Gravity Field Tutorial

The formulae and derivations in the following Chapters 1 to 3 are based on Heiskanen and Moritz (1967) and Lambeck (1990).

1. Expansion of the gravitational potential into spherical harmonics

The stationary part of the **Earth's gravitational potential U at any point $P(r,\varphi,\lambda)$** on and above the Earth's surface is expressed on a global scale conveniently by summing up over degree and order of a spherical harmonic expansion. The spherical harmonic (or Stokes') coefficients represent in the spectral domain the global structure and irregularities of the geopotential field or, more generally spoken, of the gravity field of the Earth. The equation relating the spatial and spectral domain of the geopotential is as follows:

$$U(r,\varphi,\lambda) = \frac{GM}{R}\left[\frac{R}{r}\overline{C}_{00} + \sum_{l=1}^{l_{max}} \sum_{m=0}^{l} \left(\frac{R}{r}\right)^{l+1} \overline{P}_{lm}(\sin\varphi)\left(\overline{C}_{lm}\cos m\lambda + \overline{S}_{lm}\sin m\lambda\right)\right] \qquad (1.1)$$

where r,φ,λ - spherical geocentric coordinates of computation point (radius, latitude, longitude)

R - reference length (mean semi-major axis of Earth)

GM - gravitational constant times mass of Earth

l,m - degree, order of spherical harmonic

$\overline{P}_{lm}$ - fully normalized Lengendre functions

$\overline{C}_{lm},\overline{S}_{lm}$ - Stokes' coefficients (fully normalized)

The $\overline{C}_{00}$-term is close to 1 and scales the value GM. The degree 1 spherical harmonic coefficients $(\overline{C}_{10},\overline{C}_{11},\overline{S}_{11})$ are related to the geocentre coordinates and zero if the coordinate systems' origin coincides with the geocentre. The coefficients $(\overline{C}_{21},\overline{S}_{21})$ are connected to the mean rotational pole position that is a function of time.

Subtracting from the low-degree zonal coefficients (order 0) the corresponding Stokes' coefficients $(\overline{C}_{00}^{ell},\overline{C}_{20}^{ell},\ldots\overline{C}_{80}^{ell})$ of an **ellipsoidal 'normal' potential $V(r,\varphi)$** leads to the mathematical representations of the **disturbing potential $T(r,\varphi,\lambda)$** in spherical harmonics, related to a conventional ellipsoid of revolution that approximates the Earth's parameters. At the Earth surface with $r = R$ (in spherical approximation) the disturbing potential reads:

$$T(R,\varphi,\lambda) = U(R,\varphi,\lambda) - V(R,\varphi) \qquad (1.2a)$$

$$T(R,\varphi,\lambda) = \frac{GM}{R}\left[\overline{C}'_{00} + \sum_{l=1}^{l_{max}} \sum_{m=0}^{l} \overline{P}_{lm}(\sin\varphi)\left(\overline{C}'_{lm}\cos m\lambda + \overline{S}_{lm}\sin m\lambda\right)\right], \qquad (1.2b)$$

with $\overline{C}' = \overline{C} - \overline{C}^{ell}$ and T defined on the geoid. Note, that $\overline{C}'_{00}$ is close to zero.

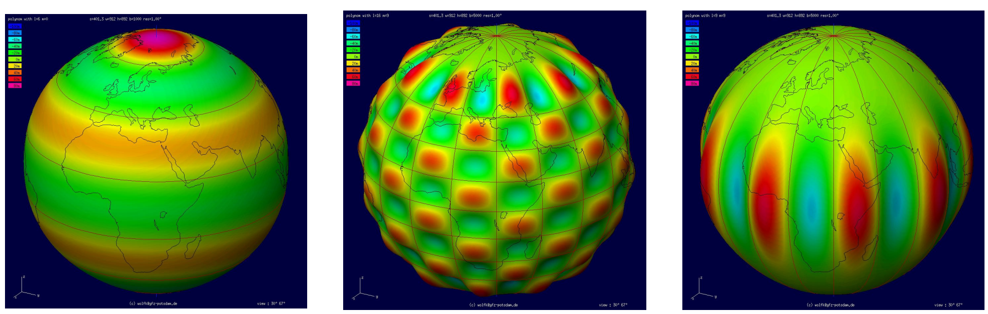
The maximum degree l_{max} of the expansion in Equation (1.1) correlates to the spatial resolution at the Earth surface by

