1. The theta term. and Axion Electrodynamics

In relativistic notation, the Maxwell action for electromagnetism takes a wonderfully compact form:

$$S_{Max} = \int d^4x - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \int d^4x \, \frac{1}{2} (\vec{E}^2 - \vec{B}^2)$$

Here:

$$\begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ E_i = F_{0i} \\ F_{ij} = -\epsilon_{ijk} B_k \end{cases}$$

One reason that the Maxwell action is so simple is that there is very little else we can write down that is both gauge invariant and Lorentz invariant. There is, however one term that we can add to the Maxwell action is also both gauge invariant and Lorentz invariant. which we call it theta term:

$$S_\theta = \frac{\theta e^2}{4\pi^2} \cdot \int d^4x \, \frac{1}{4} {}^\star F^{\mu\nu} \cdot F_{\mu\nu} = -\frac{\theta e^2}{4\pi^2} \int d^4x \, \vec{E} \cdot \vec{B}$$

which ${}^\star F^{\mu\nu}$ is the dual tensor:

$${}^\star F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \cdot F_{\rho\sigma}$$

and θ is a constant parameter.

However, the theta term is simple to be check that ~~the~~ it can be written as a total derivative:

$$\int d^4x \frac{1}{4} {}^*F^{\mu\nu} F_{\mu\nu} = -\frac{1}{2}\int d^4x \cdot \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \cdot F_{\rho\sigma} \cdot F_{\mu\nu}$$

$$= -\frac{1}{2}\int d^4x \cdot \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \cdot (\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= -\frac{1}{2}\int d^4x \cdot \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} [(\partial_\mu A_\nu \partial_\rho A_\sigma + \partial_\nu A_\mu \partial_\sigma A_\rho)$$

$$\cancel{-(\partial_\mu \partial_\nu A_\mu \partial_\sigma A_\rho + \partial}$$

$$-(\partial_\mu A_\nu \partial_\sigma A_\rho + \partial_\nu A_\mu \partial_\rho A_\sigma)]$$

$$= -\frac{1}{2}\int d^4x \cdot \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\partial_\mu A_\nu \partial_\rho A_\sigma \cancel{+ \partial_\nu A_\mu \partial_\sigma A_\rho})$$

Now let's proof a useful equation:

$$\epsilon^{\mu\nu\rho\sigma} \cdot A_\nu \partial_\mu \rho A_\sigma = 0$$

Because:

$$\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\mu \rho A_\sigma = \epsilon^{\rho\nu\mu\sigma} \cancel{A_{\rho\mu} A_\sigma} A_\nu \partial_\rho \mu A_\sigma$$

$$= \epsilon^{\rho\nu\mu\sigma} \cancel{A_\rho} A_\nu \partial_\mu \rho A_\sigma$$

$$= -\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\mu \rho A_\sigma.$$

So we can get:

$$2\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\mu \rho A_\sigma = 0 \Rightarrow \epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\mu \rho A_\sigma = 0$$

So the quadratic part of theta term become:

$$\int d^4x \, \frac{1}{4}{}^*F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\int d^4x \cdot \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \cdot (\partial_\mu A_\nu \partial_\rho A + A_\nu \partial_\mu \partial_\rho A_\sigma)$$

$$= \frac{1}{2}\int d^4x \, \epsilon^{\mu\nu\rho\sigma} \cdot \partial_\mu (A_\nu \partial_\rho A_\sigma)$$

The total derivative form of theta therm is:

$$S_\theta = \frac{\theta e^2}{8\pi^2} \int d^4x \, \partial_\mu (\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma)$$

We say that the theta term is topological. It depends only ~~the~~ on boundary information. However, when ~~deriving~~ using the ~~princeple~~ principle of least action to derive the field's equation of motion, the values of field ~~is~~ are fixed ~~invariant~~ on the infinite boundary. Therefore, the ~~sh~~ upshot is that the theta term does not change the equations of motion and, it would seem, can have little effect on the physics. ~~But~~.

But under some situations that involve subtle interplay between quantum mechanics and topology, there ~~as~~ are a number of interesting phenomena of physics which are ~~leaded~~ led by theta term.

