3. 电磁场的动能量

1. 能量

存在电荷与电流分布，空间中存在电磁场，电磁场对电荷作功 ⇒ 电磁场能量减小

$\vec{F}\cdot d\vec{l} = \vec{F}\cdot\vec{v}\,dt = q(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v}\,dt = q\vec{E}\cdot\vec{v}\,dt.$

$\therefore$ 在全空间中，$q=\int\rho\,dV$，电磁场作功，$\rho\vec{v}=\vec{J}$，即认为电流由ρ运动引起

$dW=\int_V \vec{F}\,d\vec{l} = (\int_V \rho\,\vec{E}\cdot\vec{v}\,dV)\cdot dt$

$= (\int_V \vec{J}\cdot\vec{E}\cdot dV)\,dt.$

$\therefore \frac{dW}{dt} = \int_V \vec{J}\cdot\vec{E}\,dV.$

ρ是自由的话：

$\vec{J} = \cancel{\nabla\times\vec{H}}\ \nabla\times\vec{H} - \frac{\partial\vec{D}}{\partial t}.$

$\therefore \vec{J}\cdot\vec{E} = (\nabla\times\vec{H})\cdot\vec{E} - \frac{\partial\vec{D}}{\partial t}\cdot\vec{E}.$

$\nabla(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$

$\vec{H}\cdot(\nabla\times\vec{E}) - \nabla(\vec{E}\times\vec{H})$

$= \vec{H}\cdot -\frac{\partial\vec{B}}{\partial t} - \nabla(\vec{E}\times\vec{H})$

$\therefore \vec{J}\cdot\vec{E} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} - \nabla(\vec{E}\times\vec{H})$

$\therefore \frac{dW}{dt} = -\int_V \nabla(\vec{E}\times\vec{H})\,dV - \int(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t})\,dV$ ⇒ 电磁场作的功为电磁场流出的能量加上其减少的能量

$\Downarrow$

$-\oint_S(\vec{E}\times\vec{H})\cdot d\vec{S}$

$\therefore \vec{S} = \vec{E}\times\vec{H}$ 为坡印亭矢量，也为能流密度.

$\frac{\partial w}{\partial t} = \vec{H}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$ 为单位时间内变化的电磁场能量密度.

$\therefore \frac{dW}{dt} + \oint \vec{S} \cdot d\vec{S} + \int \frac{\partial w}{\partial t} dV = 0.$

当 $\frac{dW}{dt} = 0$，(无电荷时)

$$\oint \vec{S} \cdot d\vec{S} + \int \frac{\partial w}{\partial t} dV = 0$$

$$\therefore \nabla \cdot \vec{S} + \frac{\partial w}{\partial t} = 0$$

能量连续性方程.

2、动量.

f 为单位体积所受力：

$$f = \rho \vec{E} + \vec{J} \times \vec{B}$$

$$= \varepsilon_0 (\nabla \cdot \vec{E}) \vec{E} + \left(\frac{1}{\mu_0} \nabla \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \times \vec{B}$$

$$\frac{\partial}{\partial t}(\vec{E} \times \vec{B}) = \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \frac{\partial \vec{B}}{\partial t}.$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}.$$

$$-\frac{\partial \vec{E}}{\partial t} \times \vec{B} = \vec{E} \times \frac{\partial \vec{B}}{\partial t} - \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

$$= -\vec{E} \times (\nabla \times \vec{E}) - \frac{\partial}{\partial t}(\vec{E} \times \vec{B}).$$

$$f = \varepsilon_0 (\nabla \cdot \vec{E}) \vec{E} + \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} - \vec{E} \times (\nabla \times \vec{E}) - \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

$$\vec{E} \times (\nabla \times \vec{E}) = \frac{1}{2} \nabla (\vec{E})^2 - (\vec{E} \cdot \nabla) \vec{E}$$

$$\therefore f = \varepsilon_0 [(\nabla \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \nabla) \vec{E}] + \frac{1}{\mu_0} [(\nabla \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \nabla) \vec{B}]$$

