Part I
Feynman Path Integral

Chapter 1
Feynman Path Integral

The version of quantum mechanics:

1. Schrödinger's wavefuction (operator form):

$$\hat{H}|\psi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle \tag{1.1}$$

2. Feynman's Path Integral (Common number form):

$$\mathrm{i}G(\text{Green's function}) \propto \int \mathcal{D}(x,t)\mathrm{e}^{\mathrm{i}\int \mathcal{L}\mathrm{d}t} \tag{1.2}$$

There are many advantages of Feynman Path Integral:

1. Make the double-slit experiment more understandable.
2. The classical limit "$\hbar \to 0$" is "tractable": quantum $\xrightarrow{\hbar\to 0}$ classical.
3. Provide a semi-classical picture. for. quantum mechanics.
4. "Quantum fluctuations" are more "understandable".
5. A natural route. to low energy effective theory of quantum many-body systems.
6. A natural language for describing topological properties of quantum many-body systems.

But the practical calculation in the path-integral representation of simple quantum mechanical problem many be notoriously difficult and lengthy.

1.1 Propagators

Consider a quantum partical confined in a one-dimensional space:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \tag{1.3}$$

and the canonical pair: $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = \mathrm{i}\hbar$

The Schrödinger's equation is:

$$\mathrm{i}\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle \tag{1.4}$$

This first-order nature allows us to define a time evolution operator $\hat{U}(t, t_0)$ which propagates the state vector from an initial time t_0 to a final time t:

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \tag{1.5}$$

Assuming the Hamiltonian H is not explicitly depend on time, the formal solution of $\hat{U}$ is:

$$\hat{U} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_0)} \tag{1.6}$$

A crucial property of $\hat{U}$ is the "chain-like" rule, or composition property. For any intermediate time t' such that $t > t' > t_0$:

$$\hat{U}(t, t_0) = \hat{U}(t, t')\hat{U}(t', t_0) \tag{1.7}$$

This property is the key to the entire path integral derivation. And $\hat{U}(t, t_0)$ is unitary:

$$\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \hat{\mathbb{I}} \tag{1.8}$$

where $\hat{\mathbb{I}}$ is the identity operator.

In the position representation, we can obtain matrix elements:

$$\begin{aligned} U(x, t; x_0, t_0) &= \langle x|\hat{U}(t, t_0)|x_0\rangle \\ &= \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}(t-t_0)}|x_0\rangle \end{aligned} \tag{1.9}$$

We can define a propagators (Green's function) of the quantum system by using the matrix elements:

$$\mathrm{i}G(x, t; x_0, t_0) = U(x, t; x_0, t_0) \tag{1.10}$$

Using the matrix elements, $\psi(x, t)$ can be reformulated as:

$$\begin{aligned} \psi(x, t) &= \langle x|\hat{U}(t, t_0)|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_0 \langle x|\hat{U}(t, t_0)|x_0\rangle\langle x_0|\psi(t_0)\rangle \\ &= \int \mathrm{d}x_0 U(x, t; x_0, t_0)\psi(x_0, t_0) \end{aligned} \tag{1.11}$$

Also, the propagator also satisfies the Schrödinger's equation:

$$\boxed{\mathrm{i}\hbar\frac{\partial}{\partial t}G(x,t;x_0,t_0) = \hat{H}G(x,t;x_0,t_0)} \tag{1.12}$$

And the initial condition is:

$$G(x,t_0;x_0,t_0) = -\mathrm{i}\langle x|\hat{U}(t_0,t_0)|x_0\rangle = -\mathrm{i}\delta(x-x_0) \tag{1.13}$$

$\delta(x-x_0)$ is Dirac function.

Example 1.1.1. For free particle: $\hat{H} = \frac{1}{2m}\hat{p}^2$, in position representation:

$$\hat{p} \longrightarrow -\mathrm{i}\hbar\frac{\partial}{\partial x} \tag{1.14}$$

So the PDE is:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}G(x,t;x_0,t_0) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x,t;x_0,t_0) \tag{1.15}$$

Solve the PDE:
Use Fourier Transform: (we use $G(x,t)$ instead of $G(x,t;x_0,t_0)$). We solve the free-particle Green's function by transforming to momentum space:

$$\begin{cases} G(x,t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int \mathrm{d}k\cdot\tilde{G}(k,t)\mathrm{e}^{\mathrm{i}kx} \\ \tilde{G}(k,t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int \mathrm{d}x\ G(x,t)\mathrm{e}^{-\mathrm{i}kx} \end{cases} \tag{1.16}$$

With these conventions, spatial derivatives become algebraic in k-space while the time derivative remains unchanged:

$$\begin{cases} \mathcal{F}\left\{\mathrm{i}\hbar\dfrac{\partial G(x,t)}{\partial t}\right\} = \mathrm{i}\hbar\dfrac{\partial\tilde{G}(k,t)}{\partial t} \\ \mathcal{F}\left\{-\dfrac{\hbar^2}{2m}\dfrac{\partial^2 G}{\partial x^2}\right\} = -\dfrac{\hbar^2}{2m}[-k^2\tilde{G}(k,t)] = \dfrac{\hbar^2k^2}{2m}\tilde{G}(k,t) \end{cases} \tag{1.17}$$

Applying the transform to the PDE yields an ordinary differential equation in time for each k:

$$\begin{cases} \mathrm{i}\hbar\dfrac{\partial}{\partial t}\tilde{G}(k,t) = \dfrac{\hbar^2k^2}{2m}\tilde{G}(k,t) \\ \dfrac{\mathrm{d}\tilde{G}}{\tilde{G}} = -\mathrm{i}\dfrac{\hbar k^2}{2m}\mathrm{d}t \end{cases} \tag{1.18}$$

Integrating in time gives the logarithm of the solution up to a k-dependent constant:

$$\ln\tilde{G} = -\mathrm{i}\frac{\hbar k^2}{2m}t + C(k) \tag{1.19}$$

So we can get the solution:

$$\tilde{G}(k,t) = A(k)\mathrm{e}^{-\mathrm{i}\frac{\hbar k^2}{2m}t}, \quad A(k) = \mathrm{e}^{C(k)} \tag{1.20}$$

To determine $A(k)$, impose the initial condition at time t_0 in position space:

$$\begin{aligned}\tilde{G}(k,t_0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x G(x,t_0)\mathrm{e}^{-\mathrm{i}kx} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x \delta(x-x_0)\mathrm{e}^{-\mathrm{i}kx}\end{aligned} \tag{1.21}$$

Using the Fourier transform of the Dirac delta, we find:

$$\tilde{G}(k,t_0) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0} \tag{1.22}$$

Matching at t_0 fixes the k-space amplitude:

$$A(k) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0} \cdot \mathrm{e}^{\mathrm{i}\frac{\hbar k^2}{2m}t_0}. \tag{1.23}$$

Therefore, for general time t we have:

$$\tilde{G}(k,t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\mathrm{i}kx_0}\mathrm{e}^{-\mathrm{i}\frac{\hbar k^2}{2m}(t-t_0)} \tag{1.24}$$

Finally, inverse-transform back to position space to obtain the integral representation of the propagator:

$$\begin{aligned}G(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}k \cdot \tilde{G}(k,t)\mathrm{e}^{\mathrm{i}kx} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k(x-x_0)} \cdot \mathrm{e}^{-\mathrm{i}\frac{\hbar(t-t_0)}{2m}k^2}\end{aligned} \tag{1.25}$$

This is a standard Gaussian integral of the form:

$$\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-ak^2+bk} = \sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}}. \tag{1.26}$$

Let's identify the coefficients:

$$\begin{cases} a = \mathrm{i}\dfrac{\hbar(t-t_0)}{2m} \\ b = \mathrm{i}(x-x_0) \end{cases} \tag{1.27}$$

So we can get the solution:

$$\mathrm{i}G(x,t) = \left[\frac{m}{2\pi\hbar\mathrm{i}(t-t_0)}\right]^{\frac{1}{2}} \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\cdot\frac{m(x-x_0)^2}{2(t-t_0)}} \tag{1.28}$$

Also, we can solve this PDE via definition:

$$\begin{aligned}\mathrm{i}G &= \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}}|x_0\rangle \\ &= \int \langle x|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}}|p\rangle\langle p|x_0\rangle\, \mathrm{d}p \\ &= \int \mathrm{d}p\, \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)p^2}{2m\hbar}}\langle x|p\rangle\langle p|x_0\rangle \\ &= \frac{1}{2\pi\hbar}\int \mathrm{d}p\, \mathrm{e}^{-\frac{\mathrm{i}(t-t_0)p^2}{2m\hbar}+\mathrm{i}\frac{(x-x_0)}{\hbar}p}.\end{aligned} \tag{1.29}$$

we use $P = \hbar k$ and can get the same equation as the Fourier Transform Method.

