A Sensitivity Analysis Method for a Class of Cyber-Physical Systems and Its Application to Parameter Optimization

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***Abstract*— Sensitivity analysis is fundamental and essential in the analysis and design of any system. Sensitivities are usually defined as derivatives of system variables or system performance with respect to its parameters. This paper discusses a sensitivity analysis method for a class of cyber-physical systems. We consider the following system as a class of cyber-physical systems at the first step of the research: a hybrid system in which a continuous-time system and a discrete-time system are connected through A/D and D/A interfaces. They are represented by block diagrams in which arbitrary elements including nonlinear elements are arbitrarily connected. It is known that the sensitivity analysis based on Tellegen's theorem has become a standard method for electric circuits. We extend Tellegen's theorem to the hybrid system and drive a method for computing sensitivities of any signal with respect to any parameter in the system. We also propose a method to apply the proposed method to parameter optimization of the systems. In order to evaluate the performance of the proposed method, some numerical experiments are conducted. We apply the proposed method to a cyber-physical system whose parameter sensitivities can be derived analytically in order to estimate its accuracy. It is shown through numerical experiments that the proposed method can obtain sensitivities with sufficient accuracy. Furthermore, we apply it to an optimization problem of a continuous and discrete-time hybrid system. It is shown that the proposed method makes it possible to obtain optimal parameters as the hybrid system.**

***Index Terms*— Sensitivity analysis, cyber-physical system, hybrid system, parameter optimization.**

I. Introduction

RECENTLY, with the remarkable development of information and communication technology, cyber-physical systems, which are the combination of physical systems and information systems, are becoming more and more common in our daily lives and are found in almost all fields. Such systems are attracting a great deal of attention, and it is strongly desired to develop versatile methods for their analysis, design and control. Several studies on cyber-physical systems have been done for several aspects [1], [2], [3], [4]. Researches on cyber-physical systems have also become active in the areas of control, estimation, and identification (for example [5], [6], [7], [8], [9], [10], [11], [12]). Meanwhile, for any system, sensitivity analysis, that is, to investigate how its behavior, characteristics or performance change when parameters in the system vary by a small amount is essential and necessary in various situations in system analysis and design and is becoming a very important issue. Furthermore, given a certain performance index or objective function for the behavior or response of a system, consider the problem of determining the parameters that minimize or maximize it, i.e., the parameter optimization problem. Many problems in the areas of control, estimation, identification, and so on are formulated as parameter optimization problems. In solving these problems by gradient-based optimization algorithms, for which various useful algorithms are available, to compute derivatives of the performance index or objective function with respect to the parameters, that is, sensitivity analysis is also required and is becoming an important issue. The purpose of this paper is to propose a sensitivity analysis method for a class of cyber-physical systems and to apply it to parameter optimization problems.

In this paper, we consider the following systems as a class of cyber-physical systems at the first step in our research: the hybrid systems in which a continuous-time nonlinear dynamical system and a discrete-time nonlinear dynamical system are connected through A/D and D/A interfaces [4]. Note that the continuous-time system corresponds to the physical system and the discrete-time system to the cyber system. In both the continuous-time and discrete-time systems, arbitrary elements, including nonlinear elements, are arbitrarily connected. For such systems, it is almost impossible to derive sensitivities analytically, and therefore, it necessary to develop a computer-aided method for analyzing sensitivities.

Sensitivity analysis of systems is not new and has quite a history, and there have been many studies done [13], [14], [15]. It is known that there have been two approaches in deriving parameter sensitivities: one is based on the sensitivity equation (forward equation), and the other is based on the adjoint equation (backward equation) [16], [17]. The latter is known

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to be more computationally efficient than the former; that is, in the former, the calculation cost increases linearly as the number of parameters increases, however in the latter, it does not. We adopt the latter approach in this paper. In deriving the adjoint equation for the sensitivity analysis, the methods proposed so far are classified into three types. The first is a method to obtain the adjoint equation from the sensitivity equation using the inner product relationship [17]. The second is a method to obtain it by applying the variational method to the Lagrangian relaxation method [18]. The third is the method based on Tellegen's theorem. In the sensitivity analysis of electric circuits, the adjoint network approach based on Tellegen's theorem is known to be a useful method and has become a standard method. This is because it can be applied to any nonlinear electric circuit and can be easily implemented by using a general-purpose circuit simulator [19], [20].

There have been several studies of sensitivity analysis methods for cyber-physical systems, and almost all of them have been for a class of hybrid systems in which a continuous-time system and a discrete-event system are coupled. In the sensitivity analysis of such hybrid systems, the problem is how to handle the system's mode changes due to the changes of the events in the discrete-event system. However, in studies of sensitivity analysis for the time interval other than the time instants of mode changes, the method based on the sensitivity equations or that based on the adjoint equations is used. Those studies are classified from this perspective as follows. In [23], [24], [25], [26], and [27] the method based on the sensitivity equation is used and in [28] and [29] the method based on the adjoint equation is used. In the derivation of the latter method, [28] uses the method via the sensitivity equations, and [29] uses the method based on the Lagrangian relaxation method, while there seems to be no study of the derivation method based on Tellegen's theorem yet.

Meanwhile, typical examples of this class of hybrid systems are power electronic systems and spiking neural networks. The former becomes a hybrid system of this class due to the switching function of power semiconductors, and the latter due to the firing function of spiking neurons. In [21] and [22], one of the authors of this paper and his colleagues proposed a sensitivity analysis method for spiking neural networks for deriving their learning methods, the derivation of which is based on sensitivity equations in [21] and is based on adjoint equations in [22]. They also proposed a sensitivity analysis method for power electronic systems, the derivation method of which is the adjoint network approach based on Tellegen's theorem [30], [31].

The contributions and originality of this paper are as follows.

1) There are a lot of studies of sensitivity analysis methods for the class of hybrid systems mentioned above. However, the sensitivity analysis has not yet been studied for the systems considered in this paper, that is, the class of hybrid systems in which a continuous-time system and a discrete-time system are coupled through A/D and D/A interfaces.
2) We extend Tellegen's theorem to the cyber-physical systems described above and propose a computer-aided sensitivity analysis method. The proposed method can derive sensitivities of any signal of the systems with respect to any parameter.
3) We also propose a method to apply the proposed method to parameter optimization problems of the systems. The proposed method makes it possible to obtain parameters which can optimize systems as cyber-physical systems without any approximations and transformation.

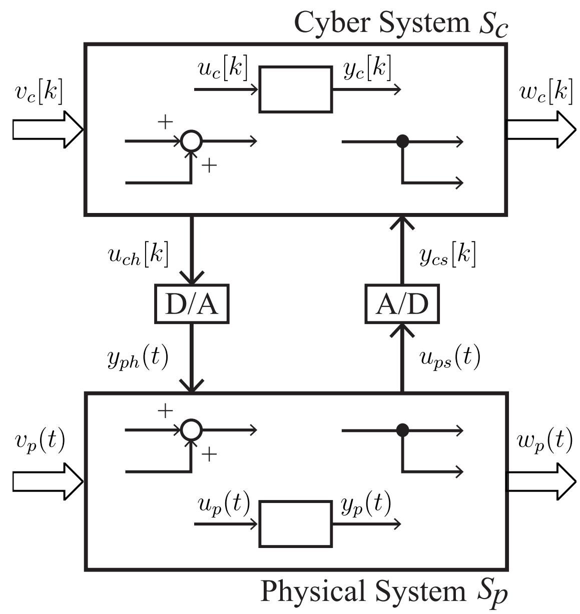


Fig. 1. Block diagram of a class of cyber-physical systems S.

In order to evaluate the effectiveness and performance of the proposed method, we conduct numerical experiments from two aspects: accuracy and applicability. First, to evaluate the accuracy of the proposed method, we apply the proposed method to a cyber-physical system whose sensitivities can be obtained analytically. It is confirmed that the proposed method makes it possible to obtain the sensitivities with enough accuracy. Furthermore, in order to demonstrate the applicability of the proposed method, we apply it to a parameter optimization problem and demonstrate its effectiveness.

II. Target System

We consider in this paper a class of cyber-physical systems represented by the block diagram as shown in Fig.1. The system, denoted by S, consists of a continuous-time system and a discrete-time system, and they are coupled through A/D and D/A interfaces. Note that the continuous-time system corresponds to the physical system S_p and the discrete-time system to the cyber system S_c. The behavior of physical systems such as electrical and mechanical systems, for example, automobiles and aircraft, can be modeled as continuous-time systems represented by differential equations. On the other hand, embedded computers that control these physical systems acquire necessary information and change control inputs through sensors and actuators. They are done with a specific sampling width through A/D and D/A interfaces. Embedded computers can thus be modeled as discrete-time systems.

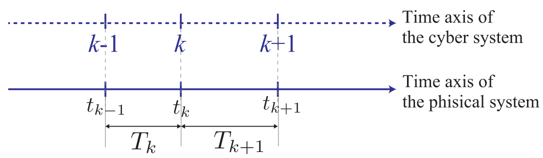


Fig. 2. Time axis of the cyber-physical systems S.

Figure 1 represents all such control systems in general. In this paper, we describe a signal x in the physical and cyber system as $x(\cdot)$ and $x[\cdot]$, respectively.

The physical system S_p consists of several kinds of block elements whose connection relationship is expressed by summing points and take-off points. We assume that an arbitrary number of those block elements, summing points and take-off points are arbitrarily connected in S_p. For block elements, integral elements, multi-input multi-output linear and nonlinear elements are considered, and their mathematical models will be given in the next section. By combining any number of these elements, any nonlinear continuous-time dynamical system represented by differential and algebraic equations can be represented. S_p represents such systems. S_p also has an arbitrary number of external inputs and outputs, i-th external input and output is denoted by v_{p_i} and w_{p_i}, respectively. S_p has an arbitrary number of inputs from D/A interfaces and outputs to A/D interfaces, and input from i-th D/A interface and output to i-th A/D interface is denoted by y_{ph_i} and u_{ps_i}, respectively.

