THE UNIVERSITY OF MEMPHIS
CENTER FOR EARTHQUAKE RESEARCH AND INFORMATION

Homework 3

"Finite Difference Approximation of 1D Wave Equation"

DATA ANALYSIS IN GEOPHYSICS

CERI 8104

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INSTRUCTIONS:

Solve the unidirectional wave equation in 1D:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

Which can be approximated numerically as:

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right) \tag{1}$$

over the spatial domain $\Omega = [0, 1]$ km given the initial conditions:

$$\text{IC: } u(x, t = 0) = g(x) = A_0 \exp\left(-\frac{(x-\mu)^2}{\sigma^2}\right). \text{ and}$$
$$\text{BC: } u(0, t) = 0, u(N, t) = 0, \text{ with } i = 1, \ldots, N$$

The initial conditions describe a Gaussian amplitude profile with peak amplitude A_0, location, μ and scale, σ. You can use the following numerical values for these parameters:

$$A_0 = 1.5, \mu = 0.3 \text{ and } \sigma = 0.005$$

The wave speed, c is 0.5 km/s.

Use a grid spacing of $\Delta x = 0.01$ ($n_\text{x} = x_\text{max}/\text{d}x + 1$) and a time step of $\Delta t = 0.5\Delta x/c$. Note that the explicit solution becomes unstable for large time steps. The stability criterion for the numerical solution is given by: $c\Delta t/\Delta x \leq 0.5$.

To solve 1, you have to implement a nested for-loop. Create an equally spaced time vector between 0 and $t_\text{max} = 1$ in Δt increments. Solve for the amplitude profile at each new time step, with the amplitudes at the first time-step being equal to the initial conditions.

a. Plot the amplitude profile at every time step and describe your observations!

b. In which direction does the wave propagate?

c. What happens to the amplitude of the signal as it propagates? (This is an artifact of the numerical method, not a property of the solution.)

d. Create a matrix that holds the amplitude profiles at each time step (i.e. dim= $nx \times nt$). Store the amplitude vector at each time step in this matrix. We will now use this matrix to compute the slowness: $s = 1/c$.

To visualize the propagating matrix, you can use:

```
[m_x, m_t] = meshgrid( a_x, a_t);
plot3 = pcolor(ax3, m_x, m_t, flipud(m_u'));
shading interp;
axis on
```

Use the resulting figure to graphically estimate the slowness. Discuss your results!

RESULTS:

I used the provided MATLAB script template to solve the 1D wave equation numerically.
The initial amplitude profile as a Gaussian pulse centered at $x = 0.3$ km with a peak amplitude of 1.5 and standard deviation (scale) of 0.005 km can be seen in the following figure:

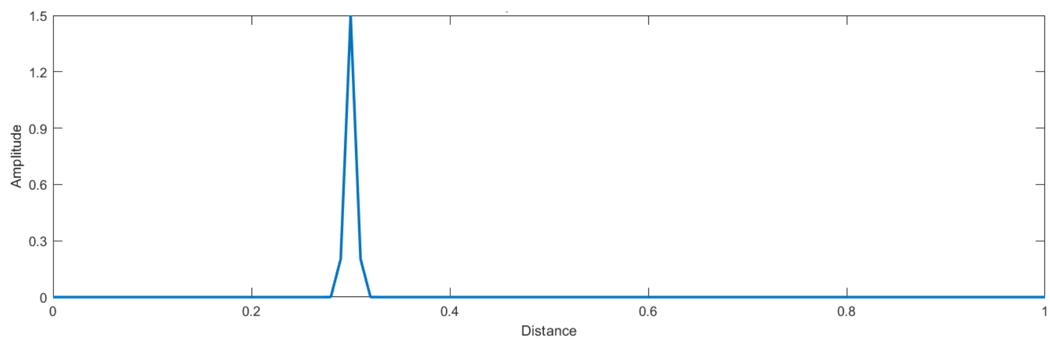

Figure 1 – *The initial Gaussian amplitude profile at $t = 0$. The peak is centered at $x = 0.3$ km with an amplitude of 1.5.*

After this, for plotting and visualizing the evolution of the wave amplitude profile over time, I created a .gif animation that shows how the initial Gaussian pulse propagates through the spatial domain. The wave clearly moves from left to right, consistent with the governing equation.
The the animation can be found here: **Wave Equation Animation**.
Next, I generated a figure with four subplots showing the amplitude profiles at the first, middle (50th), last time steps, and all time steps superimposed.

