

3. The path integral for many-spin system

3.1 General theory

Consider a spin system in a d -dimensional lattice which is called Heisenberg Model

$$\hat{H} = J \cdot \sum_{\langle i,j \rangle} \hat{S}_i \cdot \hat{S}_j$$

where J is the exchange coupling and \hat{S} are quantum spin operators. The $\langle i,j \rangle$ denotes a summation over nearest-neighbor pairs

The partition function of the spin system is:

$$Z = \text{Tr} e^{-\beta \hat{H}} \\ = \int d\vec{n}_0 \cdot \langle \vec{n}_0 | e^{-\beta \hat{H}} | \vec{n}_0 \rangle$$

where

$|\vec{n}_0\rangle = \bigotimes_{i=1}^N |\vec{n}_0^i\rangle$ is the tensor product of all the state on the lattice.

In the previous chapters, we have seen that we can change real time t to imaginary time τ :

$$\langle n_f | e^{-i\hat{H}(t_f - t_i)} | n_i \rangle \xrightarrow{i(t_f - t_i) \rightarrow \tau} \langle n_f | e^{-\hat{H}\tau} | n_i \rangle$$

In this version, ~~K~~ propagator K can be

written as (for closed path).

$$K = \int \mathcal{D}[\vec{n}(t)] e^{S_E}$$

where, $S_E = -iS S_{WZ} = i \cdot S \cdot A_T$ for $\vec{H} = 0$ and

$$S_E = -iS \cdot S_{WZ} + \int_{t_i}^{t_f} \langle \vec{n}(t) | \hat{H} | \vec{n}(t) \rangle dt \text{ for } \vec{H} \neq 0.$$

Because the Hamiltonian diagonal element of partition function and propagator have the same mathematical form, so we can use the same method to solve the diagonal elements.

let's make ΔT equal to $\frac{\beta}{N}$ (N is the slice number) whose mean is that β play a same role in the diagonal liking T in the propagator. $(t_i \rightarrow t_f \implies 0 \rightarrow \beta)$

But in many-spin system, the identity operator for the entire system is the tensor product of the identities for each site:

$$\hat{1}_{\text{total}} = \bigotimes_{i=1}^N \hat{1}_i = \int \left(\prod_{i=1}^N d^2 \vec{n}_i \right) |\vec{n}_1\rangle \langle \vec{n}_1| \otimes |\vec{n}_2\rangle \langle \vec{n}_2| \otimes \dots \otimes |\vec{n}_N\rangle \langle \vec{n}_N|$$

So the diagonal can be written as a mathematical form like propagator:

$$\langle \vec{n}_0 | e^{-\beta \hat{H}} | \vec{n}_0 \rangle = \int D(\vec{n}) \cdot e^{-S_E}$$

where $D(\vec{n}) = \prod_i \prod_t d^2 \vec{n}_t^i$. Because diagonal likes the propagator with close path ($\langle \vec{n}_0 | e^{-\beta \hat{H}} | \vec{n}_0 \rangle$ and not $\langle \vec{n}_i | e^{-\beta \hat{H}} | \vec{n}_j \rangle$ for $i \neq j$), S_E can be written as:

$$S_E = \int_0^\beta d\tau \langle \vec{n} | \frac{\partial}{\partial \tau} | \vec{n} \rangle + \int_0^\beta d\tau \langle \vec{n} | \hat{H} | \vec{n} \rangle$$

For the energy term:

$$\begin{aligned} S_E &= \int_0^\beta d\tau \langle \vec{n} | \hat{H} | \vec{n} \rangle \\ &= \int_0^\beta d\tau \sum_{\langle i,j \rangle} \langle \vec{n} | \vec{S}_i \cdot \vec{S}_j | \vec{n} \rangle \end{aligned}$$

Use the formula:

$$\langle \vec{n} | \vec{S} | \vec{n} \rangle = S_{\text{spin}} \vec{n}$$

we can get:

$$S_E = \int_0^\beta d\tau \sum_{\langle i,j \rangle} S_{\text{spin}}^2 \vec{n}_i(\tau) \cdot \vec{n}_j(\tau)$$

For the ~~Wize~~ - geometric term:

$$S_g = \int_0^\beta d\tau (\otimes |\vec{n}\rangle) \frac{\partial}{\partial \tau} (\otimes |\vec{n}\rangle)$$

$$= \int_0^\beta d\tau (\otimes \langle \vec{n} |) \left[(\otimes |\vec{n}\rangle + d(\otimes |\vec{n}\rangle) - \otimes |\vec{n}\rangle \right]$$

let's ~~retain~~ retain the first order term:

$$S_g = \int_0^\beta d\tau (\otimes \langle \vec{n} |) \left[(\otimes |\vec{n}\rangle + \sum_i \sum_{j \neq i} (\otimes |\vec{n}_j\rangle d\vec{n}_i) - \otimes |\vec{n}\rangle \right]$$

