UNIVERSITY OF KHARTOUM

GeiodApp: A Unified Framework for Geoid Computations

by

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A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the
Faculty of Engineering
Surveying Engineering

November 2016

Declaration of Authorship

I, Mohamed Yousif and Mohamed Jaafar, declare that this thesis titled, 'GeoidApp: A Unified Framework for Geoid Computations' and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
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"Write a funny quote here."

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Abstract

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Figure of the earth remains one of the most crucial aspects of geodesy. Modern studies of the figure of the earth have started with Gauss **?** who call it 'geoid'. The geoid according to Gauss is the surface that approximates the sea level. Another approach of describing the figure of the earth was proposed by Molodnskey **?**. His proposal was to treat the earth's figure as 'a boundary value problem'. We're interested in the later approach as we believe that it's economically feasible, yet accurate enough.

Acknowledgements

The acknowledgements and the people to thank go here, don't forget to include your project advisor...

Contents

List of Figures

List of Tables

Abbreviations

GGM	**G**lobal **G**eopotentail **M**odel
BVP	**B**oundary **V**alue **P**roblem
GBVP	**G**eodetic **B**oundary **V**alue **P**roblem
GPS	**G**lobal **P**ositioning **S**ystem
GNSS	**G**lobal **N**avigation **S**atellite **S**ystem

Physical Constants

Speed of Light $c = 2.997\ 924\ 58 \times 10^{8}\ \mathrm{ms}^{-\mathrm{S}}$ (exact)

Symbols

a	distance	m
P	power	W (Js^{-1})
ω	angular frequency	$rads^{-1}$
σ	standard deviation	meters

For/Dedicated to/To my...

Chapter 1

Introduction

Historically Sudan has two gravimetric geoid models **?**. The first one (Geoid91) was computed by Fashir (1991) in 1991 using Geodetic reference System GRS80. The free-air co-geoid was computed from the combination of surface gravity data using a modified Stokes's kernel and GODDARD EARTH MODEL (GEM-T1) Fashir (1991). GEM-T1, a satellite-only model, is complete to degree and order of 36 **?**. The second model is known as 'KTH-SDG08' was computed in 2008 using optimum least-squares modification of Stoke's kernels, which is widely known as KTH method Abdalla (2009). EIGEN-GRACE02S satellite-only model was adopted for KTHSDG08 final computation at spherical harmonic degree and order 120. Adam was the first to attempt to compute the geoid for Sudan in 1967 **?**. Due to lack of data from neighboring countries and large un-surveyed areas in Northwest and Southwest parts of Sudan, Adam found that the information was insufficient to determine the accurate geoid in Sudan and recommended to fill the gabs over there Abdalla (2009).

1.1 Objectives of the thesis

The main objective of this thesis is to evaluate recent grace/goce models over Sudan using asterogeodetic data and GPS/Leveling data. As a result we also propose a general software framework for various geoid components computations. In particular, it can be used to compute "geoid-height", "geoid-undulation'', and "geoid-disturbance". To the best of our knowledge, we could not find a software for geoid computations that is

- Supported. Most of geoid computations libraries are no longer supported, their links are dead.

- Strong back-end, and simple front-end. It is common to use a low-level programming language in the calculation of geoid componenets e.g. geoid height. Low-level languages are very fast compared to high-level languages, but they are much harder. We built a framework on top of C/C++ libraries, with a very simple interface, in particular Java and Matlab.
- Customizable. Because we know that users need to explore different models with different degree. We offer them just that. GeoidApp is designed such that the user can easily modify any parameter.

In particular our contribution is

- Evaluate recent Grace/Goce money in Sudan
- GeoidApp: A unified software framework for geoid computations

1.2 Thesis structure

Following this introduction, the thesis is divided into five chapters

- Chapter 2
  Glances the theory behind GGMs, and summarizes dedicated satellites missions.
- Chapter 3
  Details about the input data
- Chapter 4
  Methodology

Chapter 2

Input Data

In this chapter we will introduce our datasets that we use to evaluate recent Goce/Grace models in Sudan.

2.1 Astrogeodetic data

In 1967 Adam (1967) during his studies at Cornell University has conducted a study to compute the geoid in Sudan. His data consists of 46 points for different locations in Sudan. Due to the lack of data from neighboring countries, and the huge gaps between measurements inside. He suggested to fill the gaps to have a reliable datum. For a detailed discussion about the attempts to compute datums for Sudan, reader is referred to (Abdalla et al., 2012).

We have mentioned that Adam (1967) used an astro-geodetic observation, hence their datum is astro-geodetic datum. Abdalla (2009) used gravity data to compute gravimetric datum for Sudan. The astro-geodetic datum, or *geodetic datum* is a geoid computed by astronomically determined deflection of the vertical, while the gravimetric geoid is a geoid referred to the geocentric datum, hence the use of satellite data. A good discussion about the differences between the geoid types can be found in Vanek (1975). A sample of this data is provided in Table 2.1. While the full data is provided in the appendix. The distribution of (Adam, 1967) work is introduced in Figure 2.2.

2.2 GPS/Leveling Data

GPS/leveling data are used to evaluate GGMs results by deriving the geoid height from them. The data were collected over Khartoum area.

Table 2.1: Astro-geodetic data for latitude longitude and geoid height

station	ϕ°	λ°	*geoid height* (m)
zv	22.1686	31.489	10
12	20.1361	30.662	9.556
28	18.4728	30.840	10.198
43	17.0511	31.272	10.199
53	14.4962	30.251	14.180
70	13.8316	29.654	15.833
76	13.2329	30.110	15.354
79	12.8660	29.956	15.647
80	12.7763	30.853	14.453
85	11.6038	30.411	15.387

GPS/Leveling data are two separate heights systems (orthometric and ellipsoidal height) for the same points (in terms of latitude and longitude). GPS data is ellipsoidal height data that is computed by means of GPS/GNSS systems. Leveling data on the other hand, is the orthometric height that is computed using spirit leveling. The geoid height can then be computed using Eq. (**??**)

