Numerical PDEs

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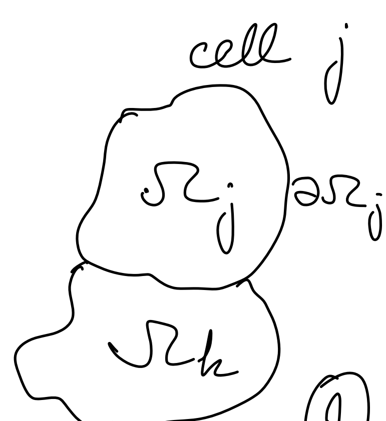
Finite Volume Methods

The bible on this topic is book "FVM for hyperbolic problems" R. LeVeque
– freely available as PDF to you

Key idea: Break up domain into a grid of cells, and use as variables the average of u over each cell

$$\bar{u}_j = \frac{1}{|\Omega_j|} \int_{\Omega_j} u \, dr$$



Conservation law

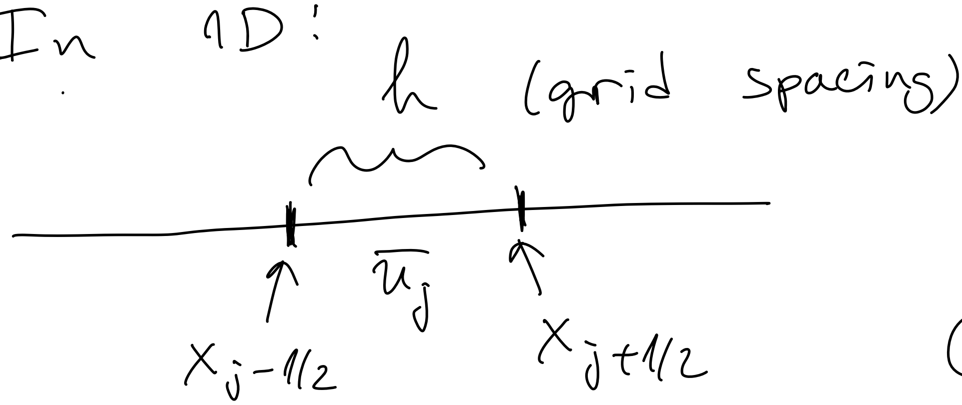
$$\int_{\Omega_j} \frac{\partial u}{\partial t}\, dr = -\int_{\Omega_j} (\nabla \cdot \vec{f})\, dr$$

$$|\Omega_j| \frac{d}{dt} \bar{u}_j = -\int_{\partial\Omega_j} \vec{f}\, dA$$

$$\frac{d\bar{u}_j}{dt} = -\frac{1}{|\Omega_j|} \int_{\partial\Omega_j} \vec{f}\, dA$$

which is a system of ODEs

In 1D:



In <u>1D adv-diff</u>:

$$h \cdot \frac{d}{dt} \bar{u}_j = -\left(f_{j+1/2} - f_{j-1/2}\right) =$$

$$-\left[a(x_{j+1/2})u(x_{j+1/2}) - a(x_{j-1/2})u(x_{j-1/2})\right]$$

$$+\left[d(x_{j+1/2})u_x(x_{j+1/2}) - d(x_{j-1/2})u_x(x_{j-1/2})\right]$$

This is a weak form of PDE and not (yet) a discretization, i.e., it is exact.

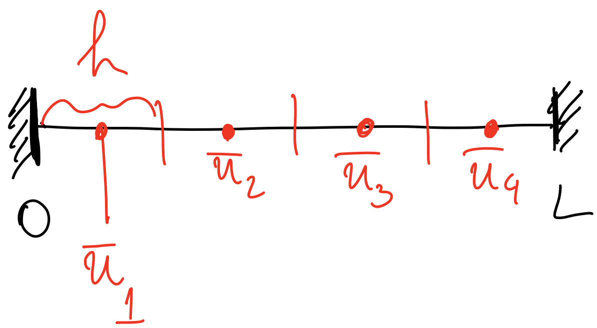
To make it into a scheme we need to figure out the fluxes in terms of the cell averages.