$$\lambda_{min} \approx 40000\ km / (l_{max} + 0.5)\ , \tag{1.3}$$

where λ_{min} is the minimum wavelength (or twice the pixel side length) of gravity field features that are resolved by the $l_{max} \cdot (l_{max}+1) \approx (l_{max} + 0.5)^2$ parameters $\overline{C}_{lm}, \overline{S}_{lm}$.

Equation (1.1) contains the upward-continuation of the gravitational potential at the Earth's surface for $r > R$ and reflects the attenuation of the signal with altitude through the factor $(R/r)^{l+1}$.

Figure 1.1 gives examples for the three different kinds of spherical harmonics $P_{lm}(\sin\varphi) \cdot \cos m\lambda$: (a) zonal with $l \neq 0,\ m = 0$, (b) tesseral with $l \neq 0,\ m \neq 0 \neq l$ and (c) sectorial harmonic with $l = m$. Amplitudes and phase of the individual spherical harmonics then are determined by multiplication with the C_{lm} and S_{lm} coefficients.



zonal: l=6, m=0　　tesseral: l=16, m=9　　sectorial: l=9, m=9

Figure 1.1: *Examples for spherical harmonics* $P_{lm}(\sin\varphi) \cdot \cos m\lambda$ *[from –1 (blue) to +1 (violet)].*

2. Functionals of the disturbing gravitational potential

The **geoid undulation** $\boldsymbol{N}$ (Figure 2.1) is the distance between the special equipotential surface $U(R,\varphi,\lambda)$ = const that is close to the mean sea level and the surface of the conventional ellipsoid of revolution. As such the geoid is derived from the disturbing potential T applying *Bruns formula*

$$N = \frac{T}{\gamma}\quad , \tag{2.1}$$

where γ is 'normal' gravity on the surface of the ellipsoid. With $\gamma = GM/R^2$ in spherical approximation, the geoid undulations (or geoid heights) can be computed from the spherical harmonic coefficients in Equation (1.2) by

$$N(R,\varphi,\lambda) = \frac{R^2}{GM} \cdot T(R,\varphi,\lambda)\quad , \tag{2.2a}$$

$$N(R,\varphi,\lambda) = R\left[\overline{C}'_{00} + \sum_{l=1}^{l_{max}} \sum_{m=0}^{l} \overline{P}_{lm}(\sin\varphi)\left(\overline{C}'_{lm}\cos m\lambda + \overline{S}_{lm}\sin m\lambda\right)\right] \quad . \tag{2.2b}$$

The negative of the vertical derivative of the disturbing potential $\delta g = -\frac{\partial T}{\partial r}$ is called **gravity disturbance** $\boldsymbol{\delta g}$ (Figure 2.2) that is equal to gravity at a point *P* (negative of vertical derivative of *U*) minus 'normal' gravity at point *P* (negative of vertical derivative of *V*). On the geoid and in spherical approximation *(r = R)* the gravity disturbance then is expressed by

$$\delta g(R,\varphi,\lambda) = \frac{GM}{R^2}\left[\overline{C}'_{00} + \sum_{l=1}^{l_{max}} (l+1) \sum_{m=0}^{l} \overline{P}_{lm}(\sin\varphi)\left(\overline{C}'_{lm}\cos m\lambda + \overline{S}_{lm}\sin m\lambda\right)\right]. \tag{2.3}$$

The difference between gravity at a point *P* on the geoid and 'normal' gravity at the corresponding point *Q* on the ellipsoid is called **gravity anomaly** $\boldsymbol{\Delta g}$ (Figure 2.3) and related to the disturbing potential by