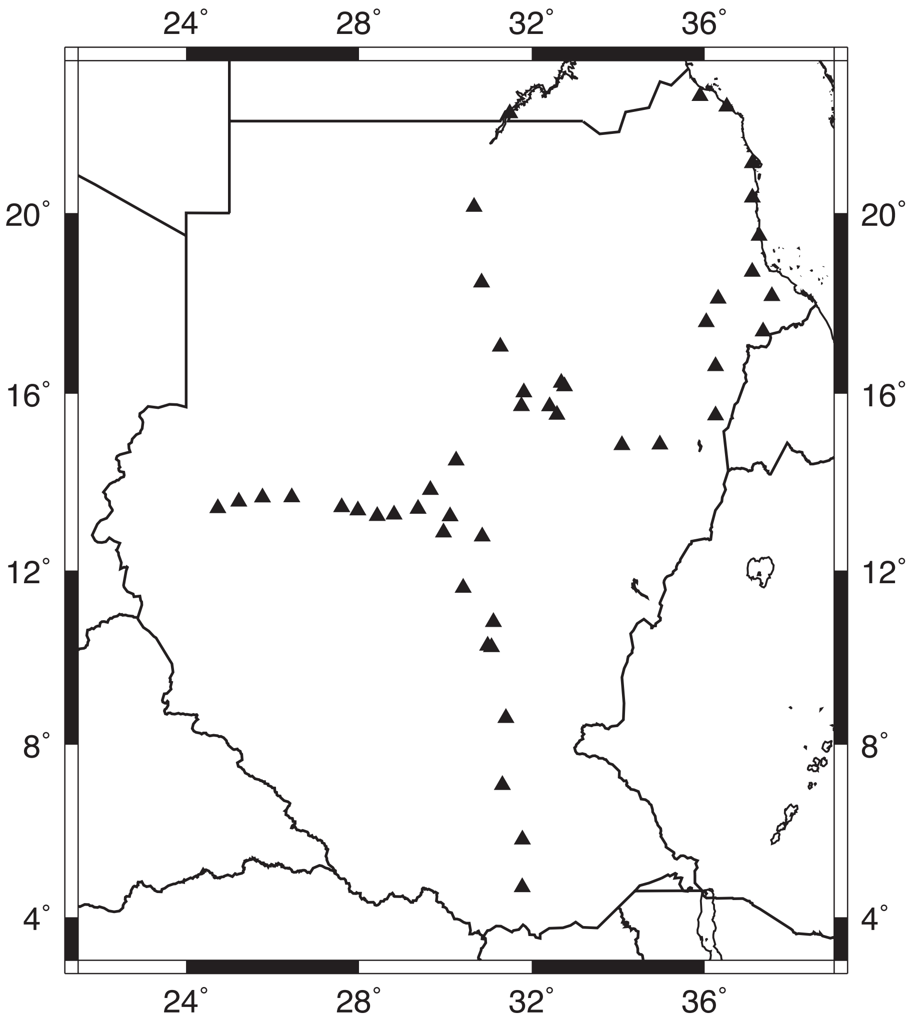
$$N = h - H \tag{2.1}$$

The GPS measurements were performed using dual frequency GPS receivers LEICA 1200, LEICA RS500 and Trimple 5700, and choke rings antennas ASH701945E_M from Ashtech, LEICA AT504 used from Abdalla (2009) work. The accuracy in leveling data is a bit problem because some of them were taken from a geodetic network of 3^{rd} order, which drastically affect our evaluations. The distribution of GPS/leveling results are shown in Figure **??**.

2.3 Global Geopotential Models

A global Geopotential Model is a mathematical function approximates the real gravity potential of the Earth. From such an approximation, all related gravity field functionals can be computed (e.g., gravity potential, gravity vector). However, other gravity field functionals e.g., *geoid height, gravity anomaly and gravity disturbance, and the second radial derivative* cannot be computed without a defined reference system (Geodetic Reference System 1980). As the centrifugal part can be modeled easily and accurately. It is usually beneficial to select a global geopotential model (and degree) that is best fit to the local gravity field as base for a regional gravimetric model. This will reduce the

Figure 2.1: Distribution of (Adam, 1967) points in Sudan. It is clear the huge gap that Adam (1967) suggested to be filled



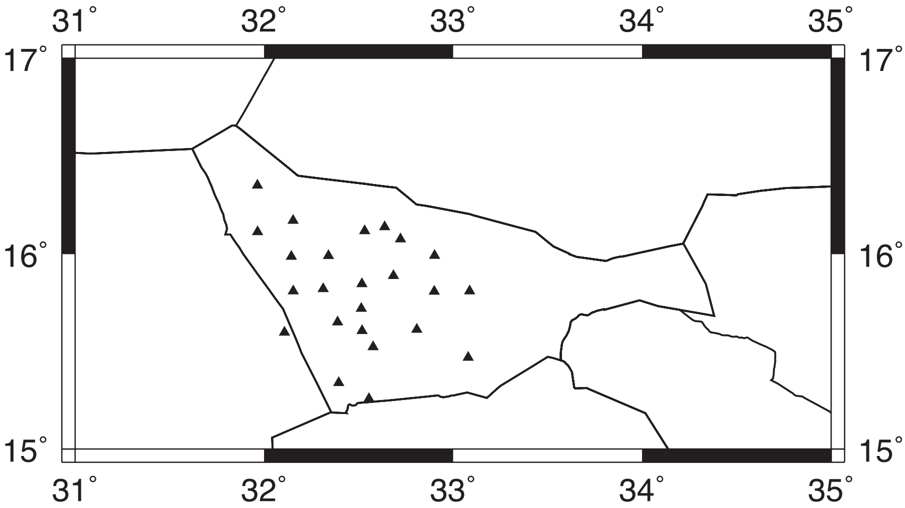
error by Stokes' formula by reducing the amount of the geoid contribution to the total error.

2.3.1 Satellite-only models

they are derived solely from the analysis of the satellite orbit. They use laser and Doppler measurements to track the satellites **?** Historically, these models were known to have a low precision due to

1. the inability to track complete satellite orbits using ground-based stations
2. imprecise modeling of atmospheric drag attraction
3. the incomplete sampling of the global gravity due to the limited number of satellites

Figure 2.2: Distribution of GPS/leveling data over Khartoum



These issues were fixed by the new dedicated satellite missions in the earlier 2000s Rummel et al. (2002). In our study, we used two satellite-only models ITU_GCC16 and ITU_GRACE16 of 280, 180 degree and order respectively.

2.3.2 Combined Models

These models are combined from different source. A GGM is combined with land and shiptrack gravity observations, and marine gravity anomalies Amos and Featherstone (2003). It is often that combined models have higher degrees than satellite-only models. Even with the added source of e.g., terrestrial data, combined models are also limited in their precision because of their use of satellite-only models. In our study we have tested three different combined models namely EGM2008, EIGEN-6C4 (2014), and GECO (2015). They are all up to degree 2190, and we tested them up-to their maximum degree. One source of error caused by the combination of different regional (local) datums and the offset between them. Heck (1990) reported that the magnitude of errors due to the inconsistency of regional datums with global datum is underestimated. The source of error was due to the simplified free-air reduction procedure and of different kinds of height system. Heck (1990) in their work have show that the corresponding systematic errors in gravity anomalies are maximum in mid-latitudes. The error due to datums inconsistency is a systematic error.

2.3.3 Tailored Models

A combination of the aforementioned models designed (tailored) for a specific area. That actually against the term global in GGM. None of these models were tested in our

Table 2.2: GPS/Leveling data for Khartoum area

id	$\phi°$	$\lambda°$	*geoid height* (m)
1	16.1719143	32.15278395	3.573
2	15.822078561111	32.312747819444	2.998
3	15.889747230556	32.683920419444	3.619
4	16.076534180556	32.721614461111	3.119
5	15.810990861111	33.086944869444	2.175
6	15.809063111111	32.899570888889	2.285
7	15.613491838889	32.808035077778	2.02
8	16.351884494444	31.964771408333	4.174
9	15.847591619444	32.517391980556	3.078
10	16.118530788889	32.531269330556	2.263
11	15.99188345	32.3404394	3.21
12	15.810295125	32.154373358333	3.21
13	16.11349295	31.965449913889	3.197
14	15.992628230556	32.9012736	3.254
15	15.987956075	32.144070430556	3.447
16	15.469951816667	33.079683027778	2.297
17	15.651378211111	32.388177761111	2.81
18	15.721159511111	32.514142069444	2.655
19	16.139034477778	32.637590772222	2.4199
20	15.607008355556	32.518298555556	2.6729
21	15.599259138889	32.107760216667	2.5378
22	15.524018605556	32.576891125	2.5993
23	15.258628422222	32.554645944444	2.2311
24	15.340130033333	32.394058058333	2.6918

experiments.

