1 The theta term and Axion Electrodynamics

In relativistic notation, the Maxwell action for electromagnetism takes a wonderfully compact form:

$$S_{Max} = \int \mathrm{d}^4x - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \int \mathrm{d}^4x \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) \tag{1}$$

Here:

$$\begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \quad = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \\ E_i = F_{0i} \\ F_{ij} = -\epsilon_{ijk}B_k \end{cases} \tag{2}$$

One reason that the Maxwell action is so simple is that there is very little else we can write down that is both gauge invariant and Lorentz invariant. There is, however one term that we can add to the Maxwell action is also both gauge invariant and Lorentz invariant, which we call it theta term:

$$S_\theta = \frac{\theta e^2}{4\pi^2} \int \mathrm{d}^4x \frac{1}{4} {}^\star F^{\mu\nu}F_{\mu\nu} = -\frac{\theta e^2}{4\pi^2} \int \mathrm{d}^4x \boldsymbol{E} \cdot \boldsymbol{B} \tag{3}$$

which ${}^\star F^{\mu\nu}$ is the dual tensor:

$$\begin{aligned} {}^\star F^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \\ &= \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \end{aligned} \tag{4}$$

and θ is a constant parameter.

However, the theta term is simple to check that it can be written as a total derivative:

$$\begin{aligned} \int \mathrm{d}^4x \frac{1}{4} {}^\star F^{\mu\nu}F_{\mu\nu} &= \frac{1}{2}\int \mathrm{d}^4x \cdot \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}F_{\mu\nu} \\ &= \frac{1}{8}\int \mathrm{d}^4x \cdot \epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \frac{1}{2}\int \mathrm{d}^4x \cdot \epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu \partial_\rho A_\sigma) \end{aligned} \tag{5}$$

Now let's proof a useful equation:

$$\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\mu\partial_\rho A_\sigma = 0 \tag{6}$$

Because:

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\mu\partial_\rho A_\sigma &= \epsilon^{\rho\nu\mu\sigma}A_\nu\partial_\rho\partial_\mu A_\sigma \\ &= \epsilon^{\rho\nu\mu\sigma}A_\nu\partial_\mu\partial_\rho A_\sigma \\ &= -\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\mu\partial_\rho A_\sigma\end{aligned} \tag{7}$$

So we can get:

$$2\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\mu\partial_\rho A_\sigma = 0 \Rightarrow \epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\mu\partial_\rho A_\sigma = 0 \tag{8}$$

So the quadratic part of theta term become:

$$\begin{aligned}\int \mathrm{d}^4x\frac{1}{4}{}^\star F^{\mu\nu}F_{\mu\nu} &= \frac{1}{2}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu\partial_\rho A_\sigma + A_\nu\partial_\mu\partial_\rho A_\sigma) \\ &= \frac{1}{2}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu\partial_\rho A_\sigma)\end{aligned} \tag{9}$$

The total derivative form of theta term is:

$$S_\theta = \frac{\theta e^2}{8\pi^2}\int \mathrm{d}^4x\partial_\mu(\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma) \tag{10}$$

We say that the theta term is topological. It depends only on boundary information. However, when deriving using the principle of least action to derive the field's equation of motion, the values of field are fixed on the infinite boundary. Therefore, the upshot is that the theta term does not change the equations of motion and, it would seem, can have little effect on the physics.

But under some situations that involve subtle interplay between quantum mechanics and topology, there are a number of interesting phenomena of physics which are led by theta term.