Note that for low-order (1^{st} or 2^{nd}) schemes there is really not that much difference between finite difference & finite volume - it is more of a matter of mental picture.

Key to FVM is to write fluxes <u>not</u> divergence of fluxes.

Another difference is with boundary conditions:

<u>Physical boundaries should overlap</u> with cell boundaries for FVM schemes.



FVM

We know the cell averages, not a function $u(x)$.

Constructing an approximation $u(x)$ from $\bar{u}$'s is called reconstruction in FVM.

As with finite difference (FD) methods, there are two main approaches:

- MOL (method of lines): write ODEs for $\bar{u}_j$ and solve
- Space time schemes: write $(\bar{u}_j^{n+1} - \bar{u}_j^{n})/\Delta t$

For space time:

$$\frac{\bar{u}_j^{n+1} - \bar{u}_j^{n}}{\Delta t} = \frac{1}{h} \int_{n\Delta t}^{(n+1)\Delta t} \left(f_{j+\frac{1}{2}}(t) - f_{j-1/2}(t) \right) dt$$

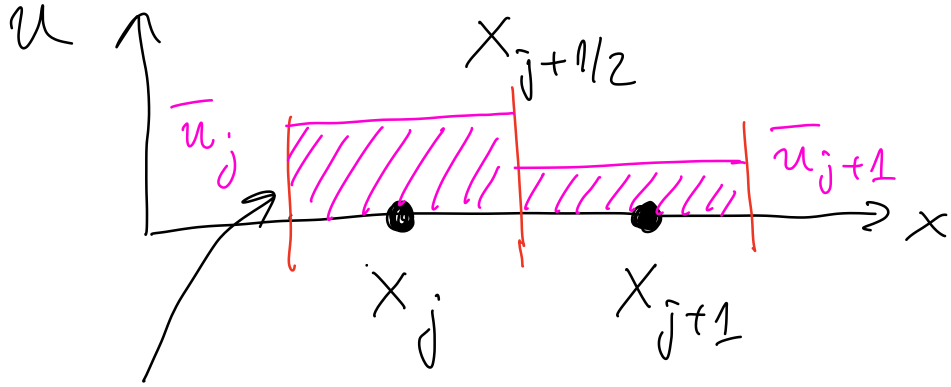
So we need a way to approximate the total or average flux over a time step.

For MOL, we need to estimate:

a) Advective flux

$$f_{j+1/2}^{(a)} = \underset{\substack{\uparrow \\ \text{"easy" (evaluate } a(x))}}{a}(x_{j+1/2})\, u(x_{j+1/2})$$

But how do we get $u(x_{j+1/2})$ from $\bar{u}$'s? This is the key complexity for advection: extrapolate from cell centers to faces.



Piecewise constant reconstruction

$$u(x_{j-1/2} < x < x_{j+1/2}) = \bar{u}_j + O(h^2)$$

This is discontinuous at $x_{j+1/2}$: Do we set ?

$$f_{j+1/2} = \begin{cases} a_{j+1/2}\,\bar{u}_j & \underline{\text{or}} \\ a_{j+1/2}\,\bar{u}_{j+1} & \end{cases}$$

If $a_{j+1/2} > 0$ then we know solution moves to the right, i.e., information comes from the left: upwind!

$$f_{j+1/2} = \begin{cases} a_{j+1/2}\,\bar{u}_j & \text{if } a_{j+1/2} > 0 \\ a_{j+1/2}\,\bar{u}_{j+1} & \text{otherwise} \end{cases}$$

<u>Upwind</u> flux

For diffusive flux,

"obviously":

$$f^d_{j+1/2} = d(x_{j+1/2})\, u_x(x_{j+1/2})$$

$$\simeq d(x_{j+1/2}) \left(\frac{\bar{u}_{j+1} - \bar{u}_j}{h} \right)$$

to $O(h^2)$, and this is what is most often used in CFD codes.

The real challenge in CFD is handling advection, i.e., handling hyperbolic conservation laws more generally. The physical reason for this is:

Advection is non-dissipative & simple advection is "non-dispersive"

Dissipation stabilizes numerical methods, but we don't want it.

