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We begin by recognizing the familiar conversion from rectangular to spherical coordinates (http://planetmath.org/node/34491) (note that ϕ is used to denote the azimuthal angle (http://planetmath.org/node/34491), whereas θ is used to denote the polar angle (http://planetmath.org/node/34491))

$$x=r\sin(\theta)\cos(\phi),y=r\sin(\theta)\sin(\phi),z=r\cos(\theta), \tag{1}$$

and conversely (http://planetmath.org/node/39554) from spherical to rectangular coordinates (http://planetmath.org/node/36016)

$$r=\sqrt{x^2+y^2+z^2},\theta=\arccos\left(\frac{z}{r}\right),\phi=\arctan\left(\frac{y}{x}\right). \tag{2}$$

Now, we know that the Laplacian (http://planetmath.org/node/33030) in rectangular coordinates is defined[1] in the following way (http://planetmath.org/node/31384)

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}. \tag{3}$$

We also know that the partial derivatives (http://planetmath.org/node/30841) in rectangular coordinates can be expanded (http://planetmath.org/node/40380) in the following way by using the chain rule (http://planetmath.org/node/32561)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}+\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}, \tag{4}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}+\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial y}, \tag{5}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z}+\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial z}. \tag{6}$$

The next step is to convert the right (http://planetmath.org/node/36456) -hand side (http://planetmath.org/node/41602) of each of the above three equations (http://planetmath.org/node/37330) so that it only has partial derivatives in terms (http://planetmath.org/node/33376) of r, θ and ϕ. We can do this by substituting the following values (which are easily derived from (2)) in their respective places (http://planetmath.org/node/36640) in the above three equations

$$\begin{aligned}
&\frac{\partial r}{\partial x}=\sin(\theta)\cos(\phi),\frac{\partial \theta}{\partial x}=\frac{1}{r}\cos(\theta)\cos(\phi),\frac{\partial \phi}{\partial x}=-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)},\\
&\frac{\partial r}{\partial y}=\sin(\theta)\sin(\phi),\frac{\partial \theta}{\partial y}=\frac{1}{r}\cos(\theta)\sin(\phi),\frac{\partial \phi}{\partial y}=\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)},\\
&\frac{\partial r}{\partial z}=\cos(\theta),\frac{\partial \theta}{\partial z}=-\frac{1}{r}\sin(\theta),\frac{\partial \phi}{\partial z}=0.
\end{aligned} \tag{7}$$

After the substitution (http://planetmath.org/node/41772), equation (4) looks like the following

$$\frac{\partial f}{\partial x} = \sin(\theta)\cos(\phi)\frac{\partial f}{\partial r}+\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial f}{\partial \theta}-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial f}{\partial \phi}. \tag{8}$$

Assuming that f is a sufficiently differentiable function (http://planetmath.org/node/32919), we can replace f by $\frac{\partial f}{\partial x}$ in the above equation and arrive at the following

$$\frac{\partial^2 f}{\partial x^2} = \sin(\theta)\cos(\phi)\frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}\right]+\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial x}\right]-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial}{\partial \phi}\left[\frac{\partial f}{\partial x}\right]. \tag{9}$$

Now the trick is to substitute equation (8) *into* equation (9) in order (http://planetmath.org/node/31727) to eliminate any partial derivatives with respect to x. The result is the following equation

$$\frac{\partial^2 f}{\partial x^2} = \sin(\theta)\cos(\phi)\frac{\partial}{\partial r}\left[\sin(\theta)\cos(\phi)\frac{\partial f}{\partial r}+\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial f}{\partial \theta}-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial f}{\partial \phi}\right]+$$

$$\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial}{\partial\theta}\left[\sin(\theta)\cos(\phi)\frac{\partial f}{\partial r}+\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial f}{\partial\theta}-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}\right]-$$
$$\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial}{\partial\phi}\left[\sin(\theta)\cos(\phi)\frac{\partial f}{\partial r}+\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial f}{\partial\theta}-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}\right].$$