Also, we can look at the situations where θ affects the dynamics classically. This occurs when θ is not constant, but instead varies in space and time.

$$\theta = \theta(\boldsymbol{x}, t) \tag{11}$$

So we can write down the axion electrodynamics action:

$$S = \int \mathrm{d}^4x(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{e^2}{16\pi^2}\theta(\boldsymbol{x}, t)^\star F^{\mu\nu}F_{\mu\nu}) \tag{12}$$

We use Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\nu(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}) = 0 \tag{13}$$

to get the equations of axion electrodynamics

Because $\mathcal{L}$ doesn't depend on A_μ, so the equations of motion are:

$$\frac{\partial\mathcal{L}}{\partial A_\mu} = 0 \Rightarrow \partial_\nu[\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}] = 0 \tag{14}$$

and the $\mathcal{L}$ of axion electrodynamics is ($\alpha = \frac{e^2}{4\pi}$):

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{Max} + \mathcal{L}_{\theta} \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\alpha}{4\pi}\theta(\boldsymbol{x},t)^{\star}F^{\mu\nu}F_{\mu\nu}\end{aligned} \tag{15}$$

First, let's see $\mathcal{L}_{\theta}$:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_{\nu}A_{\mu})} &= \frac{\partial}{\partial(\partial_{\nu}A_{\mu})}\cdot\frac{\alpha}{4\pi}\theta(\boldsymbol{x},t)\cdot\frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} \\ &= \frac{\alpha}{8\pi}\{[\frac{\partial}{\partial(\partial_{\nu}A_{\mu})}\theta(\boldsymbol{x},t)]\cdot\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} + \theta(\boldsymbol{x},t)[\frac{\partial\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}}{\partial(\partial_{\nu}A_{\mu})}]\}\end{aligned} \tag{16}$$

let's see the first term:

$$\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}\frac{\partial}{\partial(\partial_{\nu}A_{\mu})}\theta(\boldsymbol{x},t) = 0 \tag{17}$$

then let's see the second term:

$$\begin{aligned}\theta(\boldsymbol{x},t)\cdot\frac{\partial}{\partial(\partial_{\nu}A_{\mu})}\cdot\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} &= 4\theta(\boldsymbol{x},t)(\frac{\partial}{\partial(\partial_{\nu}A_{\mu})}\partial_{\alpha}A_{\beta}\partial_{\rho}A_{\sigma})\epsilon^{\alpha\beta\rho\sigma} \\ &= 4\theta(\boldsymbol{x},t)(\delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\partial_{\rho}A_{\sigma} + \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma}\partial_{\alpha}A_{\beta})\cdot\epsilon^{\alpha\beta\rho\sigma} \\ &= 4\theta(\boldsymbol{x},t)\cdot\epsilon^{\nu\mu\rho\sigma}(\partial_{\rho}A_{\sigma} - \partial_{\sigma}A_{\rho}) \\ &= 8\theta(\boldsymbol{x},t)^{\star}F^{\nu\mu}\end{aligned} \tag{18}$$

next, we can get:

$$\partial_{\nu}[\theta(\boldsymbol{x},t)^{\star}F^{\nu\mu}] = {}^{\star}F^{\nu\mu}\partial_{\nu}\theta(\boldsymbol{x},t) + \theta(\boldsymbol{x},t)\cdot\partial_{\nu}{}^{\star}F^{\nu\mu} \tag{19}$$

using the Bianchi identities:

$$\partial_{\mu}{}^{\star}F^{\mu\nu} = 0 \tag{20}$$

so we can get:

$$\partial_{\nu}\theta(\boldsymbol{x},t)^{\star}F^{\nu\mu} = {}^{\star}F^{\nu\mu}\partial_{\nu}\theta(\boldsymbol{x},t) \tag{21}$$

Finally, we can get:

$$\partial_{\nu}[\frac{\partial\mathcal{L}_{\theta}}{\partial(\partial_{\nu}A_{\mu})}] = \frac{\alpha}{\pi}{}^{\star}F^{\nu\mu}\cdot\partial_{\nu}\theta(\boldsymbol{x},t) \tag{22}$$

And the Lagrangian of Maxwell part:

$$\partial_{\nu}[\frac{\partial\mathcal{L}_{Max}}{\partial(\partial_{\nu}A_{\mu})}] = -\partial_{\nu}F^{\nu\mu} \tag{23}$$