The cyber system S_c consists of several kinds of block elements whose connection relationship is expressed by summing points and take-off points. We assume that an arbitrary number of those elements are arbitrarily connected in S_c. For block elements, delay elements, multi-input multi-output linear and nonlinear elements are considered, and their mathematical models will be given in the next section. By combining any number of these elements, any nonlinear discrete-time dynamical system represented by difference and algebraic equations can be represented. S_c represents such systems. S_c also has an arbitrary number of external inputs and outputs, i-th external input and output is denoted by v_{c_i} and w_{c_i}, respectively. S_c has inputs from A/D interfaces and outputs to D/A interfaces, and input from i-th A/D interface and output to i-th D/A interface is denoted by y_{cs_i} and u_{ch_i}, respectively.

In this paper, we discuss a method for analyzing the sensitivities of arbitrary signals with respect to arbitrary parameters in the system S described as above. In the following, for the sake of simplicity, the subscript i of the signals is omitted when this does not cause any confusion.

It is assumed that the A/D interfaces and the D/A interfaces operate synchronously. Figure 2 shows the time axis of the physical system $t \in \mathbb{R}$ and the cyber system $k \in \mathbb{Z}$ where $\mathbb{R}$ is the real number and $\mathbb{Z}$ is the integer. In the figure, t_k is k-th sampling instant defined by $t_k = \Sigma_{n=1}^{k} T_n$ and T_n is n-th sampling period. Note that the sampling period T_n is different for each sample interval. The mathematical models of A/D and D/A interfaces are given as follows.

The A/D interface is assumed to be an ideal sampler. Let the output of the A/D interface at time instant k be $y_{cs}[k]$, and the input of the A/D interface be $u_{ps}(t)$ at time t. The mathematical model of the A/D interface is expressed as follows:

$$y_{cs}[k] = u_{ps}(t_k-), \quad \text{for} \quad k = 0, 1, 2, \cdots \tag{1}$$

where $u_{ps}(t_k-) = \lim_{\varepsilon \to 0} u_{ps}(t_k - \varepsilon)$, and $\varepsilon > 0$.

The D/A interface is assumed to be a zero-order hold. Let the output of the D/A interface at time t be $y_{ph}(t)$, and the input of the D/A interface be $u_{ch}[k]$ at time instant k. The mathematical model of the D/A interface is expressed as follows:

$$y_{ph}(t) = u_{ch}[k], \quad t \in [t_k, t_{k+1}), \text{ for } \quad k = 0, 1, 2, \cdots . \tag{2}$$

III. Extended Tellegen's Theorem and Adjoint System

It is well known that in the sensitivity analysis the adjoint network approach based on Tellegen's theorem has been a standard method for electric circuits [19], [20]. In general, the mathematical model of an electric circuit can be derived by writing down Kirchhoff's voltage law and current law, which expresses the connection relationship of all branches in it, and the equations that express characteristics of all branch elements. Tellegen's theorem is a simple and fundamental theorem that is derived only from the network topology, that is, the connection relations of all branches in the electric circuit, and it does not depend on the characteristics of the branch elements. Due to its simplicity, it is possible to derive a versatile method of sensitivity analysis for electric circuits. For a target electric circuit, introducing any circuit having the same topology as that of the original circuit is introduced, Tellegen's theorem holds between them. The sensitivity analysis method can be derived systematically by appropriately specifying the element characteristic of each branch of the introduced circuit by using the fundamental relationship of Tellegen's theorem. The circuit thus introduced is called the adjoint circuit.

Tellegen's theorem has been extended to be applicable to systems represented by block diagrams, and it is called the extended Tellegen's theorem [32]. A sensitivity analysis method based on the extended Tellegen's theorem has also been proposed for block diagram systems. In this paper, we propose a sensitivity analysis method for a class of cyber-physical systems represented by the block diagram shown in Fig. 1 by extending the theorem. In such systems, it becomes a problem how to handle the fact that continuous-time signals and discrete-time signals coexist in the system and how to handle A/D and D/A interfaces. In order to solve the problem, we replace the discrete-time signals with equivalent continuous-time signals as follows so as to treat all the signals in the system S in the same manner.

Let all the discrete signals x_c of the cyber system S_c be virtually piecewise-constant continuous-time signals $x_c(t)$ as follows:

$$x_c(t) := x_c[k], \quad t \in [t_k, t_{k+1}) \quad \text{for} \quad k = 0, 1, 2, \cdots . \tag{3}$$

By the replacement of (3) all the discrete-time signals of the system S of Fig. 1 become piecewise-constant continuous-time signals. Note that the output signals of the A/D interfaces

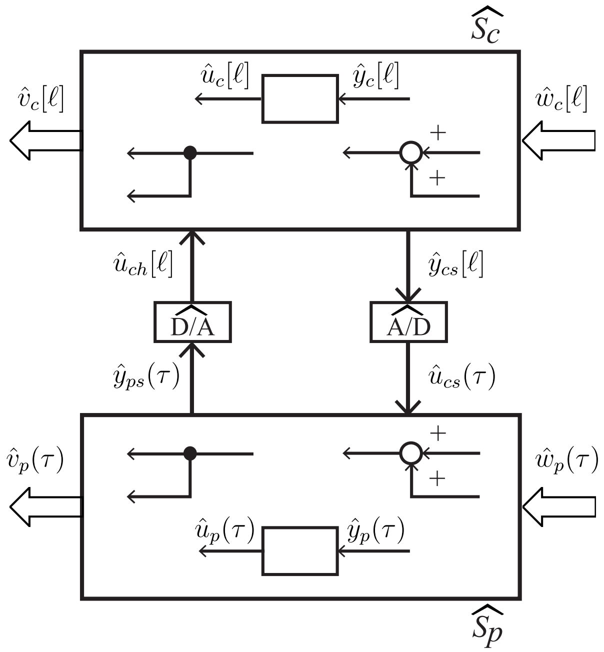


Fig. 3. Block diagram of adjoint system $\hat{S}$ of the cyber-physical system S.

$y_{cs}[k]$ and the input signals of the D/A interfaces $u_{ch}[k]$ are respectively replaced by continuous-time signals $y_{cs}(t)$, and $u_{cs}(t)$ according to (3). Let inputs and outputs of block elements in S_p be u_p and y_p, respectively, and let those of block elements in S_c be u_c and y_c. We now define the adjoint system for the system of Fig. 1 as follows.

Definition 1 (Definition of adjoint system $\hat{S}$ of system S): For the system S, the system constructed in the following manner is called the adjoint system of S, which we denote $\hat{S}$.

1) S and $\hat{S}$ have the same topology, but the direction of each signal in $\hat{S}$ is opposite to that of the corresponding signal in S.
2) Each summing point in S is replaced by a take-off point in $\hat{S}$ and each take-off point in S is replaced by a summing point in $\hat{S}$.
3) Each external input, v_p in S_p and v_c in S_c, is replaced by an external output, denoted by $\hat{v}_p$ in $\hat{S}_p$ and $\hat{v}_c$ in $\hat{S}_c$. Also, each external output, w_p in S_p and w_c in S_c, is replaced by an external input, denoted by $\hat{w}_p$ in $\hat{S}_p$ and $\hat{w}_c$ in $\hat{S}_c$.
4) Each block element, with input/output signals u_p/y_p in S_p and u_c/y_c in S_c, is replaced by a block element with input/output signals $\hat{y}_p/\hat{u}_p$ in $\hat{S}_p$ and $\hat{y}_c/\hat{u}_c$ in $\hat{S}_c$.
5) Each A/D interface in S with input/output signals u_{ps}/y_{cs} is replaced by a block element with input/output signals $\hat{y}_{cs}/\hat{u}_{ps}$ in $\hat{S}$. Also, each D/A interface in S with input/output signals u_{ch}/y_{ph} is replaced by a block element with input/output signals $\hat{y}_{ph}/\hat{u}_{ch}$ in S.

The adjoint system $\hat{S}$ constructed according to the above definition is represented as shown in Fig. 3. It should be noted that the input-output characteristics of the block elements in the adjoint system are not specified at this moment, that is, they are arbitrary. They are appropriately specified in order to derive the sensitivity analysis method, which will be discussed in the next section.

For all signals of the original system S and the adjoint system $\hat{S}$ defined above, the following theorem, which is similar to Tellegen's theorem for electric circuits, holds.

Theorem 1: For all signals of the system S and its adjoint system $\hat{S}$, the following relation holds:

$$\begin{aligned}&\sum_i v_{p_i}\hat{v}_{p_i} + \sum_i v_{c_i}\hat{v}_{c_i} + \sum_i y_{p_i}\hat{y}_{p_i} + \sum_i y_{c_i}\hat{y}_{c_i}\\ &\quad + \sum_i y_{cs_i}\hat{y}_{cs_i} + \sum_i y_{ph_i}\hat{y}_{ph_i}\\ &= \sum_i u_{p_i}\hat{u}_{p_i} + \sum_i u_{c_i}\hat{u}_{c_i} + \sum_i u_{ps_i}\hat{u}_{ps_i} + \sum_i u_{ch_i}\hat{u}_{ch_i}\\ &\quad + s\sum_i w_{p_i}\hat{w}_{p_i} + \sum_i w_{c_i}\hat{w}_{c_i}. \end{aligned} \tag{4}$$

Proof of this theorem is given in Appendix A.

IV. Sensitivity Analysis Method

A. Problem Statement

In this section, we propose a method for deriving sensitivities of arbitrary signals with respect to arbitrary parameters in S by using the definition of $\hat{S}$ (Definition 1). In general, sensitivity of a signal to a system parameter is defined as the derivative of the signal with respect to that parameter. Let the signals in S whose sensitivities are to be analyzed be taken as the external output signals.

Consider a behavior of the system S starting at the initial time $t = 0$ and finishing at any certain time $t = t_f$. Let K_{p_i} and K_{c_i} be i-th parameter in S_p and S_c, respectively. We will derive a method to compute the sensitivities of those signals at $t = t_f$ with respect to the system parameters K_{p_i} and K_{c_i}, that is,

$$\frac{\partial w_{p_j}(t_f)}{\partial K_{p_i}},\quad \frac{\partial w_{p_j}(t_f)}{\partial K_{c_i}},\quad \frac{\partial w_{c_j}[k_f]}{\partial K_{p_i}} \text{ and } \frac{\partial w_{c_j}[k_f]}{\partial K_{c_i}}$$

where w_{p_j} is the j-th external output of S_p and w_{c_j} is the j-th external output of S_c, and k_f is the integer satisfying

$$\sum_{n=1}^{k_f} T_n \le t_f < \sum_{n=1}^{k_f+1} T_n.$$

The method of derivation of the sensitivity can be obtained by appropriately defining the characteristics of the block elements of the adjoint $\hat{S}$ system of Fig. 3 and using the extended Tellegen's theorem (4).