Also, we can look at the situations where θ affects the dynamics classically. This occurs when θ is not constant, but instead varies in space and time.

$$\theta = \theta(\vec{x}, t).$$

So we can ~~wit~~ write down the axion electrodynamics action:

$$S = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{e^2}{16\pi^2}\cdot\theta(\vec{x},t)\cdot{}^*F^{\mu\nu}\cdot F_{\mu\nu}\right)$$

We use Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0$$

to get the equactions of axion electrodynamics

Because $\mathcal{L}$ doesn't depend on A_μ, so the equactions of motion are:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \Rightarrow \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0$$

and the $\mathcal{L}$ of axion electrodynamics is ($\alpha = \frac{e^2}{4\pi}$):

$$\mathcal{L} = \mathcal{L}_{Max} + \mathcal{L}_\theta$$

$$= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\alpha}{4\pi}\cdot\theta(\vec{x},t)\cdot{}^*F^{\mu\nu}F_{\mu\nu}$$

First, let's see $\mathcal{L}_\theta$:

$$\frac{1}{2}\cdot\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{\partial}{\partial(\partial_\nu A_\mu)}\cdot\frac{\alpha}{4\pi}\,\theta(\vec{x},t)\cdot\cancel{\epsilon^{\mu\nu\rho\sigma}\cdot F_{\rho\sigma}F_{\mu\nu}}$$

$$= \frac{\alpha}{8\pi}\left\{\left[\frac{\partial}{\partial(\partial_\nu A_\mu)}\cdot\theta(\vec{x},t)\right]\cdot\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} + \theta(\vec{x},t)\left[\frac{\partial\,\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}}{\partial(\partial_\nu A_\mu)}\right]\right\}$$

let's see the first term.

$$\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}\frac{\partial}{\partial(\partial_\nu A_\mu)}\theta(\vec{x},t) = 0$$

then let's see the second term:

$$\theta(\vec{x},t)\cdot\frac{\partial}{\partial(\partial_\nu A_\mu)}\cdot\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}\ \cancel{=\theta(\vec{x},t)}\cdot$$

$$= 4\theta(\vec{x},t)\left(\frac{\partial}{\partial(\partial_\nu A_\mu)}\cdot\partial_\alpha A_\beta\partial_\rho A_\sigma\right)\epsilon^{\alpha\beta\rho\sigma}.$$

$$= 4\theta(\vec{x},t)\left(\delta^\nu_\alpha\delta^\mu_\beta\cdot\partial_\rho A_\sigma + \delta^\nu_\rho\delta^\mu_\sigma\cdot\partial_\alpha A_\beta\right)\cdot\epsilon^{\alpha\beta\rho\sigma}.$$

$$= 4\theta(\vec{x},t)\cdot\epsilon^{\nu\mu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$= 8\theta(\vec{x},t)\,{}^*F^{\nu\mu}$$

next, we can get:

$$\partial_\nu\theta(\vec{x},t)\,{}^*F^{\nu\mu} = {}^*F^{\nu\mu}\partial_\nu\theta(\vec{x},t) + \theta(\vec{x},t)\cdot\partial_\nu\,{}^*F^{\nu\mu}$$

using the Bianchi identities:

$$\partial_\mu\,{}^*F^{\mu\nu} = 0$$

so we can get:

$$\partial_\nu \theta(\vec{x},t)\, {}^*F^{\nu\mu} = {}^*F^{\nu\mu}\, \partial_\nu \theta(\vec{x},t)$$

~~Finally, we can~~ Finally, we can get:

$$\partial_\nu\left[\frac{\partial \mathcal{L}_\theta}{\partial(\partial_\nu A_\mu)}\right] = \frac{\alpha}{\pi}\, {}^*F^{\nu\mu} \cdot \partial_\nu \theta(\vec{x},t)$$

~~th~~ And the Lagrangian of Maxwell part:

$$\partial_\nu\left[\frac{\partial \mathcal{L}_{max}}{\partial(\partial_\nu A_\mu)}\right] = -\partial_\nu F^{\nu\mu}.$$

So the equactions of axion electrodynamics is.

