5. 静电场与静磁场

由 $\nabla \vec{E} = \frac{\rho}{\varepsilon_0}$

$\nabla \times \vec{E} = 0$

可以定义定电势：

~~$\varphi = -\nabla$~~

$\vec{E} = -\nabla\varphi$.

$\therefore \nabla^2\varphi = -\frac{\rho}{\varepsilon_0}$. (泊松方程)

Poisson.

当 $\rho = 0$,

$\nabla^2\varphi = 0$, (Laplace 方程)

在线性介质中：

$\nabla \cdot \vec{D} = \rho_f$

$\therefore \nabla \varepsilon \vec{E} = \rho_f$

$\therefore \nabla^2\varphi = \frac{\rho_f}{\varepsilon}$.

边界条件：$\vec{E}^{\parallel}_{up} = \vec{E}^{\parallel}_{down}$

$$\begin{cases} D^{\perp}_{up} - D^{\perp}_{down} = \sigma_f. \Rightarrow -\varepsilon_{up}\frac{\partial\varphi_{up}}{\partial n} + \varepsilon_{down}\frac{\partial\varphi_{down}}{\partial n} = \sigma_f \\ \cancel{\varphi_1 = \varphi_2}\ \varphi_1 = \varphi_2. \end{cases}$$

对于导体，等势体有：$\varphi_{down} = Const$, $D^{\perp}_{up} = -\varepsilon_{up}\frac{\partial\varphi_{up}}{\partial n} = \sigma_f$

总能量：$W = \frac{1}{2}\int \vec{E}\cdot\vec{D}\, dV$.

$= \frac{1}{2}\int_V \rho\varphi\, dV$.

5.2 唯一性定理.

V内电势要确定，即给出 $\nabla^2\varphi=-\frac{\rho}{\varepsilon_0}$ 的φ有唯一解(差常数、

除了给出 ρ 外，还给给出. $\varphi|_{\partial V}$. 或 $\frac{\partial\varphi}{\partial n}|_S$.

若区域中包含导体，若在上述条件上加上给定每个导体总电荷，电~~势~~场唯一确定.

在球坐标中，Laplace方程解为：

$$\varphi(R,\theta,\varphi)=\sum(a_{nm}R^n+\frac{b_{nm}}{R^{n+1}})P_n^m(\cos\theta)\cdot\cos m\varphi+$$
$$\sum(c_{nm}R^n+\frac{d_{nm}}{R^{n+1}})P_n^m(\cos\theta)\sin n\varphi.$$

轴对称时，$\varphi(R,\theta)=\sum(a_nR^n+\frac{b_n}{R^{n+1}})P_n^0(\cos\theta).$

$$\begin{cases}P_1=1\\P_1(x)=x\\P_2(x)=(3x^2-1)\cdot\frac{1}{2}\end{cases}$$

5.3 电多极矩：

$$\varphi=\frac{1}{4\pi\varepsilon_0}\int\rho(\vec r)dV\cdot\frac{1}{|\vec R-\vec r|}.$$

$\frac{1}{|\vec R-\vec r|}$. 在 $\vec r=0$ 附近展开.

$$\frac{1}{|\vec R-\vec r|}=\frac{1}{R}+(-\vec r)\cdot\nabla\frac{1}{R}+\frac{1}{2!}(-\vec r)(\nabla\nabla)\frac{1}{R}(-\vec r)+\cdots$$

$$\varphi^{(0)}=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec r)}{R}dV.$$
$$\varphi^{(1)}=\frac{1}{4\pi\varepsilon_0}\int\rho(\vec r)(-\vec r)\cdot\nabla\frac{1}{R}dV=\frac{\nabla\frac{1}{R}}{4\pi\varepsilon_0}\int\rho(\vec r)\cdot(-\vec r)dV$$
$$\varphi^{(2)}=\frac{1}{4\pi\varepsilon_0}\int\rho(\vec r)\cdot\frac{1}{2!}(-\vec r)(\nabla\nabla)\frac{1}{R}(-\vec r)dV=.$$

$\because Q = \int \rho(\vec{r}) dV$

$\vec{p} = \int_V \rho(\vec{r}) \cdot \vec{r} \, dV$.

$\overleftrightarrow{\mathscr{D}} = \int_V 3 \vec{r}\vec{r} \rho(\vec{r}) dV$.

$\therefore \varphi^{(2)} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{6} \cdot \overleftrightarrow{\mathscr{D}} : \nabla\nabla \frac{1}{R}$.