So the equations of axion electrodynamics is:

$$\begin{cases}\partial_{\nu}F^{\nu\mu} = \frac{\alpha}{\pi}{}^{\star}F^{\nu\mu}\cdot\partial_{\nu}\theta(\boldsymbol{x},t) \\ \partial_{\mu}{}^{\star}F^{\mu\nu} = 0\end{cases} \tag{24}$$

In the end, the deformed Maxwell equations which are the equations of axion electrodynamics are:

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \dfrac{\alpha}{\pi}(\nabla\theta) \cdot \boldsymbol{B} \\ \nabla \times \boldsymbol{B} = \dfrac{\partial \boldsymbol{E}}{\partial t} - \dfrac{\alpha}{\pi}(\dot{\theta}\boldsymbol{B} + \nabla\theta \times \boldsymbol{E}) \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \end{cases} \tag{25}$$

The first equation tells us that in regions of space where θ varies, a magnetic field $\boldsymbol{B}$ acts like an electric charge density. The second equation tells us that the combination $(\dot{\theta}\boldsymbol{B} + \nabla\theta \times \boldsymbol{E})$ acts like a current density

1.1 The periodicity of theta and its values under Parity or Time - Reversal Symmetry

In quantum theory, θ is a periodic variable: it lies in the range:

$$\theta \in [0, 2\pi) \tag{26}$$

After imposing appropriate boundary conditions, S_θ can only take values of the form:

$$S_\theta = \theta N \quad \text{with } N \in \mathbb{Z} \tag{27}$$

This means that the theta angle contributes to the partition function as

$$e^{\mathrm{i}S_\theta} = e^{\mathrm{i}N\theta} \tag{28}$$

To show that S_θ must take the form, we consider a compact Euclidean spacetime which we take to T^4 and we take each of the circles in the torus to have radii R.

We consider easy case:

$$\begin{cases} \boldsymbol{E} = (0, 0, E_z),\ E_z = F_{03} = \partial_0 A_3 - \partial_3 A_0 \\ \boldsymbol{B} = (0, 0, B_z),\ B_z = F_{21} = \partial_2 A_1 - \partial_1 A_2 \end{cases} \tag{29}$$

The integral of S_θ is:

$$I = \int_{T^4} \mathrm{d}^4 x E_z \cdot B_z = \int_{T^4} \mathrm{d}x^0 \mathrm{d}x^3 \cdot E_z \cdot \int_{T^4} \mathrm{d}x^1 \mathrm{d}x^2 \cdot B_z \tag{30}$$

The gauge field A_μ must be well defined on the underlying torus, which will put restrictions on the allowed values of E_z and B_z. So the integral can't take any value.

Now let consider the restrictions of A_μ if A_μ be defined on torus. When a direction of space, say x^1, is periodic with radius R, the physics state in original point ($x^1 = 0$) must be identical to the final physics state:

$$\text{state}[A_1(0)] = \text{state}[A_1(2\pi R)] \tag{31}$$

Because $B_z = \partial_2 A_1, (A_2 = 0) = \text{constant}$, so $A_1 = B_z x^2 + C$. The simplest case is $A_1 = 0$ but this case is trivial. So we should find a gauge transformations to connect $A(0)$ and $A(2\pi R)$, which will make two different values of A_1 correspond to the same physics state.

$$A_1(2\pi R) = A_1(0) + G \tag{32}$$

let's find the gauge transformation that we are interested:

$$A_1 \to A_1 + \partial_1 \chi \tag{33}$$

We do not insist that $\chi(x)$ is single valued. Instead, we require only that $e^{\mathrm{i}e\chi}$ is single-valued, since this is what acts on the wavefunction. So χ can have the simple form followed:

$$\chi = \frac{nx^1}{eR} \quad \text{with } n \in \mathbb{Z} \tag{34}$$

Because in the same point $(2\pi NR + X)$, $e^{\mathrm{i}e\chi}$ is the same state:

There are sometimes called large gauge transformations, a name which reflects the fact that they cannot be continuously deformed to the identity. Under such a gauge transformation, we can see that:

$$A_1 \to A_1 + \partial_1 \chi = A_1 + \frac{n}{eR} \tag{35}$$

the two different values of A_1 correspond the same physics state. From this transformation, we can determine the expression of A_1,

$$\begin{aligned} &A_1(0) = A_1(2\pi R) + \frac{n}{eR} \\ &\Rightarrow A_1 = \frac{nx^2}{2\pi e R^2} \end{aligned} \tag{36}$$

And we can write down the expression of B_z:

$$B_z = \partial_2 A_1 = \frac{n}{2\pi e R^2} \tag{37}$$

So the integral of magnetic field is:

$$\int_{T^2} \mathrm{d}x^1 \mathrm{d}x^2 B_z = \frac{2\pi n}{e} \quad \text{with } n \in \mathbb{Z} \tag{38}$$

We can also apply exactly the same argument to the electric field

$$E_z = \partial_0 A_3 \quad (A_0 = 0) \tag{39}$$

So the integral of electric field is:

$$\int_{T^2} \mathrm{d}x^0 \mathrm{d}x^3 E_z = \frac{2\pi n'}{e} \quad \text{with } n' \in \mathbb{Z} \tag{40}$$

Armed with two integrals above, the S_θ becomes:

$$\begin{aligned} S_\theta &= -\frac{\theta e^2}{4\pi^2}\int_{T^4} \mathrm{d}^4x \boldsymbol{E}\cdot\boldsymbol{B} \\ &= -\frac{\theta e^2}{4\pi^2}\frac{4\pi^2}{e^2}\cdot nn' \quad (\text{let } N = -nn') \\ &= N\theta \quad \text{with } N \in \mathbb{Z} \end{aligned} \tag{41}$$

It's pretty straightforward to generalise it to non-constant $\boldsymbol{E}$ and $\boldsymbol{B}$ fields:

$$\begin{aligned} \int_{T^2} \mathrm{d}x^1\mathrm{d}x^2 B_z &= \int_{T^2} \mathrm{d}x^1\mathrm{d}x^2 \cdot \partial_2 A_1 \\ &= 2\pi R[A_1(2\pi R) - A_1(0)] \\ &= \frac{2\pi n}{e} \end{aligned} \tag{42}$$

So we can see that θ is a periodic variable in quantum theory.

Although the theta term is gauge invariant and Lorentz invariant, it is not invariant under certain discrete symmetries, and does not preserve the same symmetries as the Maxwell term. These discrete symmetries are parity $\mathcal{P}$ and time reversal $\mathcal{T}$.

$$\begin{cases} \mathcal{P}: \boldsymbol{x} \to -\boldsymbol{x} \\ \mathcal{T}: t \to -t \end{cases} \tag{43}$$

If we act on both $\boldsymbol{E}$ and $\boldsymbol{B}$

$$\begin{cases} \mathcal{P}: \boldsymbol{E}(\boldsymbol{x}, t) \to -\boldsymbol{E}(-\boldsymbol{x}, t) \text{ and } \mathcal{P}: \boldsymbol{B}(\boldsymbol{x}, t) \to \boldsymbol{B}(-\boldsymbol{x}, t) \\ \mathcal{T}: \boldsymbol{E}(\boldsymbol{x}, t) \to \boldsymbol{E}(\boldsymbol{x}, -t) \text{ and } \mathcal{T}: \boldsymbol{B}(\boldsymbol{x}, t) \to -\boldsymbol{B}(\boldsymbol{x}, -t) \end{cases} \tag{44}$$

So $\boldsymbol{E}$ is odd under parity and even under time reversal. $\boldsymbol{B}$ is even under parity and odd under time reversal.

