Also, we can look at the situations where θ affects the dynamics classically. This occurs when θ is not constant, ~~but~~ instead, varies in space and time.

$$\theta = \theta(\vec{x}, t).$$

So we can ~~wit~~ write down the axion ~~d~~ electrodynamics action:

$$S = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{e^2}{16\pi^2}\cdot\theta(\vec{x},t)\cdot {}^{*}F^{\mu\nu}\cdot F_{\mu\nu}\right)$$

We use Euler-Lagrange equations: ~~to~~

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0$$

to get the equactions of axion electrodynamics

Because $\mathcal{L}$ doesn't depend on A_μ, so the equactions of motion are:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \Rightarrow \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0$$

and the $\mathcal{L}$ of axion ~~dr~~ electrodynamics is ($\alpha = \frac{e^2}{4\pi}$):

$$\mathcal{L} = \mathcal{L}_{Max} + \mathcal{L}_\theta$$

$$= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\alpha}{4\pi}\cdot\theta(\vec{x},t)\cdot {}^{*}F^{\mu\nu}F_{\mu\nu}$$

First, let's see $\mathcal{L}_\theta$:

$$\frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{\partial}{\partial(\partial_\nu A_\mu)}\cdot\frac{\alpha}{4\pi}\,\theta(\vec{x},t)\cdot\cancel{\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}F_{\mu\nu}}$$

$$= \frac{\alpha}{\cancel{8}\pi}\left\{\left[\frac{\partial}{\partial(\partial_\nu A_\mu)}\cdot\theta(\vec{x},t)\right]\cdot\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} + \theta(\vec{x},t)\left[\frac{\partial\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}}{\partial(\partial_\nu A_\mu)}\right]\right\}$$

let's see the first term.

$$\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}\frac{\partial}{\partial(\partial_\nu A_\mu)}\theta(\vec{x},t) = 0$$

then let's see the second term:

$$\theta(\vec{x},t)\frac{\partial}{\partial(\partial_\nu A_\mu)}\cdot\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}\ \cancel{=\theta(\vec{x},t)}$$

$$= 4\theta(\vec{x},t)\left(\frac{\partial}{\partial(\partial_\nu A_\mu)}\cdot\partial_\alpha A_\beta\partial_\rho A_\sigma\right)\epsilon^{\alpha\beta\rho\sigma}$$

$$= 4\theta(\vec{x},t)\left(\delta^\nu_\alpha\delta^\mu_\beta\cdot\partial_\rho A_\sigma + \delta^\nu_\rho\delta^\mu_\sigma\cdot\partial_\alpha A_\beta\right)\cdot\epsilon^{\alpha\beta\rho\sigma}$$

$$= 4\theta(\vec{x},t)\cdot\epsilon^{\nu\mu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$= 8\theta(\vec{x},t)\,{}^*F^{\nu\mu}$$

next, we can get:

$$\partial_\nu\theta(\vec{x},t)\,{}^*F^{\nu\mu} = {}^*F^{\nu\mu}\,\partial_\nu\theta(\vec{x},t) + \theta(\vec{x},t)\cdot\partial_\nu\,{}^*F^{\nu\mu}$$

using the Bianchi identities:

$$\partial_\mu\,{}^*F^{\mu\nu} = 0$$

so we can get:

$$\partial_\nu \theta(\vec{x},t)\, {}^*F^{\nu\mu} = {}^*F^{\nu\mu}\, \partial_\nu \theta(\vec{x},t)$$

~~Finally, we can~~ Finally, we can get:

$$\partial_\nu \left[\frac{\partial \mathcal{L}_\theta}{\partial(\partial_\nu A_\mu)}\right] = \frac{\alpha}{\pi}\, {}^*F^{\nu\mu} \cdot \partial_\nu \theta(\vec{x},t)$$

~~th~~ And the Lagrangian of Maxwell part:

$$\partial_\nu \left[\frac{\partial \mathcal{L}_{max}}{\partial(\partial_\nu A_\mu)}\right] = -\partial_\nu F^{\nu\mu}.$$

So the equations of axion electrodynamics is.

$$\begin{cases} \partial_\nu F^{\nu\mu} = \frac{\alpha}{\pi}\, {}^*F^{\nu\mu} \cdot \partial_\nu \theta(\vec{x},t) \\ \partial_\mu {}^*F^{\mu\nu} = 0 \end{cases}$$

In the end, the deformed Maxwell equations which are ~~axi~~ the ~~e~~ equations of axion electrodynamics are:

$$\begin{cases} \nabla \cdot \vec{E} = \frac{\alpha}{\pi}(\nabla \theta) \cdot \vec{B} \\ \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} - \frac{\alpha}{\pi}(\dot{\theta}\vec{B} + \nabla\theta \times \vec{E}) \\ \nabla \cdot \vec{B} = 0 \\ \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0 \end{cases}$$