2.3.4 Difficulties in modeling the Earth from the space

(Rummel et al., 2002) in their work reported that there are two problems in efficiently model the Earth using satellite observations. In particular they report that 1) satellites can be tracked from the ground only over short intervals and, as a consequence, the gravity signal printed onto the orbit can only be extracted where it produces an orbit signal of large size such as at or close to orbit resonances, and 2) satellite motion is not determined by gravitation alone but disturbed by several types of surface forces of non-gravitational origin. The main objectives of the new satellite missions is to tackle these two fundamental problems.

2.4 New Satellite Missions

Because of the mentioned limitations of satellite missions, new dedicated satellite missions was launched in early 2000s to tackle those problems. There are three satellite missions launched for that purpose

2.4.1 CHAMP

CHAMP **CHA**llenging **M**inisatellite **P**ayload is a German small satellite mission for geoscientific and atmospheric research and applications, managed by GFZ. The orbit of CHAMP is almost circular (inclination 87degree) which gives it an advantage of getting homogeneous and complete global coverage. With an altitude of 454 km, it guarantees that the satellite can work in severe atmospheric conditions. It also has a direct on-board measurement to account for the non-gravitational orbit perturbations. . CHAMP uses satellite-to-satellite tracking working on high-to-low mode, i.e., GNSS satellites are tracking the spaceship position, and it uses an accelerometer to compute the geoid functionals (cf. REIGBER et al. (1999) for detailed discussions about CHAMP mission). Clearly, the aforementioned issues with previous satellite mission was resolved in the new dedicated satellite mission. A detailed description of CHAMP, and other new satellite missions e.g., GOCE and GRACE will be presented in Table **??**.

2.4.2 GRACE

Gravity **R**ecovery and **C**limate **E**xperiment is the second satellite mission that was launched on March 2002. GRACE consists of two identical satellites separated by 220 km from each others, and at 500 km above the Earth surface. Unlike CHAMP, GRACE is uses satelite-to-satellite tracking to precise position determination, but it uses low-low mode to compute the gravity functionals i.e., the gravity is measured by the change of the distance between the twin satellites. Areas of slightly stronger gravity (greater masses) will affect the leading satellite first. An accelerometer will be used to measure the non-gravitational acceleration–common errors due to atmospheric attraction–such that only the accelerations caused by the gravity are measured.
It is common to find a model with data from different satellite mission–each one of them contribute to specific range of spherical harmonics degrees.

2.4.3 GOCE

The last mission in the new gravity measurement satellites era is GOCE. **G**ravity Field and Steady-State **O**cean **C**irculation **E**xplorer. It is specifically designed for the determination of the stationary gravity field geoid and gravity anomalies to high accuracy and spatial resolution Rummel et al. (2002). As reported by (**?**)rummer), GOCE is the only satellite mission that was able to solve the problem of gravity observations from satellites that was introduced in Section 2.3.4. In summary, GOCE has tackled the classic problems of using satellite to compute the gravity as follow

1. Uninterrupted tracking in three spatial dimensions
2. Measurement or compensation of the effect of non-gravitational forces
3. Orbit altitude as low as possible

2.5 Our models

For our study we have tested five models. Two of them are satellite-only models (ITU_GCC16, and ITU_GRACE16), three are combined models (EGM2008, EIGEN-6C4, and GECO). The choice of these models are based on their release date. The choice of the number of models is totally arbitrary.

1. ITU_GCC16. It was released on 2016 with degree and order up-to 280. A detail description about the mission can be found here
2. ITU_GRACE16. It was also released on 2016, with degree and order of 180. You can check their website
3. EGM2008. The popular new version of EGM series, the previous ones were EGM84 and EGM96. We use this one for up-to degree of 2190. Released on 2008. For interested readers you can check their online website at EGM2008
4. EIGEN-6C4. Composed of data from different EIGEN series missions, in particular it uses data from EIGEN*6C, EIGEN6C2, and EIGEN6C3. It was released on 2014, and it is the last mission of EIGEN series. The model is up-to to degree and order of 2190. More details about the model can be found here
5. GECO. It was released 2015, and it is the latest models of high o/d models (those above 2190 d/o). It combined data from GOCE mission as well as EGM2008. No available details about GECO model from ICGEM official website (at the time of writing this)

TABLE 2.3: Comparsion of new dedicated satellite missions

mission	*inclination angle* °	*altitude (km)*	*release date*	status
CHAMP	87.2777	454		stopped-2010
GRACE	89	500		still operating
GOCE	96.7	268		still operating

TABLE 2.4: Breakthrough in satellite observations

Mission	*Technology*	mode	*missions*
SST-hl	Accelerometer	high-low	CHAMP
SST-ll	inter-satellite link	low-low	GRACE
SST-ll	Gradiometer	low-low	GOCE

2.6 An overview about this chapter

In this chapter we have introduced the new method of computing the geoid, or more generally measuring the surface of the Earth. While in the classic conservative approach to problems of physical geodesy as described by **?**. The advantage of this approach is that the geoid as a reference surface has a very simple definition in terms of the physically meaningful and geodetically important, Potential V. The disadvantage, however, is that the potential V inside the earth depends on the density ϱ because of the Poisson's equation Eq. (2.2).

$$\triangle W = -4\pi G\varrho \tag{2.2}$$

The mathematical aspects of GGM are reserved for the next chapter. It is clearly impossible that we cannot measure the density at each point in the earth, hence there should be some approximations. The other, or the modern approach is what proposed by Molodensky in 1945. He was able to show that the physical surface of the earth can be determined from geodetic measurements alone, without the need of the density of earth's crust. That means the concept of the geoid should be abandoned. This disadvantage is that the concept of the reference surface–which was easily defined by the geoid–will be more abstract, and the mathematical method will be more difficult.
The development of satellite missions (that uses Molodensky's theory) was going such that with every new model there is a possible increase in degrees, and hence the accuracy. Table 2.4 summarizes the major technical details regarding the new dedicated satellite missions. Table 5.3 summarizes the models that were used in our study.

TABLE 2.5: Summary of GGM that were used in our study

Model Name	Type	Max Degree °	*Tested models* (m)
ITU_GGC16	satellite-only	280	270
ITU_GRACE16	satellite-only	180	170
EGM2008	combined	2190	500
EIGEN-6C4	satellite-only	2190	200
GECO	combined	2190	500

Chapter 3

Geoid Determination from GGM

In this chapter we will present the mathematical and physical interpretations of computing the geoid from GGMs. In the previous chapter we said that one issue of classical way of solving geodetic problems is the that the density of the body (in this case the Earth), should be known. Which is clearly impossible. Our problem is that we want to know the gravitational field in outer space without the knowing the density structure of the Earth, but with the knowledge of the potential o the boundary. This kind of problem is called “Boundary Value Problem” or BVP. In our case, even the shape of that body e.g., the Earth must be considered unknown. Which to a special type of BVP called “Geodetic Boundary Value Problem” or GBVP.