1.2 Path-Integral

When $t > t_1 > t_0$, and t_1 is an arbitrarily selected intermediate time, we can write:

$$\begin{aligned} \mathrm{i}G(x,t;x_0,t_0) &= \langle x|\hat{U}(t,t_0)|x_0\rangle \\ &= \langle x|\hat{U}(t,t_1)\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_1 \langle x|\hat{U}(t,t_1)|x_1\rangle\langle x_1|\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_1\, \mathrm{i}G(x,t;x_1,t_1)\cdot \mathrm{i}G(x_1,t_1;x_0,t_0) \end{aligned} \tag{1.30}$$

This intergral over x_1 means "superposition" of all possible "path" that connect x and x_0. Next, we try to "smooth" the path along time directly. We can insert more time slices between x and x_0. If we insert infinite time slices, the path become smooth.

Firstly, let's discretize time. domain $[t_0, t]$ into N pieces of equal length $\Delta t = \frac{t-t_0}{N}$:

$$\begin{aligned} \mathrm{i}G(x,t;x_0,t_0) &= \langle x|\hat{U}(t,t_{N-1})\hat{U}(t_{N-1},t_{N-2})\cdots\hat{U}(t_1,t_0)|x_0\rangle \\ &= \int \mathrm{d}x_{N-1}\cdots\mathrm{d}x_1 \prod_{l=1}^{N} \mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) \end{aligned} \tag{1.31}$$

let $\mathcal{D}_x = \prod_{l=1}^{N-1}\mathrm{d}x_l$. Consider $N\to\infty$, so $\Delta t = \frac{t-t_0}{N} \to 0$, which means $t_l - t_{l-1} = \Delta t$.

$$\mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) = \langle x_l|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\Delta t}|x_{l-1}\rangle \tag{1.32}$$

Because Δt is small, we can approximate the exponential function by its Taylor series:

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}\Delta t} \approx \hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\hat{H}\Delta t = \hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\Delta t\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right] \tag{1.33}$$

Substitute (1.33) into (1.32):

$$\begin{aligned} \mathrm{i}G(x_l,t_l;x_{l-1},t_{l-1}) &= \int \mathrm{d}p_l \langle x_l|p_l\rangle\langle p_l|\hat{\mathbb{I}} - \frac{\mathrm{i}}{\hbar}\Delta t\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]|x_{l-1}\rangle \\ &= \int \mathrm{d}p_l \langle x_l|p_l\rangle\langle p_l|x_{l-1}\rangle\left[1 - \frac{\mathrm{i}}{\hbar}\left(\frac{p_l^2}{2m} + V(x_{l-1})\right)\Delta t\right] \end{aligned} \tag{1.34}$$

With the approximations $V(x_l) \approx V(x_{l-1})$:

$$\left[1 - \frac{\mathrm{i}}{\hbar}\left(\frac{p_l^2}{2m} + V(x_{l-1})\right)\Delta t\right] \approx \left(1 - \frac{\mathrm{i}}{\hbar}H_l\Delta t\right) \approx \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H_l\Delta t} \tag{1.35}$$

So $\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1})$ can be written as:

$$\begin{aligned} \mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1}) &= \int \mathrm{d}p_l \frac{1}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} p_l (x_l - x_{l-1})} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H_{cl} \Delta t} \\ &= \frac{1}{2\pi\hbar} \int \mathrm{d}p_l e^{\frac{\mathrm{i}}{\hbar}[p_l (x_l - x_{l-1}) - H_l \Delta t]} \\ &= \frac{1}{2\pi\hbar} \int \mathrm{d}p_l \mathrm{e}^{\frac{\mathrm{i}}{\hbar}[p_l (\frac{x_l - x_{l-1}}{\Delta t}) - H_l] \Delta t} \end{aligned} \tag{1.36}$$

where, H_l is the classical Hamiltonion as a function of p_l and x_l.

When $\Delta t \to 0$:

$$\frac{x_l - x_{l-1}}{\Delta t} = \dot{x}_l \tag{1.37}$$

So we can get:

$$p_l \dot{x}_l - H_l = \mathcal{L}_l. \tag{1.38}$$

where, $\mathcal{L}_l$ is the classical Lagrangian.

So $\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1})$ can be written as the form with Lagrangian:

$$\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1}) = \int \mathrm{d}p_l \cdot \frac{1}{2\pi\hbar} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \mathcal{L}_l \Delta t}. \tag{1.39}$$

Substitute $\mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1})$ into the path integral:

$$\prod_{l=1}^{N} \mathrm{i}G(x_l, t_l; x_{l-1}, t_{l-1}) = \int \frac{\mathrm{d}p_N}{2\pi\hbar} \cdots \frac{\mathrm{d}p_1}{2\pi\hbar} \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \sum_{l=1}^{N} \mathcal{L}_l \cdot \Delta t} \tag{1.40}$$

let $\mathcal{D}_p = \prod_{l=1}^{N} \frac{\mathrm{d}p_l}{2\pi\hbar}$, when $\Delta t \to 0$, which means:

$$\sum_{l=1}^{N} \mathcal{L}_l \Delta t = \int_{t_0}^{t} \mathrm{d}\tau \cdot \mathcal{L}[p(\tau), x(\tau)] \tag{1.41}$$

Finally, we can get the propagators by the path integral:

Theorem 1.2.1. *The propagators path integral:*

$$\mathrm{i}G(x, t; x_0, t_0) = \int \mathcal{D}_x \mathcal{D}_p \cdot \mathrm{e}^{\frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} \mathrm{d}\tau \cdot \mathcal{L}[p(\tau), x(\tau)]} \tag{1.42}$$

where, the pair of $p(t)$ and $\dot{x}(t)$ characterizes a path in the px phase space.

1.3 Gaussian Integration

If the functional integration over p is Gaussian, we can exactly integrate out p. For example, $H = \frac{p^2}{2m} + V$, so $\mathcal{L} = p\dot{x} - H = p\dot{x} - \frac{p^2}{2m} - V(x)$,we can get:

$$\mathrm{i}G = \int \mathcal{D}p\mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_t \left(p_t\dot{x}_t - \frac{p_t^2}{2m} - V(x_t)\right)\Delta t\right] \tag{1.43}$$

Let:

$$\boldsymbol{p} = \begin{pmatrix} p_l \\ \vdots \\ p_1 \end{pmatrix}, \quad \dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_l \\ \vdots \\ \dot{x}_1 \end{pmatrix} \tag{1.44}$$

So we can rewrite the integral as:

$$\mathrm{i}G = \int \mathcal{D}x \cdot \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}(-V(x_l))\Delta t\right] \cdot \int \mathcal{D}p \cdot \exp\left[\frac{\mathrm{i}}{2m\hbar}(-\boldsymbol{p}^T\boldsymbol{p} + 2m\boldsymbol{p}^T\dot{\boldsymbol{x}})\Delta t\right] \tag{1.45}$$

We have an useful formula for Gaussian integral (Proof in A):

$$\boxed{\int \prod_{n=1}^{N} \mathrm{d}x_n \exp\left[-\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{y}\right] = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \exp\left[\frac{1}{2}\boldsymbol{y}^T A^{-1}\boldsymbol{y}\right]} \tag{1.46}$$

where, $\boldsymbol{x}, \boldsymbol{y}$ are real vectors and A is real symmetric matrix.

Let:

$$\begin{aligned} I &= \int \mathcal{D}p \exp\left[\frac{\mathrm{i}}{2m\hbar}(-\boldsymbol{p}^T\boldsymbol{p} + 2m\boldsymbol{p}^T\dot{\boldsymbol{x}})\Delta t\right] \\ &= \left(\frac{1}{2\pi\hbar}\right)^N \int \prod_{n=1}^{N} \mathrm{d}p_n \cdot \exp\left[-\frac{1}{2}\boldsymbol{p}^T A\boldsymbol{p} - \boldsymbol{p}^T\dot{\boldsymbol{x}}'\right] \end{aligned} \tag{1.47}$$

where $A = \frac{\mathrm{i}\Delta t}{m\hbar}\mathbb{I}_{N\times N}$ and $\dot{\boldsymbol{x}}' = -\frac{\mathrm{i}\Delta t}{\hbar}\dot{\boldsymbol{x}}$, $\mathbb{I}_{N\times N}$ is the $N \times N$ identity matrix.