In the hopes of simplifying the above equation, we operate the derivates on the operands and get

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} = {} & \sin(\theta)\cos(\phi)\left[\sin(\theta)\cos(\phi)\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\cos(\theta)\cos(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial^2 f}{\partial r\,\partial\theta}+\right.\\ & \left.\frac{1}{r^2}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\phi\,\partial r}\right]\\ & \sin(\theta)\cos(\phi)\frac{\partial^2 f}{\partial r\,\partial\theta}-\frac{1}{r}\sin(\theta)\cos(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r}\frac{\sin(\phi)\cos(\theta)}{\sin^2(\theta)}\frac{\partial f}{\partial\phi}-\\ & \left.\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\theta\,\partial\phi}\right]\\ & \frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial^2 f}{\partial\theta\,\partial\phi}-\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\phi^2}\right]\end{aligned}$$

After further simplifying the above equation, we arrive at the following form

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} = {} & \left[\sin^2(\theta)\cos^2(\phi)\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\cos(\theta)\sin(\theta)\cos^2(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\cos(\theta)\sin(\theta)\cos^2(\phi)\frac{\partial^2 f}{\partial r\,\partial\theta}+\right.\\ & \left.\frac{1}{r^2}\sin(\phi)\cos(\phi)\frac{\partial f}{\partial\phi}-\frac{1}{r}\sin(\phi)\cos(\phi)\frac{\partial^2 f}{\partial\phi\,\partial r}\right]\\ & \frac{1}{r}\sin(\theta)\cos(\theta)\cos^2(\phi)\frac{\partial^2 f}{\partial r\,\partial\theta}-\frac{1}{r^2}\sin(\theta)\cos(\theta)\cos^2(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r^2}\cos^2(\theta)\cos^2(\phi)\frac{\partial^2 f}{\partial\theta^2}+\\ & \left.\frac{1}{r^2}\frac{\sin(\phi)\cos(\phi)\cos^2(\theta)}{\sin^2(\theta)}\frac{\partial f}{\partial\phi}-\frac{1}{r^2}\frac{\cos(\theta)\sin(\phi)\cos(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\theta\,\partial\phi}\right]\\ & \frac{1}{r}\sin(\phi)\cos(\phi)\frac{\partial^2 f}{\partial r\,\partial\phi}+\frac{1}{r^2}\frac{\cos(\theta)\sin^2(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\theta}-\frac{1}{r^2}\frac{\cos(\theta)\cos(\phi)\sin(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\theta\,\partial\phi}+\\ & \left.\frac{1}{r^2}\frac{\sin(\phi)\cos(\phi)}{\sin^2(\theta)}\frac{\partial f}{\partial\phi}+\frac{1}{r^2}\frac{\sin^2(\phi)}{\sin^2(\theta)}\frac{\partial^2 f}{\partial\phi^2}\right]\end{aligned}$$

Notice that we have derived the first term of the right-hand side of equation (3) (i.e. $\frac{\partial^2 f}{\partial x^2}$) in terms of spherical coordinates. We now have to do a similar (http://planetmath.org/node/32278) arduous derivation (http://planetmath.org/node/42513) for the rest of the two terms (i.e. $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial z^2}$). Lets do it!

After we substitute the values of (7) into equation (5) we get

$$\frac{\partial f}{\partial y} = \sin(\theta)\sin(\phi)\frac{\partial f}{\partial r}+\frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}. \tag{10}$$

Again, assuming that f is a sufficiently differentiable function, we can replace f by $\frac{\partial f}{\partial y}$ in the above equation and arrive at the following

$$\frac{\partial^2 f}{\partial y^2} = \sin(\theta)\sin(\phi)\frac{\partial}{\partial r}\left[\frac{\partial f}{\partial y}\right]+\frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial}{\partial\theta}\left[\frac{\partial f}{\partial y}\right]+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial}{\partial\phi}\left[\frac{\partial f}{\partial y}\right]. \tag{11}$$