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}T \quad . \tag{2.4}$$

On the geoid this becomes (note: no degree 1 terms appear in Equation 2.5)

$$\Delta g(R,\varphi,\lambda) = \frac{GM}{R^2}\left[-\overline{C}'_{00} + \sum_{l=2}^{l_{max}} (l-1) \sum_{m=0}^{l} \overline{P}_{lm}(\sin\varphi)\left(\overline{C}'_{lm}\cos m\lambda + \overline{S}_{lm}\sin m\lambda\right)\right], \tag{2.5a}$$

thus

$$\Delta g = \delta g - \frac{2}{R}T \quad . \tag{2.5b}$$

The second derivatives of the disturbing potential leads to the **gravity-gradient tensor**. The most important vertical gradient $g_r = \frac{\partial^2 T}{\partial r^2}$ of the tensor component can be represented as

$$g_r = -\frac{GM}{R^3}\left[\left(\frac{R}{r}\right)^3 2\overline{C}'_{00} + \sum_{l=1}^{l_{max}} \sum_{m=0}^{l} (l+1)(l+2)\left(\frac{R}{r}\right)^{l+3} \overline{P}_{lm}(\sin\varphi)\left(\overline{C}'_{lm}\cos m\lambda + \overline{S}_{lm}\sin m\lambda\right)\right]. \tag{2.6}$$

Once the spherical harmonic coefficients $\overline{C}_{lm}, \overline{S}_{lm}$ of a global gravity field model are given, the quantities of the various functionals described above can be computed in its geographical distribution. If computed in terms of gravity disturbances or anomalies and gravity gradients, the higher frequency regional to local content is emphasised through the degree-dependent factors *(l+1), (l-1)* and *(l+1)(l+2),* respectively, whereas the potential

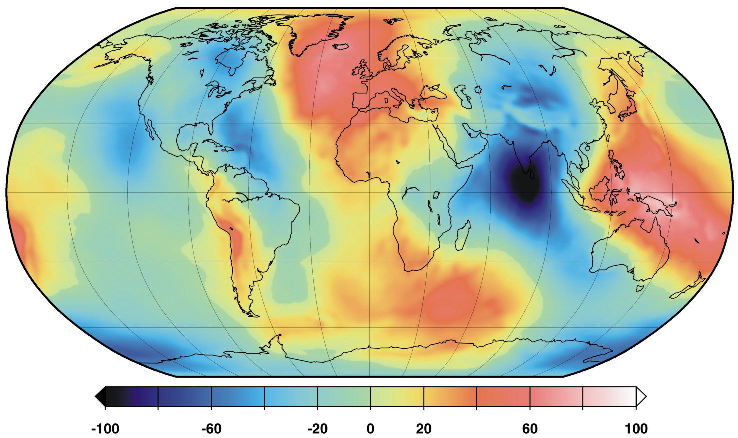


Figure 2.1: *Geoid undulations N [m]: resolution λ=500 km, rms (N $\sqrt{\cos\varphi}$) = 30.6 m.*

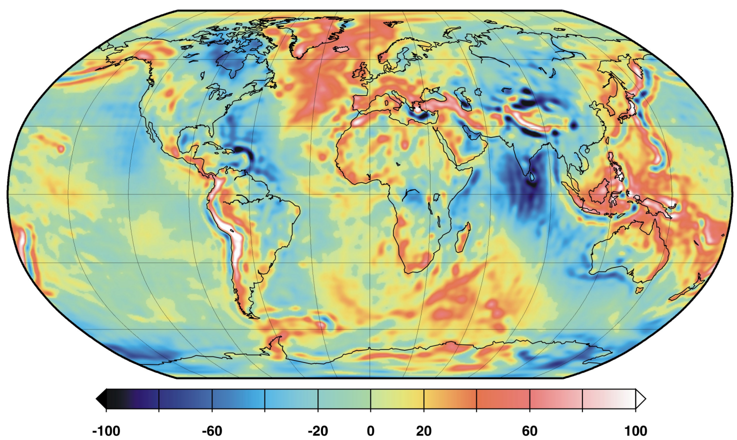


Figure 2.2: *Gravity disturbances δg [mgal]: resolution λ=500 km, rms (δg $\sqrt{\cos\varphi}$) = 27.2 mgal.*