So we can get:

$$\begin{cases} (\det A)^{-\frac{1}{2}} = \left(\frac{\mathrm{i}\Delta t}{m\hbar}\right)^{-\frac{N}{2}} \\ A^{-1} = \frac{m\hbar}{\mathrm{i}\Delta t}\mathbb{I}_{N\times N} \end{cases} \tag{1.48}$$

The exponent term is:

$$\begin{aligned} \frac{1}{2}\dot{\boldsymbol{x}}'^T A^{-1}\dot{\boldsymbol{x}}' &= \frac{1}{2} \cdot \frac{m\hbar}{\mathrm{i}\Delta t} \cdot \left(-\frac{\mathrm{i}\Delta t}{\hbar}\right)^2 \sum_{l=1}^{N} \dot{x}_l^2 \\ &= \frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t \end{aligned} \tag{1.49}$$

So:

$$\begin{aligned} I &= \left(\frac{1}{2\pi\hbar}\right)^N \cdot (2\pi)^{\frac{N}{2}} \cdot \left(\frac{\mathrm{i}\Delta t}{m\hbar}\right)^{-\frac{N}{2}} \cdot \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right] \\ &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right] \end{aligned} \tag{1.50}$$

So we can get the integration without p:

$$\begin{aligned} \mathrm{i}G &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t - V(x_l)\Delta t\right] \\ &= \left(\frac{m}{\mathrm{i}2\pi\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}\tau\, \mathcal{L}(x,\dot{x})\right] \end{aligned} \tag{1.51}$$

So the path-integral is proportional to :

$$\boxed{\mathrm{i}G \propto \int \mathcal{D}x \cdot e^{\frac{\mathrm{i}}{\hbar}S[x(t)]}} \tag{1.52}$$

where, $S[x(t)] = \int_{t_0}^{t} \mathrm{d}t\, \mathcal{L}(x,\dot{x})$ is action and $\mathcal{L}(x,\dot{x}) = \frac{m\dot{x}^2}{2} - V(x)$ is Lagrangean.

From the Path-integral in real space-time, we can get some information about Physics Picture:

(1) Each path is weighted with a $U(1)$ phase factor $e^{\frac{\mathrm{i}}{\hbar}S}$. The Quantum interference effect between different paths.

(2) Since $\hbar \sim 10^{-34}\,\mathrm{J\cdot s}$, any "small change" in S (we change S to $S+\delta S$), will drastically lead to quantum destructive interference. So only the paths that satisfy $\delta S = 0$ make dominant contributions to the path-integral.

(3) Remarkably, $\delta S = 0$ is exactly Hamilton's Principle in classical mechanics. So the classical paths ($\delta S = 0$) dominate the path integral in the limit $\hbar \to 0$. In other words, in classical mechanics, as $\hbar \to 0$, it neglects the contribution of the integral over all other paths near the path with $\delta S = 0$. So we can get the conclusion:

$$\text{Quantum system} \xrightarrow{\hbar\to 0} \text{classical system}$$

Example 1.3.1. Free particles' Hamiltonian is $H = \frac{p^2}{2m}$.
Using this Hamiltonian, we can get the path-integral:

$$\mathrm{i}G = \left(\frac{m}{i2\hbar\Delta t}\right)^{\frac{N}{2}} \int \mathcal{D}x \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{l=1}^{N}\frac{m}{2}(\frac{x_l - x_{l-1}}{\Delta t})^2\Delta t\right]. \tag{1.53}$$

we let:

$$I = \int \mathcal{D}x \exp\left[\frac{im}{2\hbar\Delta t}\sum_{l=1}^{N}(x_l^2 - 2x_l x_{l-1} + x_{l-1}^2)\right] \tag{1.54}$$

In order to use Gaussian integral:

$$\boxed{\int \prod_{n=1}^{N} \mathrm{d}x_n\, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{y}} = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}} e^{\frac{1}{2}\boldsymbol{y}^T A^{-1} \boldsymbol{y}}} \tag{1.55}$$

we should rewrite the form of $\exp[\frac{im}{2\hbar\Delta t}\sum(x_l^2-2x_lx_{l-1}+x_{l-1}^2)]$, so we let:

$$A=\frac{-\mathrm{i}m}{\hbar\Delta t}\begin{bmatrix}2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2\end{bmatrix}_{(N-1)\times(N-1)} \tag{1.56}$$

and:

$$\boldsymbol{x}=\begin{bmatrix}x_{N-1}\\ \vdots \\ x_1\end{bmatrix},\quad \boldsymbol{y}=\frac{-\mathrm{i}m}{\hbar\Delta t}\begin{bmatrix}-x_N\\ 0\\ \vdots\\ \text{all zero}\\ \vdots\\ 0\\ x_0\end{bmatrix} \tag{1.57}$$

We get the new form of the exponent term:

$$\exp\left[\frac{\mathrm{i}m}{2\hbar\Delta t}\sum_{l=1}^{N-1}(x_l^2-2x_lx_{l-1}+x_{l-1}^2)\right]=\exp(-\frac{1}{2}\boldsymbol{x}^TA\boldsymbol{x}-\boldsymbol{x}^T\boldsymbol{y})e^{\frac{\mathrm{i}m}{2\hbar\Delta t}(x_N^2+x_0^2)} \tag{1.58}$$

Because of $\mathcal{D}x=\prod_{l=1}^{N-1}\mathrm{d}x_l$ without x_N and x_0, so:

$$\mathrm{i}G=\left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}}\cdot e^{\frac{im}{2\hbar\Delta t}(x_N^2+x_0^2)}\int\mathcal{D}x\, e^{-\frac{1}{2}\boldsymbol{x}^TA\boldsymbol{x}-\boldsymbol{x}^T\boldsymbol{y}} \tag{1.59}$$

So we can compute $(\det A)=N\cdot\left(\frac{-im}{\hbar\Delta t}\right)^N$ (Proof in B) and get:

$$\mathrm{i}G=\left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}}\cdot e^{\frac{im}{2\hbar\Delta t}(x_N^2+x_0^2)}(2\pi)^{\frac{N}{2}}N^{-1/2}\left(\frac{-\mathrm{i}m}{\hbar\Delta t}\right)^{-\frac{N}{2}}\cdot e^{\frac{1}{2}\boldsymbol{y}^TA^{-1}\boldsymbol{y}} \tag{1.60}$$

Although A^{-1} is difficult to compute, we notice that $\boldsymbol{y}=\begin{bmatrix}-x_N\\ 0\\ \vdots\\ 0\\ x_0\end{bmatrix}$ only have two non-zero elements, which locate in the first row and the last row respectively. So we only need to calculate the first and last columns of matrix A^{-1}, denoted $\boldsymbol{A}_1^{-1}$ and $\boldsymbol{A}_{N-1}^{-1}$, respectively (Proof in B):

$$\boldsymbol{A}_1^{-1}=\frac{\mathrm{i}\hbar\Delta t}{mN}\begin{bmatrix}N-1\\ N-2\\ \vdots\\ 1\end{bmatrix},\quad \boldsymbol{A}_{N-1}^{-1}=\frac{\mathrm{i}\hbar\Delta t}{mN}\begin{bmatrix}1\\ 2\\ \vdots\\ N-1\end{bmatrix} \tag{1.61}$$

The last term of the propagator(1.60) is:

$$
\begin{aligned}
-\frac{1}{2}\boldsymbol{y}^T A^{-1}\boldsymbol{y} &= \frac{1}{2}\boldsymbol{y}^T [x_N \boldsymbol{A}_1^{-1} + x_0 \boldsymbol{A}_{N-1}^{-1}] \\
&= \frac{\mathrm{i}m}{2\hbar\Delta t}\left[-(x_N^2 + x_0^2) + \frac{(x_N - x_0)^2}{N}\right]
\end{aligned} \tag{1.62}
$$

So we can get the complete integral :

$$
\begin{aligned}
\mathrm{i}G &= \left(\frac{m}{i2\pi\hbar\Delta t}\right)^{\frac{N}{2}} (2\pi)^{\frac{N}{2}} N^{-\frac{1}{2}} \left(\frac{-\mathrm{i}m}{\hbar\Delta t}\right)^{-\frac{N-1}{2}} e^{\frac{\mathrm{i}}{\hbar}\frac{m(x_N - x_0)^2}{2N\Delta t}} \\
&= \left(\frac{m}{i2\pi\hbar\Delta t N}\right)^{\frac{1}{2}} e^{\frac{\mathrm{i}}{\hbar}\frac{m(x_N - x_0)^2}{2N\Delta t}}
\end{aligned} \tag{1.63}
$$

where $t - t_0 = N\Delta t$, $x = x_N$. So the free particle's propagator is:

$$
\boxed{\mathrm{i}G = \left[\frac{m}{i2\pi\hbar(t - t_0)}\right]^{\frac{1}{2}} \cdot e^{\frac{\mathrm{i}}{\hbar}\cdot\frac{m(x - x_0)^2}{2(t - t_0)}}} \tag{1.64}
$$

Chapter 2

Stationary Phase Approximation

Part II

Quantum Spins, Coherent-state Path Integral, and Topological Terms

Chapter 3

Quantum Spin

We begin by considering the Hilbert space $\mathcal{H}$ for a single quantum spin-1/2 particle. This is a two-dimensional complex vector space.