B. Mathematical Models of Block Elements

We first show the mathematical models of the input-output characteristics of the block elements of the system S. We consider that the system S consists of the following elements.

[Block element of S]

Physical system S_p for $t \in [0, t_f]$:

$$\text{Integral element:}\quad \frac{d}{dt}y_p(t) = u_p(t),\quad y_p(0) = y_{p0} \tag{5}$$

$$\text{Parameter element:}\quad y_p(t) = K_p u_p(t) \tag{6}$$

$$\text{Nonlinear element:}\quad y_p(t) = f_p(u_p(t)) \tag{7}$$

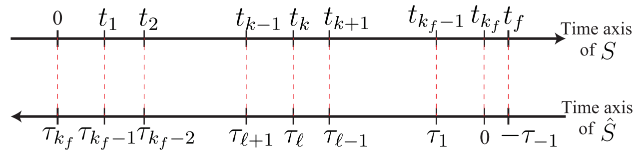


Fig. 4. Time axis for the and the cyber-physical system S and the adjoint system $\hat{S}$.

Cyber system S_c for $k = 0, 1, \cdots, k_f$:

$$\text{Delay element:} \quad y_c[k+1] = u_c[k], \quad y_c[0] = y_{c0} \tag{8}$$

$$\text{Parameter element:} \quad y_c[k] = K_c u_c[k] \tag{9}$$

$$\text{Nonlinear element:} \quad y_c[k] = f_c(u_c[k]) \tag{10}$$

Note that it is enough to consider only those elements given above so as that S_p and S_c represent any continuous-time and discrete-time nonlinear dynamical systems.

For the block elements of the system S represented above, we define the block elements of the adjoint system $\hat{S}$ as follows. As shown in Fig. 4, we define the time axis of the adjoint system $\hat{S}$ as $\tau := t_{k_f} - t$ in $\hat{S}_p$ and $\ell := k_f - k$ in $\hat{S}_c$. Let τ_ℓ be $\tau_\ell := t_f - t_k$, τ_{-1} be $\tau_{-1} := t_{k_f+1} - t_{k_f}$ and T_{k_f+1} be $T_{k_f+1} := |t_{k_f+1} - t_{k_f}|$ where $t_{k_f+1} := t_f$.

[Block element of the adjoint system $\hat{S}$]

Adjoint physical system $\hat{S}_p$ for $\tau \in [-\tau_{-1}, t_f]$:

$$\text{Integral element:} \quad \begin{cases} \dfrac{d}{d\tau}\hat{u}_p(\tau) = \hat{y}_p(\tau) \\ \hat{u}_p(0) = \hat{u}_{p0} \end{cases} \tag{11}$$

$$\text{Parameter element:} \quad \hat{u}_p(\tau) = K_p \hat{y}_c(\tau) \tag{12}$$

$$\text{Nonlinear element:} \quad \hat{u}_p(\tau) = \frac{\partial f_p}{\partial u_p}\hat{y}_p(\tau) \tag{13}$$

Adoint cyber system $\hat{S}_c$ for $\ell = 0, 1, 2, \cdots, k_f$:

$$\text{Delay element:} \quad \begin{cases} \hat{u}_c[\ell+1] = \dfrac{T_{k_f-\ell+1}}{T_{k_f-\ell}}\hat{y}_c[\ell] \\ \hat{u}_c[0] = \hat{u}_{c0} \end{cases} \tag{14}$$

$$\text{Parameter element:} \quad \hat{u}_c[\ell] = K_d \hat{y}_c[\ell] \tag{15}$$

$$\text{Nonlinear element:} \quad \hat{u}_c[\ell] = \frac{\partial f_c}{\partial u_c}\hat{y}_c[\ell] \tag{16}$$

Furthermore, we define the characteristics of the block elements of the adjoint system $\hat{S}$ corresponding to the A/D interface and the D/A interface in the system S as follows.

Adjoint A/D interface for $\tau \in (\tau_{\ell-1}, \tau_\ell]$:

$$\hat{u}_{ps}(\tau) = \delta_+(\tau - \tau_\ell)T_{k_f-\ell+1} \quad \hat{y}_{cs}[\ell], \ \ \ell = 0, 1, \cdots, k_f \tag{17}$$

where δ_+ satisfies the following equation.

$$\int_a^b f(\tau)\delta_+(\tau - \tau_0)d\tau = \begin{cases} f(\tau_0+) & (\tau_0 \in [a, b]) \\ 0 & (\tau_0 \notin [a, b]) \end{cases} \tag{18}$$

where

$$f(\tau_0+) = \lim_{\varepsilon \to 0} f(\tau_0 + \varepsilon) \tag{19}$$

and $\varepsilon > 0$.

Adjoint D/A interface for $\ell = 0, 1, \cdots, k_f$:

$$\hat{u}_{dh}[\ell] = \frac{1}{T_{k_f-\ell+1}}\int_{\tau_{\ell-1}}^{\tau_\ell} \hat{y}_{ch}(\tau)d\tau, \quad \ell = 0, 1, \cdots, k_f \tag{20}$$

Remark 1: So far, all block elements in the original system S have been assumed to be single-input and single-output elements. However, our proposed method can deal with cases where they are multi-input multi-output elements. Consider, for example, multi-input multi-output nonlinear elements whose input-output characteristics are given by $\boldsymbol{y}_p(t) = \boldsymbol{f}_p(\boldsymbol{u}_p(t))$ in S_p and $\boldsymbol{y}_c[k] = f_c(\boldsymbol{u}_c[k])$ in S_c. In this case, the input and output signals of them and their adjoint elements are vectors, and the 3rd and 4th terms of the left-hand side of (4) become

$$\boldsymbol{y}_{p_i}^\top \hat{\boldsymbol{y}}_{p_i} + \boldsymbol{y}_{c_i}^\top \hat{\boldsymbol{y}}_{c_i}$$

and the 1st and 2nd terms of the right-hand side of (4) become

$$\boldsymbol{u}_{p_i}^\top \hat{\boldsymbol{u}}_{p_i} + \boldsymbol{u}_{c_i}^\top \hat{\boldsymbol{u}}_{c_i}$$

where $^\top$ denotes vector or matrix transposition. Their adjoint elements are defined

$$\text{Nonlinear element:} \quad \hat{\boldsymbol{u}}_p(\tau) = \frac{\partial \boldsymbol{f}_p}{\partial \boldsymbol{u}_p}^\top \hat{\boldsymbol{y}}_p(\tau) \tag{21}$$

in $\hat{S}_p$ and

$$\text{Nonlinear element:} \quad \hat{\boldsymbol{u}}_c[\ell] = \frac{\partial \boldsymbol{f}_c}{\partial \boldsymbol{u}_c}^\top \hat{\boldsymbol{y}}_c[\ell] \tag{22}$$

in $\hat{S}_c$. Note that $\partial \boldsymbol{f}_p / \partial \boldsymbol{u}_p$ and $\partial \boldsymbol{f}_c / \partial \boldsymbol{u}_c$ are the Jacobian matrices.

Remark 2: As mentioned in Section II, the A/D interface is assumed to be an ideal sampler, and its mathematical model is given by (1). Quantizations are not considered in this model. In general, there may be quantizations in converting analog signals to digital signals. Such signal quantizations can be expressed using a floor function, and quantizations can be considered by including such a function as a nonlinear element (7) in S_p or (10) in S_c. However, such a function has many discontinuous and not differentiable points. The adjoint model of the nonlinear element is defined by (13) or (16); therefore, the nonlinear element should be differentiable and not include such a floor function. Consequently, we assume that the errors due to such quantizations are negligible in this paper.

C. Derivation of Sensitivity Analysis Method

We now derive a method of obtaining the sensitivities using the adjoint system defined above. Firstly, similar to the signals in the cyber system S_c of the original system S, we let all the signals $\hat{x}_c$ in the adjoint cyber system $\hat{S}_c$ of the adjoint system $\hat{S}$ virtually be piecewise-constant continuous-time signals $\hat{x}_c(\tau)$ as follows.

$$\hat{\boldsymbol{x}}_c(\tau) := \hat{\boldsymbol{x}}_c[\ell], \quad \tau \in (\tau_{\ell-1}, \ \tau_\ell], \ \text{for } \ell = 0, 1, \cdots, k_f \tag{23}$$

Suppose that the parameters $\{K_c, K_d\}$ of the original system S vary by $\{\Delta K_c, \Delta K_d\}$ and become $\{K_c + \Delta K_c, K_d + \Delta K_d\}$.

The resultant system with parameters $\{K_c + \Delta K_c, K_d + \Delta K_d\}$ is denoted by $S + \Delta S$. The following theorem holds.

Theorem 2: By setting all the initial conditions of the adjoint system to zero, $\hat{u}_{p0} = 0$, $\hat{u}_{c0} = 0$ and by using the mathematical models of characteristics of the block elements (integral element, delay element, parameter element, nonlinear element) of the system S and its adjoint system $\hat{S}$, and also those of the A/D and the D/A interfaces, we obtain the following equation:

$$\begin{aligned}\sum_i \int_0^{t_f} \Delta w_{p_i}(t)\hat{w}_{p_i}(\tau)dt + \sum_i \int_0^{t_f} \Delta w_{c_i}(t)\hat{w}_{c_i}(\tau)dt \\ = \sum_i \int_0^{t_f} u_{p_i}(t)\hat{y}_{p_i}(\tau)dt\Delta K_{p_i} \\ + \sum_i \sum_{k=0}^{k_f} T_{k+1}u_{c_i}[k]\hat{y}_{c_i}[\ell]\Delta K_{c_i} + O(\Delta^2). \end{aligned} \tag{24}$$

Proof: Proof of this theorem is given in Appendix B. ■

From the theorem, we can obtain the sensitivities of the external outputs of the continuous-time system as follows.