For dispersion, read appendix E.3.9 in FD book of LeVeque (it is excellent!)

Basic idea: In Fourier space

$$\hat{u}(k,t) = e^{-i\omega(k)t}\,\hat{u}(0,t)$$

where k is the wave number

The equation relating ω to k is called the dispersion relation. It can be found by putting

$$u(x,t) = e^{-i\omega t} e^{ikx}$$

into the PDE for simple constant-coefficient equations.

$$c_p(k) = \frac{\omega(k)}{k} \quad \text{is phase velocity}$$

$$c_g(k) = \frac{d\omega(k)}{dk} \quad \text{is group velocity}$$

If $\omega(k)$ is <u>real</u> then the PDE is non-dissipative.

Special case is simple advection:

$$c_p = c_g = a = \text{const.}$$

"non-dispersive"

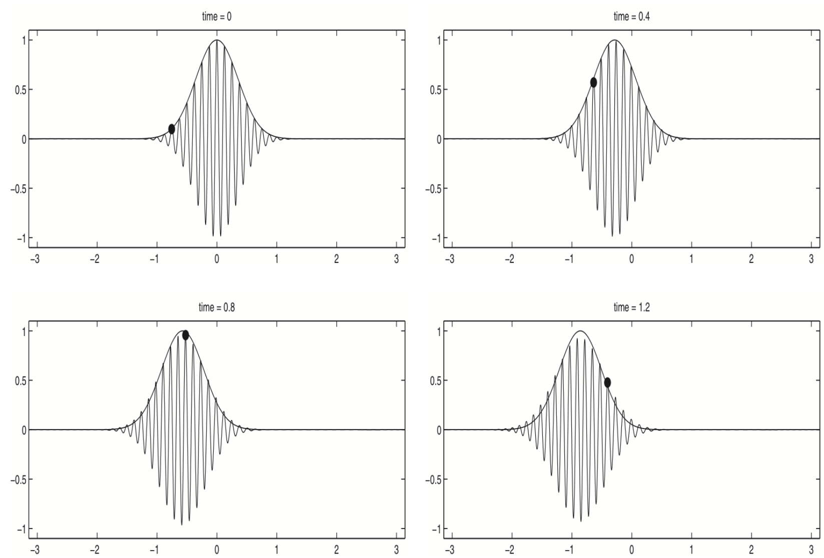


Figure E.2. *The oscillatory wave packet satisfies the dispersive equation* $u_t + au_x + bu_{xxx} = 0$. *Also shown is a black dot attached to one wave crest, translating at the phase velocity* $c_p(\xi_0)$, *and a Gaussian that is translating at the group velocity* $c_g(\xi_0)$. *Shown for a case in which* $c_g(\xi_0) < 0 < c_p(\xi_0)$.

c_g determines the speed of propagation of the envelope of the wave packet, while c_p of an individual peak/crest.

E.g. $u_t + a_1 u_x + a_2 u_{xx} + a_3 u_{xxx} + a_4 u_{xxxx} = 0 \Rightarrow$

$$\omega(k) = a_1 k + i a_2 k^2 - a_3 k^3 - i a_4 k^4$$

$$\hat{u}(k,t) = e^{(a_2 k^2 - a_4 k^4)t} e^{i(a_1 k - a_3 k^3)t} \hat{u}(k,0)$$

Dissipative
for $a_2 < 0$, $a_4 > 0$

For $a_2 = a_4 = 0$,

$$c_p(k) = a_1 - a_3 k^2$$

$$c_g(k) = a_1 - 3a_3 k^2$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k,0) e^{ik(x - c_p(k)t)} dk$$

For a numerical method, find
numerical dispersion relation
by plugging into method wave:

$$\overline{u}_j^n = e^{-i\omega n \Delta t} e^{ikjh}$$

An alternative, which gives us intuition quickly but is <u>not</u> rigorous, is to look at modified equations: find a PDE that the method solves to <u>higher</u> order than it does the PDE we want to solve.