The conventional approach is to use an orthonormal basis formed by the eigenvectors of the spin operator along a chosen axis, typically the z-axis, denoted $\hat{S}_z$.

$$\hat{S}_z|\uparrow\rangle = +\frac{\hbar}{2}|\uparrow\rangle \quad \text{and} \quad \hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle \tag{3.1}$$

Here, $|\uparrow\rangle$ represents the "spin up" state and $|\downarrow\rangle$ represents the "spin down" state. These two states form a complete orthonormal basis, satisfying:

- **Orthogonality:** $\langle\uparrow|\downarrow\rangle = 0$
- **Normalization:** $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$
- **Completeness:** $|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| = \hat{\mathbb{I}}$

where $\hat{\mathbb{I}}$ is the identity operator in $\mathcal{H}$.

3.1 Spin Coherent States

A general, normalized state $|\psi\rangle$ in the spin-1/2 Hilbert space can be written as a complex linear combination of the basis states:

$$|\psi\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle \tag{3.2}$$

where $z_1, z_2 \in \mathbb{C}$ are complex coefficients.

3.1.1 Degrees of Freedom and Normalization

The normalization condition $\langle\psi|\psi\rangle = 1$ imposes a constraint on these coefficients:

$$\langle\psi|\psi\rangle = (|z_1|^2 + |z_2|^2) = 1 \tag{3.3}$$

A complex number $z = x + \mathrm{i}y$ has two real parameters. Therefore, the pair (z_1, z_2) is defined by four real parameters. The normalization condition $|z_1|^2 + |z_2|^2 = 1$ removes one degree of freedom, leaving three.

Furthermore, in quantum mechanics, the overall phase of a state vector is unphysical. The states $|\psi\rangle$ and $\mathrm{e}^{\mathrm{i}\gamma}|\psi\rangle$ (for any real γ) represent the same physical state (i.e., they belong to the same ray in Hilbert space). This "gauge freedom" removes one more degree of freedom.

This leaves $4 - 1 - 1 = 2$ real, physical degrees of freedom. This is a crucial observation: the state space of a spin-1/2 particle is topologically equivalent to the surface of a 2D sphere, which is also parameterized by two angles (like latitude and longitude).

3.1.2 Parametrization

We can explicitly parameterize z_1 and z_2 using two angles, θ and ϕ, which will map directly to the surface of a sphere. A standard (but not unique) parametrization for the spin coherent state, labeled by a unit vector $\boldsymbol{n}$, is:

$$|\boldsymbol{n}\rangle \equiv |\theta, \phi\rangle = \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.4}$$

Here, the spherical coordinate angles have the domains $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. We can easily verify that this state is normalized:

$$\langle\boldsymbol{n}|\boldsymbol{n}\rangle = \left|\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\right|^2 + \left|\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\right|^2 = \cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1 \tag{3.5}$$

This set of states $\{|\boldsymbol{n}\rangle\}$ is continuously parameterized by the angles (θ, ϕ), addressing the first drawback of the discrete basis.

3.2 Physical Interpretation: The Bloch Sphere

To understand the physical meaning of θ and ϕ, we compute the expectation value of the vector spin operator $\hat{\boldsymbol{S}}$ in the state $|\boldsymbol{n}\rangle$. We will set $\hbar = 1$ from here on for simplicity. The spin operator is $\hat{\boldsymbol{S}} = \frac{1}{2}\hat{\boldsymbol{\sigma}}$, where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.6}$$

In the $|\uparrow\rangle, |\downarrow\rangle$ basis, $|\boldsymbol{n}\rangle$ is represented by the column vector:

$$|\boldsymbol{n}\rangle = \begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \quad \Longrightarrow \quad \langle\boldsymbol{n}| = \begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} & \sin(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix} \tag{3.7}$$

Expectation value of $\hat{S}_z$:

$$\begin{aligned} \langle\hat{S}_z\rangle &= \langle\boldsymbol{n}|\left(\frac{1}{2}\hat{\sigma}_z\right)|\boldsymbol{n}\rangle = \frac{1}{2}\begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} & \sin(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \\ &= \frac{1}{2}\left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}\cos(\theta) \end{aligned} \tag{3.8}$$

Expectation value of $\hat{S}_x$:

$$\begin{aligned} \langle\hat{S}_x\rangle &= \langle\boldsymbol{n}|\left(\frac{1}{2}\hat{\sigma}_x\right)|\boldsymbol{n}\rangle = \frac{1}{2}\left(\dots\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(\dots\right) \\ &= \frac{1}{2}\left(\cos(\theta/2)\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} + \sin(\theta/2)\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2}\right) \\ &= \frac{1}{2}\cos(\theta/2)\sin(\theta/2)\left(\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}\right) = \left(\frac{1}{2}\sin\theta\right)\left(\frac{\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}}{2}\right) = \frac{1}{2}\sin\theta\cos\phi \end{aligned} \tag{3.9}$$

Expectation value of $\hat{S}_y$:

$$\begin{aligned} \langle\hat{S}_y\rangle &= \langle\boldsymbol{n}|\left(\frac{1}{2}\hat{\sigma}_y\right)|\boldsymbol{n}\rangle = \frac{1}{2}\left(\dots\right)\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\left(\dots\right) \\ &= \frac{1}{2}\left(\cos(\theta/2)(-\mathrm{i})\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} + \sin(\theta/2)(\mathrm{i})\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2}\right) \\ &= \frac{1}{2}\cos(\theta/2)\sin(\theta/2)\left(-\mathrm{i}\mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\right) = \left(\frac{1}{2}\sin\theta\right)\left(\frac{\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi}}{2\mathrm{i}}\right) = \frac{1}{2}\sin\theta\sin\phi \end{aligned} \tag{3.10}$$

3.2.1 Conclusion: The Classical Spin Vector

Combining these results, the expectation value of the spin vector is:

$$\langle\hat{\boldsymbol{S}}\rangle = \frac{1}{2}\left(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\right) \tag{3.11}$$

This is a vector of length $S = 1/2$ pointing in the direction specified by the unit vector $\boldsymbol{n}$:

$$\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{3.12}$$

Thus, the state $|\boldsymbol{n}\rangle$ is the quantum state that "points" in the classical direction $\boldsymbol{n}$. This direction vector lives on the surface of a unit sphere, known as the **Bloch Sphere**.

This formalism treats all directions $\boldsymbol{n}$ on an equal footing, making the SU(2) rotational symmetry manifest. This addresses the second drawback of the discrete basis.

3.2.2 Coherent State as an Eigenvector

The formula presented in the original notes, $\langle \boldsymbol{n}|\hat{\boldsymbol{S}}\cdot\boldsymbol{n}|\boldsymbol{n}\rangle = \frac{1}{2}|\boldsymbol{n}\rangle$, appears to contain a typographical error, as an expectation value (a scalar) cannot be equal to a state vector.

The more fundamental property, which is likely intended, is the eigenvector equation:

$$(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}|\boldsymbol{n}\rangle \tag{3.13}$$

This equation signifies that the coherent state $|\boldsymbol{n}\rangle$ is, by definition, the "spin up" eigenvector of the spin operator projected along its own pointing direction $\boldsymbol{n}$, with the eigenvalue $+1/2$ (with $\hbar = 1$).

Proof: We first construct the operator $\hat{\boldsymbol{S}}\cdot\boldsymbol{n}$ in matrix form:

$$\begin{aligned}
\hat{\boldsymbol{S}}\cdot\boldsymbol{n} &= \hat{S}_x n_x + \hat{S}_y n_y + \hat{S}_z n_z \\
&= \frac{1}{2}(\hat{\sigma}_x n_x + \hat{\sigma}_y n_y + \hat{\sigma}_z n_z) \\
&= \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta\right] \\
&= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - \mathrm{i}\sin\phi) \\ \sin\theta(\cos\phi + \mathrm{i}\sin\phi) & -\cos\theta \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix}
\end{aligned} \tag{3.14}$$

Now, we apply this operator to the coherent state vector $|\boldsymbol{n}\rangle$:

$$(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} \\ \sin\theta\,\mathrm{e}^{\mathrm{i}\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos(\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi/2} \\ \sin(\theta/2)\,\mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \tag{3.15}$$

We compute the top and bottom components of the resulting vector separately.