Theorem 3: Let us choose the external inputs $\hat{w}_p$ and $\hat{w}_c$ of the adjoint system $\hat{S}$ as follows.

$$\hat{w}_{p_j}(\tau) = \begin{cases} \delta(\tau) & \text{for} \quad j = i \\ 0 & \text{for} \quad j \neq i \end{cases}, \tau \in [0, t_f] \tag{25}$$

$$\hat{w}_{c_j}[\ell] = 0 \quad \text{for} \quad {}^\forall j, \quad \ell = 0, 1, \cdots, k_f \tag{26}$$

where $\delta(\tau)$ is the Dirac's delta function. With the use of the solution of the system S and that of the adjoint system $\hat{S}$ with the initial condition $\hat{u}_{p0} = 0$, $\hat{u}_{c0} = 0$ and with the external inputs (25) and (26), the sensitivities $\partial w_{p_j}(t_f)/\partial K_{p_i}$ and $\partial w_{p_j}(t_f)/\partial K_{c_i}$ are expressed as follows:

$$\frac{\partial w_{p_j}(t_f)}{\partial K_{p_i}} = \int_0^{t_f} u_{p_i}(t)\hat{y}_{p_i}(\tau)dt \tag{27}$$

$$\frac{\partial w_{p_j}(t_f)}{\partial K_{c_i}} = \sum_{k=0}^{k_f} T_{k+1}u_{c_i}[k]\hat{y}_{c_i}[\ell]. \tag{28}$$

Proof: According to (23), let the external input $\hat{w}_{c_j}[\ell]$ in (26) of the adjoint cyber system virtually be piecewise-constant continuous-time signals $\hat{w}_{c_j}(\tau)$. Substituting (25) and $\hat{w}_{c_j}(\tau)$ into (24) gives the following equation:

$$\begin{aligned}\Delta w_{p_i}(t_f) = \sum_i \int_0^{t_f} u_{p_i}(t)\hat{y}_{p_i}(\tau)dt\Delta K_{p_i} \\ + \sum_i \sum_{k=0}^{k_f} T_{k+1}u_{dc_i}[k]\hat{y}_{c_i}[\ell]\Delta K_{c_i} + O(\Delta^2).\end{aligned} \tag{29}$$

From this equation, the theorem is immediately obtained. ■

Note that it is found from the theorem that, by solving the target system S and its adjoint system $\hat{S}$ once, the sensitivities with respect to all the parameters of the system can be simultaneously obtained.

From Theorem 2 we can obtain the sensitivities of the external outputs of the cyber system as follows.

Theorem 4: Let us choose the external inputs $\hat{w}_p$ and $\hat{w}_c$ of the adjoint system $\hat{S}$ as follows:

$$\hat{w}_{p_j}(\tau) = 0 \quad \text{for} \ {}^\forall j, \quad \tau \in [0, t_f] \tag{30}$$

$$\hat{w}_{c_j}[\ell] = \begin{cases} \dfrac{1}{T_{k_f+1}}\delta[\ell] & \text{for} \ j = i \\ 0 & \text{for} \ j \neq i \end{cases}, \ \ell = 0, 1, \cdots, k_f \tag{31}$$

where $\delta[\ell]$ is the function satisfying

$$\delta[\ell] = \begin{cases} 1, & \ell = 0 \\ 0, & \ell \neq 0. \end{cases} \tag{32}$$

With the use of the solution of the system S and that of the adjoint system $\hat{S}$ with the initial condition $\hat{u}_{p0} = 0$, $\hat{u}_{c0} = 0$ and with the external inputs (30) and (31), the sensitivities $\partial w_{c_j}[k_f]/\partial K_{c_i}$ and $\partial w_{c_j}[k_f]/\partial K_{d_i}$ are expressed as follows:

$$\frac{\partial w_{c_j}[k_f]}{\partial K_{p_i}} = \int_0^{t_f} u_{p_i}(t)\hat{y}_{p_i}(\tau)dt \tag{33}$$

$$\frac{\partial w_{c_j}[k_f]}{\partial K_{c_i}} = \sum_{k=0}^{k_f} T_{k+1}u_{c_i}[k]\hat{y}_{c_i}[\ell]. \tag{34}$$

Proof: According to (23), let the external input $\hat{w}_{d_j}[\ell]$ in (31) of the discrete-time adjoint system virtually be piecewise-constant continuous-time signals $w_{c_j}(\tau)$. Substituting (30) and $\hat{w}_j(\tau)$ into (24) gives the following equation:

$$\begin{aligned}\Delta w_{c_i}(t_{k_f}) = \sum_i \int_0^{t_f} u_{p_i}(t)\hat{y}_{p_i}(\tau)dt\Delta K_{p_i} \\ + \sum_i \sum_{k=0}^{k_f} T_{k+1}u_{c_i}[k]\hat{y}_{c_i}[\ell]\Delta K_{c_i} + O(\Delta^2).\end{aligned} \tag{35}$$

From this equation, the theorem is immediately obtained. ■

Next, let us consider the case where an objective function evaluates the performance of the system S is given. We discuss the method for deriving the sensitivities of the objective function with respect to the system parameters. Suppose that an objective function J which evaluates the external output signals w_{p_i} of the physical system S_p is given as follows:

$$J = \int_0^{t_f} L(w_{p_1}, w_{p_2}(t), \cdots, w_{p_i}(t), \cdots)dt \tag{36}$$

where L a nonlinear scalar function of all the external signals $w_{p_i}(t)$ in S_p and $L > 0$. Consider the problem of determining values of the parameters K_c of the cybers ystem S_c that minimize the objective function J. The usual way to solve this problem is to utilize an appropriate optimization method, typical examples of which are gradient-based optimization algorithms. One of the key issues in applying these algorithms is how to calculate the gradient $\partial J/\partial K_c$ efficiently. In the following, we will derive an efficient method to calculate it. Supposing that the parameters K_c vary by ΔK_c to $K_c + \Delta K_c$, the corresponding variation of the objective function ΔJ is derived as follows:

$$\Delta J = \int_0^{t_f} \left\{ \sum_i \frac{\partial L}{\partial w_{p_i}} \Delta w_{p_i}(t) \right\} dt + O(\Delta^2). \tag{37}$$

From this, the following theorem can be obtained.

Theorem 5: Let us choose the external inputs $\hat{w}_p$ and $\hat{w}_c$ of the adjoint system $\hat{S}$ as follows:

$$\hat{w}_{p_i}(\tau) = \frac{\partial L}{\partial w_{p_i}} \quad \text{for} \quad {}^\forall i, \quad \tau \in [\tau_{-1}, t_f] \tag{38}$$

$$\hat{w}_{c_i}[\ell] = 0 \;\; \text{for} \;\; {}^\forall i, \;\; \ell = 0, 1, \cdots, k_f. \tag{39}$$

With the use of the solution of the system S and the solution of the adjoint system $\hat{S}$ with the initial condition $\hat{u}_{p0} = 0$, $\hat{u}_{c0} = 0$ and with the external inputs (38) and (39), the sensitivities $\partial J / \partial K_{c_i}$ are expressed as follows:

$$\frac{\partial J}{\partial K_{c_i}} = \sum_{k=0}^{k_f} T_{k+1} u_{c_i}[k] \hat{y}_{c_i}[\ell] \;\; \text{for} \; {}^\forall i. \tag{40}$$

Proof: According to (23), let the external input $\hat{w}_{c_j}[\ell]$ in (39) of the adjoint cyber system virtually be piecewise-constant continuous-time signals $\hat{w}_{c_j}(\tau)$. Substituting (38) and $\hat{w}_{c_j}(\tau)$ to the left side of (24) and using (37), we obtain the following equation:

$$\Delta J = \sum_i \sum_{k=0}^{k_f - 1} T_{k+1} u_{c_i}[k] \hat{y}_{c_i}[\ell] + O(\Delta^2). \tag{41}$$

From this, the theorem is immediately obtained. ■

It can be seen from the theorem that the sensitivity $\partial J / \partial K_{c_i}$ can be calculated similarly to Theorem 3 by replacing the input signal of the adjoint system by (38) and (39).

Furthermore, we consider the case where the objective function (36) is given as the sum of square errors between the external outputs $w_{p_i}(t)$ and their desired values:

$$J = \frac{1}{2} \sum_i \int_0^{t_f} q_i \left\{ w_{p_i}(t) - \bar{w}_{p_i}(t) \right\}^2 dt, \quad q_i > 0 \tag{42}$$

where $\bar{w}_{p_i}$ are desired values of w_{p_i} and q_i are weight coefficients. In this case the external input (38) and (39) of the adjoint system $\hat{S}$ become:

$$\hat{w}_{p_i}(\tau) = q_i \left\{ w_{p_i}(t) - \bar{w}_{p_i}(t) \right\} \;\; \text{for} \;\; {}^\forall i, \; , \tau \in [\tau_{-1}, t_f] \tag{43}$$

$$\hat{w}_{c_i}[\ell] = 0, \;\; \text{for} \;\; {}^\forall i, \;\; \ell = 0, 1, \cdots, k_f. \tag{44}$$

Therefore Theorem 5 becomes: with the use of the solution of the system S and the solution of the adjoint system $\hat{S}$ with the initial condition $\hat{u}_{p0} = 0$, $\hat{u}_{c0} = 0$ and with inputs (43) and (44), the sensitivities $\partial J / \partial K_{d_i}$ are expressed by (40).

V. Performance Verification of Proposed Method

In order to evaluate the performance of the proposed method we apply it to a cyber-physical system and calculate its sensitivities. In particular, in order to estimate the numerical accuracy we take up as an example a nonlinear continuous-time and discrete-time hybrid system whose exact sensitivities can be derived analytically, and we compare the sensitivities obtained by the proposed method with the analytically obtained ones. Figure 5 shows a block diagram of a cyber-physical system we take up as an example. In the system the mathematical model of the continuous-time system is given by a constant coefficient Riccati differential equation and the discrete-time system contains only a parameter element K, and they are coupled by A/D and D/A interfaces. The mathematical model of the whole system is given as follows:

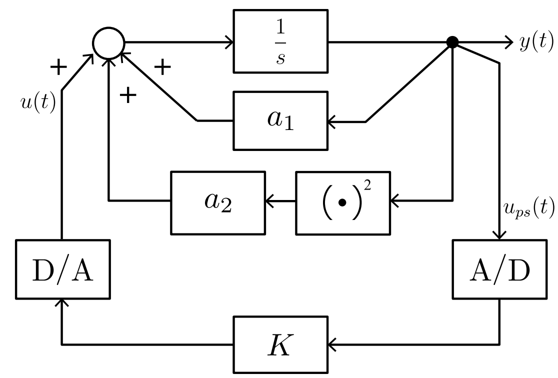


Fig. 5. An example of the cyber-physical system whose sensitivities can be derived analytically.