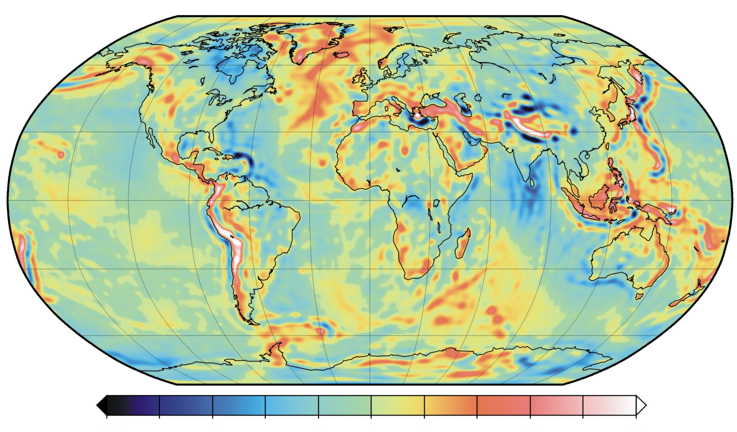


Figure 2.3: *Gravity anomalies Δg [mgal]: resolution λ=500 km, rms (Δg $\sqrt{\cos\varphi}$) = 20.6 mgal.*

and geoid representations of the gravity field show the broad and generalized features of the gravity field. Vice versa, a gradiometer measuring gravity gradients is capable to better resolve detailed structures of the gravity field rather than the long wavelength part.

The fully normalized spherical harmonic coefficients in Equation (1.1) are related to the mass distribution within the Earth by

$$(2l+1)\,\overline{C}_{lm} = \frac{1}{MR^l} \iiint\limits_{Earth} r^l\, \overline{P}_{lm}(\sin\varphi)\cos m\lambda\, dM \qquad (2.7a)$$

$$(2l+1)\,\overline{S}_{lm} = \frac{1}{MR^l} \iiint\limits_{Earth} r^l\, \overline{P}_{lm}(\sin\varphi)\sin m\lambda\, dM \qquad (2.7b)$$

with the mass element $dM{=}dM\,(r,\varphi,\lambda)$.

Figure 2.4 depicts the geopotential distribution of gravity anomalies over Europe derived from spherical harmonic coefficients complete to l_{max} equal to 10, 50, 100, 300, respectively, in order to demonstrate the relation between spectral and spatial resolution according to Equation (1.1).