Top component:

$$\begin{aligned}
&\frac{1}{2}\left[\cos\theta\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2} + \sin\theta\mathrm{e}^{-\mathrm{i}\phi}\sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi/2}\right] \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\phi/2}\left[\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right] \\
&= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\phi/2}\left[\cos(\theta - \theta/2)\right] \quad \text{(using } \cos(A-B) \text{ identity)} \\
&= \frac{1}{2}\cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi/2}
\end{aligned} \tag{3.16}$$

This is precisely $\frac{1}{2}$ times the top component of $|\boldsymbol{n}\rangle$.

Bottom component:

$$\begin{aligned}
&\frac{1}{2}\left[\sin\theta e^{i\phi}\cos(\theta/2)e^{-i\phi/2} - \cos\theta\sin(\theta/2)e^{i\phi/2}\right] \\
&= \frac{1}{2}e^{i\phi/2}\left[\sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2)\right] \\
&= \frac{1}{2}e^{i\phi/2}\left[\sin(\theta-\theta/2)\right] \quad \text{(using } \sin(A-B) \text{ identity)} \\
&= \frac{1}{2}\sin(\theta/2)e^{i\phi/2}
\end{aligned} \tag{3.17}$$

This is precisely $\frac{1}{2}$ times the bottom component of $|\boldsymbol{n}\rangle$.

Combining both components, we have shown:

$$(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})|\boldsymbol{n}\rangle = \frac{1}{2}\begin{pmatrix}\cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2}\end{pmatrix} = \frac{1}{2}|\boldsymbol{n}\rangle \tag{3.18}$$

This completes the proof. The expectation value $\langle\hat{\boldsymbol{S}}\cdot\boldsymbol{n}\rangle = \langle\boldsymbol{n}|(\hat{\boldsymbol{S}}\cdot\boldsymbol{n})|\boldsymbol{n}\rangle = \langle\boldsymbol{n}|(\frac{1}{2}|\boldsymbol{n}\rangle) = \frac{1}{2}\langle\boldsymbol{n}|\boldsymbol{n}\rangle = \frac{1}{2}$ follows directly.

3.3 Gauge Choice and Topological Singularities

The parametrization in Eq. (3.4) is not unique, and it hides a subtle topological problem.

- **At the North Pole** ($\theta = 0$): The direction $\boldsymbol{n}$ is $(0,0,1)$. The angle ϕ is ill-defined. Our formula gives $|\theta=0\rangle = \cos(0)e^{-i\phi/2}|\uparrow\rangle + \sin(0)\cdots = e^{-i\phi/2}|\uparrow\rangle$. The state vector itself depends on the meaningless angle ϕ. This is a **singularity**.
- **At the South Pole** ($\theta = \pi$): The direction $\boldsymbol{n}$ is $(0,0,-1)$. Our formula gives $|\theta=\pi\rangle = \cos(\pi/2)\cdots + \sin(\pi/2)e^{i\phi/2}|\downarrow\rangle = e^{i\phi/2}|\downarrow\rangle$. This is also singular.

This is analogous to the problem of creating a flat map of the Earth: you cannot do so without singularities (e.g., at the poles) or cuts.

We can "fix" the singularity at one pole by making a ϕ-dependent gauge choice (i.e., multiplying by an overall phase $e^{i\gamma(\phi)}$).

Choice 1: Regular at North Pole. Let's choose an overall phase $\gamma = \phi/2$. The new state, $|\boldsymbol{n}\rangle_N$, is:

$$|\boldsymbol{n}\rangle_N = e^{i\phi/2}|\boldsymbol{n}\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|\downarrow\rangle \tag{3.19}$$

- At the North Pole ($\theta = 0$): $|\boldsymbol{n}\rangle_N = \cos(0)|\uparrow\rangle + \sin(0)\cdots = |\uparrow\rangle$. This is now regular and well-defined.
- At the South Pole ($\theta = \pi$): $|\boldsymbol{n}\rangle_N = \cos(\pi/2)|\uparrow\rangle + \sin(\pi/2)e^{i\phi}|\downarrow\rangle = e^{i\phi}|\downarrow\rangle$. The singularity has been "pushed" to the South Pole.

Choice 2: Regular at South Pole. Let's choose $\gamma = -\phi/2$. The new state, $|\boldsymbol{n}\rangle_S$, is:

$$|\boldsymbol{n}\rangle_S = \mathrm{e}^{-\mathrm{i}\phi/2}|\boldsymbol{n}\rangle = \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle \tag{3.20}$$

This state is regular at the South Pole ($|\boldsymbol{n}\rangle_S = |\downarrow\rangle$) but singular at the North Pole.

This unavoidable singularity is topological in nature and is the origin of the **Berry Phase**, or the "topological term," in the coherent-state path integral.

3.4 Over-Completeness and Orthogonality

The set of all coherent states $\{|\boldsymbol{n}\rangle\}$ for all $\boldsymbol{n}$ on the sphere is an **over-complete** basis. The Hilbert space is only 2-dimensional, but we have an infinite, continuous set of states. This means the states are not, in general, orthogonal.

$$\langle\boldsymbol{n}'|\boldsymbol{n}\rangle \neq 0 \quad \text{for } \boldsymbol{n}' \neq \boldsymbol{n} \text{ and } \boldsymbol{n}' \neq -\boldsymbol{n} \tag{3.21}$$

A special exception, as noted in the text, is for antipodal states.

3.4.1 Orthogonality of Antipodal States

Let us prove that $\langle-\boldsymbol{n}|\boldsymbol{n}\rangle = 0$. The antipodal point $-\boldsymbol{n}$ corresponds to the angles $(\theta', \phi') = (\pi - \theta, \phi + \pi)$.

We write the state $|-\boldsymbol{n}\rangle$ using Eq. (3.4):

$$\begin{aligned}|-\boldsymbol{n}\rangle &= \cos\left(\frac{\pi-\theta}{2}\right)\mathrm{e}^{-\mathrm{i}(\phi+\pi)/2}|\uparrow\rangle + \sin\left(\frac{\pi-\theta}{2}\right)\mathrm{e}^{\mathrm{i}(\phi+\pi)/2}|\downarrow\rangle \\ &= \cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\pi/2}|\uparrow\rangle + \sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\pi/2}|\downarrow\rangle\end{aligned} \tag{3.22}$$

Using $\cos(\pi/2 - x) = \sin(x)$, $\sin(\pi/2 - x) = \cos(x)$, $\mathrm{e}^{-\mathrm{i}\pi/2} = -\mathrm{i}$, and $\mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}$:

$$|-\boldsymbol{n}\rangle = \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}(-\mathrm{i})|\uparrow\rangle + \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}(\mathrm{i})|\downarrow\rangle \tag{3.23}$$

Now we compute the inner product $\langle-\boldsymbol{n}|\boldsymbol{n}\rangle$:

$$\begin{aligned}&\langle-\boldsymbol{n}|\boldsymbol{n}\rangle \\ &= \left(\mathrm{i}\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\langle\uparrow| + (-\mathrm{i})\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\langle\downarrow|\right)\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle\right) \\ &= \mathrm{i}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}\mathrm{e}^{-\mathrm{i}\phi/2} + (-\mathrm{i})\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}\mathrm{e}^{\mathrm{i}\phi/2} \\ &= \mathrm{i}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \mathrm{i}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ &= 0\end{aligned} \tag{3.24}$$

This confirms that antipodal states are orthogonal, as expected. For example, $|\boldsymbol{n} = \hat{z}\rangle = |\uparrow\rangle$ is orthogonal to $|\boldsymbol{n} = -\hat{z}\rangle = |\downarrow\rangle$.

Distinction Between $|-\boldsymbol{n}\rangle$ and $-|\boldsymbol{n}\rangle$

It is a common point of confusion to mistake the antipodal state $|-\boldsymbol{n}\rangle$ for the state $-|\boldsymbol{n}\rangle$. We must justify that, in general, $|-\boldsymbol{n}\rangle \neq -|\boldsymbol{n}\rangle$.