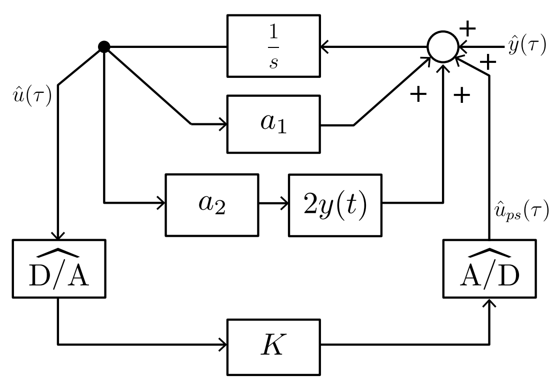


Fig. 6. The adjoint system of the cyber-physical system shown in Fig. 5.

$$\begin{cases} \dfrac{d}{dt} y(t) = a_2 y(t)^2 + a_1 y(t) + u(t) \\ y(0) = y_0 \\ u(t) = K y(kT-) \qquad (kT \leq t < (k+1)T). \end{cases} \tag{45}$$

The first equation of (45) is the Riccati differential equation whose exact solution can be derived analytically [33].

Suppose that the target system operates starting at the initial time $t = 0$ and finishing at a certain time $t = t_f$. We calculate the sensitivities of the external output of the system at $t = t_f$, $y(t_f)$, with respect to the parameters a_1 and a_2 of the continuous time system, i.e., $\partial y(t_f)/\partial a_1$ and $\partial y(t_f)/\partial a_2$, and the parameter K of the discrete-time system i.e., $\partial y(t_f)/\partial K$. According to the discussion in the previous section, the adjoint system of the target system for the sensitivity analysis can be derived as shown in Fig. 6, the mathematical model of which

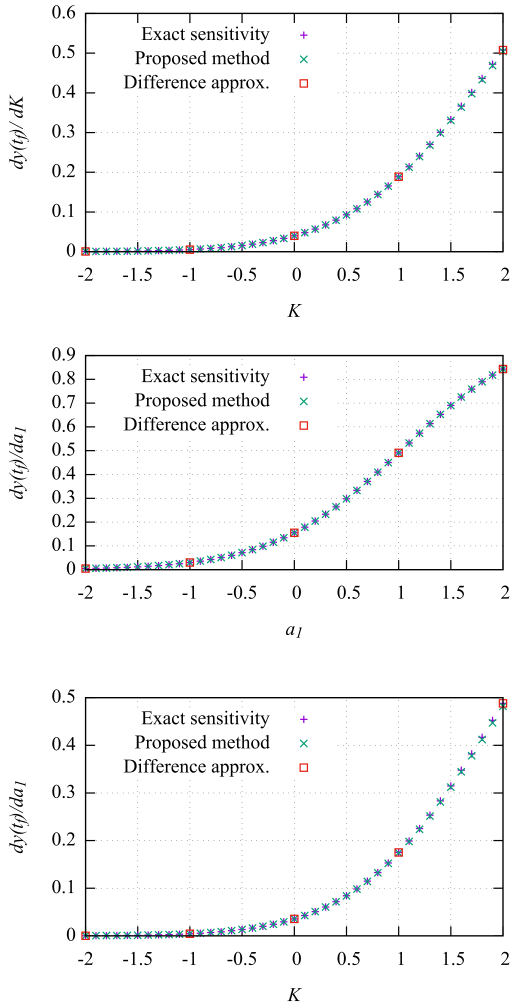


Fig. 7. Values of sensitivities obtained by three methods: Exact sensitivity, Proposed method and Difference approximation with changing the values of the parameters K (upper), a_1 (middle) and a_2 (lower).

is given by:

$$\begin{cases} \dfrac{d}{d\tau}\hat{u}(\tau) = 2a_2 y(t)\hat{u}(\tau) + a_1\hat{u}(\tau) + \hat{u}_{ps}(\tau) \\ \hat{u}(0) = 0 \\ \hat{u}_{ps}(\tau) = K\delta_+(\tau - \ell T)\displaystyle\int_{(\ell-1)T}^{\ell T} \hat{u}(\tau)d\tau. \end{cases} \tag{46}$$

From Theorem 3, the sensitivities $\partial y(t_f)/\partial a_1$, $\partial y(t_f)/\partial a_2$, and $\partial y(t_f)/\partial K$ can be calculated by giving the external input $\hat{y}(\tau)$ of the adjoint system Fig. 6 as follows:

$$\hat{y}(0) = \delta(0). \tag{47}$$

Note that the exact solution of the target system described by (45) can be analytically derived and the analytical expressions of the exact sensitivities can be obtained from the solutions. For comparison, we calculate those sensitivities by three methods, the proposed method, the method of deriving from the exact solutions and the difference approximation method with variously changing the values of the parameters a_1, a_2 and K. The results are shown in Fig. 7. In the figure, the horizontal axis represents the value of each parameter and the vertical axis represents the value of the obtained sensitivities. In each graph, the values of the sensitivities obtained by the above-mentioned three methods are plotted with respect to the parameters K (upper), a_1 (middle) and a_2 (lower). In calculating the sensitivities, we let the initial state of the system represented by (45) be $y_0 = 3$, the final time t_f be $t_f = 2.05$ and the sample period be $T_k = 0.1$ ($k = 1, 2, \cdots$). It can be seen from the figure that the values of the sensitivities obtained by the three method are almost the same, which implies that the proposed method can provide enough numerical accuracy.

VI. Application to Parameter Optimization

The proposed method is applied to an optimization problem and its results are shown in this section. Consider the system represented by the block diagram of Fig. 8, which is a speed control system of a DC motor implemented by a digital controller. In the system the feedback gains K_1, K_2 and K_3 are parameters to be optimized under the performance index which is given as follows:

$$J = \int_0^{t_f} \Big[q_1(\omega(t) - \bar{\omega})^2 + q_2(i_a(t) - \bar{i}_a)^2 + q_3(u(t) - \bar{u})^2 \Big] dt \tag{48}$$

where $q_1 \geq 0$, $q_2 \geq 0$ and $q_3 \geq 0$ are weighting coefficients, and ω, i_a and u are the angular velocity, the armature current, and the input voltage of the DC motors, respectively. $\bar{\omega}$ is the target reference angular velocity, and $\bar{i}_a$ and $\bar{u}$ are the steady-state values of the corresponding armature current and input voltage. The problem is to find values of the feedback gains K_1, K_2 and K_3 which minimize the performance index (48). We solve the optimization problem with gradient-based algorithms where the gradients are calculated by the proposed method of Theorem 5.

In numerical experiment we choose values of the parameters in the figure as: $L_a = 6.40\times10^{-3}$H, $R_a = 8.35\times10^{-2}\ \Omega$, $J = 7.50\times10^{-1}$kg$\cdot$m^2, $K_m = 1.07$, $K_r = 2.31\times10$. We set values of weighting coefficients q_1, q_2 and q_3 as $q_1 = 5.0$, $q_2 = 5.0$, $q_3 = 1.0$, and the sampling period T_k and the final time t_f as $T_k = 0.1$s ($k = 1, 2, \cdots$) and $t_f = 2.05$s. The target reference speed is set to $\bar{\omega} = 30$rad/s. We apply the steepest descent method to the optimization problem in which the sensitivity analysis based on Theorem 5 is implemented. The obtained optimum gains denoted by K_1^*, K_2^* and K_3^* and the optimum value of the performance index denoted by $J_{\min}$ are:

$$\left[K_1^*, K_2^*, K_3^*\right] = \left[3.97\times10^{-2}, -4.79\times10^{-3}, -4.34\times10^{-2}\right] \tag{49}$$

$$J_{min} = 7.62\times10^2. \tag{50}$$

In order to confirm the validity and superiority of the result, we further consider the following. One of the conventional methods of designing digital controllers for a continuous-time plant is to derive discrete equivalent of the plant model and then design a discrete-time controller for it [34]. The above optimization problem can be reduced to the discrete-time LQR (linear quadratic regulator) problem by deriving a discrete equivalent model of the DC motor and to the steady-state

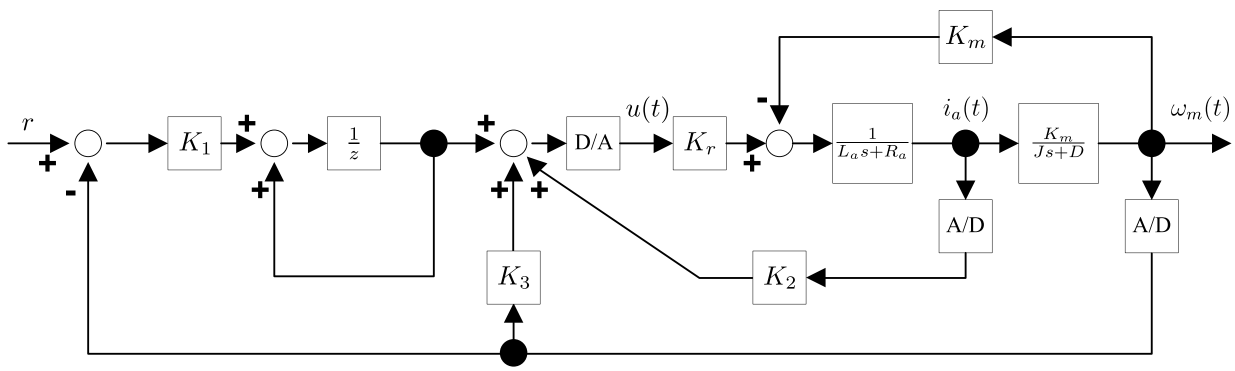


Fig. 8. Block diagram of digital speed control system of DC motor.

optimal control by letting $t_f = \infty$, $(k_f = \infty)$. It is known that the solution (optimal feedback gain) can be obtained by solving the so-called the algebraic Riccati equation [34]. Note that the discrete equivalent model is built such that its response is equal to that of the original continuous-time system at the sampling instances. Let the discrete equivalent optimal gains be K_1^{r*}, K_2^{r*} and K_3^{r*}, which are obtained by solving the algebraic Riccati equation of the LQR steady-state optimal control. The optimal gain parameters thus obtained are:

$$\begin{aligned}&\left[K_1^{r*}, K_2^{r*}, K_3^{r*}\right]\\&\quad = \left[5.22 \times 10^{-2}, -9.08 \times 10^{-4}, -5.90 \times 10^{-2}\right]. \end{aligned} \tag{51}$$

The step responses of the control system with the optimal parameters obtained by the proposed method and those by the discrete-time LQR are shown in Fig. 9. From the step response, we calculated the values of the performance index (48) of the closed control system with the optimal parameters (51), denoted by $J_{\min}^r$, and the result was

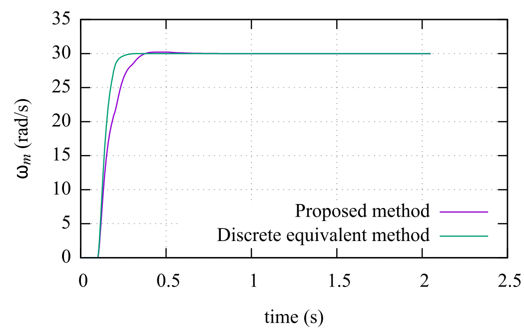
$$J_{\min}^r = 7.91 \times 10^2.$$

Note that $J_{\min}^r > J_{\min}$, and in the result of the discrete equivalent method in Fig. 9 it is observed that the larger armature current i_a flows with a rush between the sampling instants compared with that in the result of the proposed method. This is due to the fact that the discrete equivalent method neglects the inter-sample behavior of the system. On the other hand, the proposed optimization method makes it possible to find the optimal parameters in the digital controller as hybrid continuous-tine and discrete-time systems and to determine optimal gains by considering the inter-sample behavior of the system.