$$\begin{cases} \partial_\nu F^{\nu\mu} = \frac{\alpha}{\pi}\, {}^*F^{\nu\mu} \cdot \partial_\nu \theta(\vec{x},t) \\ \partial_\mu {}^*F^{\mu\nu} = 0 \end{cases}$$

In the end, the deformed Maxwell equations which are ~~axi~~ the ~~e~~ equations of axion electrodynamics ~~s~~ are:

$$\begin{cases} \nabla\cdot\vec{E} = \frac{\alpha}{\pi}(\nabla\cdot\theta)\cdot\vec{B} \\ \nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t} - \frac{\alpha}{\pi}(\dot{\theta}\vec{B} + \nabla\theta\times\vec{E}) \\ \nabla\cdot\vec{B} = 0 \\ \frac{\partial\vec{B}}{\partial t} + \nabla\times\vec{E} = 0 \end{cases}$$

The first equation tells us that in regions of space where θ varies, a magnetic field $\vec{B}$ acts like an electric charge density. The second equation tells us that ~~tha~~ the combination $(\dot{\theta}\vec{B}+\nabla\theta\times\vec{E})$ acts like a current density

1.1 ~~θ~~ The ~~predi~~ periodicity of θ and its values under Parity or Time-Reversal Symmetry.

In quantum theory, θ is a periodic variable: it lies in the range:

$$\theta \in [0, 2\pi)$$

After imposing appropriate boundary conditions, S_θ can only take values of the form:

$$S_\theta = \theta N \quad \text{with } N \in \mathbb{Z}$$

This means that the theta angle contributes to the partition function as

$$e^{iS_\theta} = e^{iN\theta}$$

To show that S_θ must take the form, we consider a compact Euclidean spacetime which we take to T^4 and we take each of the circles in the torus to have radii R.

We consider easy case:

$$\begin{cases} \vec{E} = (0, 0, E_z), & E_z = F_{03} = \partial_0 A_3 - \partial_3 A_0 \\ \vec{B} = (0, 0, B_z). & B_z = F_{21} = \partial_2 A_1 - \partial_1 A_2 \end{cases}$$

The integral of S_θ is:

$$I = \int_{T^4} d^4x \, E_z \cdot B_z = \int_{T^4} dx^0 dx^3 \cdot E_z - \int_{T^4} dx^1 dx^2 \cdot B_z$$

The ~~gu~~gauge field A_μ must be will defind on the underlying tour, which will put restrictions on the allow values of E_z and B_z.

So the integral can't take any value.

Now let consider the restrictions of A_μ if A_μ be defined on tour. When a direction of space, say x^1, is periodic with radius R, the physics state in original point ($x^1 = 0$) must be identical to the final physics state:

$$\text{state}[A_1(0)] = \cancel{A_1(2\pi R)} \; \cancel{\text{state}[A_1(0)]} \; \text{state}[A_1(2\pi R)]$$

Because $B_z = \partial_2 A_1$ ($A_2 = 0$) = constant, so $A_1 = B_z x^2 + C$. The simplest case is $A_1 = 0$ but this case is trival. So we should find a gauge transformations to connect $A(0)$ and $A(2\pi R)$, which will make two different values of A_1 correspond to the same physics state:.

$$A_1(2\pi R) = A_1(0) + G$$

let's find the gauge transformation that we are insterested:

$$A_1 \to A_1 + \partial_1 \chi$$

We do not insist that $\chi(x)$ is single valued. Instead, we require only that $\cancel{e^{\frac{ie}{\hbar}\chi}}$ $e^{ie\chi}$ is single-valued, since this is what acts on the wavefunction. So χ can have the simple form followed:

$$\chi = \frac{n x'}{eR} \quad \text{with } n \in \mathbb{Z}$$

Because in the same point $(2\pi N R + x)$, $e^{ie\chi}$ is the same state:

$$e^{ie\chi} = e^{ie\chi} e^{i\frac{x}{R}} \cdot e^{i2\pi N n}$$

$$= e^{i\frac{x}{R}}.$$

There are sometimes called large gauge transformations, a name which reflects the fact that they cannot be ~~continuon~~ ~~continuslu~~ continuously deformed to the identity. Under such a gauge transformation, we can see that:

$$A_1 \to A_1 + \partial_1 \chi = A_1 + \frac{n}{eR}$$

the two different values of A_1 correspond the same physics state. From this transformation, we can determine the expression of A_1

$$A_1(0) = A_1(2\pi R) + \frac{n}{eR}.$$

$$\Rightarrow A_1 = \frac{n x^1}{2\pi e R^2}$$

And we can write down the expression of B_z:

$$B_z = \partial_2 A_1 = \frac{n}{2\pi e R^2}$$

~~So the mg magnetic field part of integral is:~~

So the integral of magnetic field is:

$$\int_{T^2} dx^1 dx^2 B_z = \frac{2\pi n}{e} \quad \text{with } n \in \mathbb{Z}$$

We can also apply exactly the same argument to the electric field

$$E_z = \partial_0 A_3 \quad (A_0 = 0)$$

So the integral of electric field is:

$$\int_{T^2} dx^0 dx^3 E_z = \frac{2\pi n'}{e} \quad \text{with } n' \in \mathbb{Z}$$

Armed with two integral above, the S_θ become:

$$S_\theta = -\frac{\theta e^2}{4\pi^2}\cdot\int_{T^4} d^4x\, \vec{E}\cdot\vec{B}$$

$$= -\frac{\theta e^2}{4\pi^2}\ \frac{4\pi^2}{e^2}\cdot nn' \quad (\text{let } N=-nn')$$

$$= N\theta \quad \text{with } N\in\mathbb{Z}$$

It's pretty straightforward to generalise it to non-constant $\vec{E}$ and $\vec{B}$ fields:

$$\int_{T^2} dx^1 dx^2 B_z = \int_{T^2} dx^1 dx^2 \cdot \partial_2 A_1$$

$$= 2\pi R\left[A_1(2\pi R) - A_1(0)\right]$$

$$= \frac{2\pi n}{e}$$

So we can see that θ is a periodic variable in quantum theory.

1.2. Parity, Time-Reversal and $\theta=\pi$

Although the theta term is gauge invariant and Lorentz invariant, it is not invariant under certain discrete symmetries and does not preserve the same symmetries as the Maxwell term.

These discrete symmetries are parity P and time reversal T.

$$\begin{cases} P: \vec{x} \to -\vec{x} \\ T: t \to -t. \end{cases}$$

If we act on both $\vec{E}$ and $\vec{B}$

$$\begin{cases} P: \vec{E}(\vec{x},t) \to -E(-\vec{x},t) \text{ and } P: \vec{B}(\vec{x},t) \to B(-\vec{x},t) \\ T: \vec{E}(\vec{x},t) \to E(\vec{x},-t) \text{ and } T: \vec{B}(\vec{x},t) \to -B(\vec{x},-t) \end{cases}$$

So $\vec{E}$ is odd under parity and even under time reversal
$\vec{B}$ is even under parity and odd under time reversal.

This means that, the theta term $S_\theta \propto \int d^4x\, \vec{E}\cdot\vec{B}$ breaks both parity and time ~~rese~~ reversal invariance, which means $S_\theta \to -S_\theta$ under P and T. But because of the periodicity of θ, there are two exceptions:

$$S_\theta = -S_\theta + 2\pi N.$$

$$\Rightarrow \cancel{\theta = N\pi}\ 2N'\theta = 2\pi N. \text{ (modulo } 2\pi)$$

~~So we can know that. wh~~

In order to make the equation hold for any integer values of N, θ must equal to 0 or π $[\theta \in [0, 2\pi)]$

So we can know that. ~~when $\theta = 0$ or $\theta = \pi$~~, the theory is invariant under P and T symmetries when $\theta = 0$ or $\theta = \pi$

In subsequent sections, we can see that the topological ~~ins~~ insulator said to have a $\mathbb{Z}_2$ classification has $\theta = \pi$ because these materials are defined to be in time-reversal invariant. Most materials ~~are boring~~ have $\theta = 0$. But some materials have a band structure which is twisted in a particular way, which results in $\theta = \pi$.

1.3 The Witten effect