Legendre's equation and polynomials, the Laplacian and spherical harmonics

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1 Introduction

The following notes treat a combination of things - one being the approach to solving Laplace's equation (or equations including the Laplacian operator) in spherical coordinates. This includes the so-called spherical harmonics, which I have added after discussions with Qichuan Bai, one of my students at Penn State.

Another treated in detail is Legendre's equation, and how the Legendre polynomials arise from it. Those notes are taken from a Math 405 class I taught at Penn State in Fall 2006. I am indebted to my students in that class for their reactions on how helpful they found these lectures, and I hope that these notes will be of further benefit to more students.

As much as I like the historical perspective in mathematics, here I will follow a more nuts-and-bolts approach to the subject, in part because I imagine it will be helpful to other students trying to sort out Legendre's equation and the resulting polynomials for another class. I will add to these notes in time with later sections, to discuss details and give additional information. If you have any comments please email them to me at alb18@psu.edu.

2 The Laplacian in Spherical Coordinates

The Laplacian operator ∇^2 is a differential operator for scalar functions, defined for a function u as

$$\nabla^2 u = \nabla \cdot (\nabla u) = \text{div grad } u$$

This may also be written as Δu. It appears in many physically important equations, such as the Schrödinger equation in quantum mechanics

$$\frac{-h^2}{2m}\nabla^2\psi + V\psi = E\psi$$

where ψ is the wavefunction, or Poisson's equation for the electric potential ϕ

$$\nabla^2 \phi = 4\pi\rho$$

where ρ is the charge density. Then of course there is Laplace's equation itself: $\nabla^2 u = 0$, which is simplest in Cartesian coordinates (x, y, z), where

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Our interest here is in spherical coordinates (3D); let's consider the function $u = u(r, \theta, \phi)$. In this case we have term by term:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

Taking the usual separation of variables approach, we look for solutions of the form

$$u(r, \theta, \phi) = F(r)G(\theta, \phi)$$

Using this in Laplace's equation in spherical coordinates $\nabla^2 u = 0$ leads after some rearrangement to

$$\frac{r^2}{F}\left(F'' + \frac{2}{r}F'\right) = -\frac{1}{G}\left(\frac{\partial^2 G}{\partial \phi^2} + \cot\phi\frac{\partial G}{\partial \phi} + \frac{1}{\sin^2\phi}\frac{\partial^2 G}{\partial \theta^2}\right) \tag{1}$$

As always with separation of variables we conclude that, as the LHS depends only on r, and the RHS only on ϕ and θ, this can only be true if each side independently is equal to some arbitrary constant. One way to see this is to take $\partial/\partial r$ of both sides of Eq. 1, which would make the RHS equal to zero - then integrate and obtain the arbitrary constant.

Let's call this constant k. We now have two equations, one from each side, each coupled to the other via k. The equation in r

$$r^2 F'' + 2rF' - kF = 0,$$

is in Euler-Cauchy form, and can be solved directly. Note however that it is an eigenvalue problem for k (and $k = 0$ is a possibility which we must treat).

The other equation we obtain is in the angular variables:

$$\frac{\partial^2 G}{\partial \phi^2} + \cot\phi \frac{\partial G}{\partial \phi} + \frac{1}{\sin^2\phi}\frac{\partial^2 G}{\partial \theta^2} + kG = 0$$

We next specialize our approach to axisymmetric problems: i.e. problems with no θ (azimuthal) dependence. Thus $\partial G/\partial\theta = 0$, and we obtain a second order nonlinear coefficient ODE

$$\frac{d^2 G}{d\phi^2} + \cot\phi \frac{dG}{d\phi} + kG = 0$$

How to deal with this nonlinearity? The first hope is to eliminate it - perhaps with a change of variables? (I remind you that the following trick results - as all such tricks do - from the technique of Try Everything Until Something Works).