3.1 Boundary Value Problem

Earlier, we have introduced Poisson’s equation 2.2, but without any further details about it. For the sake of convenience, we will write the equation again

$$\triangle W = -4\pi G\varrho \tag{3.1}$$

This equation is a general case of a more familiar equation in geodesy called ‘Laplace’s equation’. Laplace’s equation is a special kind of Poisson equation, it can be derived by setting the ϱ to zero outside the masses

$$\triangle W = 0 \tag{3.2}$$

In a compact form

$$\triangle W = \begin{cases} -4\pi G\varrho & \text{inside, Poisson} \\ 0 & \text{outside, Laplace} \end{cases} \tag{3.3}$$

3.1.1 Existence and Uniqueness

The first step after developing our BVP is to prove their *existence* and *uniqueness*. We basically aim to prove that our BVP has a solution, and it is unique.

3.1.1.1 Uniqueness

Assuming the BVP is not unique, and we were able to find two different solutions W_1 and W_2. And let us say that their different is called U. That is $W_1 - W_1 = U$. Using Green's 1^{st} identity we can prove the existence of our BVP

So either U, or its normal derivative is zero on the boundary and since the integrand is always positive, ∇U must be zero. And that prove the uniqueness of our 1^{st} BVP. For interested readers in the development and prove for the rest of BVPs refer to **?**.
We have proved that the solution of BVP is unique, but we did not derive that solution yet.

3.2 Solving Laplace's equation

We will start simply by solving Laplace's equation $\triangle W = 0$ in the Cartesian case, and then we will solve the problem in the spherical case. Both solutions will lead into a series of orthogonal base functions that can be solved by 1) Fourier series in the case of Cartesian solution, and 2) spherical harmonics in the case of spherical solution. The former solution (Cartesian one) is generally used in regional application, while the latter (spherical) serves more as a global solution. Hence the use of GGM.

3.2.1 Cartesian coordinates

Our task is to solve $\triangle W(x, y, z) = 0$ for $z > 0$. We start our solution by separating the variables

$$\triangle W(x, y, z) = \triangle f(x) \triangle g(y) \triangle h(z) = 0 \tag{3.4}$$

remember that $\triangle f$ is just a short hand notation for $\nabla \cdot \nabla f$

$$\triangle f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Applying the chain rule gives us

$$f''gh + fg''h + fgh'' \tag{3.5}$$

we substitute the second partial derivative symbol by $''$ notation, just for clarification. Dividing by fgh will lead us to

$$\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} = 0 \tag{3.6}$$

The separation of variables lead to separate differential equations

$$\frac{f''}{f} = -n^2 \qquad : f'' + n^2 f = 0 \tag{3.7}$$

$$\frac{g''}{g} = -m^2 \qquad : f'' + m^2 g = 0 \tag{3.8}$$

$$\frac{h''}{h} = n^2 + m^2 \qquad : h'' - (n^2 + m^2) = 0 \tag{3.9}$$

We can easily solve these equation (they are second order ODEs), each part will have two solutions

$$\begin{aligned} f_1(x) &= \cos nx & f_2(x) &= \sin nx \\ g_1(y) &= \cos my & g_2(y) &= \sin my \\ h_1(z) &= e^{-\sqrt{n^2+m^2 z}} & h_2(z) &= e^{n^2+m^2 z} \end{aligned}$$

The general solution is a combination of all possible solutions. That is, for each n and m we get a new solution. However, due to the regularity condition, we discard all terms with amplifying upward continuation (For further discussions cf. **?**). That leads us to

$$W(x,y,z) = \sum_{m=0}^{\infty}\sum_{m=0}^{\infty}(p_{nm}\cos nx \cos my + q_{nm}\cos nx \sin my + \\ r_{nm}\sin nx \cos my + s_{nm}\sin nx \sin my) \quad (3.10)$$

Now we have to develop our BVP in terms of 2D Fourier series

$$\frac{\partial W(x,y,z)}{\partial x}\Big|_{z=0} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(p_{nm}\cos nx \cos my + \\ q_{nm}\cos nx \sin my + r_{nm}\sin nx \cos my + s_{nm}\sin nx \sin my) \quad (3.11)$$

We have solved the Laplace's equation in the sphere case. With slight modifications, we can derive Laplace's equation in the spherical case. Separation of variables gives us

$$\triangle(r,\phi,\lambda) = r^2\frac{f''}{f} + 2r\frac{f'}{f} + \frac{\triangle_S Y}{Y} = 0 \quad (3.12)$$

Again, that leads to the known solutions

$$\begin{aligned} f_1(r) &= r^{-(l+1)} & f_2 &= r^l \\ h_1(\lambda) &= \cos m\lambda & h_2(\lambda) &= \sin m\lambda \\ g_1(\theta) &= P_{lm}(\cos\theta) & g_2(\theta) &= Q_{lm}(\cos\theta) \end{aligned}$$

That will leave us with two different types of base functions. **Solid and spherical harmonic functions**. Solid spherical harmonics are

$$\begin{Bmatrix} r^{-(l+1)} \\ r^l \end{Bmatrix} = P_{lm}(\cos\theta)\begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \quad (3.13)$$

Where surface spherical harmonics are

$$Y_{lm}(\phi,\lambda) = P_{lm}(\cos\theta)\begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \quad (3.14)$$

Where l is the degree of the spherical harmonic, and n is the order. This inconsistency in notations is a bit tricky, but it is the one adopted by ICGEM, so we decided to go with it. In other literature, degree and order, of the model are ofter represented by n and m respectively.
Fourier's series cannot be used in spherical coordinates, so another types of base functions should be introduced. Hence the use of spherical harmonics.

$$W(r,\phi,\lambda) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_{lm}(\sin\theta)(a_{lm}\cos m\lambda + b_{lm}\sin m\lambda)r^{-(l+1)}$$
$$+P_{lm}(\cos\theta)(c_{lm}\cos m\lambda + d_{lm}\sin m\lambda)r^{l}$$

Our solution should follow the general form as follows

$$W(r,\phi,\lambda) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_{lm}(\sin\theta)(a_{lm}\cos m\lambda + b_{lm}\sin m\lambda)R^{-(l+1)} \tag{3.15}$$

Comparing our solution with the general form yields

$$W(r,\phi,\lambda) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_{lm}(\sin\theta)(u_{lm}\cos m\lambda + u_{lm}\sin m\lambda)\left(\frac{R}{r}\right)^{l+1} \tag{3.16}$$

3.2.2 Legendre Polynomials

We need Legendre polynomials as a base function to solve our BVP. For interested readers, a thorough discussion of Legendre polynomials can be found at **?**.
We will begin our discussion of Legendre polynomials by *analytical recipe*. In particular Rodrigues and Ferrers formulas. The only different between them is that Ferres' is a general form of Rodrigues. Rodrigues works on zonal areas. That is the order (or m) is zero. We begin our derivation by assuming that $t = \cos\theta$

$$P_l = \frac{d^l(t^2-1)^l}{2^l\ l!\ dt^l} \qquad \text{(Rodrigues)} \tag{3.17}$$
$$P_{lm} = \frac{(1-t^2)^{\frac{m}{2}} d^m P_l(t)}{dt^m} \qquad \text{(Ferres)} \tag{3.18}$$