$\varphi^{(1)} = \frac{1}{4\pi\varepsilon_0} \vec{p} \cdot \nabla \frac{1}{R} = \frac{\vec{p} \cdot \vec{R}}{4\pi\varepsilon_0 R^3}$

$\varphi^{(2)} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{6} \sum \mathscr{D}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{R}$., $\mathscr{D}_{ij} = \int 3 x_i x_j \rho(\vec{r}) dV$

能量

$W = \int \rho \varphi_e dV$

$\varphi_e(\vec{r})$在0处展开:

$\varphi_e(\vec{r}) = \varphi_e(0) + \vec{r} \cdot \nabla \varphi_e|_{\vec{r}=0} + \frac{1}{2!} \sum x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} \varphi_e|_{\vec{r}=0}$

$\therefore W^{(0)} = \int \rho(x) \varphi_e(0) dV = Q \varphi_e(0)$

$W^{(1)} = \int \rho(x) \vec{r} \cdot \nabla \varphi_e|_{\vec{r}=0} dV$

$= \vec{p} \cdot \nabla \varphi_e|_{\vec{r}=0} = -\vec{p} \cdot \vec{E}(0)$

$W^{(2)} = \frac{1}{6} \int \sum 3 x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} \varphi|_{\vec{r}=0} \rho(\vec{r}) dV$

$= \frac{1}{6} \overleftrightarrow{\mathscr{D}} : \nabla\nabla \varphi_e|_{\vec{r}=0}$.

5.2

5.4 磁矢势

由 $\nabla\cdot\vec{B}=0$

引入 $\vec{B}=\nabla\times\vec{A}$.

$$\nabla\times\vec{B}=\nabla\times(\nabla\times\vec{A})$$

$$=\nabla(\nabla\cdot\vec{A})-\nabla^2 A=\mu_0\vec{J}$$

$\because \nabla\times\nabla\varphi=0$

$\vec{A}'=\vec{A}+\nabla\varphi$

$\vec{B}'=\vec{B}$

$\nabla\vec{A}+\nabla^2\varphi=0$

找到 $\nabla\vec{A}=-\nabla^2\varphi$ 时,

$\nabla\cdot(\nabla\cdot\vec{A}')=0$

$\therefore \nabla^2\vec{A}'=\mu_0\vec{J}$

分量满足泊松方程.

$$\therefore \vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{\Delta r}dV$$

对线电流

$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{I\cdot d\vec{l}}{\Delta r}$$

面电流

$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{K}\,ds}{\Delta r}$$

边界条件:

$$\begin{cases} B^{\perp}_{up} - B^{\perp}_{down} = 0 \\ \vec{B}^{\parallel}_{up} - \vec{B}^{\parallel}_{down} = \mu_0 \vec{K} \times \vec{n}. \end{cases}$$

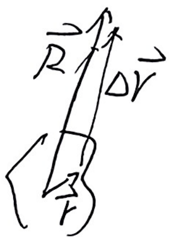
$$(\vec{H}^{\parallel}_{up} - \vec{H}^{\parallel}_{down}) = \vec{K}_f \times \vec{n}$$

$$\Rightarrow \begin{cases} \vec{A}_{up} = \vec{A}_{down} \\ e_n \times \left(\frac{1}{\mu_2}\nabla\times\vec{A}_2 - \frac{1}{\mu_1}\nabla\times\vec{A}_1\right) = \vec{\alpha}. \end{cases}$$

5.5 磁多极矩.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\, dV' \frac{1}{r}.$$

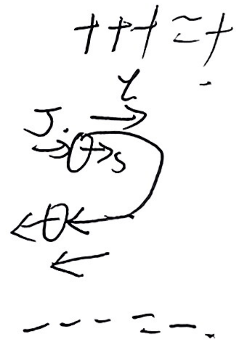
$$= \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\, dV \left[\frac{1}{R} + (-\vec{r}')\cdot\nabla\frac{1}{R} + \frac{1}{2!}\sum x_i' x_j' \frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{R} + \cdots\right]$$



$$\vec{A}^{(0)} = \frac{\mu_0}{4\pi R}\int \vec{J}(\vec{r}')\, dV.$$

$$= \frac{\mu_0}{4\pi R}\oint I\, d\vec{l}$$

$$= 0$$



$$\vec{A}^{(1)} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\, dV \cdot (-\vec{r}')\cdot\nabla\frac{1}{R}.$$

$$\nabla\frac{1}{R} = \frac{\vec{R}}{R^3}.$$

$$\vec{A}^{(1)} = \frac{\mu_0}{4\pi}\cdot\frac{\vec{R}}{R^3}\cdot\int \vec{J}(\vec{r}')\, dV \cdot (-\vec{r}')$$

$$0=\oint d[(\vec{x}'\cdot\vec{R})\vec{x}']=\oint_L(\vec{x}'\cdot\vec{R})d\vec{l}+\oint_L(d\vec{l}\cdot\vec{R})\cdot\vec{x}'.$$

$$\int\vec{r}\cdot\vec{R}\cdot d\vec{l}=\frac{1}{2}\int[(\vec{r}\cdot\vec{R})d\vec{l}-(d\vec{l}\cdot\vec{R})\cdot\vec{r})$$

$$=\frac{1}{2}\oint(\vec{r}\times d\vec{l})\times\vec{R}.$$

$$\therefore\vec{A}^{(1)}=\frac{\mu_0}{4\pi R^3}\int(-\vec{r})\cdot\vec{R}\,\vec{J}(\vec{r})dV=\frac{\mu_0}{4\pi}\cdot\vec{R}\times\frac{1}{2}\int\vec{r}\times\vec{J}(\vec{r})dV.$$