The change of variables to $w = \cos\phi$ modifies the ODE but does not make it linear. Here we go thru the details - for the first derivative:

$$\frac{dG}{d\phi} = \frac{dG}{dw}\frac{dw}{d\phi} = -\sin\phi \frac{dG}{dw}$$

and for the second:

$$\frac{d^2G}{d\phi^2} = \frac{d}{d\phi}\left(-\sin\phi\frac{dG}{dw}\right) = -\cos\phi\frac{dG}{dw} - \sin\phi\frac{d}{d\phi}\left(\frac{dG}{dw}\right) = -\cos\phi\frac{dG}{dw} + \sin^2\phi\frac{d^2G}{dw^2}$$

Using these two rules and changing completely to w, we find that the ODE is modified to

$$(1-w^2)G''(w) - 2wG'(w) + kG(w) = 0$$

where we now write prime for differentiation with respect to w, and we note that at this point k is still arbitrary. This is Legendre's Equation.

3 Solving Legendre's Equation

To connect with our text [1] we change the names of variables $G(w) \to y(x)$. Thus Legendre's equation becomes

$$(1-x^2)y'' - 2xy' + ky = 0 \tag{2}$$

which means that at $x = \pm 1$, we have a regular singular point. Rewriting this ODE in the form $y'' + py' + qy = 0$ we have

$$p(x) = \frac{-2x}{1-x^2} \qquad q(x) = \frac{k}{1-x^2}$$

which means that we can write the solution as a power series around $x = 0$, expecting a radius of convergence $R = 1$. Thus we require $|x| < 1$. As usual we posit the form of the solution as a power series

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

and need to find the coefficients, the a_m. We first substitute into Eq.1 and obtain

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2ma_m x^m + \sum_{m=0}^{\infty} ka_m x^m = 0$$

Looking at the powers of x to see how we can combine this all into one sum, we see that only the first term needs adjusting: we change to $s = m - 2$, which means $m = s + 2$.

We then peel off the terms corresponding to $s = 0$ and 1 (which not every sum has), leading to

$$2a_2 + ka_0 = 0 \qquad \rightarrow \qquad a_2 = -\frac{k}{2}a_0$$

and

$$6a_3 + (k-2)a_1 = 0 \qquad \rightarrow \qquad a_3 = \frac{2-k}{6}a_1$$

Already we see something interesting here: if $k = 2$, then $a_3 = 0$. Recall that k is arbitrary, so this is certainly possible.

Then combining all four sums for $s \geq 2$, we obtain the recursion relation

$$a_{s+2} = \frac{s(s+1) - k}{(s+2)(s+1)}\, a_s \tag{3}$$

The structure of this relation, which is $a_{s+2} \sim a_s$, immediately tells us two things:
1) For the odd s terms, $a_7 \sim a_5 \sim a_3 \sim a_1$, which means that all of those will ultimately be proportional to a_1. Similarly all even s coefficients will be proportional to a_0. Thus the solution is going to look like:

$$y(x) = a_0(\text{even terms}) + a_1(\text{odd terms})$$

2) There is the possibility of a truncated series: if for some n, $a_n = 0$, then all higher (odd or even) terms will also have their coefficients equal to zero. Thus the infinite series in x will be a polynomial (in this example, of order $n-2$). This leads to the next topic.

4 Legendre Polynomials

The so-called Legendre polynomials arise precisely in this case of truncation. But how could the series truncate? We see from Eq. 3 that this is only possible if for some s we have $s(s+1) = k$. This means that truncation could only occur if the arbitrary constant k were to be an integer. For instance, if $k = 30$, then $a_7 = 0$ since $30 = 5 \cdot 6$, which corresponds to $s = 5$ (and if $a_7 = 0$ then all higher odd terms are also zero).

Note that not just any integer would do: for instance, if $k = 5$, there is no integer s for which this would work; but it would for $k = 6$... So k should not be prime - but $k = 18$ will also not work!

To guarantee that k is an integer of the correct form, such that there will always be an s to truncate the series, we write the arbitrary constant in a particular form:

$$k = n(n+1)$$

where n is some arbitrary integer. This choice for k leads to n-th order polynomial solutions $y = P_n(x)$, which are known as the Legendre polynomials. Here are the first few:

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

References

[1] E. Kreyszig , *Advanced Engineering Mathematics*, 9th edition, (Wiley, 2006).