Now we substitute equation (10) into equation (11) in order to eliminate any partial derivatives with respect to y. The result is the following

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} = {} & \sin(\theta)\sin(\phi)\frac{\partial}{\partial r}\left[\sin(\theta)\sin(\phi)\frac{\partial f}{\partial r}+\frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}\right]+\\ & \frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial}{\partial\theta}\left[\sin(\theta)\sin(\phi)\frac{\partial f}{\partial r}+\frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}\right]+\\ & \frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial}{\partial\phi}\left[\sin(\theta)\sin(\phi)\frac{\partial f}{\partial r}+\frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}\right].\end{aligned}$$

Now we operate the operators (http://planetmath.org/node/31982) and get

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} = {} & \sin(\theta)\sin(\phi)\left[\sin(\theta)\sin(\phi)\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\cos(\theta)\sin(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial^2 f}{\partial r\,\partial\theta}-\right.\\ & \left.\frac{1}{r^2}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial r\,\partial\phi}\right]\\ & \cos(\theta)\sin(\phi)\frac{\partial f}{\partial r}-\frac{1}{r}\sin(\theta)\sin(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial^2 f}{\partial\theta^2}-\frac{1}{r}\frac{\cos(\phi)cos(\theta)}{\sin^2(\theta)}\frac{\partial f}{\partial\phi}+\end{aligned}$$

$$\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\theta\,\partial\phi}\Bigg]$$

$$\frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial^2 f}{\partial\theta\,\partial\phi}-\frac{1}{r}\frac{\sin(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\phi}+\frac{1}{r}\frac{\cos(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\phi^2}\Bigg]$$

and after some simplifications

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \Bigg[\sin^2(\theta)\sin^2(\phi)\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\sin(\theta)\cos(\theta)\sin^2(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\sin(\theta)\cos(\theta)\sin^2(\phi)\frac{\partial^2 f}{\partial r\,\partial\theta}-\\ &\frac{1}{r^2}\sin(\phi)\cos(\phi)\frac{\partial f}{\partial\phi}+\frac{1}{r}\sin(\phi)\cos(\phi)\frac{\partial^2 f}{\partial r\,\partial\phi}\Bigg]\\ &\frac{1}{r}\cos^2(\theta)\sin^2(\phi)\frac{\partial f}{\partial r}-\frac{1}{r^2}\sin(\theta)\cos(\theta)\sin^2(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r^2}\cos^2(\theta)\sin^2(\phi)\frac{\partial^2 f}{\partial\theta^2}-\\ &\frac{1}{r^2}\frac{\sin(\phi)\cos(\phi)\cos^2(\theta)}{\sin^2(\theta)}\frac{\partial f}{\partial\phi}+\frac{1}{r^2}\frac{\cos(\theta)\sin(\phi)\cos(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\theta\,\partial\phi}\Bigg]\\ &\frac{1}{r}\cos(\phi)\sin(\phi)\frac{\partial^2 f}{\partial r\,\partial\phi}+\frac{1}{r^2}\frac{\cos(\theta)\cos^2(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\theta}+\frac{1}{r^2}\frac{\cos(\theta)\cos(\phi)\sin(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\theta\,\partial\phi}-\\ &\frac{1}{r^2}\frac{\sin(\phi)\cos(\phi)}{\sin^2(\theta)}\frac{\partial f}{\partial\phi}+\frac{1}{r^2}\frac{\cos^2(\phi)}{\sin^2(\theta)}\frac{\partial^2 f}{\partial\phi^2}\Bigg]\end{aligned}$$

Now its time to derive $\frac{\partial^2 f}{\partial z^2}$. After our substitution of value in (7) into equation (6) we get

$$\frac{\partial f}{\partial z} = \cos(\theta)\frac{\partial f}{\partial r}-\frac{1}{r}\sin(\theta)\frac{\partial f}{\partial\theta}. \qquad (12)$$

