We want to find the classical path that minimizes the action. However, simply varying S_M is incorrect because the variable $\vec{n}$ is not free; it is constrained to the surface of a sphere:

$$\vec{n}^2 - 1 = 0$$

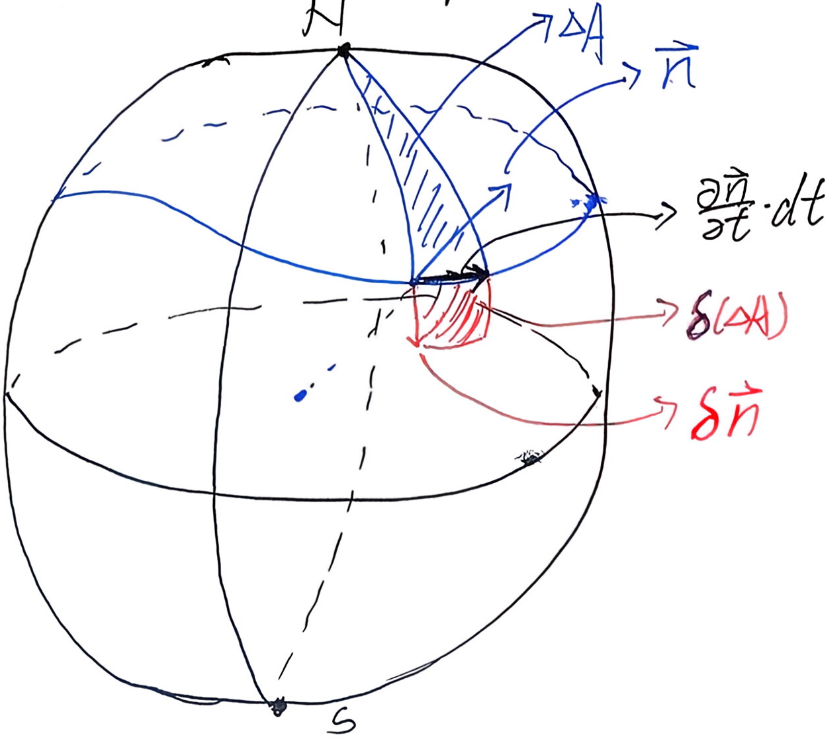
To handle this, we introduce a Lagrange multiplier field λ. So the total action becomes:

$$S_{tot} = S_M + \int d^d x \, dt \cdot \frac{\lambda}{2} (n^2 - 1)$$

Now, we can safely solve the variation equation below:

$$\delta S_{tot} = 0$$

We first need to variation of the Wess-Zumino term δS_{WZ}. Geometrically, the S_{WZ} is proportional to the area swept out by the spin vector on the unit sphere



From the picture, we can see that the variation of this area element $\delta(\Delta A)$ with respect to a small change in path $\delta\vec{n}$ is given by:

$$\vec{n}\cdot\delta(\Delta A) = \delta\vec{n}\times\left(\frac{\partial\vec{n}}{\partial t}\right)dt$$

we product $\vec{n}$ on the both sides of the equation:

$$\delta(\Delta A) = \vec{n}\left(\delta\vec{n}\times\frac{\partial\vec{n}}{\partial t}\right)dt.$$
$$= -\delta\vec{n}\left(\vec{n}\times\frac{\partial\vec{n}}{\partial t}\right)dt.$$

Therefore, the variation of the Wess-Zumion part of the action is:

$$\cancel{\delta\int\frac{S}{x}}\ \delta\left(\frac{S}{a_0^d}\cdot\int d^dx\, S_{WZ}(\vec{n})\right) = \delta\left[\frac{S}{a_0^d}\int d^dx\cdot -A_\gamma(\vec{n})\right]$$
$$= \frac{S}{a_0^d}\int d^dx\left[\delta\vec{n}\cdot\left(\vec{n}\times\frac{\partial\vec{n}}{\partial t}\right)\right]$$

This implies the functional derivative is:

$\cancel{\delta S_W}\ \cancel{\delta S_g}$

$$\frac{\delta S_g}{\delta\vec{n}} = \frac{S}{a_0^d}\cdot\left(\vec{n}\times\frac{\partial\vec{n}}{\partial t}\right)$$

