NOTES ON PHYSICAL GEO

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Dedicated to those who appreciate LaTeX

and the work of Edward R. Tufte and Donald E. Knuth.

Introduction

This work is very beta. A lot of errors are expected.

Laplace's Equation

We will begin our discussion about Laplace's equation.

$$\triangle V = 0 \tag{1}$$

and notice that this differential operator $\triangle(\cdot)$ is just $\nabla \cdot \nabla(\cdot)$. Solving Laplace's equation is not easy. The common way of doing that is solving it at some coordinate system; Cartesian system, spherical system, etc. In physical geodesy, we use Laplace's equation to compute the potential of a field i.e., potential of the gravity field. From that potential, one can compute other functions e.g., gravity disturbance, geoid height, gravity anomaly, etc. That's why we are studying it.

Laplace's equation in Cartesian Coordinate System

Recall that from Eqn. (1), Laplace's equation is

$$\triangle V = 0$$

, replacing V by its values yields,

$$\nabla \cdot \nabla V = \left(\frac{\partial^2}{\partial X^2} \cdot \frac{\partial^2}{\partial Y^2} \cdot \frac{\partial^2}{\partial Z^2} \right)$$

. So, we basically using the dot product property to separate these functions. Applying the chain rule to compute the second derivative of these functions give us,

$$\triangle V = \frac{\partial^2 X}{\partial x^2} YZ + \frac{\partial^2 Y}{\partial x^2} XY + \frac{\partial^2 Z}{\partial x^2} YX = 0 \tag{2}$$

. And because we want to separate between the x in the derivative, and that of the vector, let us rearrange the equation as following

$$\frac{\partial^2 X(\cdot)}{\partial x^2} YZ + \frac{\partial^2 Y(\cdot)}{\partial x^2} XY + \frac{\partial^2 Z(\cdot)}{\partial x^2} YX = 0 \tag{3}$$

Because each of these variables (X, Y, Z) are actually a function. Now let us divide Eqn. (3) by XYZ, to get

$$\frac{\frac{\partial^2 X}{\partial x^2}}{X(\cdot)} + \frac{\frac{\partial^2 Y}{\partial y^2}}{Y(\cdot)} + \frac{\frac{\partial^2 Z}{\partial z^2}}{Z(\cdot)} = 0 \tag{4}$$

This is indeed a partial ordinary differential equation. In order for the solution to be true for all X, Y, Z, each of these terms should be a constant [1]. For the base solution for this differential equation, we choose Let us rearrange Eqn. (4) to be like the following, and only taking the X part,

[1] Well, as you can see we are taking the derivative for these functions. For the answer to be equals to zero, each of these terms should be a constant != 0. You get the idea?

$$\frac{\partial^2 X}{\partial x^2} = -k_1^2 X, \text{ where } k_1 \text{ is a constant} \tag{5}$$

and similarly for the $Y(\cdot)$ and $Z(\cdot)$ part,

$$\frac{\partial^2 Y}{\partial y^2} = -k_2^2 Y, \tag{6}$$

$$\frac{\partial^2 Z}{\partial y^2} = (k_1^2 + k_2^2) Z \tag{7}$$

For the base solution [2] we have,

[2] Other base solutions can be chosen e.g., $\sin mx, \cos my$. This solution is a common one in the literature.

$$X(x) = e^{\pm i k_1 x}, \tag{8}$$
$$Y(y) = e^{\pm i k_2 y}, \tag{9}$$
$$Z(z) = e^{\pm z\sqrt{k_1^2+k_2^2}} \tag{10}$$

to get,

$$V_{k_1,k_2} = e^{\left(\pm(ik_1x+ik_2y)\pm z\sqrt{k_1^2+k_2^2}\right)} \tag{11}$$

. And recall that from the beginning we have restricted ourselves in a "Cartesian system", well the reason for that is because we want to have some boundary values to solve our problem! Defining the boundary condition for this BVP might be a little bit hard. In Fig. 1, we have a box with dimensions of L_1, L_2, L_3, we have these set of conditions,

$$V(0,Y,Z) = V(X,0,Z) = V(X,Y,0) = 0 \tag{12}$$
$$V(L_1,Y,Z) = V(X,L_2,Z) = 0 \tag{13}$$
$$V(X,Y,L_3) = f(X,Y) \tag{14}$$

Then, the boundary conditions can be rewritten as follows,

$$X(0) = X(L_1) = 0 \tag{15}$$
$$Y(0) = Y(L_2) = 0 \tag{16}$$
$$Z(0) = 0 \tag{17}$$

It follows that the only k_1, k_2 that fits these conditions are,

$$k_1 = \frac{\pi j}{L} \tag{18}$$
$$k_2 = \frac{\pi k}{L} \tag{19}$$

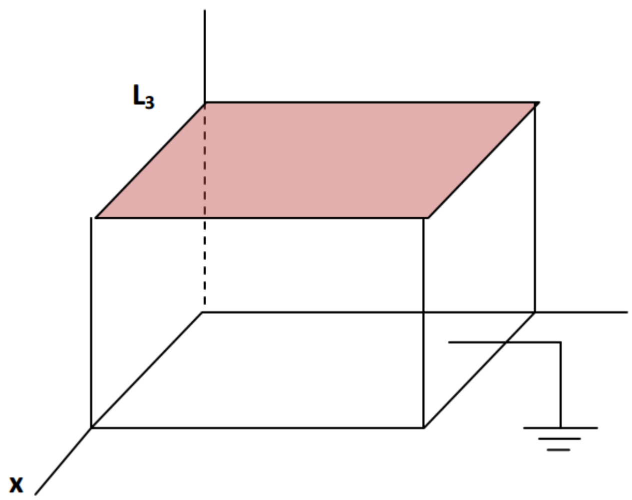

Figure 1: Think of this box as our world e.g., the Earth. The size of this box is L. Credits of Professor D. K. Ghosh.

and the sine functions are the only suitable functions for this.

$$V_{i,k}(x,y,z) = \sin(\pi\frac{jx}{L})\sin(\pi\frac{ky}{L})\; e^{\left(\pm\pi\sqrt{(j^2+k^2)}\frac{z}{L}\right)} \tag{20}$$

This solution may now be generalized for different values of $j, k = \pm1, \pm2 \pm 3 \ldots$. Notice that the sine of zero is zero, in this case for $j, k = 0$ the solution is 0. Also, for $n \in \mathbb{R}$ the solution is identical. The zero-level for Eqn. (20), is the familiar (?) Fourier sine expansion,

$$V(x,y,0) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} v_{jk}\sin\left(\pi\frac{jx}{L}\right)\sin\left(\pi\frac{ky}{L}\right) \tag{21}$$

Bibliography