From our derivation in the previous section, the antipodal state is:

$$|-\boldsymbol{n}\rangle = \mathrm{i}\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle - \mathrm{i}\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.25}$$

In contrast, the state $-|\boldsymbol{n}\rangle$ is:

$$-|\boldsymbol{n}\rangle = -\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-\mathrm{i}\phi/2}|\uparrow\rangle - \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{\mathrm{i}\phi/2}|\downarrow\rangle \tag{3.26}$$

By simple inspection, these two state vectors are clearly not identical. They are, in fact, orthogonal to each other, as we just proved $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle = 0$. If $|-\boldsymbol{n}\rangle$ were equal to $-|\boldsymbol{n}\rangle$, then we would have $\langle -\boldsymbol{n}|\boldsymbol{n}\rangle = \langle -\boldsymbol{n}| - (-\boldsymbol{n})\rangle = -1 \cdot \langle -\boldsymbol{n}| - \boldsymbol{n}\rangle = -1$, which contradicts our result of 0 (unless the state is null, which is not the case).

The state $|-\boldsymbol{n}\rangle$ represents a spin pointing in the *opposite direction* (e.g., spin down), while $-|\boldsymbol{n}\rangle$ represents the *same physical state* as $|\boldsymbol{n}\rangle$ but with a phase shift of π (since $\mathrm{e}^{\mathrm{i}\pi} = -1$).

3.5 Completeness Relation

Despite being over-complete, the spin coherent states provide a resolution of the identity operator $\hat{\mathbb{I}}$.

Let's call the integral $J = \int \mathrm{d}\Omega\, |\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta\, |\boldsymbol{n}\rangle\langle\boldsymbol{n}|$.

1. Integrate over ϕ: The off-diagonal terms depend on $\mathrm{e}^{\pm\mathrm{i}\phi}$.

$$\int_0^{2\pi} \mathrm{e}^{\pm\mathrm{i}\phi}\, \mathrm{d}\phi = 0 \tag{3.27}$$

The diagonal terms are independent of ϕ.

$$\int_0^{2\pi} 1\, \mathrm{d}\phi = 2\pi \tag{3.28}$$

After integrating over ϕ, the matrix J becomes diagonal:

$$J = \int_0^{\pi} \sin\theta\, \mathrm{d}\theta \begin{pmatrix} 2\pi\cos^2(\theta/2) & 0 \\ 0 & 2\pi\sin^2(\theta/2) \end{pmatrix} \tag{3.29}$$

2. Integrate over θ: Both integrals evaluate to 2π. Thus, the full integral is:

$$J = \int \mathrm{d}\Omega\, |\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \begin{pmatrix} 2\pi & 0 \\ 0 & 2\pi \end{pmatrix} = 2\pi\, \hat{\mathbb{I}} \tag{3.30}$$

Dividing by 2π, we arrive at the completeness relation:

$$\frac{1}{2\pi} \int \mathrm{d}\Omega\, |\boldsymbol{n}\rangle\langle\boldsymbol{n}| = \hat{\mathbb{I}} \tag{3.31}$$

This relation is the foundation for the coherent-state path integral. It allows us to insert the identity operator at infinitesimally small time steps, t_j, as an integral over the Bloch sphere: $\hat{\mathbb{I}} = \int \frac{\mathrm{d}\Omega_j}{2\pi} |\boldsymbol{n}_j\rangle\langle\boldsymbol{n}_j|$. Summing over all paths becomes an integral over all $\boldsymbol{n}_j$ at all times t_j.

Part III

Appendix

Appendix A

Multivariate Gaussian Integral

The multivariate Gaussian integral:

$$\boxed{I = \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left(-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{y}\right) = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right)} \tag{A.1}$$

where:

- $\boldsymbol{x}$ and $\boldsymbol{y}$ are N-dimensional column vectors.
- A is an $N \times N$ real, symmetric, and positive-definite matrix.
- The notation $\int \prod_{n=1}^{N} \mathrm{d}x_n$ denotes integration over all components of $\boldsymbol{x}$ from $-\infty$ to $+\infty$.

A.1 Proof of the Multivariate Gaussian Integral

The proof relies on the assumptions that A is symmetric ($A^\top = A$) and positive-definite (all eigenvalues are positive), which ensures the integral converges. The proof proceeds in several key steps.

Step 1: Completing the Square

The primary technique is to complete the square for the quadratic form in the exponent. We want to rewrite the argument of the exponential, $-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y}$, into the form $-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^\top A(\boldsymbol{x} - \boldsymbol{x}_0) + C$, where $\boldsymbol{x}_0$ and C are constants with respect to $\boldsymbol{x}$.

Expanding this target form, we get:

$$\begin{aligned} -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^\top A(\boldsymbol{x} - \boldsymbol{x}_0) &= -\frac{1}{2}(\boldsymbol{x}^\top - \boldsymbol{x}_0^\top)A(\boldsymbol{x} - \boldsymbol{x}_0) \\ &= -\frac{1}{2}(\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top A\boldsymbol{x}_0 - \boldsymbol{x}_0^\top A\boldsymbol{x} + \boldsymbol{x}_0^\top A\boldsymbol{x}_0) \end{aligned} \tag{A.2}$$

Since A is symmetric ($A = A^\top$), the scalar term $\boldsymbol{x}_0^\top A\boldsymbol{x}$ is equal to its own transpose: $(\boldsymbol{x}_0^\top A\boldsymbol{x})^\top = \boldsymbol{x}^\top A^\top \boldsymbol{x}_0 = \boldsymbol{x}^\top A\boldsymbol{x}_0$. Thus, the two cross-terms are equal.

$$-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^\top A(\boldsymbol{x} - \boldsymbol{x}_0) = -\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} + \boldsymbol{x}^\top A\boldsymbol{x}_0 - \frac{1}{2}\boldsymbol{x}_0^\top A\boldsymbol{x}_0 \tag{A.3}$$

Comparing this to the original exponent, $-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y}$, we can equate the terms linear in $\boldsymbol{x}$:

$$-\boldsymbol{x}^\top\boldsymbol{y} = \boldsymbol{x}^\top A\boldsymbol{x}_0 \implies A\boldsymbol{x}_0 = -\boldsymbol{y} \tag{A.4}$$

Since A is positive-definite, it is invertible. We can solve for $\boldsymbol{x}_0$:

$$\boldsymbol{x}_0 = -A^{-1}\boldsymbol{y} \tag{A.5}$$

With this definition of $\boldsymbol{x}_0$, the original exponent can be written as:

$$-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y} = -\frac{1}{2}(\boldsymbol{x} + A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x} + A^{-1}\boldsymbol{y}) + \frac{1}{2}(A^{-1}\boldsymbol{y})^\top A(A^{-1}\boldsymbol{y}) \tag{A.6}$$

Let's simplify the constant term (the term not involving $\boldsymbol{x}$):

$$\begin{aligned}\frac{1}{2}(A^{-1}\boldsymbol{y})^\top A(A^{-1}\boldsymbol{y}) &= \frac{1}{2}\boldsymbol{y}^\top(A^{-1})^\top AA^{-1}\boldsymbol{y} \\ &= \frac{1}{2}\boldsymbol{y}^\top A^{-1}AA^{-1}\boldsymbol{y} \quad (\text{since } (A^{-1})^\top = (A^\top)^{-1} = A^{-1}) \\ &= \frac{1}{2}\boldsymbol{y}^\top IA^{-1}\boldsymbol{y} = \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\end{aligned} \tag{A.7}$$

So, the exponent is:

$$-\frac{1}{2}\boldsymbol{x}^\top A\boldsymbol{x} - \boldsymbol{x}^\top\boldsymbol{y} = -\frac{1}{2}(\boldsymbol{x} + A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x} + A^{-1}\boldsymbol{y}) + \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y} \tag{A.8}$$

Step 2: Change of Variables (Translation)

Substituting the completed square back into the integral:

$$I = \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left[-\frac{1}{2}(\boldsymbol{x} + A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x} + A^{-1}\boldsymbol{y}) + \frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right] \tag{A.9}$$

The term $\exp(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y})$ is constant with respect to $\boldsymbol{x}$ and can be factored out of the integral:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \int \prod_{n=1}^{N} \mathrm{d}x_n \, \exp\left[-\frac{1}{2}(\boldsymbol{x} + A^{-1}\boldsymbol{y})^\top A(\boldsymbol{x} + A^{-1}\boldsymbol{y})\right] \tag{A.10}$$

Now, we perform a change of variables. Let $\boldsymbol{z} = \boldsymbol{x} + A^{-1}\boldsymbol{y}$. This is a simple translation of the coordinate system. The differential element $\prod \mathrm{d}x_n$ transforms as $\prod \mathrm{d}z_n$, as the Jacobian of this transformation is 1. The limits of integration remain from $-\infty$ to $+\infty$. The integral becomes:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \int \prod_{n=1}^{N} \mathrm{d}z_n \, \exp\left(-\frac{1}{2}\boldsymbol{z}^\top A\boldsymbol{z}\right) \tag{A.11}$$