Next, we calculate the variation of energy term:

$$\delta\left[\int d^dx\,dt\,(\nabla\vec{n})^2\right] = \int d^dx\,dt\sum_i^d\delta\left(\frac{\partial\vec{n}}{\partial x_i}\right)^2$$
$$= \int d^dx\,dt\cdot\sum_i^d 2\cdot\left(\frac{\partial\vec{n}}{\partial x_i}\right)\cdot\delta\left(\frac{\partial\vec{n}}{\partial x_i}\right)$$

and we use the partial integration method:

$$\int d^d x \sum_i^d 2\left(\frac{\partial \vec{n}}{\partial x_i}\right)\cdot \delta\left(\frac{\partial \vec{n}}{\partial x_i}\right) \xrightarrow{\frac{\partial \vec{n}}{\partial x_i} = \partial_i \vec{n}} 2\int \cancel{d^d x} \sum_i^d (\partial_i \vec{n}) \cdot \cancel{\partial_i(\delta \vec{n})}\, \partial^d(\delta \vec{n})$$

$$= 2\left\{ \sum_i^d (\partial_i \vec{n})\, \delta \vec{n} \Big| - \int d^d x \sum_i^d \delta \vec{n} \cdot \partial_i^2 \vec{n} \right\}$$

$$= -2 \int d^d x \cdot \sum (\nabla^2 \vec{n}) \cdot \delta \vec{n}$$

So the variation of energy term is

$$\frac{\delta S_e}{\delta \vec{n}} = \frac{|J| S^2}{a_0^{d-2}} (\nabla^2 \vec{n})$$

Finally, we calculate the variation of λ term:

$$\delta \cdot \int d^d x\, dt\, \frac{\lambda}{2}(\vec{n}^2 - 1) = \int d^d x\, dt \cdot \lambda \vec{n}\, \delta \vec{n}$$

So the variation of λ term is:

$$\frac{\delta S_\lambda}{\delta \vec{n}} = \lambda \vec{n}$$

Now we can get the ~~total~~ action's variation:

$$\delta S_{tot} = 0$$

$$\Rightarrow \frac{S}{a_0^d}\left(\vec{n} \times \frac{\partial \vec{n}}{\partial t}\right) + \lambda \vec{n} = -\frac{|J|S^2}{a_0^{d-2}} \nabla^2 \vec{n}.$$

To find the value of the Lagrange multiplier, take the dot product of the above ~~equaction~~ equation with $\vec{n}$:

$$\underbrace{\frac{S}{a_0^d}\vec{n}\cdot\left(\vec{n} \times \frac{\partial \vec{n}}{\partial t}\right)}_{= \frac{S}{a_0^d}\frac{\partial \vec{n}}{\partial t}(\vec{n}\times\vec{n}) = 0} + \lambda \underbrace{\vec{n}\cdot\vec{n}}_{=1} = -\frac{|J|S^2}{a_0^{d-2}} (\vec{n}\cdot\nabla^2\vec{n})$$

Thus: $$\lambda = -\frac{|J|S^2}{a_0^{d-2}} (\vec{n}\cdot\nabla^2\vec{n}).$$

Now, ~~sus~~ substitude the λ back into equation:

$$\frac{S}{a_0^d}\cdot\left(\vec{n} \times \frac{\partial \vec{n}}{\partial t}\right) - \left[\frac{-|J|S^2}{a_0^{d-2}}(\vec{n}\cdot\nabla^2\vec{n})\right]\vec{n} = -\frac{|J|S^2}{a_0^{d-2}} \nabla^2 \vec{n}.$$

Rearrange to group the derivative terms on the right:

$$\frac{S}{a_0^d}\left(\vec{n} \times \frac{\partial \vec{n}}{\partial t}\right) = -\frac{|J|S^2}{a_0^{d-2}}\left[\nabla^2\vec{n} - (\vec{n}\cdot\nabla^2\vec{n})\cdot\vec{n}\right]$$

If we look at: $\vec{n} \times (\vec{n} \times \nabla^2 \vec{n})$:

$$\vec{n} \times (\vec{n} \times \nabla^2 \vec{n}) = \vec{n} \cdot (\vec{n} \cdot \nabla^2 \vec{n}) - \nabla^2 \vec{n} \, (\vec{n} \cdot \vec{n})$$

$$= \vec{n} \, (\vec{n} \cdot \nabla^2 \vec{n}) - \nabla^2 \vec{n}.$$

So:

$$\frac{S}{a_0^d} (\vec{n} \times \partial_t \vec{n}) = \frac{|J| S^2}{a_0^{d-2}} \left[\vec{n} \times (\vec{n} \times \nabla^2 \vec{n}) \right]$$

Simplified to:

~~We co~~

$$\vec{n} \times \overset{\partial_t \vec{n}}{\cancel{\partial_t \vec{n}}} = \vec{n} \times \left[\cancel{\frac{|J| S^2}{a_0}} |J| S^2 a_0^2 (\nabla^2 \vec{n}) \right]$$

We can get the Landau-Lifshitz equation:

$$\cancel{\partial_t \vec{n} =} \frac{\partial \vec{n}}{\partial t} = |J| S^2 a_0^2 (\vec{n} \times \nabla^2 \vec{n}).$$