Once more, assuming that f is a sufficiently differentiable function, we can replace f by $\frac{\partial f}{\partial z}$ in the above equation which gives [us (http://planetmath.org/node/42525)](http://planetmath.org/node/42525) the following

$$\frac{\partial^2 f}{\partial z^2} = \cos(\theta)\frac{\partial}{\partial r}\left[\frac{\partial f}{\partial z}\right]-\frac{1}{r}\sin(\theta)\frac{\partial}{\partial\theta}\left[\frac{\partial f}{\partial z}\right]. \qquad (13)$$

Now we substitute equation (12) into equation (13) in order to eliminate any partial derivatives with respect to z and we arrive at

$$\begin{aligned}\frac{\partial^2 f}{\partial z^2} &= \cos(\theta)\frac{\partial}{\partial r}\left[\cos(\theta)\frac{\partial f}{\partial r}-\frac{1}{r}\sin(\theta)\frac{\partial f}{\partial\theta}\right]-\\ &\frac{1}{r}\sin(\theta)\frac{\partial}{\partial\theta}\left[\cos(\theta)\frac{\partial f}{\partial r}-\frac{1}{r}\sin(\theta)\frac{\partial f}{\partial\theta}\right].\end{aligned}$$

After operating the operators we get

$$\begin{aligned}\frac{\partial^2 f}{\partial z^2} &= \cos(\theta)\left[\cos(\theta)\frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2}\sin(\theta)\frac{\partial f}{\partial\theta}-\frac{1}{r}\sin(\theta)\frac{\partial^2 f}{\partial r\,\partial\theta}\right]-\\ &\frac{1}{r}\sin(\theta)\Bigg[-\sin(\theta)\frac{\partial f}{\partial r}+\cos(\theta)\frac{\partial^2 f}{\partial r\,\partial\theta}-\frac{1}{r}\cos(\theta)\frac{\partial f}{\partial\theta}-\\ &\frac{1}{r}\sin(\theta)\frac{\partial^2 f}{\partial\theta^2}\Bigg]\end{aligned}$$

and then simplifying

$$\begin{aligned}\frac{\partial^2 f}{\partial z^2} &= \left[\cos^2(\theta)\frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2}\cos(\theta)\sin(\theta)\frac{\partial f}{\partial\theta}-\frac{1}{r}\cos(\theta)\sin(\theta)\frac{\partial^2 f}{\partial r\,\partial\theta}\right]+\Bigg[\frac{1}{r}\sin^2(\theta)\frac{\partial f}{\partial r}-\\ &\frac{1}{r}\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial r\,\partial\theta}+\frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial f}{\partial\theta}+\frac{1}{r^2}\sin^2(\theta)\frac{\partial^2 f}{\partial\theta^2}\Bigg]\end{aligned}$$

Now that we have all three terms of the [right hand side (http://planetmath.org/node/37330)](http://planetmath.org/node/37330) of equation (3)(i.e. $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial z^2}$), we add them all together (because of equation (3)) to get the laplacian in terms of r, θ and ϕ

$$\begin{aligned}\nabla^2 f &= \Bigg[\sin^2(\theta)\cos^2(\phi)\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\cos(\theta)\sin(\theta)\cos^2(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r}\cos(\theta)\sin(\theta)\cos^2(\phi)\frac{\partial^2 f}{\partial r\,\partial\theta}+\\ &\frac{1}{r^2}\sin(\phi)\cos(\phi)\frac{\partial f}{\partial\phi}-\frac{1}{r}\sin(\phi)\cos(\phi)\frac{\partial^2 f}{\partial\phi\,\partial r}\Bigg]\\ &\frac{1}{r}\sin(\theta)\cos(\theta)\cos^2(\phi)\frac{\partial^2 f}{\partial r\,\partial\theta}-\frac{1}{r^2}\sin(\theta)\cos(\theta)\cos^2(\phi)\frac{\partial f}{\partial\theta}+\frac{1}{r^2}\cos^2(\theta)\cos^2(\phi)\frac{\partial^2 f}{\partial\theta^2}+\\ &\frac{1}{r^2}\frac{\sin(\phi)\cos(\phi)\cos^2(\theta)}{\sin^2(\theta)}\frac{\partial f}{\partial\phi}-\frac{1}{r^2}\frac{\cos(\theta)\sin(\phi)\cos(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\theta\,\partial\phi}\Bigg]\\ &\frac{1}{r}\sin(\phi)\cos(\phi)\frac{\partial^2 f}{\partial r\,\partial\phi}+\frac{1}{r^2}\frac{\cos(\theta)\sin^2(\phi)}{\sin(\theta)}\frac{\partial f}{\partial\theta}-\frac{1}{r^2}\frac{\cos(\theta)\cos(\phi)\sin(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial\theta\,\partial\phi}+\end{aligned}$$

