The first equation tells us that in regions of space where θ varies, a magnetic field $\vec{B}$ acts like an electric charge density. The second equation tells us that ~~tha~~ the combination $(\dot{\theta}\vec{B} + \nabla\theta \times \vec{E})$ acts like a current density

1.1 ~~θ~~ The ~~predi~~ periodicity of θ and its values under Parity or Time-Reversal Symmetry.

In quantum theory, θ is a periodic variable: it lies in the range:

$$\theta \in [0, 2\pi)$$

After imposing appropriate boundary conditions, S_θ can only take values of the form:

$$S_\theta = \theta N \quad \text{with } N \in \mathbb{Z}$$

This means that the theta angle contributes to the partition function as

$$e^{iS_\theta} = e^{iN\theta}$$

To show that S_θ must take the form, we consider a compact Euclidean spacetime which we take to $\mathbf{T}^4$ and we take each of the circles in the torus to have radii R.

We consider easy case:

$$\begin{cases} \vec{E} = (0, 0, E_z), \quad E_z = F_{03} = \partial_0 A_3 - \partial_3 A_0 \\ \vec{B} = (0, 0, B_z). \quad B_z = F_{21} = \partial_2 A_1 - \partial_1 A_2 \end{cases}$$

The integral of S_θ is:

$$I = \int_{T^4} d^4x \, E_z \cdot B_z = \int_{T^4} dx^0 dx^3 \cdot E_z \cdot \int_{T^4} dx^1 dx^2 \cdot B_z$$

The ~~gu~~gauge field A_μ must be will defind on the underlying tour, which will put restrictions on the allow values of E_z and B_z.

So the integral can't take any ~~k~~ value.

Now let consider the restrictions of A_μ if A_μ be defined on tour. When a direction of space, say x^1, is periodic with radius R, the physics state in original point ($x^1 = 0$) must be identical to the final physics state:

$$\text{state}[A_1(0)] = \cancel{A_1(2\pi R)}) \; \text{state}[A_1(0)]$$