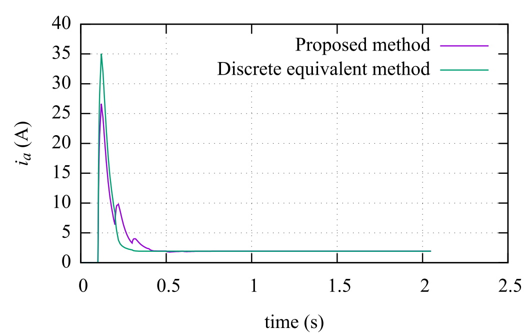
Although the proposed method can be applied to nonlinear systems, the above DC motor control system is linear. Next, we consider the problem of optimal controller design for a nonlinear system. We consider the saturation of the amateur current $i_a(t)$ of the DC motor in Fig. 8. Let the mathematical model of the saturation be

$$f(x) = sat(x) = i_{sat} \tanh\left(\frac{x}{i_{sat}}\right) \tag{52}$$

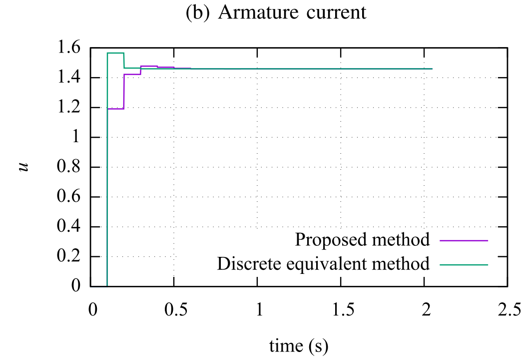
where i_{sat} is the saturation values of $i_a(t)$. As shown in Fig. 10 the block diagram of the saturation is obtained by replacing the block on the left by that on the right. We apply the



(a) Rotation speed



(b) Armature current



(c) Controller output

Fig. 9. Step responses of the obtained optimal DC motor control system.

proposed method to the nonlinear system in order to obtain optimal values of the control parameters K_1, K_2, and K_3 which minimize (48). We again set the $q_1 = 5.0$, $q_2 = 5.0$, $q_3 = 1.0$, $T_k = 0.1$s $(k = 1, 2, \cdots)$ and $t_f = 2.05$s. The target speed is set to $\bar{\omega} = 30$rad/s. and the saturation values of $i_a(t)$ is

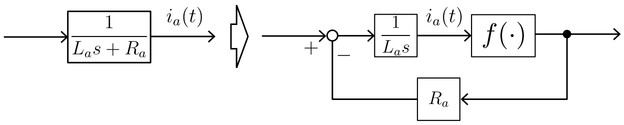
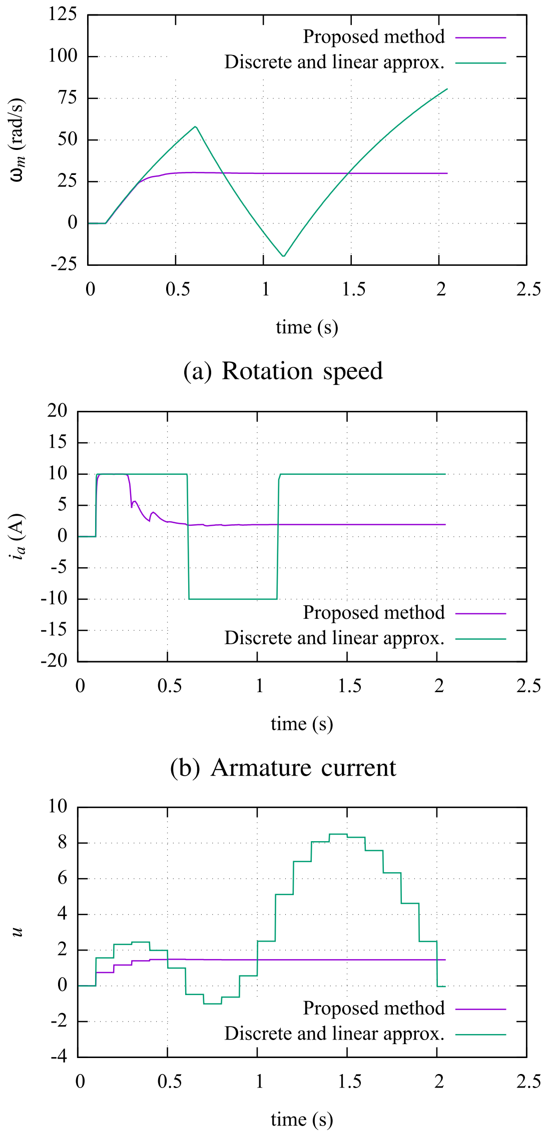


Fig. 10. Saturation of the armature current i_a of DC motor.



(a) Rotation speed

(b) Armature current

(c) Controller output

Fig. 11. Step responses of the obtained optimal DC motor control system with the saturation of the armature current.

$i_{sat} = 10$ A. The obtained optimal parameters, denoted by K^*_{sat1}, K^*_{sat2} and K^*_{sat3}, and the values of the performance index (48), denoted by J^*_{sat} are as follows:

$$\begin{aligned}&[K^*_{sat1}, K^*_{sat2}, K^*_{sat3}]\\ &\quad = \left[2.49 \times 10^{-2}, -7.51 \times 10^{-3}, -1.90 \times 10^{-2}\right] \quad (53)\\ &J^*_{sat} = 8.66 \times 10^2. \quad (54)\end{aligned}$$

The step responses of the control system with the optimal parameters obtained by the proposed method are shown in Fig. 11. Note that the discrete-time LQR theory can not be applied to nonlinear systems as it is. The conventional way to design a controller for a nonlinear system is to derive an approximate linearized model for it and apply a linear control theory. For comparison, in Fig. 11, the step responses of the control system with the optimal parameters obtained by the discrete-time LQR theory to the approximate linearized model (discrete and linear approximation) are also shown. It can be seen from the step responses of the proposed method that the output quickly follows its step reference input. On the other hand, in the step response of the discrete and linear approximation the output cannot follow its step reference input and even diverge, which implies that the closed system becomes unstable. It is therefore concluded that the proposed method works validly and is useful also for nonlinear systems.

VII. Conclusion

In this paper, we proposed a computer-aided method of sensitivity analysis for a class of cyber-physical system. The target system is a hybrid system in which a nonlinear continuous-time system and a nonlinear discrete-time system are connected through A/D and D/A interfaces. The proposed method is based on the extended Tellegen's theorem and makes it possible to perform sensitivity analysis only by giving characteristic equations of all the elements and their connection relationship in target systems without deriving their model equations such as the state space models. We applied the proposed method to a cyber-physical system whose parameter sensitivities can be derived analytically in order to estimate its accuracy. It was confirmed that sensitivities can be obtained with sufficient accuracy by the proposed method. Furthermore, we applied the proposed method to an optimization problem of a continuous-time and discrete-time hybrid system. It was confirmed that it works properly and can obtain optimal gains as the hybrid system.

Appendix A
Proof of Theorem 1

Figures 1 and 3 can be redrawn as shown in Figs. 12 and 13. Note that in these figures inside the large block drawn by the dashed line there are only summing points and take-off points which represent the system topology, that is, the connection relation among all the block elements and interfaces included in the systems S and $\hat{S}$. For all the signals in the system S, the following relation holds:

$$\boldsymbol{x} = \boldsymbol{\Theta} \boldsymbol{z} \quad (55)$$

where $\boldsymbol{x}$ is the vector consisting of all the signals going out of the block drawn by the dashed lines:

$$\boldsymbol{x} = [\boldsymbol{u}_p^\top, \boldsymbol{u}_c^\top, \boldsymbol{u}_{ps}^\top, \boldsymbol{u}_{ch}^\top, \boldsymbol{w}_p^\top, \boldsymbol{w}_c^\top]^\top$$

and $\boldsymbol{z}$ is the vector consisting of all the signals going into the block drawn by the dashed lines:

$$\boldsymbol{z} = [\boldsymbol{v}_p^\top, \boldsymbol{v}_c^\top, \boldsymbol{y}_p^\top, \boldsymbol{y}_c^\top, \boldsymbol{y}_{cs}^\top, \boldsymbol{y}_{ph}^\top]^\top$$

as shown in Fig. 12. $\boldsymbol{\Theta}$ is the matrix consisting of all the elements of which take only the value 1 or 0 and represents the connection relation among all those signals. On the other

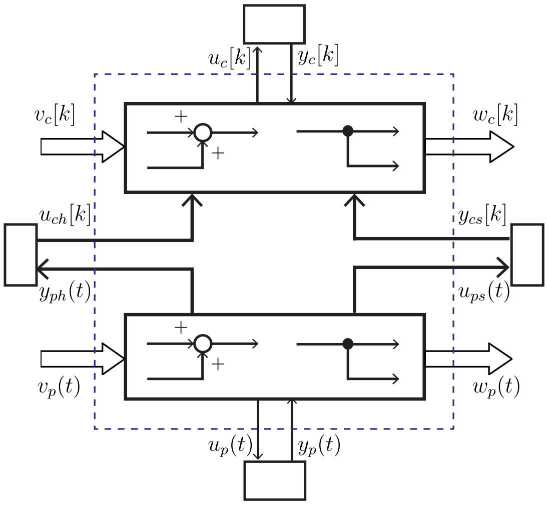


Fig. 12. Redrawed equivalent block-diagram of the target cyber-physical system S shown in Fig. 1.