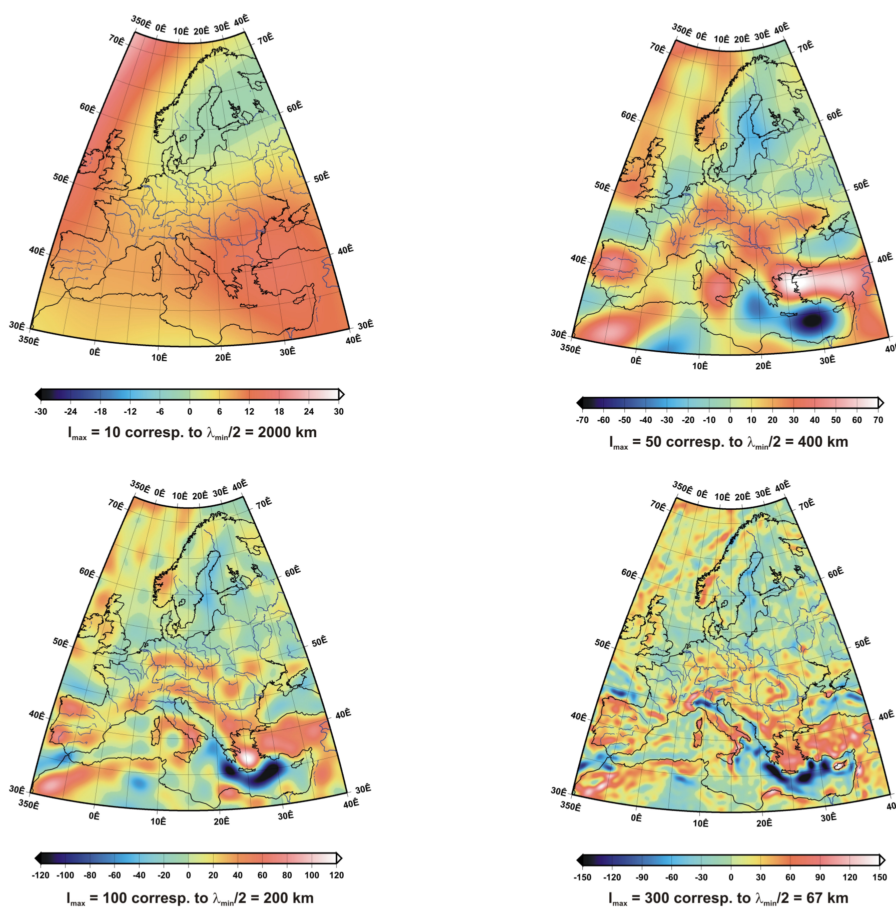


Figure 2.4: *Geographical distribution of gravity anomalies [mgal] over Europe with different spectral (l_{max}) and spatial resolution pixel size ($\lambda_{min}/2$).*

3. The power spectrum of the Earth's gravity field

Given the fully normalized Stokes' coefficients $\overline{C}'_{lm}, \overline{S}_{lm}$ of a specific degree l over orders m $(m=0...l)$ the **signal degree amplitudes** σ_l (or square root of power per degree l) of functions of the disturbing potential $T\,(R,\varphi,\lambda)$ at the Earth's surface are readily computed by

$$\sigma_l = \sqrt{\sum_{m=0}^{l} (\overline{C}'^2_{lm} + \overline{S}^2_{lm})} \qquad \text{in terms of unitless coefficients} \qquad (3.1a)$$

$$\sigma_l(T) = \frac{GM}{R} \cdot \sigma_l \qquad \text{in terms of disturbing potential values } (m^2/s^2) \qquad (3.1b)$$

$$\sigma_l(N) = R \cdot \sigma_l \qquad \text{in terms of geoid heights (m)} \qquad (3.1c)$$

$$\sigma_l(\delta g) = \frac{GM}{R^2}(l+1) \cdot \sigma_l \qquad \text{in terms of gravity disturbances } (m/s^2) \qquad (3.1d)$$

$$\sigma_l(\Delta g) = \frac{GM}{R^2}(l-1) \cdot \sigma_l \qquad \text{in terms of gravity anomalies } (m/s^2) \qquad (3.1e)$$

$$\sigma_l(g_r) = \frac{GM}{R^3}(l+1)(l+2) \cdot \sigma_l \qquad \text{in terms of vertical gravity gradients } (1/s^2) \qquad (3.1f)$$

where the $\overline{C}'_{lm}, \overline{S}_{lm}$ are related to the 'normal' potential. The SI units of the physical gravitational quantities are given in parenthesis. Following Kaula's 'rule of thumb' (Kaula, 1966) the power law follows approximately

$$\sigma_l \approx \sqrt{(2l+1) \cdot \frac{10^{-10}}{l^4}} \quad . \qquad (3.2)$$

Examples for signal degree amplitudes are given in Figure 3.1.

If the estimation errors of the Stokes' coefficients in a global gravity field model are known, the **error degree amplitudes** (error spectrum) are computed accordingly replacing the coefficients in Equation (3.1) by their standard deviations. **Difference degree amplitudes**, representing the agreement of two different gravity field models per degree, are readily computed replacing the coefficients in Equation (3.1) by the coefficients' differences between the two models. Examples for difference degree amplitudes are given in Figure 3.2.

