$$\delta_f - \delta_i = 2\pi \cdot N \quad (N \in \mathbb{Z}).$$

So for a close path, $\dot{S}[n(t)]$, to be more precise, the geometric phase factor: ~~$e^{iS'} = e^{iS + i\Delta\delta} = e^{iS + 2\pi N i} = e$~~

$$e^{iS'} = e^{iS + i\Delta\delta} = e^{iS + 2\pi N \cdot i} = e^{iS}$$

remains unchanged, or that the geometric. phase S is invariant under the ~~model~~ module 2π operation.

2.2 Wess-Zumino term and Witten extension

The Wess-Zumino term is S_{WZ}, which satisfies:

$$S = \frac{1}{2} \iint_{\partial A r} d\Omega = \frac{1}{2} S_{WZ} = S_{spin} \cdot S_{WZ}.$$

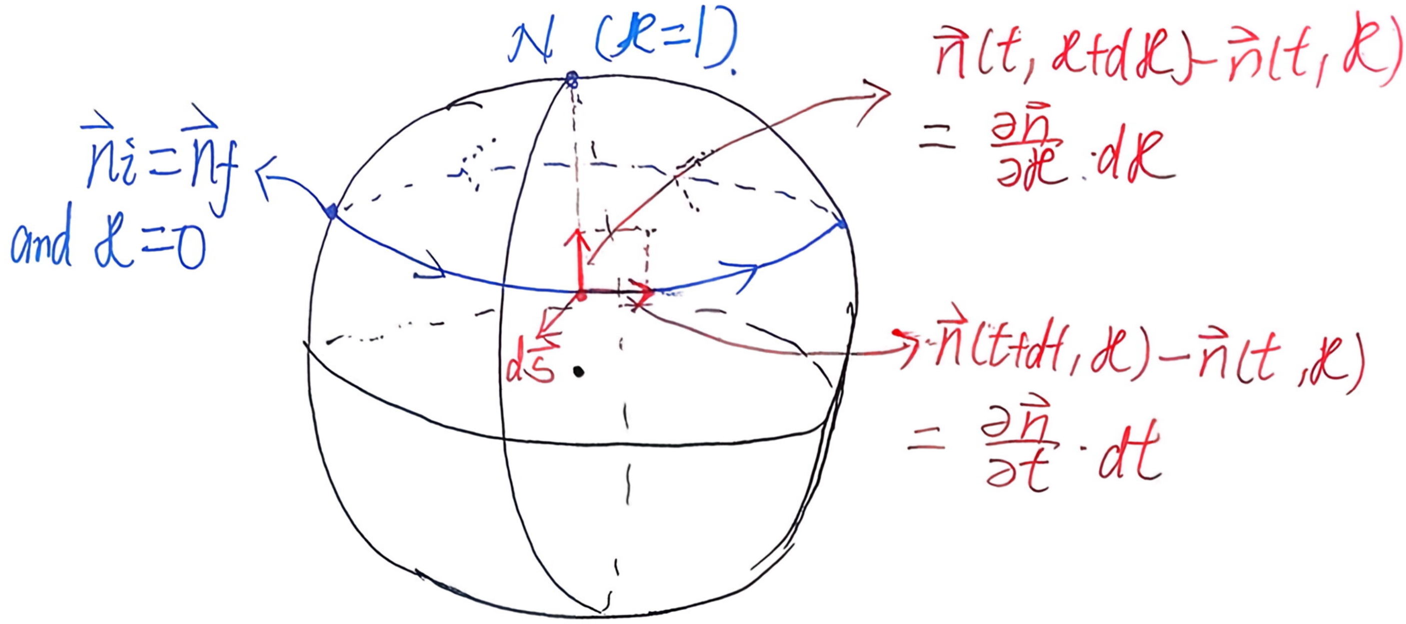
It means that:

$$S_{WZ} = -\iint_{\partial A r} d\Omega = -\iint_{A r} \vec{n} \cdot d\vec{S}$$

$$\cancel{= -i \int_{t_i}^{t_f} dt \langle n(t) | \frac{d}{dt} | n(t) \rangle}$$

~~let's rewrite the integral with $|n(t)\rangle = \sum \vec{n} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$~~

~~and. $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ which are coodinates on Bloch Sphere.~~



2.3. Emergence of U(1) gauge degrees of freedom.

$$S = i \int_\gamma dt \langle \vec{n}(t) | \partial_t | \vec{n}(t) \rangle = i \int_\gamma dt\, \vec{z}^\dagger \cdot \partial_t \vec{z}$$

where, $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = |\vec{n}\rangle$.