Associated Legendre Polynomials are then provided by this equation

TABLE 3.1: first 3 values for Legendre Polynomials

l	m	$P_{lm}(t)$
0	0	1
1	0	t
	1	$\sqrt{1-t^2}$
2	0	$\frac{1}{2}(3t^2-1)$
	1	$3t\sqrt{1-t^2}$
	2	$3(1-t^2)$
3	0	$\frac{1}{2}(5t^2-3)$
	1	$\frac{3}{2}(5t^2-1)\sqrt{1-t^2}$
	2	$15(1-t^2)t$
	3	$15(1-t^2)^{3/2}$

$$P_l^m(t) = \frac{(1-t)^{m/2}\ d^m\ P_l(t)}{t} \tag{3.19}$$

Note that $P_l^0(t) = P_l(t)$. Plugging Legendre function to our previous equation result in

$$W(r, \phi, \lambda) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} \bar{P_{lm}}(\sin\theta)(a_{lm}\cos m\lambda + b_{lm}\sin m\lambda) \tag{3.20}$$

Combining the above steps, and solving the potential for the Earth yields

$$W(r, \phi, \lambda) = \frac{GM}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{l+1}\sum_{m=0}^{l} \bar{P_{lm}}(\sin\theta)(\bar{C_{lm}}\cos m\lambda + \bar{S_{lm}}\sin m\lambda) \tag{3.21}$$

In Table 3.1 we present the first three degrees of Legendre polynomials. Note that there is a different solution for each degree.

3.3 Our work

We derived the geoid height using the popular Brums fromula Eq. 3.22

$$N = \frac{T}{\gamma} \tag{3.22}$$

where γ is the normal gravity, T is the disturbance potential. The disturbance potential T in spherical harmonics is given by 3.23

$$T(r,\phi,\lambda)=\frac{GM}{r}\sum_{l=0}^{l_{max}}\left(\frac{R}{r}\right)^{l}\sum_{m=0}^{l}\bar{P_{lm}}(\sin\theta)\left[\bar{C_{lm}}\cos m\lambda+\bar{S_{lm}}\sin m\lambda\right] \tag{3.23}$$

The geoid height is the can be computed as follows

$$N(r,\phi,\lambda)=\frac{GM}{r\ \gamma}\sum_{l=0}^{l_{max}}\left(\frac{R}{r}\right)^{l}\sum_{m=0}^{l}\bar{P_{lm}}\sin\theta)\left[(\bar{C_{lm}}\cos m\lambda+\bar{S_{lm}}\sin m\lambda\right] \tag{3.24}$$

The last piece of our puzzle is the equation of normal gravity γ. We have used Somigliana-Pizzetti normal gravity formula as shown in Eq. 3.25.

$$\gamma(\varphi)=\frac{a\gamma_a\cos^2\phi+b\gamma_b\sin^2\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}} \tag{3.25}$$

3.4 summary

We aimed at this chapter to introduce a sufficient mathematical derivation for geopotential functionals (geoid height being the example used). Our goal was never to show complete derivation, but rather a very simple and intuitive representation of geopotential functionals. A clear and detailed derivation of the aforementioned topics can often be found in a typical geodesy textbooks. We derived the geoptential of the Earth in terms of cartesian coordinates and spherical coordinates, using spherical harmonics for the latter. Our goal of that, was to help the reader to familiarize himself with the use of spherical harmonics in geodesy context. We have derived the geoid height–one of several geopotential functionals, but arguably the most important one. The notation was a common problem, we found different notations in the literature. We followed the notations that are used by ICGEM website.

Chapter 4

Geoid Height Computation

In this chapter we will discuss the computation of the geoid height and the validation of our results based on local terrestrial data. High resolution models are required to convert GPS leveling data (ellipsoidal height) into orthometric height. As we mentioned in the previous chapter, we have two terrestrial data 1) Astrogeodetic data provided by Adam (1967), and 2) GPS leveling data by **?**. Our results show that high degrees will often result in a lower standard deviation. Combined models e.g., EGM2008, EIGEN-6C4, and Geco have a very similar trend–it is expected because both EIGEN-6C4 and Geco share some degrees with EGM2008. Interestingly, ITU GCC has shown a peak in higher degrees (above 150), we believe that is one of the problems of satellite-only models in high degrees.

4.1 Historical Background about Geoid Computations in Sudan

There are several attempts was done through the years to compute the geoid for Sudan. The first was committed by Adam (1967) in 1967. He was done it during his MSc in Cornell University. He used astrogeodetic data i.e., astrogeodetic geoid, it's common to use astrogeodetic observations when there is a lack of data. Even with that, Osman recommended to fill the gaps due to large un-surveyed areas (largely on Northwest and Southwest parts of Sudan), and lack of data from neighboring countries. He used Clarke 1880 as a reference ellipsoid which is reasonable because the well-known WGS wasn't established at that time. Another attempt was done by Fashir (1991) in 1991. Unlike Osman's geoid, he used a gravimeteric data i.e., a gravimetric geoid. He covered a grid of $(5^\circ \leq \phi \leq 22^\circ, 22^\circ \leq \lambda \leq 38^\circ)$. Fashir introduced the use of GGMs in computing the long wavelength components of the Earth gravity. He used Goddard Earth Model

(GEM-T1) with a modified Stoke's kernel to compute the geoid height.
More recently Abdalla (2009) proposed a new gravimetric datum for Sudan KTH-SDG08. Ahmed's model was based on the new dedicated satellite missions (Goce/Grace, and CHAMP). New satellite mission have an improved results over the previous satellite missions, thanks to the gradiometer and the precise positioning system from GNSS. The new gravimetric geoid model (KTH-SDG08) has been determined over the whole country of Sudan at 5 x 5 grid for area ($4° \leq \phi \leq 23°, 22° \leq \lambda \leq 38°$). The optimum method provides the best agreement with GPS/levelling estimated to 29 cm while the agreement for the relative geoid heights to 0.493 ppm Abdalla (2009). Ahmed used GRACE02S gravitational model (for the long wavelength part) and 30x 30 SRTM DEM (for the short wavelength part of Earth's gravity field), beside the terrestrial data for the medium wavelength part. For a fair comparsion, we compared only the results of Abdalla (2009) with our results. It clearly shows that the std has dropped from 0.576 (in KTH-SD08) to 0.349 in our case. We gained a 40% accuracy without any terrestrial observations. We wanted to compare our results with that of Fashir (1991), but as indicated by Abdalla (2009) Doppler data of Fashir (1991) is not available. From Abdalla (2009), Fashirs model looks smoother than KTH-SDG08. The drastic values of the geoidal height in the north-west corner and south-east corner are 14 m and -10 m for Fashirs model while 20 m and -14 m for KTH-SDG08, respectively. The fitting with the reference ellipsoid is similar from the northwest to south-east. Fashirs model covers the reference ellipsoid over a large area (approximately 75%). On the contrary KTH-SDG08 model apparently keeps the same fitting with reference ellipsoid as in the original area before resizing.

4.2 Computing the Geoid

We developed a software application called GeoidApp for this task. GeoidApp is a huge suite of applications to not only compute geoid height (in addition to gravity anomaly and gravity disturbance), but it also has analysis and interactive visualization features to help the users get the most out of it. For that reason we will discuss the core features about GeoidApp. To compute the geoid height you need to account for these details

- The data. GGMs are provided as *.gfc files from ICGEM. They contains in addition to the coefficients text contents about the authors, acknowledgments, etc. They also include model parameters, such as the maximum degree, the model name, the GM, and a few others. The idea is to store the coefficients in an array (for computations efficiency).