This means that, the theta term $S_\theta \propto \int \mathrm{d}^4x \boldsymbol{E}\cdot\boldsymbol{B}$ breaks both parity and time reversal invariance, which means $S_\theta \to -S_\theta$ under $\mathcal{P}$ and $\mathcal{T}$. But because of the periodicity of θ, there are two exceptions:

$$S_\theta = -S_\theta + 2\pi N \tag{45}$$

In order to make the equation hold for any integer values of N, θ must equal to 0 or π ($\theta \in [0, 2\pi)$).

So we can know that when $\theta = 0$ or $\theta = \pi$, the theory is invariant under $\mathcal{P}$ and $\mathcal{T}$ symmetries.

In subsequent sections, we can see that the topological insulator said to have a $\mathbb{Z}_2$ classification has $\theta = \pi$ because these materials are defined to be time-reversal invariant. Most materials have $\theta = 0$. But some materials have a band structure which is twisted in a particular way, which results in $\theta = \pi$.

1.2 Continuity Conditions

Let's consider a equivalent way to describing the physics above. We introduce the electric displacement:

$$\boldsymbol{D} = \boldsymbol{E} - \frac{\alpha\theta}{\pi} \cdot \boldsymbol{B} \tag{46}$$

because the divergence of $\boldsymbol{B}$ equal to zero, so:

$$\nabla \boldsymbol{D} = \nabla \boldsymbol{E} - \frac{\alpha}{\pi}(\nabla \cdot \theta)\boldsymbol{B} = 0. \tag{47}$$

Comparing to the expression of $\boldsymbol{D}$, we see that a magnetic field $\boldsymbol{B}$ act like polarisation in a topological insulator. When the θ varies, we have a varying polarisation resulting in bound charge. Similarly, we define the magnetising field

$$\boldsymbol{H} = \boldsymbol{B} + \frac{\alpha}{\pi}\theta \cdot \boldsymbol{E} \tag{48}$$

Using the relationship between $\boldsymbol{B}$ and $\boldsymbol{E}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \tag{49}$$

so we can get:

$$\begin{aligned}
\nabla \times \boldsymbol{H} &= \nabla \times \boldsymbol{B} + \frac{\alpha}{\pi}[(\nabla\theta) \times \boldsymbol{E} + \theta(\nabla \times \boldsymbol{E})] \\
&= \frac{\partial \boldsymbol{D}}{\partial t} + \frac{\alpha}{\pi}\theta \cdot (\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E}) \\
&= \frac{\partial \boldsymbol{D}}{\partial t}
\end{aligned} \tag{50}$$

In topological insulator, $\boldsymbol{E}$ act like magnetisation. When θ varies, we get a varying magnetisation which results in bound currents.

With these definitions, the equations of axion electrodynamics take the usual form of Maxwell equations in matter:

$$\begin{cases}
\nabla \cdot \boldsymbol{D} = 0 \\
\nabla \times \boldsymbol{H} - \dfrac{\partial \boldsymbol{D}}{\partial t} = 0 \\
\nabla \cdot \boldsymbol{B} = 0 \\
\dfrac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0
\end{cases} \tag{51}$$

This means that if we introduce the normal vector to the surface $\hat{\boldsymbol{n}}$, we have the boundary conditions:

$$\begin{cases} \hat{\boldsymbol{n}} \cdot \Delta \boldsymbol{D} = 0 \\ \hat{\boldsymbol{n}} \times \Delta \boldsymbol{E} = 0 \\ \hat{\boldsymbol{n}} \cdot \Delta \boldsymbol{B} = 0 \\ \hat{\boldsymbol{n}} \times \Delta \boldsymbol{H} = 0 \end{cases} \tag{52}$$

1.3 The Witten effect

The effect of the θ term is to endow the magnetic monopole with an electric charge, which is known as the Witten effect

