1. The theta term. and Axion Electrodynamics

In relativistic notation, the Maxwell action for electromagnetism takes a wonderfully compact form:

$$S_{Max} = \int d^4x - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \int d^4x \, \frac{1}{2} (\vec{E}^2 - \vec{B}^2)$$

Here:

$$\begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ E_i = F_{0i} \\ F_{ij} = -\epsilon_{ijk} B_k \end{cases}$$

One reason that the Maxwell action is so simple is that there is very little else we can write down that is both gauge invariant and Lorentz invariant. There is, however one term that we can add to the Maxwell action is also both gauge invariant and Lorentz invariant. which we call it theta term:

$$S_\theta = \frac{\theta e^2}{4\pi^2} \cdot \int d^4x \, \frac{1}{4} {}^\star F^{\mu\nu} \cdot F_{\mu\nu} = -\frac{\theta e^2}{4\pi^2} \int d^4x \, \vec{E} \cdot \vec{B}$$

which ${}^\star F^{\mu\nu}$ is the dual tensor:

$${}^\star F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \cdot F_{\rho\sigma}$$

and θ is a constant parameter.

However, the theta term is simple to be check that ~~the~~ it can be written as a total derivative:

$$\int d^4x \frac{1}{4} {}^*F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}\int d^4x \cdot \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \cdot F_{\rho\sigma} \cdot F_{\mu\nu}$$

$$= \frac{1}{2}\int d^4x \cdot \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \cdot (\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= \frac{1}{2}\int d^4x \cdot \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \Big[(\partial_\mu A_\nu \partial_\rho A_\sigma + \partial_\nu A_\mu \partial_\sigma A_\rho)$$

$$\cancel{(\partial_\mu \partial_\nu A_\mu \partial_\sigma A_\rho + \partial}$$

$$- (\partial_\mu A_\nu \partial_\sigma A_\rho + \partial_\nu A_\mu \partial_\rho A_\sigma)\Big]$$

$$= \frac{1}{2}\int d^4x \cdot \cancel{4}\, \epsilon^{\mu\nu\rho\sigma} (\partial_\mu A_\nu \partial_\rho A_\sigma \cancel{+ \partial_\nu A_\mu \partial_\sigma A_\rho})$$

Now let's proof a useful equation:

$$\epsilon^{\mu\nu\rho\sigma} \cdot A_\nu \partial_\mu \partial_\rho A_\sigma = 0$$

Because:

$$\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\mu \partial_\rho A_\sigma = \epsilon^{\rho\nu\mu\sigma} \cancel{A_\rho \partial_\mu A_\sigma} A_\nu \partial_\rho \partial_\mu A_\sigma$$

$$= \epsilon^{\rho\nu\mu\sigma} \cancel{A_\rho} A_\nu \partial_\mu \partial_\rho A_\sigma$$

$$= -\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\mu \partial_\rho A_\sigma.$$

So we can get:

$$2\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\mu \partial_\rho A_\sigma = 0 \Rightarrow \epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\mu \partial_\rho A_\sigma = 0$$

So the quadratic part of theta term become:

$$\int d^4x \frac{1}{4} {}^*F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\int d^4x \cdot \cancel{\tfrac{1}{4}}\, \epsilon^{\mu\nu\rho\sigma} \cdot (\partial_\mu A_\nu \partial_\rho A + A_\nu \partial_\mu \partial_\rho A_\sigma)$$

$$= \frac{1}{2}\int d^4x \, \epsilon^{\mu\nu\rho\sigma} \cdot \partial_\mu (A_\nu \partial_\rho A_\sigma)$$

The ~~total~~ derivative form of theta therm is:

$$S_\theta = \frac{\theta e^2}{8\pi^2} \int d^4x \, \partial_\mu (\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma)$$

We say that the theta term is topological. It depends only ~~the~~ on boundary information. However, when ~~deriving~~ using the ~~princeple~~ principle of least action ~~to~~ derive the field's equation of motion, the values of field are ~~is~~ fixed ~~invariant~~ on the infinity boundary. Therefore, the ~~sh~~ upshot is that the theta term does not change the equations of motion and, it would seem, can have little effect on the physics. ~~But~~.

But under some situations that involve subtle interplay between quantum mechenics and topology, there ~~as~~ are a number of interesting phenomena~~s~~ of physics which are ~~leaded~~ led by theta term.