- Helper functions for Associated Legendre Polynomials. That could tricky, solving Legendre function is non-trivial. The recursive version of Legendre function is suitable for programming environment.
- The main function for geoid computations. You have to choose whether you want to work on point mode, grid mode.

4.2.1 Working with GGMs data

Typical GGMs provided by ICGEM are raw and need to be parsed to extract the model's coefficients and other useful informations about the model. When we say a 'patter', or 'template' we mean that there are certain keywords in the *.gfc files, and it is available in all models.

4.2.1.1 Parsing the data

A typical model from ICGEM would consist of the following

- General information about the mission.
- A header with structured details about the models, and their parameters
- The coefficients part. Which is the core part of the *.gfc files

Model's summary . It includes summary about the mission its release date, mission time, etc. There is also a link for detailed informations about the mission. You can safely ignore this part of of the *.gfc file. This part of *.gfc files does not follow any pattern, you cannot build a generic tool to extract informations from it.

The header . This summarizes the previous in a template way. It will begin with the keyword *original header*. Below it, there is the name of the model, and the release date, mission duration. It also has information about GM but you can also ignore it here as it will be provided later in more structured way. A good way to think of *.gfc files is xml format. xml files are constructed as `<begin_of_tag> some_data </end_of_tag>`. For our work, we treated *.gfc files as follow

```
<begin_of_head>
   <original header>
   "product_type": "gravity_field" /* In our case we need this */,
   "model_name": "model_name",
```

```
    "radius": "6378136.30" /* It varies over models */,
    "earth_gravity_constant": "398600.4415D+09" /* It varies over models */,
    "max_degree": "n",
    "errors": "formal", /* It could take other values */,
    "tide_system": "tide_free" /* other types are available */,
    </original header>
</end_of_head>
```

Note that we use those keywords to store these variables with their corresponding values in a data structure (a dictionary, or hash table is suitable for this purpose).

The last part of parsing global geopotential models is extracting the model coefficients C_{nm}, S_{nm}. We want to construct a $(n \times n)$ matrix, where n denotes the maximum degree of our model. For each raw of our matrix will correspond to the degree of the model. Each will have no-zero elements as the index of of that raw. The first raw will always have value 1, and the rest of it are zeros. The second raw will depend on an argument that is specified by the user *subtract_normal_field*. In our evaluations we always set such that the "normal_field" will be subtracted from coefficients $c_{nm}(1:9, 1:9)$. It's important to note that indexing for global geopotential models starts from zero as oppose of one. You should account for that if your are using programming language that uses one as a base for indexing e.g., MATLAB/Octave or Julia.

$$C_{nm} = \begin{bmatrix} 1 & \dots & \dots & N_{max} \\ 0 & 0 & \dots & N_{max} \\ \vdots & \vdots & \ddots & N_{max} \\ -4.84e-04 & -3.98e-10 & 2.43e-06 & N_{max} \end{bmatrix}$$

For the S_{nm} part there the first column of it, i.e., the value of S_{nm} coefficients are always zero. That is $S_{nm}[:, 0] = 0$. Unlike C_{nm} the number of non-zero elements for each raw in S_{nm} equals the index of that raw minus 1.

$$S_{nm} = \begin{bmatrix} 0 & \dots & \dots & N_{max} \\ 0 & 0 & \dots & N_{max} \\ \vdots & \vdots & \ddots & N_{max} \\ 0 & 1.42e-09 & -1.40e-06 & N_{max} \end{bmatrix}$$

The same was applied to construct $e_{C_{nm}}, e_{S_{nm}}$ the std values for C_{nm}, S_{nm}, respectively.

4.3 Software Implementation

4.3.1 Associated Legendre Function

Solving Legendre function is not easy. It is always good to build on top of others projects, and reference them as needed. We found a few libraries that solve Legendre function. One of them is legendre function in provided by MATLAB. This function does not come with the standard version of MATLAB, so we avoided it. The implementation of it does not also meet with our use case. We have also found other implementation of Legendre that was written in either C++ or Java, both of them was not used during the development of GeoidApp. Another very recent implementation of legendre function was that of NumPy, a popular numerical library for Python. As the time of writing our code, NumPy's legendre was not yet implemented.
We had to either translate libraries from other languages, or modify that version of MATLAB. We decided to write our own implementation of Legendre. In both case we had modify something anyway, we also want to ensure that the implementation of our function should be compatible with other parts of our application. Legendre function should also be well optimized, the most expensive part of geoid computations is the computing legendre polynomials for each degree, not to forget that the computation increase with the increase in degrees. Another criteria is the stability of the solutions in high order and degrees of the model. However, when evaluated increasingly close to the poles, the ultra-high degree and order (e.g. 2700) ALFs range over thousands of orders of magnitude. This causes existing recursion techniques for computing values of individual ALFs and their derivatives to fail Holmes and Featherstone (2002). We followed Holmes and Featherstone (2002) implementation to make our Legendre function.

4.3.2 Geoid height function

We will begin by discussing various implementation techniques regarding computing the geoid. We assume that the reader is familiar with the mathematical details about computing the geoid. Hence, our discussion here will be about the efficiency, and our design decision for GeoidApp. We will begin our discussion by an overview of the existing libraries and tools to compute the geoid, and compare them–in terms of design and efficiency–with our work. We have tried to contact the authors of EGMLab, but the did not respond, neither we did able to find any valid link for it. In Kiamehr and Eshagh (2008) they have a model reader, which was not very clear from their description how it was being implemented. Their model reader does basic things like converting the column coefficients into a matrix form, but they did not mention, and extracting the model name. Then they multiply the parsed coefficients by the GM (as a fixed value here, but they also said that GM might be dynamically changed.) Their model, which is a critical part in our comparison only works within maximum degree and order of 720 (based on EGM2008). They also report the use of Horner and Clenshaw algorithms algorithm will expand the maximum degree and order up to 2170 as shown in Holmes and Featherstone (2002). Their investigation on Clenshaw and Horner algorithm using A Java program on official ICGEM website indicates that in order to use a model with a degree up to 2160, the result will take at least an hour! In our work we show that we can compute geoid height (or other geoid components) within a few minutes (particularly 6 minutes). In terms of efficiency, it is clearl that our work surpasses Kiamehr and Eshagh (2008) work by many order of magnitude. Also, because of Clenshaw and Horner, our work is able to account for any future models, as long as they are within 2170 range, which is not going to be exceeded too soon. In terms of software design, Kiamehr and Eshagh (2008) works in point-wise mode as well as in area mode. Our work does also provide that feature i.e., you can compute the geoid height for a point, as well as a grid. It is also important to mention that Kiamehr and Eshagh (2008) in their work report that their results surpassed that of official EGM96 results. Which indicates the same for our work.