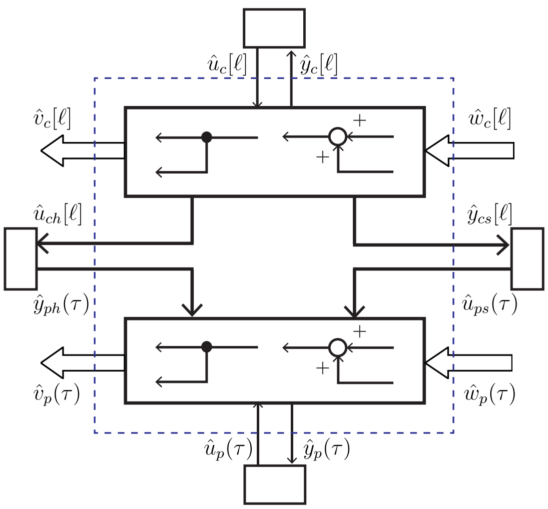


Fig. 13. Redrawed equivalent block-diagram of the adjoint system $\hat{S}$ shown in Fig. 3.

hand, for all the signals in the adjoint system $\hat{S}$, the following relation holds:

$$\hat{\boldsymbol{z}} = \boldsymbol{\Theta}^\top \hat{\boldsymbol{x}} \tag{56}$$

where $\hat{z}$ is the vector consisting of all the signals going out of the block drawn by the dashed lines:

$$\hat{\boldsymbol{z}} = [\hat{\boldsymbol{v}}_p^\top \hat{\boldsymbol{v}}_c^\top \hat{\boldsymbol{y}}_p^\top \hat{\boldsymbol{y}}_c^\top \hat{\boldsymbol{y}}_{cs}^\top \hat{\boldsymbol{y}}_{ph}^\top]^\top$$

and $\hat{x}$ is the vector consists of all the signals going into the block drawn by the dashed lines:

$$\hat{\boldsymbol{x}} = [\hat{\boldsymbol{u}}_p^\top \hat{\boldsymbol{u}}_c^\top \hat{\boldsymbol{u}}_{ps}^\top \hat{\boldsymbol{u}}_{ch}^\top \hat{\boldsymbol{w}}_p^\top \hat{\boldsymbol{w}}_c^\top]^\top$$

as shown in Fig. 13. From (55) and (56), $\boldsymbol{z}^\top \hat{\boldsymbol{z}}$ is calculated as

$$\boldsymbol{z}^\top \hat{\boldsymbol{z}} = \boldsymbol{z}^\top \boldsymbol{\Theta}^\top \hat{\boldsymbol{x}} = (\boldsymbol{\Theta} \boldsymbol{z})^\top \hat{\boldsymbol{x}} = \boldsymbol{x}^\top \hat{\boldsymbol{x}}.$$

Rewriting this equation by breaking it down into each element, we obtain (4).

Appendix B
Proof of Theorem 2

Since the extended Tellegen's theorem (Theorem 1) holds for a pair of the systems $S + \Delta S$ and $\hat{S}$, we have

$$\begin{aligned}
&\sum_i (u_{p_i} + \Delta u_{p_i})\hat{v}_{p_i} + \sum_i (v_{c_i} + \Delta v_{c_i})\hat{v}_{c_i} \\
&\quad + \sum_i (y_{p_i} + \Delta y_{p_i})\hat{y}_{p_i} + \sum_i (y_{c_i} + \Delta y_{c_i})\hat{y}_{c_i} \\
&\quad + \sum_i (y_{cs_i} + \Delta y_{cs_i})\hat{y}_{cs_i} + \sum_i (y_{ph_i} + \Delta y_{ph_i})\hat{y}_{ph_i} \\
&= \sum_i (u_{p_i} + \Delta u_{p_i})\hat{u}_{p_i} + \sum_i (u_{c_i} + \Delta u_{c_i})\hat{u}_{c_i} \\
&\quad + \sum_i (u_{ps_i} + \Delta u_{ps_i})\hat{v}_{ps_i} + \sum_i (u_{ch_i} + \Delta u_{ch_i})\hat{u}_{ch_i} \\
&\quad + \sum_i w_{p_i} + \Delta w_{p_i})\Delta\hat{w}_{p_i} + \sum_i (w_{c_i} + \Delta w_{c_i}) w_{c_i}.
\end{aligned} \tag{57}$$

Consider the time axis for the original system S and its adjoint $\hat{S}$, that is, t and τ. Let all the signals in S_c and $\hat{S}_c$ virtually be piecewise-constant continuous-time signals as shown in (3) and (23). Subtracting both sides of (4) from those of (57) and integrating the result from $t = 0$ to $t = t_f$, we obtain the following equation.

$$\begin{aligned}
&\sum_i \int_0^{t_f} \Delta w_{p_i}(t)\hat{w}_{p_i}(\tau)dt + \sum_i \int_0^{t_f} \Delta w_{c_i}(t)\hat{w}_{c_i}(\tau)dt \\
&\quad = \sum_i I_{p_i} + \sum_i I_{c_i} + \sum_i I_{cs_i} + \sum_i I_{ph_i}
\end{aligned}$$

where

$$I_{p_i} = \int_0^{t_f} \{\Delta y_{p_i}(t)\hat{y}_{p_i}(\tau) - \Delta u_{p_i}(t)\hat{u}_{p_i}(\tau)\}dt, \tag{58}$$

$$I_{c_i} = \int_0^{t_f} \left\{\Delta y_{c_i}(t)\hat{y}_{c_i}(\tau) - \Delta u_{c_i}(t)\hat{u}_{c_i}(\tau)\right\} dt, \tag{59}$$

$$I_{pc_i} = \int_0^{t_f} \left\{\Delta y_{cs_i}(t)\hat{y}_{cs_i}(\tau) - \Delta u_{ps_i}(t)\hat{u}_{ps_i}(\tau)\right\} dt, \tag{60}$$

$$I_{cp_i} = \int_0^{t_f} \left\{\Delta y_{ph_i}(t)\hat{y}_{ph_i}(\tau) - \Delta u_{ch_i}(t)\hat{u}_{ch_i}(\tau)\right\} dt. \tag{61}$$

Each term can be calculated as follows.

Physical System S_p

Integral element in S_p

Dividing the interval of integration of the term I_{p_i} in (58) into sampling intervals and using the characteristics of the integral element (5), we have

$$I_{p_i} = \sum_{k=0}^{k_f} \int_{t_k}^{t_{k+1}} \{\Delta y_{p_i}(t)\hat{y}_{p_i}(\tau) - \frac{d}{dt}\Delta y_{p_i}(t)\hat{u}_{p_i}(\tau)\}dt.$$

By using the integral by parts formula and the characteristic of the adjoint of the integral element (11) we have:

$$\begin{aligned}
I_{p_i} = \sum_{k=0}^{k_f} \Bigg\{ &\int_{t_k}^{t_{k+1}} \Delta y_{p_i}(t)\hat{y}_{p_i}(\tau)dt - \Big[\Delta y_{p_i}(t)\hat{u}_{p_i}(\tau)\Big]_{t_k}^{t_{k+1}} \\
&+ \int_{t_k}^{t_{k+1}} \Delta y_{p_i}(t)\frac{d}{dt}\hat{u}_{p_i}(\tau)dt \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{k_f} \int_{t_k}^{t_{k+1}} \Delta y_{p_i}(t) \left\{ \hat{y}_{c_i}(\tau) - \frac{d}{dt} \hat{u}_{c_i}(\tau) \right\} dt \\
&\quad - \sum_{k=0}^{k_f} \Big\{ \Delta y_{p_i}(t_{k+1}) \hat{u}_{p_i}(t_{k_f} - t_{k+1}) \\
&\quad - \Delta y_{p_i}(t_k) \hat{u}_{p_i}(t_{k_f} - k_f) \Big\} \\
&= - \sum_{k=0}^{k_f} \Big\{ \Delta y_{p_i}(t_{k+1}) \hat{u}_{p_i}(t_f - t_{k+1}) \\
&\quad - \Delta y_{p_i}(t_k) \hat{u}_{p_i}(t_{k_f} - t_k) \Big\}
\end{aligned}
$$

Furthermore, the above equation is calculated as follows:

$$
\begin{aligned}
I_{p_i} &= - \Big\{ \cancel{\Delta y_{p_i}(t_1) \hat{u}_{p_i}(t_{k_f} - t_1)} - \Delta y_{p_i}(0) \hat{u}_{p_i}(t_{k_f}) \\
&\quad + \cancel{\Delta y_{p_i}(t_2) \hat{u}_{p_i} \left(t_{k_f} - t_2\right)} - \cancel{\Delta y_{p_i}(t_1) \hat{u}_{p_i}(t_{k_f} - t_1)} \\
&\quad \vdots \\
&\quad + \cancel{\Delta y_{p_i}(t_{k_f}) \hat{u}_{c_i}(t_{k_f} - t_{k_f})} \\
&\quad - \cancel{\Delta y_{p_i} \left(t_{k_f - 1}\right) \hat{u}_{p_i}(t_{k_f} - k_{k_f - 1})} \\
&\quad + \Delta y_{p_i}(t_f) \hat{u}_{p_i}(t_{k_f} - t_{k_f + 1}) \\
&\quad - \cancel{\Delta y_{p_i}(t_{k_f}) \hat{u}_{p_i}(t_{k_f} - t_{k_f})} \Big\} \\
&= - \left\{ \Delta y_{p_i}(t_f) \hat{u}_{c_i}(-\tau_{-1}) - \Delta y_{p_i}(0) \hat{u}_{p_i}(t_{k_f}) \right\} = 0.
\end{aligned}
$$

The last equality comes from $\Delta y_{p_i}(0) = 0$ and $\hat{u}_{p_i}(0) = 0$.