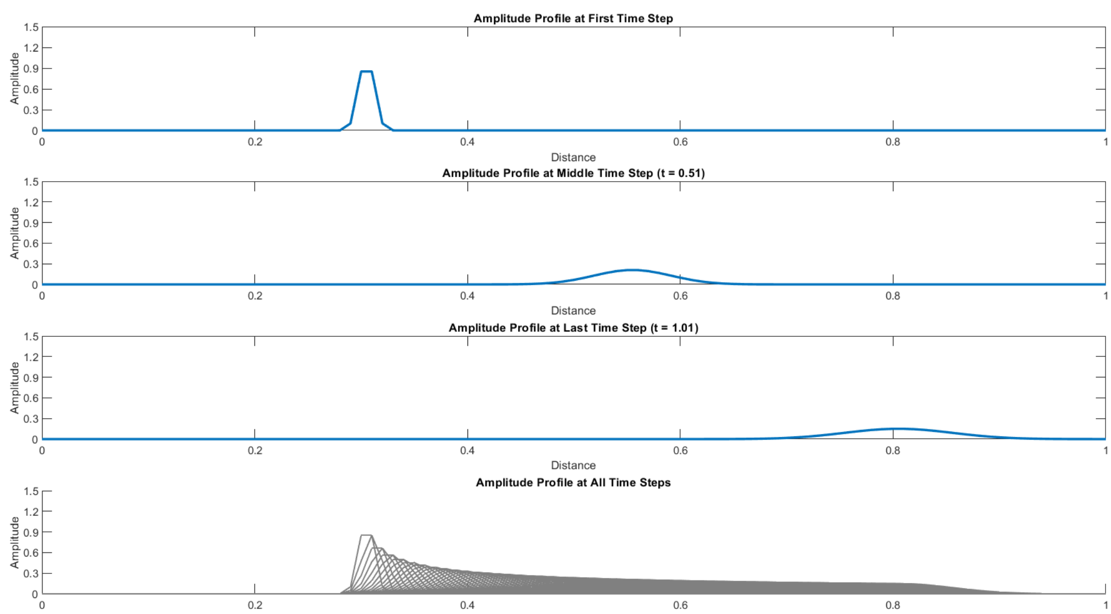

Figure 2 – *Amplitude profiles at (a) the first time step, (b) the middle time step, (c) the last time step, and (d) all time steps superimposed. The wave propagates to the right while its amplitude decreases due to numerical diffusion.*

In the figure above, we observe that the as the wave propagates, its peak amplitude decreases significantly from 1.5 to below 0.8 by the final time step. As expressed in the instructions, this amplitude reduction is a numerical artifact of the finite-difference scheme used (Forward-Time Backward-Space), because the ideal wave equation should conserve energy, and thus the amplitude should remain constant. This phenomenon is known as numerical diffusion or dissipation introduced by the used "upwind" scheme.
Finally, I created the colormap of the amplitude profiles over time to visualize the wave propagation in the x-t plane, as shown below:

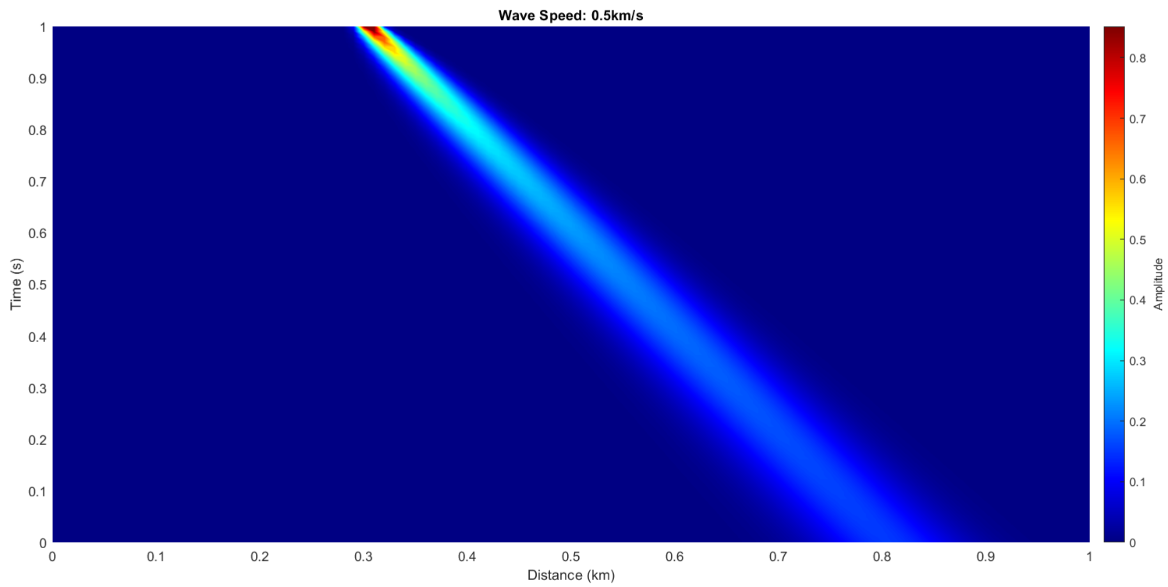

Figure 3 – *Colormap showing the wave amplitude $u(x,t)$. The bright diagonal band represents the propagating Gaussian pulse. The slope of this band is the wave speed.*

From the colormap, from the diagonal band, I graphically stimated the slowness, $s = 1/c$, by picking two points on the bright band ($x_1 = 0.5$ km at $t_1 = 0.6$ s and $x_2 = 0.7$ km at $t_2 = 0.21$ s). The points are labeled in the figure below:

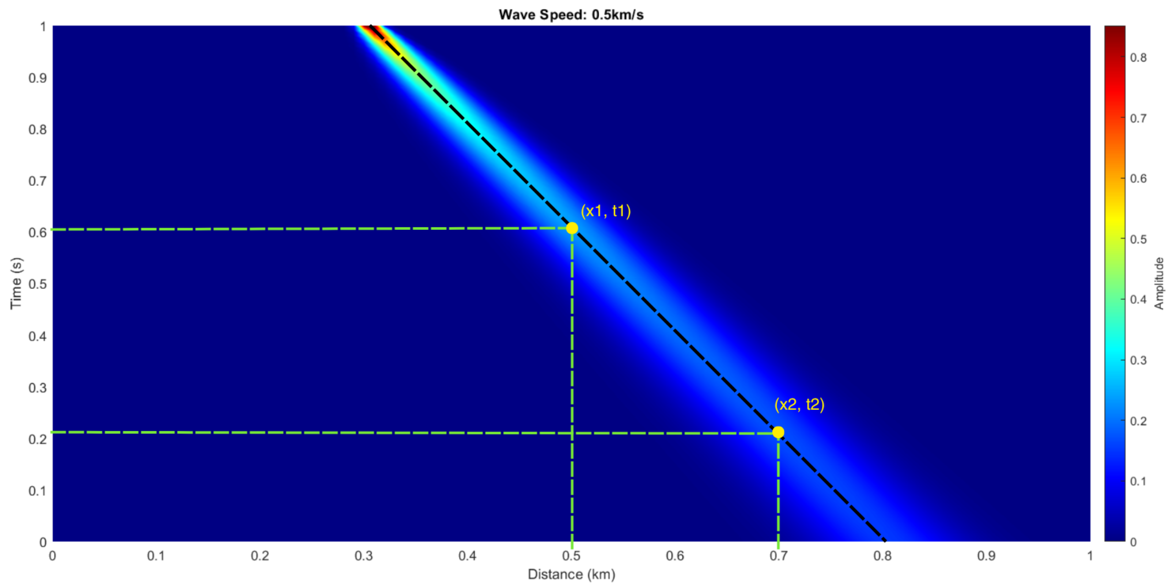

Figure 4 – *Colormap with points labeled for slowness estimation.*

However, since the we use the 'flipud' function in the pcolor command, the y-axis is flipped. Therefore, the correct coordinates for the two points are ($x_1 = 0.5$ km, $t_1 = 0.4$ s) and ($x_2 = 0.7$ km, $t_2 = 0.79$ s). Using these points, I calculated the slowness as follows:

$$\Delta x = x_2 - x_1 = 0.7\,\text{km} - 0.5\,\text{km} = 0.2\,\text{km}$$
$$\Delta t = t_2 - t_1 = 0.79\,\text{s} - 0.4\,\text{s} = 0.39\,\text{s}$$
$$\text{Velocity, } c \approx \frac{\Delta x}{\Delta t} = \frac{0.2\,\text{km}}{0.39\,\text{s}} \approx 0.513\,\text{km/s}$$
$$\text{Slowness, } s \approx \frac{1}{c} = \frac{1}{0.513\,\text{km/s}} \approx 1.95\,\text{s/km}$$

This estimated slowness is consistent with the theoretical value of $s = 1/c = 1/0.5 = 2.0$ s/km, which confirms that the simulation correctly models the wave propagation speed.