The problem is now reduced to evaluating the simpler, centered Gaussian integral:

$$I_0 = \int \prod \mathrm{d}z_n \, \exp(-\frac{1}{2}\boldsymbol{z}^\top A\boldsymbol{z}) \tag{A.12}$$

Step 3: Diagonalization

To compute I_0, we diagonalize the matrix A. Since A is a real symmetric matrix, it is orthogonally diagonalizable:

$$A = PDP^{\top} \tag{A.13}$$

where P is an orthogonal matrix ($PP^{\top} = P^{\top}P = I$) whose columns are the orthonormal eigenvectors of A, and D is a diagonal matrix whose entries are the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. Substituting this into the quadratic form $\boldsymbol{z}^{\top}A\boldsymbol{z}$:

$$\boldsymbol{z}^{\top}A\boldsymbol{z} = \boldsymbol{z}^{\top}(PDP^{\top})\boldsymbol{z} = (\boldsymbol{z}^{\top}P)D(P^{\top}\boldsymbol{z}) = (P^{\top}\boldsymbol{z})^{\top}D(P^{\top}\boldsymbol{z}) \tag{A.14}$$

We perform another change of variables. Let $\boldsymbol{w} = P^{\top}\boldsymbol{z}$. This transformation corresponds to a rotation of the coordinate system. The Jacobian determinant is $|\det(P^{\top})| = 1$, so the volume element is unchanged: $\prod \mathrm{d}z_n = \prod \mathrm{d}w_n$. The quadratic form simplifies to:

$$\boldsymbol{w}^{\top}D\boldsymbol{w} = \sum_{i=1}^{N} \lambda_i w_i^2 \tag{A.15}$$

This is because D is a diagonal matrix.

Step 4: Computing the Decoupled Integral

The integral I_0 now becomes:

$$I_0 = \int \prod_{n=1}^{N} \mathrm{d}w_n \, \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \lambda_i w_i^2\right) \tag{A.16}$$

The exponential of a sum is the product of exponentials, which allows us to separate the multi-dimensional integral into a product of N one-dimensional integrals:

$$I_0 = \int \prod_{n=1}^{N} \mathrm{d}w_n \prod_{i=1}^{N} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right) = \prod_{i=1}^{N}\left(\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right)\mathrm{d}w_i\right) \tag{A.17}$$

We use the standard formula for a 1D Gaussian integral: $\int_{-\infty}^{\infty} \exp(-au^2)\mathrm{d}u = \sqrt{\pi/a}$. In our case, $a = \lambda_i/2$.

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\lambda_i w_i^2\right)\mathrm{d}w_i = \sqrt{\frac{\pi}{\lambda_i/2}} = \sqrt{\frac{2\pi}{\lambda_i}} \tag{A.18}$$

Multiplying these N results together:

$$I_0 = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} = (2\pi)^{\frac{N}{2}} \prod_{i=1}^{N} (\lambda_i)^{-\frac{1}{2}} = (2\pi)^{\frac{N}{2}} \left(\prod_{i=1}^{N} \lambda_i\right)^{-\frac{1}{2}} \tag{A.19}$$

The determinant of a matrix is equal to the product of its eigenvalues. Thus, $\det A = \det D = \prod_{i=1}^{N} \lambda_i$.

$$I_0 = (2\pi)^{\frac{N}{2}} (\det A)^{-\frac{1}{2}} \tag{A.20}$$

Step 5: Combining the Results

Finally, we substitute the value of I_0 back into our expression for I from Step 2:

$$I = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \cdot I_0 = \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right)(2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \tag{A.21}$$

Rearranging the terms yields the final result:

$$I = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}} \exp\left(\frac{1}{2}\boldsymbol{y}^\top A^{-1}\boldsymbol{y}\right) \tag{A.22}$$

This completes the proof.

Appendix B

Calculation of the Determinant and Inverse of the a Special Matrix

Let the given $n \times n$ matrix be denoted by A_n.

$$A_n = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{n \times n} \tag{B.1}$$

B.1 Calculation of the Determinant

We will find the determinant by establishing a recurrence relation. Let $D_n = \det(A_n)$ be the determinant of the $n \times n$ version of this matrix.

Determinants for Small Sizes

We compute the determinant for small values of n to identify a pattern.

- For $n = 1$:

$$A_1 = \begin{bmatrix} 2 \end{bmatrix} \implies D_1 = \det(A_1) = 2 \tag{B.2}$$

- For $n = 2$:

$$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \implies D_2 = \det(A_2) = (2)(2) - (-1)(-1) = 3 \tag{B.3}$$

- For $n = 3$:

$$\begin{aligned} A_3 &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\ D_3 &= 2 \cdot \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - (-1) \cdot \det \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix} \\ &= 2D_2 + ((-1)(2) - (-1)(0)) = 2(3) - 2 = 4 \end{aligned} \tag{B.4}$$

The sequence of determinants $D_1 = 2, D_2 = 3, D_3 = 4$ suggests the pattern $D_n = n + 1$.

Recurrence Relation

We use cofactor expansion along the first row of A_n to derive a general recurrence relation for $D_n = \det(A_n)$.

$$D_n = 2 \cdot \det \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}_{(n-1)\times(n-1)} - (-1) \cdot \det \begin{pmatrix} -1 & -1 & 0 & \dots \\ 0 & 2 & -1 & \\ 0 & -1 & 2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}_{(n-1)\times(n-1)} \tag{B.5}$$

The first sub-determinant is simply D_{n-1}. For the second sub-determinant, we perform a cofactor expansion along its first column, which yields $-1 \cdot D_{n-2}$.

$$\begin{aligned} D_n &= 2D_{n-1} - (-1)(-1 \cdot D_{n-2}) \\ D_n &= 2D_{n-1} - D_{n-2} \end{aligned} \tag{B.6}$$

This recurrence is valid for $n \geq 3$. We check if our hypothesized formula $D_n = n + 1$ satisfies this recurrence.

$$2D_{n-1} - D_{n-2} = 2((n-1)+1) - ((n-2)+1) = 2n - (n-1) = n + 1 = D_n \tag{B.7}$$

The formula holds for the base cases and satisfies the recurrence, so it is correct by induction.

Final Result

The given matrix A has size $n = N - 1$. Therefore, its determinant is:

$$\det(A) = D_{N-1} = (N-1) + 1 = N \tag{B.8}$$

B.2 Calculation of the Inverse of the Matrix

Let $B = A^{-1}$. The j-th column of B, denoted by the vector $\boldsymbol{b}_j$, is the solution to the system $A\boldsymbol{b}_j = \boldsymbol{e}_j$, where $\boldsymbol{e}_j$ is the j-th standard basis vector. Writing this out for the i-th component (where b_{ij} is the (i, j)-th element of B) gives the system of equations:

$$-b_{i-1,j} + 2b_{i,j} - b_{i+1,j} = \delta_{ij} \quad \text{for } i, j = 1, \dots, N-1 \tag{B.9}$$

with boundary conditions $b_{0,j} = 0$ and $b_{N,j} = 0$.

Homogeneous Solution

For $i \neq j$, the equation is homogeneous:

$$-b_{i-1,j} + 2b_{i,j} - b_{i+1,j} = 0 \tag{B.10}$$

This is a linear recurrence relation with characteristic equation $r^2 - 2r + 1 = 0$, or $(r-1)^2 = 0$. This has a repeated root $r = 1$, so the general solution is linear in i:

$$b_{i,j} = C_1 + C_2 i \tag{B.11}$$

We apply this solution to two regions.

Region 1: $1 \le i \le j$. The boundary condition $b_{0,j} = 0$ implies $C_1 + C_2(0) = 0$, so $C_1 = 0$. The solution has the form:

$$b_{i,j} = C \cdot i \tag{B.12}$$

Region 2: $j \le i \le N-1$. The boundary condition $b_{N,j} = 0$ implies $D_1 + D_2(N) = 0$, so $D_1 = -D_2 N$. The solution is $b_{i,j} = -D_2 N + D_2 i = D_2(i-N)$. Letting $D = -D_2$, the solution has the form:

$$b_{i,j} = D \cdot (N - i) \tag{B.13}$$

Stitching the Solutions

The full solution is given by:

$$b_{i,j} = \begin{cases} C \cdot i & \text{if } i \le j \\ D \cdot (N-i) & \text{if } i \ge j \end{cases} \tag{B.14}$$

The constants C and D are found by satisfying two conditions at $i = j$.