Parameter element in S_p

From the characteristics of the parameter element (6), we have:

$$
\Delta y_p(t) = K_p \Delta u_p(t) + u_p(t) \Delta K_p + O(\Delta^2).
$$

By using this and the characteristic of the adjoint of the parameter element (12) we have:

$$
\begin{aligned}
&\Delta y_{p_i}(t) \hat{y}_{p_i}(\tau) - \Delta u_{p_i} \hat{u}_{p_i}(\tau) \\
&\qquad = \left(K_{p_i} \Delta u_{p_i}(t) + u_{p_i}(t) \Delta K_{p_i} \right) \hat{y}_{p_i}(\tau) \\
&\qquad \quad - \Delta u_{p_i}(t) K_{p_i} \hat{y}_{p_i}(\tau) + O(\Delta^2) \\
&\qquad = u_{p_i}(t) \hat{y}_{p_i}(\tau) \Delta K_{p_i} + O(\Delta^2)
\end{aligned}
$$

Therefore,

$$
I_{p_i} = \sum_i \int_0^{t_f} u_{p_i}(t) y_{p_i}(\tau) dt \Delta K_{p_i} + O(\Delta^2)
$$

Nonlinear element in S_p

Here we prove the case of the multi-input and multi-output nonlinear element. From the characteristic of the nonlinear element $\boldsymbol{y}_p(t) = \boldsymbol{f}_p(\boldsymbol{u}_p(t))$, we have:

$$
\Delta \boldsymbol{y}_p(t) = \frac{\partial}{\partial \boldsymbol{u}_p} f_p \left(\boldsymbol{u}_p(t) \right) \Delta \boldsymbol{u}_p(t) + O(\Delta^2).
$$

Therefore,

$$
\begin{aligned}
&\Delta \boldsymbol{y}_{p_i}(t)^\top \hat{\boldsymbol{y}}_{p_i}(\tau) - \Delta \boldsymbol{u}_{p_i}(t)^\top \hat{\boldsymbol{u}}_{p_i}(\tau) \\
&\quad = \left\{ \frac{\partial \boldsymbol{f}_p}{\partial \boldsymbol{u}_p} \Delta \boldsymbol{u}_{p_i}(t) \right\}^\top \hat{\boldsymbol{y}}_{p_i}(\tau) - \Delta \boldsymbol{u}_{p_i}(t)^\top \hat{\boldsymbol{u}}_{p_i}(\tau) + O(\Delta^2) \\
&\quad = \Delta \boldsymbol{u}_{p_i}(t)^\top \left\{ \frac{\partial \boldsymbol{f}_p}{\partial \boldsymbol{u}_p}^\top \hat{\boldsymbol{y}}_{p_i}(\tau) - \hat{\boldsymbol{u}}_{p_i}(\tau) \right\} + O(\Delta^2) \\
&\quad = O(\Delta^2).
\end{aligned}
$$

The last equality comes from the characteristic of the adjoint of the nonlinear element (21). Then we have

$$
I_{p_i} = O(\Delta^2).
$$

Cyber System S_c

Here we only show the calculations of the term I_{c_i} in (59) for the delay element and the parameter element. The term of the nonlinear element can be calculated in the same way as those in the physical system S_p

Delay element

From (3) and (23), I_{c_i} is calculated as follows:

$$
\begin{aligned}
I_{c_i} &= \sum_{k=0}^{k_f} \int_{t_k}^{t_{k+1}} \left\{ \Delta y_{c_i}(t) \hat{y}_{c_i}(\tau) - \Delta u_{c_i}(t) \hat{u}_{c_i}(\tau) \right\} dt \\
&= \sum_{k=0}^{k_f} \left\{ \Delta y_{c_i}[k] \hat{y}_{c_i}[\ell] - \Delta u_{c_i}[k] \hat{u}_{c_i}[\ell] \right\} T_{k+1}.
\end{aligned}
$$

By using the characteristics of the delay element (8) and its adjoint (14), I_{c_i} becomes

$$
I_{c_i} = \sum_{k=0}^{k_f} \left\{ \Delta y_{c_i}[k] \hat{u}_{c_i}[\ell + 1] T_k - \Delta y_{c_i}[k+1] \hat{u}_{c_i}[\ell] T_{k+1} \right\}.
$$

Furthermore I_{c_i} is calculated as follows:

$$
\begin{aligned}
I_{c_i} &= \Delta y_{c_i}[0] \hat{u}_{c_i}[k_f + 1] T_0 - \cancel{\Delta y_{c_i}[1] \hat{u}_{c_i}[k_f] T_1} \\
&\quad + \cancel{\Delta y_{c_i}[1] \hat{u}_{c_i}[k_f] T_1} - \cancel{\Delta y_{c_i}[2] \hat{u}_{c_i}[k_f - 1] T_2} \\
&\qquad \vdots \\
&\quad + \cancel{\Delta y_{c_i}[k_f - 1] \hat{u}_{c_i}[2] T_{k_f - 1}} - \cancel{\Delta y_{c_i}[k_f] \hat{u}_{c_i}[1] T_{k_f}} \\
&\quad + \Delta y_{c_i}[k_f] \hat{u}_{c_i}[1] T_{k_f} - \cancel{\Delta y_{c_i}[k_f + 1] \hat{u}_{c_i}[0] T_{k_f + 1}} \\
&= \Delta y_{c_i}[0] \hat{u}_{c_i}[k_f + 1] T_0 - \Delta y_{c_i}[k_f + 1] \hat{u}_{c_i}[0] T_{k_f + 1} \\
&= 0.
\end{aligned}
$$

The last equality is comes from $\Delta y_{c_i}[0] = 0$ and $\hat{u}_{c_i}[0] = 0$.

Parameter element

From (3) and (23), I_{c_i} is calculated as follows:

$$
\begin{aligned}
I_{c_i} &= \int_0^{t_f} \left\{ \Delta y_{c_i}(t) \hat{y}_{c_i}(\tau) - \Delta u_{c_i} \hat{u}_{c_i}(\tau) \right\} dt \\
&= \sum_{k=0}^{k_f} \left\{ \Delta y_{c_i}[k] \hat{y}_{c_i}[\ell] - \Delta u_{c_i}[k] \hat{u}_{c_i}[\ell] \right\} T_{k+1}. \qquad (62)
\end{aligned}
$$

From the characteristics of the parameter element (9), we have:

$$
\Delta y_c[k] = K_c \Delta u_c[k] + u_c[k] \Delta K_c + O(\Delta^2).
$$

Substituting this into (62) gives the following equation:

$$\begin{aligned} I_{c_i} &= \sum_{k=0}^{k_f} \Bigg\{ \left\{ K_{c_i} \Delta u_{c_i}[k] + u_{c_i}[k] \Delta K_{c_i} \right\} \hat{y}_{c_i}[\ell] \\ &\quad - u_{c_i}[k] \hat{y}_{c_i}[\ell] \Delta K_{c_i} \Bigg\} T_{k+1} + O(\Delta^2) \\ &= \sum_{k=0}^{k_f} \Bigg\{ \left\{ K_{c_i} \hat{y}_{c_i}[\ell] - \hat{u}_{c_i}[\ell] \right\} \Delta u_{c_i}[k] \\ &\quad + u_{c_i}[k] \hat{y}_{c_i}[\ell] \Delta K_{c_i} \Bigg\} T_{k+1} + O(\Delta^2). \end{aligned}$$

From the characteristic of the adjoint of the parameter element (15) we have:

$$I_{c_i} = \sum_{k=0}^{k_f} T_{k+1} u_{c_i}[k] \hat{y}_{c_i}[\ell] \Delta K_{c_i} + O(\Delta^2).$$

A/D interface

By using (3), (23) and the characteristic of the adjoint of the A/D interface (17), the term I_{c_i} in (60) is calculated as follows:

$$\begin{aligned} I_{pc_i} &= \int_0^{t_f} \left\{ \Delta y_{cs_i}(t) \hat{y}_{cs_i}(\tau) - \Delta u_{ps_i}(t) \hat{u}_{ps_i}(\tau) \right\} dt \\ &= \sum_{k=0}^{k_f} \Bigg\{ \Delta y_{cs_i}[k] \hat{y}_{cs_i}[\ell] T_{k+1} \\ &\quad - \int_{t_k}^{t_{k+1}} \Delta u_{cs_i}(t) \delta_+(\tau - \tau_\ell) T_{k_f-\ell+1} \ \hat{y}_{cs}[\ell]) dt \Bigg\}. \end{aligned}$$

By using the characteristic of the A/D interface (1), we have

$$\begin{aligned} I_{pc_i} &= \sum_{k=0}^{k_f} \Bigg\{ \Delta u_{ps_i}(t_k -) \hat{y}_{cs_i}[\ell] T_{k+1} \\ &\quad - \Delta u_{ps_i}(t_k -) \hat{y}_{cs_i}[\ell] T_{k_f-\ell+1} \Bigg\} = 0. \end{aligned}$$

D/A interface

By using the characteristic of the D/A interface (2) and (3), the term I_{cp_i} in (61) is calculated as follows:

$$\begin{aligned} I_{cp_i} &= \int_0^{t_f} \left\{ \Delta y_{ph_i}(t) \hat{y}_{ph_i}(\tau) - \Delta u_{ch_i}(t) \hat{u}_{ch_i}(\tau) \right\} dt \\ &= \sum_{k=0}^{k_f} \int_{t_k}^{t_{k+1}} \left\{ \Delta u_{ch_i}[k] \hat{y}_{ph_i}(\tau) - \Delta u_{ch_i}[k] \hat{u}_{ch_i}(\tau) dt \right\}. \end{aligned}$$

Using the characteristic of the adjoint of the D/A interface (20), we obtain

$$\begin{aligned} I_{cp_i} &= \sum_{k=0}^{k_f} \Delta u_{ch_i}[k] \int_{t_k}^{t_{k+1}} \left\{ \hat{y}_{ph_i}(\tau) - \hat{u}_{ch_i}(\tau) \right\} dt \\ &= \sum_{k=0}^{k_f} \Delta u_{ch_i}[k] \left\{ \int_{t_k}^{t_{k+1}} \hat{y}_{ch_i}(\tau) d\tau - \hat{u}_{ch_i}[\ell] T_{k+1} \right\} = 0. \end{aligned}$$

Thus we obtain (24).

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