cases}$$

We can see that:

- (1) S is an integer: the spin chain is described at low energies by the standard non-linear sigma model, without a topological term.
- (2) S is half integral S : each topological class contributes to the weight of the integral with a sign that's positive (negative) if the winding number (ℓ is even (odd))

The integer- and half-integer-spin chains fall in different universality classes. We will now see that this property implies a very important result, known as Haldane's conjecture, that states that the integer-spin chains are massive (have an energy gap), whereas the half-integer-spin chains are massless as in the spin one-half case.