Let's take a magnetic monopole with magnetic charge g and place in inside a vacuum ($\theta = 0$). Then, we surround this with a medium that has $\theta = \Theta \neq 0$

When the magnetic field crosses the interface where θ changes, it will induce an electric charge:

$$\begin{aligned} \int \nabla \cdot \boldsymbol{E} \mathrm{d}v &= \int \frac{\alpha}{\pi} (\nabla \cdot \theta) \cdot \boldsymbol{B} \mathrm{d}v. \\ \Rightarrow \quad Q \quad &= \frac{\alpha}{\pi} \Theta g \end{aligned} \tag{53}$$

This result is independent of the size of the interior region whose θ is zero. So we could shrink this region down until it is infinitesimally small, and we still find that the monopole has charge q: In fact a monopole is a dyon which carries electric charge when θ not equal to zero.

When the monopole carries the minimum allowed magnetic charge, its electric charge is given by:

$$g = \frac{2\pi}{e} \Rightarrow q = \frac{e\theta}{2\pi} \tag{54}$$

In particular, if we place a magnetic monopole inside a topological insulator, it turns into a dyon which carries half the charge of the electron.

We can also consider placing a monopole in a medium whose θ will gradually increase from zero. At the end of this process, the final electric field will be:

$$\begin{aligned} \int_0^{t_0} \frac{\partial \boldsymbol{E}}{\partial t} \mathrm{d}t &= \frac{\alpha}{\pi} \int_0^{t_0} \dot{\theta} \boldsymbol{B} \\ \Rightarrow \quad \boldsymbol{E} \quad &= \frac{\alpha}{\pi} \Theta \boldsymbol{B} \quad [\text{which } \theta(t_0) = \Theta] \\ \Rightarrow \quad q \quad &= \frac{\alpha}{\pi} \Theta g \end{aligned} \tag{55}$$

1.4 A Mirage Monopole

let's consider that we take an electric charge q and place it in the vacuum at point $\boldsymbol{x} = (0, 0, d)$. the we fill the region where z is less than zero with topological insulator. We can see that the effect produced by surface charge is equivalent to placing mirror dyons at point $\boldsymbol{x} = (0, 0, d)$ and point $\boldsymbol{x}' = (0, 0, -d)$.

We assume that the mirror dyon sitting at $\boldsymbol{x} = (0, 0, d)$ with electric and magnetic charge (q_1, g_1) and the mirror dyon sitting at $\boldsymbol{x}' = (0, 0, -d)$ with electric and magnetic charge (q_2, g_2).

Next, we can write down:

$$\begin{cases} \boldsymbol{E} = -\nabla\phi \quad (\text{because } \nabla \times \boldsymbol{E} = 0) \\ \boldsymbol{B} = -\nabla\Omega \quad (\text{because } \nabla \times \boldsymbol{B} = 0) \end{cases} \tag{56}$$

So the electric potential in the two regions is:

$$\begin{cases} \phi_+ = \dfrac{1}{4\pi}\left[\dfrac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \dfrac{q_2}{\sqrt{x^2+y^2+(z+d)^2}}\right], & z > 0 \\ \phi_- = \dfrac{1}{4\pi}\dfrac{q+q_1}{\sqrt{x^2+y^2+(z-d)^2}}, & z < 0 \end{cases} \tag{57}$$

We only need to consider the electric and magnetic field in yz plane because of symmetry, where z equal to zero

$$\begin{cases} \left.\begin{aligned} E_y^+ &= -\partial_y\phi_+ = \frac{q+q_2}{4\pi}\cdot\frac{y}{(x^2+y^2+d^2)^{\frac{3}{2}}} \\ E_z^+ &= -\partial_z\phi_+ = \frac{q_2-q}{4\pi}\cdot\frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}} \end{aligned}\right\} z = 0^+ \\ \\ \left.\begin{aligned} E_y^- &= -\partial_y\phi_- = \frac{q+q_1}{4\pi}\cdot\frac{y}{(x^2+y^2+d^2)^{\frac{3}{2}}} \\ E_z^- &= -\partial_z\phi_- = \frac{-(q+q_1)}{4\pi}\cdot\frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}} \end{aligned}\right\} z = 0^- \end{cases} \tag{58}$$