Using the chain rule where $\vec{x} = (x_1, x_2)$ are coordinates on Bloch ~~s~~ Sphere:

$$S = i \int_\gamma \vec{z}^\dagger \frac{\partial \vec{z}}{\partial \vec{x}} \cdot \frac{d\vec{x}}{dt} \cdot dt.$$

define the Gauge Potential. $\vec{A}$:

$$\vec{A} = i \vec{z}^\dagger \cdot \frac{\partial \vec{z}}{\partial \vec{x}}$$

so S can be written as

$$S = \int_\gamma \vec{A} \cdot d\vec{l}$$

where $\vec{n}$ can regrad as the field strength ~~wo~~ whose charge is 1. Because $\vec{n}$ locate in the sphere, it can be the function of two parameter t and x. which form an orthonormal busis on the sphere. So $d\vec{\xi}$ can be written as:

$$d\vec{\xi} = d\vec{n}(t+dt, x) \times d\vec{n}(t, x+dx)$$

$$= \left(\frac{\partial \vec{n}}{\partial t} \cdot dt\right) \times \left(\frac{\partial \vec{n}}{\partial x} \cdot dx\right)$$

$$= \left(\frac{\partial \vec{n}}{\partial t} \times \frac{\partial \vec{n}}{\partial x}\right) dt \cdot dx.$$

So the Wess-Zumion term have another expression that is broadly used:

$$S_{WZ} = -\int_0^1 dx \int_{t_i}^{t_f} dt \cdot \vec{n}(t, x) \cdot \left[\frac{\partial \vec{n}}{\partial t} \times \frac{\partial \vec{n}}{\partial x}\right]$$

where x is an auxillary parameter, in order to extend. $\vec{n}(t)$ to $\vec{n}(t, x)$ with:

$$\begin{cases} \vec{n}(t, 0) = \vec{n}(t) \\ \vec{n}(t, 1) = n_0 \text{ (North Pole)} \\ \vec{n}(t_i, x) = \vec{n}(t_f, x) \text{ (Closed Path)} \end{cases}$$

where $d\vec{\ell} = \frac{d\vec{x}}{dt} \cdot dt = \dot{\gamma} \cdot dt$ is the differential of path γ.

we apply a local guage transformation where the state z picks up a position-dependent phase $\phi(\vec{x})$

$$z \rightarrow z \cdot e^{i\phi}$$

So the Gauge Potential change as:

$$\vec{A}' = i \bar{z} e^{-i\phi} \left(e^{i\phi} \frac{\partial z}{\partial \vec{x}} + z \cdot \nabla \cdot \phi \right)$$

$$= \vec{A} + \nabla \cdot \phi$$

So the potential $\vec{A}$ is not gauge-invariant

From what we have discussed above, we can know that:

A single spin $\xleftrightarrow{\text{equivolent to}}$ A spherical Landau ~~prombl~~ problem with a monopole.

Because of: (for closed ~~ph~~ path)

$$S = \oint_{\gamma} \vec{A} \cdot d\vec{\ell}$$

where $\vec{A}$ can be regard as a magnetic potential that is responsible for a "magnetic monopole" at the center of Bloch Ball. Therefore, S_{WZ} can be regarded as a quantum mechanical problem: a charge particle with sufficiently small mass is confined on the unit sphere, and feels electromagnetic field generated by the "magnetic monopole" at the center of the ball.

2.4 Conclusion.

In the previews discussion, we have obtained path integral formalism of a single spin-$\frac{1}{2}$ ($S=\frac{1}{2}$).

For a generic spin-S ($S=\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$), the action is given by:

$$
\begin{cases}
S[\vec{n}(t)] = S_{spin} \cdot S_{WZ}[\vec{n}(t)] \quad (\vec{H}=0) \\
\langle \vec{n} | \vec{S} | \vec{n} \rangle = S_{spin} \cdot \vec{n}
\end{cases}
$$