$$- \frac{1}{2} \nabla \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) - \varepsilon_0 \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

$$\vec{f} = \nabla \cdot \overleftrightarrow{T} - \varepsilon_0 \frac{\partial}{\partial t}(\vec{E} \times \vec{B})$$

$$\overleftrightarrow{T} = \varepsilon_0 \vec{E}\vec{E} + \frac{1}{\mu_0}\vec{B}\vec{B} - \frac{1}{2}\overleftrightarrow{I}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

$$\vec{f} + \varepsilon_0 \frac{\partial \vec{E}\times\vec{B}}{\partial t} = \nabla\cdot\overleftrightarrow{T}$$

$$\underbrace{\int \vec{f}\,dV}_{\vec{F}} + \varepsilon_0\int \frac{\partial \vec{E}\times\vec{B}}{\partial t}\,dV = \int \nabla\cdot\overleftrightarrow{T}\,dV = \overbrace{\oint d\vec{S}\cdot\overleftrightarrow{T}}$$

~~对全空间~~

$-\overleftrightarrow{T}$ 为动量流

$$\vec{F} = \frac{d\vec{P}}{dt}.$$

$\therefore \vec{g} = \varepsilon_0 \vec{E}\times\vec{B}$ 为电磁场动量密度.

上式表示为电荷与场的动量变化为流进/流出该区域的动量.

$\therefore$ 在全空间中, $\oint d\vec{S}\cdot\overleftrightarrow{T} = 0$

$\Rightarrow \vec{f} + \frac{\partial}{\partial t}\vec{g} = 0$ 动量守恒.

PS: $\overleftrightarrow{T}$ 为应力张量: 因为在静态时, $\frac{\partial \vec{g}}{\partial t} = 0$,

$\int \vec{f}\,dV = \oint d\vec{S}\cdot\overleftrightarrow{T}$. 说明电荷受到的力为场应力张量决定

而动态时, 写为 $\vec{f} + \frac{\partial}{\partial t}\vec{g} + \nabla\cdot(-\overleftrightarrow{T}) = 0$, 表示局域的动量变化,

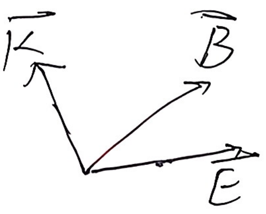
(与能量能流类似)

真空中: $\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

$\vec{g} = \varepsilon_0 \vec{E}\times\vec{B} = \varepsilon_0\mu_0\vec{S} = \frac{1}{c^2}\vec{S}$

对平面电磁波

$\vec{B} = \frac{1}{c}\vec{e}_k\times\vec{E}$.



$$\therefore \vec{g} = \frac{\varepsilon_0}{c}\vec{E}\times(\vec{e}_k\times\vec{E}) = \frac{\varepsilon_0}{c}E^2\vec{e}_k$$

平均动量密度

$\langle \vec{g} \rangle = \frac{1}{2} Re(\varepsilon_0 \vec{E}^* \times \vec{B}) = \frac{\varepsilon_0}{2c} |E_0|^2 \vec{e}_k$

对真空中：$\frac{dW}{dt} = 0$

$\therefore \nabla \cdot \vec{S} + \frac{\partial w}{\partial t} = 0.$

$\oint \vec{S} d\vec{S} + \int \frac{\partial w}{\partial t} dV = 0.$

$\vec{S} \cdot d\vec{S} + \frac{\partial w}{\partial t} \cdot c dt \cdot \cancel{dV}^{dS} = 0$

$-|S| + cw = 0$

$\therefore |S| = cw.$

$\therefore \vec{S} = cw \vec{e}_K$

$\therefore \vec{g} = \frac{w}{c} \vec{e}_k$

$w = \frac{1}{2}(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) = \frac{1}{2}(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2)$

$= \varepsilon_0 E^2.$

3. 辐射压力

$c\vec{g} = w \vec{e}_K.$