$$= \sum_i \int_0^\beta d\tau (\otimes \langle \vec{n} |) (\otimes |\vec{n}_j\rangle) d\vec{n}_i$$

because of $\langle \vec{n}_i | \vec{n}_i \rangle = 1$, so:

$$S_g = \sum_i \int_0^\beta \langle \vec{n}_i | d\vec{n}_i \rangle$$

$$= \sum_i -i S \cdot S_{WZ} [\vec{n}_i]$$

$$= -i S \sum_i S_{WZ} [\vec{n}_i]$$

let's go back to real time ($\tau \rightarrow it$):

$$-S E \rightarrow \frac{i}{\hbar} S_M$$

$$\Rightarrow S_M = \hbar S \cdot \sum_i S_{WZ} [\vec{n}_i] - S^2 \int_0^\beta dt \cdot \sum_{\langle \vec{n}_i, \vec{n}_j \rangle} \vec{n}_i \cdot \vec{n}_j$$

$$\left\{ \begin{aligned} S_M &= \hbar S \cdot \sum_i S_{WZ} [\vec{n}_i] - S^2 \int_0^\beta dt \cdot \sum_{\langle \vec{n}_i, \vec{n}_j \rangle} \vec{n}_i \cdot \vec{n}_j \end{aligned} \right.$$

So $\left\{ \sum_i S_{wz} [\vec{n}_i] \right\}$ and $\left\{ \int_0^\beta dt \sum_{\langle ij \rangle} J \cdot \vec{n}_i \cdot \vec{n}_j \right\}$ can be regarded as the phase term of action ($e^{\frac{i}{\hbar} \theta}$). Therefore,

S and $\frac{S^2}{\hbar}$ play the role of \hbar in simple action

Here, if S is very large, ~~we would~~ we called it Large- S limit ($\hbar \ll 1$, so $\frac{1}{\hbar} \approx S \gg 1$) This is a semi-classical limit.

3.2 Quantum Ferromagnetism.

Let's consider a ferromagnetism system whose J equal to $-|J|$. So the action is:

$$S_M = S \sum_i S_{wz} [\vec{n}_i] + |J| S^2 \sum_{\langle ij \rangle} \int_0^\beta dt \vec{n}_i \cdot \vec{n}_j$$

because $|\vec{n}|^2 = 1$, so we can rewrite the term $\vec{n}_i \cdot \vec{n}_j$ as:

~~$$\frac{1}{2} \vec{n}_i \cdot \vec{n}_j = \frac{1}{2} (1$$~~

$$- \vec{n}_i \cdot \vec{n}_j = \frac{1}{2} (1 - 2 \vec{n}_i \cdot \vec{n}_j + 1) - 1$$

$$= \frac{1}{2} (\vec{n}_i - \vec{n}_j)^2 - 1$$

So the energy term is

$$S_E = - \frac{|J| S^2}{2} \sum_{\langle ij \rangle} \int_0^\beta dt (\vec{n}_i - \vec{n}_j)^2 + \frac{|J| S^2}{2} \sum_{\langle ij \rangle} \int_0^\beta dt$$

The second term of S_e is the constant term (global phase), so we can drop it. and S_m can be written as:

$$S_m = S \sum_i S_{wz} [\vec{n}_i] - \frac{|J|S^2}{2} \sum_{\langle i,j \rangle} \int_0^\beta dt (\vec{n}_i - \vec{n}_j)^2$$

Next, we take the continuum limit where the lattice sums become integrals

$$\sum_i \rightarrow \int \frac{d^d x}{a_0^d} S$$

where a_0 is the lattice spacing and finite differences become gradients.

$$\begin{aligned} \sum_j (\vec{n}_i - \vec{n}_j)^2 &= \left(\sum_j \frac{\partial n_j^\alpha}{\partial x_j} \right) \left(\sum_\alpha \frac{\Delta n^\alpha}{\Delta x^\alpha} a_0, \sum_\alpha \frac{\Delta n^\alpha}{\Delta x^\alpha} a_0, \dots \right)^2 \\ &= a_0^2 (\nabla \vec{n})^2. \end{aligned}$$

where $\nabla \cdot \vec{n} = (\frac{\partial n^\alpha}{\partial x^\alpha}, \dots)$ and not $\nabla \cdot \vec{n} = (\frac{\partial n^x}{\partial x}, \dots)$

So the continuum action is:

$$\begin{aligned} S_m &= \int \frac{d^d x}{a_0^d} S \cdot S_{wz} [\vec{n}] - \frac{|J|S^2}{2} \int \frac{d^d x}{a_0^d} dt a_0^2 (\nabla \vec{n})^2 \\ &= \frac{S}{a_0^d} \int d^d x S_{wz} [\vec{n}] - \frac{|J|S^2}{2a_0^{d-2}} \int d^d x dt (\nabla \vec{n})^2 \end{aligned}$$