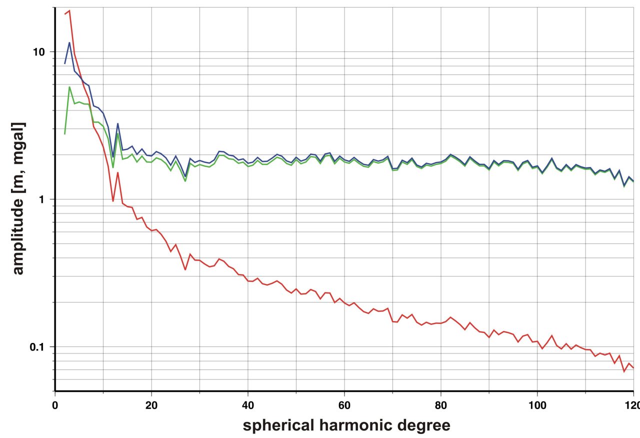


Figure 3.1: *Signal degree amplitudes for geoid undulations (red), gravity disturbances (blue) and gravity anomalies (green) in meter and mgal, respectively.*

The **degree amplitudes as a function of minimum and maximum degree *l*** displays the power (signal, error, difference) spectrum accumulated over a spectral band from l_1 to l_2:

$$\sigma_{l_1,l_2}(accumulated) = \sqrt{\sum_{l_1}^{l_2} \sigma_l^2} \tag{3.3}$$

Usually l_1=0 or 2 is taken to display the increase in overall power with increasing degree l_2. Recall that the spectral degree l is related to the spatial extension or wavelength of features in the gravity field according to Equation (1.3). Examples for difference amplitudes as a function of maximum degree l (successive accumulation of the curves in Figure 3.1) are given in Figure 3.2.

Equations (3.1) again demonstrate that the higher degree terms, i.e. the shorter wavelengths in the signal spectra, are enhanced by factors proportional to degree l for gravity anomalies and disturbances and proportional to l^2 for gravity gradients compared to the signals in the geoid and gravitational potential.

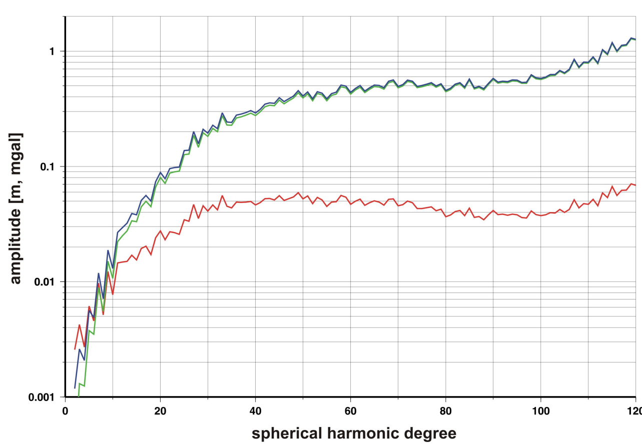


Figure 3.2: *Difference degree amplitudes (GRACE-01S vs. EGM96) in terms of geoid undulations (red), gravity disturbances (blue) and gravity anomalies (green) in meter and mgal, respectively*

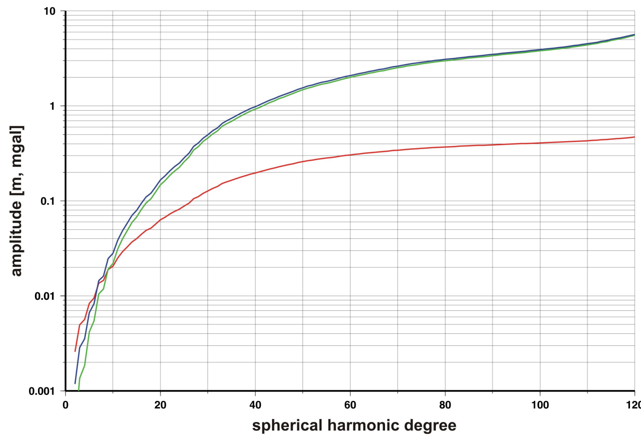


Figure 3.3: *Difference degree amplitudes (GRACE-01S vs. EGM96) as a function of maximum degree in terms of geoid undulations (red), gravity disturbances (blue) and gravity anomalies (green) in meter and mgal, respectively*

References

Heiskanen, W.A. and H. Moritz, 1967. Physical Geodesy, W.H. Freeman and Co., San Francisco.

Kaula, W.M., 1966. Theory of Satellite Geodesy, Blaisdell Publ. Company, Waltham, Mass.

Lambeck, K., 1990. Aristoteles – An ESA Mission to Study the Earth's Gravity Field, ESA Journal 14:1-21.