磁偶极矩：

$$=\frac{\mu_0}{4\pi}\cdot\frac{\vec{R}\times\vec{m}}{R^3}.$$

$$\vec{m}=\frac{1}{2}\int_V\vec{r}\times\vec{J}(\vec{r})\cdot dV.$$

$$=\frac{1}{2}\int\vec{r}\times I d\vec{l}.$$

$$=\frac{I}{2}\int\vec{r}\times d\vec{l}.$$

$$\Delta\vec{S}=\frac{1}{2}\int\vec{r}\times d\vec{l}.$$

$$\therefore\vec{m}=I\Delta\vec{S}$$

$$\therefore\vec{B}^{(1)}=\nabla\times\vec{A}^{(1)}$$

$$=\frac{\mu_0}{4\pi}\nabla\times\left(\vec{m}\times\frac{\vec{R}}{R^3}\right).$$

$$=\frac{\mu_0}{4\pi}\left[\left(\nabla\cdot\frac{\vec{R}}{R^3}\right)\cdot\vec{m}-(\vec{m}\cdot\nabla)\cdot\frac{\vec{R}}{R^3}\right).$$

$$\nabla\frac{\vec{R}}{R^3}=0\quad |R\neq 0.$$

$$\therefore B^{(1)}=-\frac{\mu_0}{4\pi}(\vec{m}\cdot\nabla)\cdot\frac{\vec{R}}{R^3}$$

$$\nabla\cdot\left(\vec{m}\cdot\frac{\vec{R}}{R^3}\right) = \left(\cancel{\nabla\cdot\vec{m}}\right)\vec{m}\times\underbrace{\left(\nabla\times\frac{\vec{R}}{R^3}\right)}^{=0} + (\vec{m}\cdot\nabla)\frac{\vec{R}}{R^3}$$

$$+\frac{\vec{R}}{R^3}\times\underbrace{(\nabla\times\vec{m})}_{=0} + \underbrace{\left(\frac{\vec{R}}{R^3}\cdot\nabla\right)\cdot\vec{m}}_{=0}.$$

$$\therefore\ \nabla\cdot\left(\vec{m}\cdot\frac{\vec{R}}{R^3}\right) = (\vec{m}\cdot\nabla)\cdot\frac{\vec{R}}{R^3}$$

$$\therefore\ B^{(1)} = -\mu_0\nabla\varphi_m^{(1)},\quad \varphi_m^{(1)} = \frac{\vec{m}\cdot\vec{R}}{4\pi R^3}.$$

磁场能量.

$$W = \frac{1}{2}\int\vec{B}\cdot\vec{H}\,dV.$$

$$\vec{B}\cdot\vec{H} = (\nabla\times\vec{A})\cdot\vec{H} = \nabla\cdot(\vec{A}\times\vec{H}) + \vec{A}\cdot(\nabla\times\vec{H})$$

$$= \cancel{\nabla\times(A}\ \nabla\cdot(\vec{A}\times\vec{H}) + \vec{A}\cdot\vec{J}_f.$$

$$\therefore\ W = \frac{1}{2}\int_V\nabla\cdot(\vec{A}\times\vec{J})\,dV + \frac{1}{2}\int_V\vec{A}\cdot\vec{J}_f\,dV$$

$$= \frac{1}{2}\int_S(\vec{A}\times\vec{J})\cdot d\vec{S}$$

无穷远处，$\vec{A}\times\vec{J}=0$

$$\therefore\ W = \frac{1}{2}\int\vec{A}\cdot\vec{J}\,dV.$$

$J = J_I + J_0$，$\vec{A} = \vec{A}_I + \vec{J}_0$

产生

$$\therefore\ W = \underbrace{\frac{1}{2}\int(\vec{J}_0\vec{A}_I + \vec{J}_I\vec{A}_0)\,dV}_{\text{相互作用}} + \frac{1}{2}\int\underbrace{\vec{J}_I\vec{A}_I\,dV + \frac{1}{2}\int\vec{A}_0\vec{J}_0\,dV}_{\text{自作用}}$$

$$\vec{A}_I = \frac{\mu}{4\pi}\int \frac{\vec{J}_I(\vec{r})\,dV}{r}$$

$$\vec{A}_I\,\vec{J}_0 = \frac{\mu}{4\pi}\int \frac{\vec{J}_I(\vec{r})\,\vec{J}_0(\vec{r})}{r}\,dV$$

同理：

$$\vec{A}_I\,\vec{J}_0 = \vec{A}_0\,\vec{J}_I$$

$$\therefore W_{int} = \int \vec{J}_I\,\vec{A}_0\,dV.$$