1. Continuity at $i = j$: The two forms must be equal.

$$C \cdot j = D \cdot (N - j) \implies D = C \frac{j}{N-j} \tag{B.15}$$

2. The inhomogeneous equation at $i = j$:

$$-b_{j-1,j} + 2b_{j,j} - b_{j+1,j} = 1 \tag{B.16}$$

Substituting the piecewise solutions into the inhomogeneous equation:

$$\begin{aligned} -C(j-1) + 2(Cj) - D(N-(j+1)) &= 1 \\ C\left[-(j-1) + 2j - \frac{j}{N-j}(N-j-1)\right] &= 1 \\ C\left[j + 1 - \frac{jN - j^2 - j}{N-j}\right] &= 1 \\ C\left[\frac{(j+1)(N-j) - (jN - j^2 - j)}{N-j}\right] &= 1 \\ C\left[\frac{jN - j^2 + N - j - jN + j^2 + j}{N-j}\right] &= 1 \\ C\left[\frac{N}{N-j}\right] &= 1 \end{aligned} \tag{B.17}$$

This gives the constants:

$$C = \frac{N - j}{N} \quad \text{and} \quad D = \left(\frac{N - j}{N}\right) \frac{j}{N - j} = \frac{j}{N} \tag{B.18}$$

Final Result

The element (i, j) of the inverse matrix A^{-1} is:

$$(A^{-1})_{ij} = \begin{cases} \frac{N-j}{N} \cdot i & \text{if } i \leq j \\ \frac{j}{N} \cdot (N - i) & \text{if } i \geq j \end{cases} \tag{B.19}$$

This can be written more compactly using min and max functions:

$$(A^{-1})_{ij} = \frac{\min(i, j) \cdot (N - \max(i, j))}{N} \tag{B.20}$$

Appendix C

Evaluation of the Fresnel Integral

The value of the **Fresnel Integral** is:

$$\boxed{\int_{-\infty}^{\infty} e^{ix^2} dx = \sqrt{\frac{\pi}{2}}(1 + i) = \sqrt{\pi} e^{i\pi/4}} \tag{C.1}$$

C.1 Derivation (Using Contour Integration)

Step 1: Define the Contour

We consider the complex function $f(z) = e^{iz^2}$, where z is a complex variable. We construct a closed path (contour) C in the complex plane. This path is a sector of a circle, composed of three parts:

1. **Path** C_1: A line segment along the real axis from 0 to R.
2. **Path** C_2: A circular arc of radius R, centered at the origin, running counter-clockwise from R to $Re^{i\pi/4}$.
3. **Path** C_3: A line segment from $Re^{i\pi/4}$ back to the origin 0.

We will eventually let $R \longrightarrow \infty$.

Step 2: Apply Cauchy's Integral Theorem

The function $f(z) = e^{iz^2}$ is analytic over the entire complex plane (it is an entire function) as it has no singularities. According to **Cauchy's Integral Theorem**, its integral over any closed path C is zero:

$$\oint_C e^{iz^2} dz = 0 \tag{C.2}$$

This closed-loop integral can be split into the sum of integrals over the three paths:

$$\int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz + \int_{C_3} e^{iz^2} dz = 0 \tag{C.3}$$

Step 3: Evaluate the Integral on Each Path

1. Integral along Path C_1 (The part we want to find)

On path C_1, we have $z = x$ (a real number) and $dz = dx$. Therefore:

$$\lim_{R\to\infty} \int_{C_1} e^{iz^2} dz = \int_0^{\infty} e^{ix^2} dx \tag{C.4}$$

This is exactly half of the integral we wish to compute, since the integrand e^{ix^2} is an even function.

2. Integral along Path C_2 (Show it vanishes as $R \to \infty$)

On path C_2, we parameterize $z = Re^{i\theta}$, where θ varies from 0 to $\pi/4$. Then $dz = iRe^{i\theta}d\theta$ and $z^2 = R^2e^{i2\theta}$. The integral becomes:

$$\begin{aligned} \int_{C_2} e^{iz^2}dz &= \int_0^{\pi/4} \exp(i(R^2e^{i2\theta})) \cdot iRe^{i\theta}d\theta \\ &= \int_0^{\pi/4} \exp(iR^2(\cos(2\theta) + i\sin(2\theta))) \cdot iRe^{i\theta}d\theta \\ &= \int_0^{\pi/4} \exp(-R^2\sin(2\theta))\exp(i(R^2\cos(2\theta) + \theta)) \cdot iRd\theta \end{aligned} \tag{C.5}$$

Taking the magnitude:

$$\left|\int_{C_2} e^{iz^2}dz\right| \le \int_0^{\pi/4} \left|\exp(-R^2\sin(2\theta))\right| \cdot Rd\theta = \int_0^{\pi/4} R\exp(-R^2\sin(2\theta))d\theta \tag{C.6}$$

On the interval $[0, \pi/4]$, 2θ is in $[0, \pi/2]$. We can use Jordan's inequality, which states $\sin(x) \ge \frac{2x}{\pi}$ for $x \in [0, \pi/2]$. Thus, $\sin(2\theta) \ge \frac{4\theta}{\pi}$.

$$\begin{aligned} \left|\int_{C_2} e^{iz^2}dz\right| &\le \int_0^{\pi/4} R\exp(-R^2(4\theta/\pi))d\theta \\ &= R\left[\frac{-\pi}{4R^2}\exp(-4R^2\theta/\pi)\right]\Big|_0^{\pi/4} \\ &= \frac{\pi}{4R}(1 - \exp(-R^2)) \end{aligned} \tag{C.7}$$

As $R \to \infty$, this expression approaches 0. Therefore:

$$\lim_{R\to\infty} \int_{C_2} e^{iz^2}dz = 0 \tag{C.8}$$

3. Integral along Path C_3 (Connection to the Gaussian Integral)

On path C_3, we parameterize $z = re^{i\pi/4}$, where r varies from R to 0. Then $dz = e^{i\pi/4}dr$ and $z^2 = (re^{i\pi/4})^2 = r^2e^{i\pi/2} = r^2i$. The integral becomes:

$$\int_{C_3} e^{iz^2}dz = \int_R^0 \exp(i(r^2i))e^{i\pi/4}dr = \int_R^0 \exp(-r^2)e^{i\pi/4}dr \tag{C.9}$$

Reversing the limits of integration and factoring out the constant:

$$= -e^{i\pi/4}\int_0^R \exp(-r^2)dr \tag{C.10}$$

As $R \to \infty$, we get the well-known Gaussian integral:

$$\lim_{R\to\infty} \int_{C_3} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z = -\mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \exp(-r^2)\mathrm{d}r \tag{C.11}$$

We know that the Gaussian integral $\int_0^\infty \exp(-x^2)\mathrm{d}x = \frac{\sqrt{\pi}}{2}$. So:

$$\lim_{R\to\infty} \int_{C_3} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z = -\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2} \tag{C.12}$$

Step 4: Combine the Results

Returning to the equation from Step 2 and taking the limit as $R \to \infty$:

$$\left(\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x\right) + (0) + \left(-\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2}\right) = 0 \tag{C.13}$$

Rearranging the terms, we find:

$$\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x = \mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2} \tag{C.14}$$

Step 5: Calculate the Final Integral

The integral we want to evaluate is from $-\infty$ to ∞. Since the integrand $\mathrm{e}^{\mathrm{i}x^2} = \cos(x^2) + \mathrm{i}\sin(x^2)$ is an even function (i.e., $f(-x) = f(x)$), we have:

$$\begin{aligned} \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x &= 2\int_0^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x \\ &= 2 \cdot \left(\mathrm{e}^{\mathrm{i}\pi/4}\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}\mathrm{e}^{\mathrm{i}\pi/4} \end{aligned} \tag{C.15}$$

Finally, we can use Euler's formula, $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, to expand $\mathrm{e}^{\mathrm{i}\pi/4}$:

$$\mathrm{e}^{\mathrm{i}\pi/4} = \cos(\pi/4) + \mathrm{i}\sin(\pi/4) = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2} = \frac{1+\mathrm{i}}{\sqrt{2}} \tag{C.16}$$

Substituting this into our result gives:

$$\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x = \sqrt{\pi} \cdot \frac{1+\mathrm{i}}{\sqrt{2}} = \sqrt{\frac{\pi}{2}}(1+\mathrm{i}) \tag{C.17}$$

C.2 Conclusion

The result of the integral is a complex number. Its real and imaginary parts correspond to two other important integrals:

$$\begin{cases} \int_{-\infty}^\infty \cos(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{2}} \\ \int_{-\infty}^\infty \sin(x^2)\mathrm{d}x = \sqrt{\frac{\pi}{2}} \end{cases} \tag{C.18}$$