Using the boundary conditions:

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = 0 \tag{59}$$

so we can get the equation:

$$E_y^+ = E_y^- \Rightarrow q_1 = q_2 = q' \tag{60}$$

Also, we can perform the same operation on the magnetic field; The magnetic scalar potential is:

$$\begin{cases} \Omega_+ = \dfrac{1}{4\pi}\cdot\dfrac{g_2}{\sqrt{x^2+y^2+(z+d)^2}}, & z > 0 \\ \Omega_- = \dfrac{1}{4\pi}\cdot\dfrac{g_1}{\sqrt{x^2+y^2+(z-d)^2}}, & z < 0 \end{cases} \tag{61}$$

So the magnetic field on yz plane where z equal to zero:

$$
\begin{cases}
\left.\begin{aligned}
B_y^+ &= -\partial_y\Omega_+ = \frac{g_2}{4\pi}\cdot\frac{y}{(x^2+y^2+d^2)^{\frac{3}{2}}}\\
B_z^+ &= -\partial_z\Omega_+ = \frac{g_2}{4\pi}\cdot\frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}}
\end{aligned}\right\} z=0^+\\
\left.\begin{aligned}
B_y^- &= -\partial_y\Omega_- = \frac{g_1}{4\pi}\cdot\frac{y}{(x^2+y^2+d^2)^{\frac{3}{2}}}\\
B_z^- &= -\partial_z\Omega_- = \frac{-g_1}{4\pi}\frac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}}
\end{aligned}\right\} z=0^-
\end{cases}
\tag{62}
$$

Using the boundary conditions:

$$
\hat{\boldsymbol{n}}\cdot\boldsymbol{B}=0 \Rightarrow B_z^- = B_z^+ \Rightarrow g_2 = -g_1 = g \tag{63}
$$

From the expression above, we can get the electric displacement:

$$
\begin{cases}
D_z^+ = E_z^+ = \dfrac{q'-q}{4\pi}\cdot\dfrac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}}\\
D_z^- = E_z^- - \dfrac{\alpha}{\pi}\theta B_z^- = \dfrac{-(q+q')+\alpha g}{4\pi}\cdot\dfrac{d}{(x^2+y^2+d^2)^{\frac{3}{2}}}
\end{cases}
\tag{64}
$$

Using the boundary conditions:

$$
\hat{\boldsymbol{n}}\cdot\boldsymbol{D}=0 \Rightarrow D_z^+ = D_z^- \Rightarrow g = \frac{-2q'}{\alpha} \tag{65}
$$

The magnetising field:

$$
\begin{cases}
H_y^+ = B_y^+ = \dfrac{g}{4\pi}\cdot\dfrac{y}{(x^2+y^2+d^2)^{\frac{3}{2}}}\\
H_y^- = B_y^- + \dfrac{\alpha}{\pi}\theta E_y^- = \dfrac{-g+\alpha(q+q')}{4\pi}\dfrac{y}{(x^2+y^2+d^2)^{\frac{3}{2}}}
\end{cases}
\tag{66}
$$

Using the boundary condition:

$$
\nabla\times\boldsymbol{H}=0 \Rightarrow H_y^+ = H_y^- \Rightarrow g = \frac{\alpha(q+q')}{2} \tag{67}
$$

Finally, we can get the electric and magnetic charges carried by these dyons:

$$
\begin{cases}
q' = \dfrac{-\alpha^2}{4+\alpha^2}\cdot q\\
g = \dfrac{2\alpha}{4+\alpha^2}\cdot q
\end{cases}
\tag{68}
$$