The other work we have considered is the official ICGEM calculation service available at ICGEM. The implementation details are hidden. They have a nice JavaScript front-end, that works not only as a wrapper for a low level language (C or C++), but also serves well their purpose of running the application in the browser. They have many application in addition to geoid height. We were inspired by many of their implementations, namely "functional", "tide_system" keywords. Their results are in a grid format. You cannot use their results in any sort of evaluations, unless you made a separate interpolation function for that purpose. You cannot also run multiple instances of ICGEM service simultaneously (3 instances simultaneously is the maximum allowed). But that should not be a problem. Our work not only does allow you to work in both point-wise and grid mode, but it also gives you a powerful application with no limitations in terms of usage or in computations. Our work also come with a tool to automatically download models from ICGEM website, so if you use it without specifying the model name, it will download the latest and the greatest model. We also have an interactive visualization tool so that we will make the process of understanding the data more easier. Another feature of our work is the automation of degree computations of the models. This feature is specially important in the case of the evaluations, where researchers often need to work with different models at different degree to find the best one. This task could be a problem if you manually tune the models. Another feature that is unique in our work, is that we try to automate the process of using the model by comparing it with freely available terrestrial data the is geographically near to the user's input data (based on

their latitude and longitude). If the user has provided an evaluation data (GPS/leveling data) that will ease and automate the process, so that the output will be the best model with the best degree. The output will also contain a file with a top ten models with their degrees based on std criteria.
The last similar work is GeographicLib which is a small C++ classes for various geodesy computations e.g., conversion between UTM and geodetic coordinates (latitude, longitude). What is similar to our work is their geoid class implementation. They only used 3 models, EGM84, EGM96, and EGM2008. You can see that it is very limited and also too outdated. Another limitation is that this library does only work in the point-wise mode. That will severely limit the use of it in large project where you have several points.

Chapter 5

Results and Discussions

We have evaluated different models in two types of datasets. The first data collected by (?) consists of 46 points. They use astrogeodetic observation to compute the geoid height. GPS/leveling data consists of 24 points collected during 2005-2008. We evaluated our models in different range of degrees (step of 5 degrees for EGM2008 and GECO, step of one degree for ITU_GGC16 and ITU_GRACE16). We used standard deviation as a measure of accuracy.

5.1 Evaluation on astrogeodetic data

Our evaluation of GGM using astrogedetic data shows very interesting results. There is a huge difference between GGM results and data of (?).

TABLE 5.1: Top 5 degrees for model ITU_GGC16

degree	std σ [m]	difference
12	5.86797	-10.7197
13	6.232	-10.0005
14	6.575	-9.9920
15	6.62	9.8420
11	6.74	-10.6067
121	6.758	-11.0029

TABLE 5.2: Top 5 degrees for model EGM2008

degree	std σ [m]	difference
14	6.57797646	10.01617876
1722	6.72616231	11.10208562
1726	6.72836585	-11.10328207
1595	6.73249422	-11.10455418
1731	6.73263864	-11.10572496

TABLE 5.3: Top 5 degrees for model EIGEN-6C4

degree	std σ [m]	difference
1729	6.71829049	-11.0980582
1718	6.72207977	-11.10011044
1609	6.72228078	-11.09995271
1598	6.72349438	-11.09978464
1872	6.72480567	-11.10075124

TABLE 5.4: Top 5 degrees for model GECO

degree	std σ [m]	difference
10	7.7916	-11.476
14	6.579	-10.016
18	7.3629	-10.144
23	7.5549	-11.196
27	6.9829	-10.997

TABLE 5.5: Top 5 degrees for model ITU_GRACE16

degree	std σ [m]	difference
12	5.8679769	-10.7196877
13	6.23255147	-10.000530
14	6.57485872	-9.992074
15	6.61925941	-9.84207
120	6.73941682	-10.9837