$$\frac{1}{r^2}\frac{\sin(\phi)\cos(\phi)}{\sin^2(\theta)}\frac{\partial f}{\partial \phi}+\frac{1}{r^2}\frac{\sin^2(\phi)}{\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}\Bigg]$$

$$\frac{1}{r^2}\sin(\theta)\cos(\theta)\sin^2(\phi)\frac{\partial f}{\partial \theta}+\frac{1}{r}\sin(\theta)\cos(\theta)\sin^2(\phi)\frac{\partial^2 f}{\partial r\,\partial \theta}-\frac{1}{r^2}\sin(\phi)\cos(\phi)\frac{\partial f}{\partial \phi}+$$

$$\frac{1}{r}\sin(\phi)\cos(\phi)\frac{\partial^2 f}{\partial r\,\partial \phi}\Bigg]$$

$$\frac{1}{r^2}\sin(\theta)\cos(\theta)\sin^2(\phi)\frac{\partial f}{\partial \theta}+\frac{1}{r^2}\cos^2(\theta)\sin^2(\phi)\frac{\partial^2 f}{\partial \theta^2}-\frac{1}{r^2}\frac{\sin(\phi)\cos(\phi)\cos^2(\theta)}{\sin^2(\theta)}\frac{\partial f}{\partial \phi}+$$

$$\frac{1}{r^2}\frac{\cos(\theta)\sin(\phi)\cos(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial \theta\,\partial \phi}\Bigg]$$

$$\frac{1}{r^2}\frac{\cos(\theta)\cos^2(\phi)}{\sin(\theta)}\frac{\partial f}{\partial \theta}+\frac{1}{r^2}\frac{\cos(\theta)\cos(\phi)\sin(\phi)}{\sin(\theta)}\frac{\partial^2 f}{\partial \theta\,\partial \phi}-\frac{1}{r^2}\frac{\sin(\phi)\cos(\phi)}{\sin^2(\theta)}\frac{\partial f}{\partial \phi}+$$

$$\frac{1}{r^2}\frac{\cos^2(\phi)}{\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}\Bigg]$$

$$\left[\frac{1}{r}\sin^2(\theta)\frac{\partial f}{\partial r}-\frac{1}{r}\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial r\,\partial \theta}+\frac{1}{r^2}\sin(\theta)\cos(\theta)\frac{\partial f}{\partial \theta}+\frac{1}{r^2}\sin^2(\theta)\frac{\partial^2 f}{\partial \theta^2}\right].$$

It may be hard to believe but the truth is that the above expression (http://planetmath.org/node/40307), after some miraculous simplifications of course, reduces to the following succinct form and we finally arrive at the Laplacian in spherical coordinates!

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{1}{r^2}\frac{1}{\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\cos(\theta)}{\sin(\theta)}\frac{\partial f}{\partial \theta}. \tag{14}$$

We can write the Laplacian in an even (http://planetmath.org/node/34703) more compact (http://planetmath.org/node/30503) form as[2]

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial f}{\partial r}\right]+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\frac{\partial f}{\partial \theta}\right]+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}. \tag{15}$$

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