Figure 5.1: std behaviour with the change of degrees

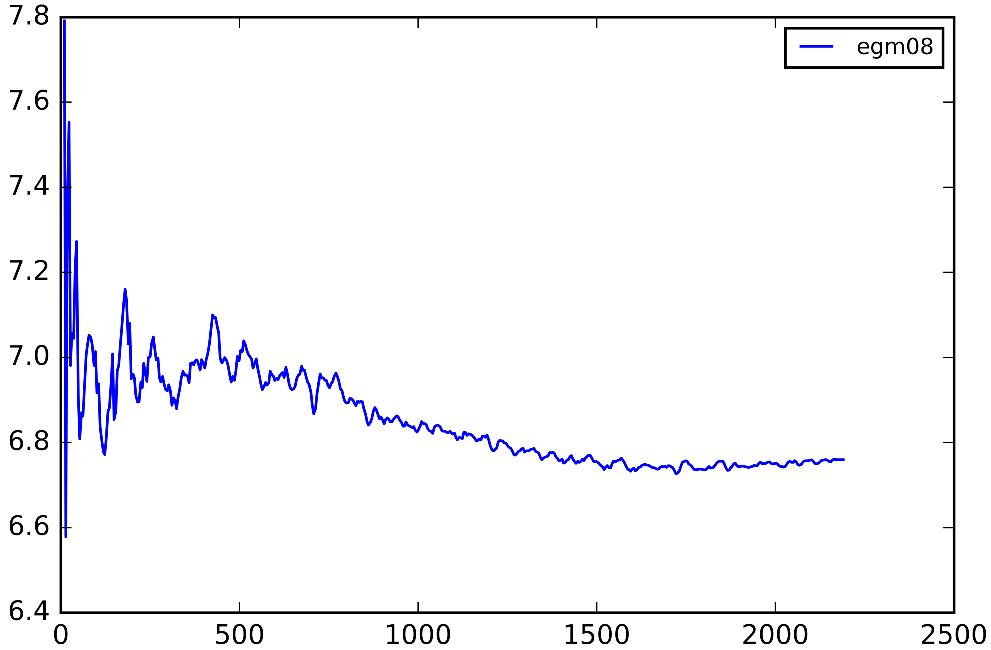


Figure 5.2: std behaviour with the change of degrees

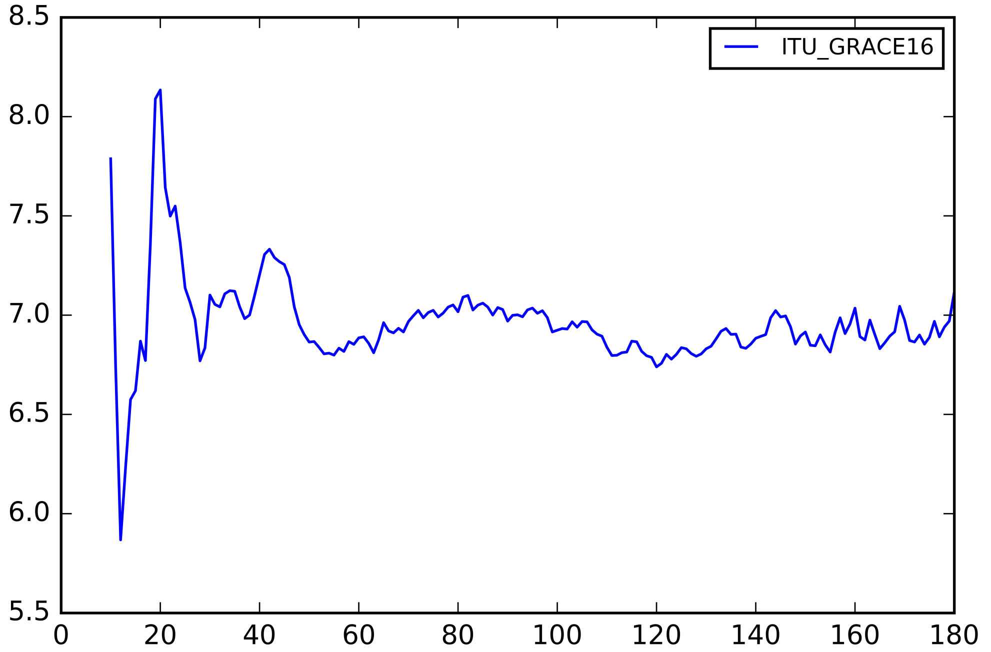


FIGURE 5.3: std behaviour with the change of degrees

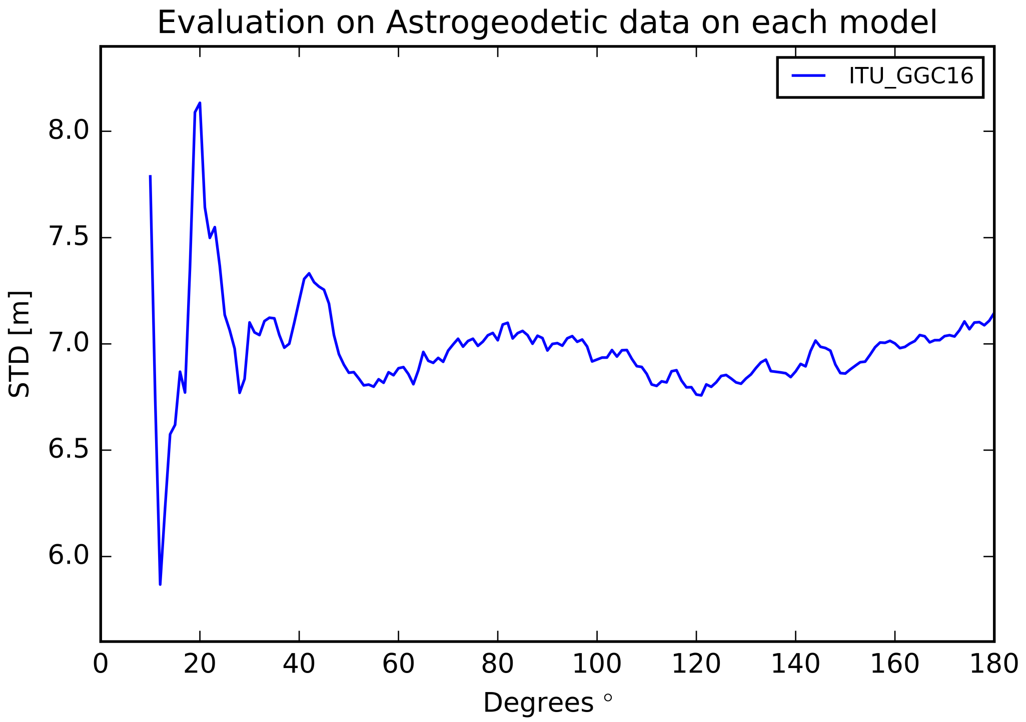


FIGURE 5.4: std behaviour with the change of degrees

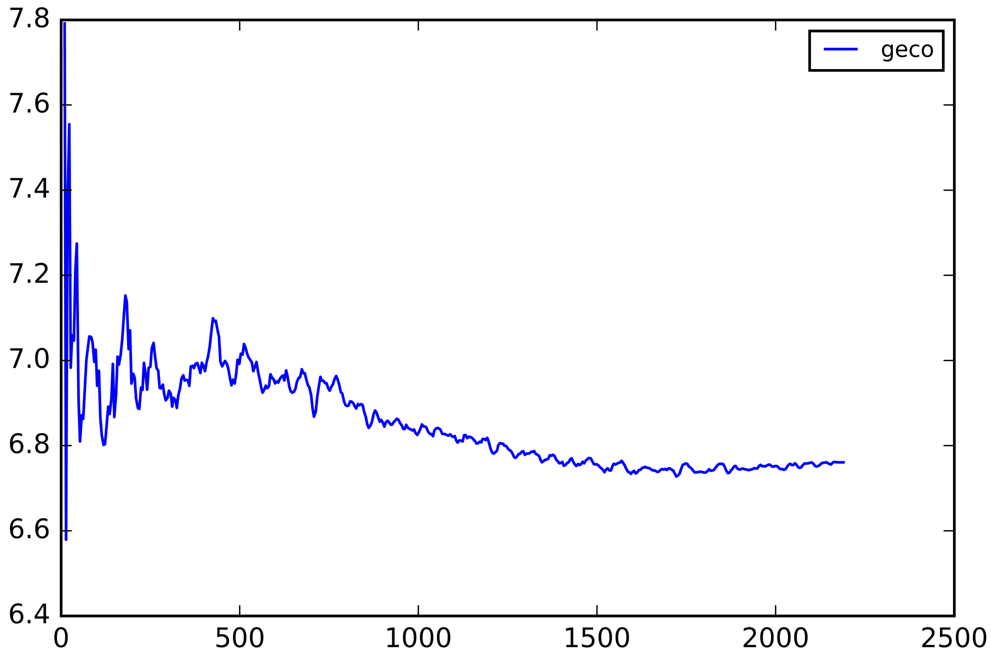
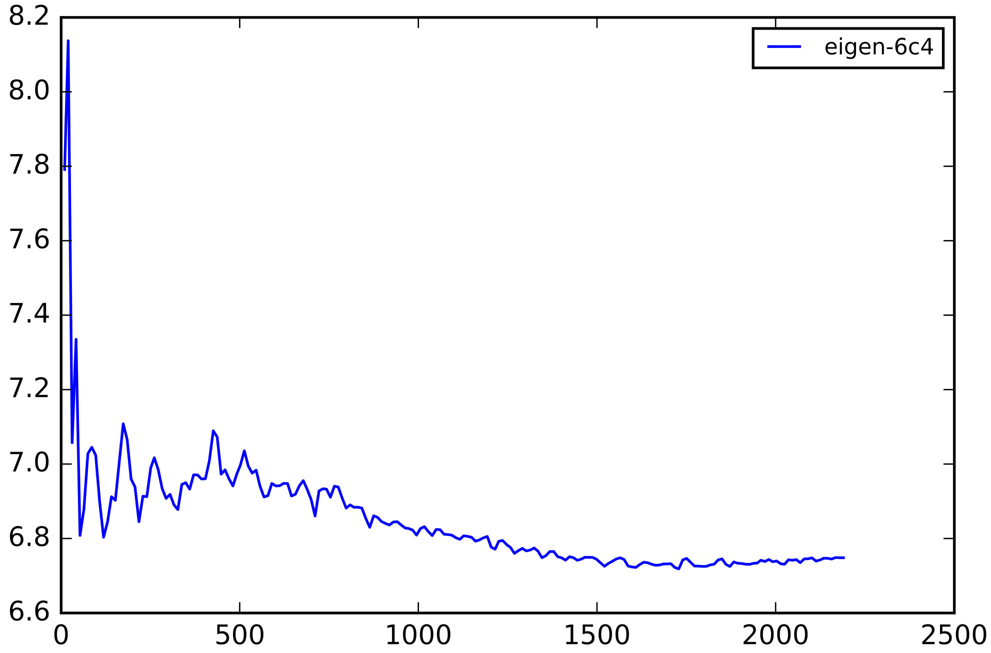


FIGURE 5.5: std behaviour with the change of degrees



Appendix A

An Appendix

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