We can do this to a MOL scheme <u>separately</u> for the spatial discretization (the rhs of the system of ODEs), separately for the temporal discretization, or, as we have to for space-time schemes, combine the spatial & temporal errors.

$$u_t + a u_x = 0 \qquad , a > 0$$

$$\frac{d\bar{u}_j}{dt} = a\left(\bar{u}_{j-1} - \bar{u}_j\right) \quad \frac{\text{upwind}}{\text{spatial}}$$

$$\bar{u}_j = u(x_j) + O(h^2)$$

Since

$$\frac{1}{h}\left(u(x-h) - u(x)\right) = -u_x(x) + \frac{h}{2} u_{xx}(x) + O(h^2)$$

the upwind difference gives a 2nd order approximation to the modified equation:

$$\tilde{u}_t + a\tilde{u}_x = \underbrace{\overbrace{\frac{ah}{2}}^{O(h)} \tilde{u}_{xx}}_{\text{artificial dissipation}} \quad \text{(upwind)}$$

This spurious dissipation makes upwinding the most robust but also least accurate scheme

Another way to see this:

$$\frac{a}{h}\left(\bar{u}_{j-1}-\bar{u}_j\right)=\frac{a}{2h}\left(\bar{u}_{j-1}-\bar{u}_{j+1}\right)$$

$$+\underbrace{\frac{ah/2}{h^2}\left(\bar{u}_{j+1}-2\bar{u}_j+\bar{u}_{j-1}\right)}_{\text{artificial diffusion}}$$

What's better?

If we used

$$u_{j+1/2}=\frac{\bar{u}_j+\bar{u}_{j+1}}{2}$$

(centered scheme)

$$\Rightarrow \frac{d\bar{u}_j}{dt}=-a\left(\frac{\bar{u}_{j+1}-\bar{u}_{j-1}}{2h}\right)$$

which is now second-order accurate in space, then we would get the modified equation

$$\tilde{u}_t + a\tilde{u}_x = \underbrace{-\frac{ah^2}{6}\tilde{u}_{xxx}}_{\text{artificial dispersion}}$$

$O(h^2)$

which has phase velocity

$$c_p = a\left(1 - O\left((kh)^2\right)\right)$$

So higher frequency (wavenumber) modes lag behind and cause unphysical oscillations in the solution.

Central dilemma is CFD:

Trade-off between (higher order) accuracy & robustness: what is the minimal artificial dissipation we need?

If needed, go through all of Ch. 10 in Finite Difference book of Le Veque.

What is a better MOL scheme?
We will "derive" this soon:

$$\bar{u}_{j+1/2} = \frac{1}{6}\left[-\bar{u}_{j-1} + 5\bar{u}_j + 2\bar{u}_{j+1}\right]$$

if $a_{j+1/2} > 0$

3rd-order upwind biased

which gives for $a = \text{const}$:

(*)

$$\frac{d\bar{u}_j}{dt} = \frac{a}{h}\left[-\frac{1}{6}\bar{u}_{j-2} + \bar{u}_{j-1} - \frac{\bar{u}_j}{2} - \frac{\bar{u}_{j+1}}{3}\right]$$

if $a > 0$

more information from upwind

For $a = \text{const}$, this upwind biased spatial discretization is 3rd order either as a FD or FV scheme. But, only 3rd order for non-constant advection as finite volume. (see Maple worksheet on webpage!)

How to show this:

$$u_t + (au)_x = u_t + u a_x + a u_x = 0$$

If Finite Difference:

$$\frac{du_j}{dt} = u_t(x_j, t) = -u(x_j)\, a_x(x_j) - a(x_j)\, u_x(x_j)$$

Compare this to Taylor series of r.h.s. of $(*)$

and you will see a mismatch of $O(h^2)$ if $a(x) \neq$ const.

If <u>Finite Volume</u>:

$$\frac{d\bar{u}_j}{dt} = \frac{1}{h}\int_{x_j-h/2}^{x_j+h/2} u_t(x,t)\,dx$$

$$= -\frac{1}{h}\int_{\mathcal{V}_j} (au)_x\,dx = -\frac{1}{h}(au)\Big|_{x_{j-1/2}}^{x_{j+1/2}}$$

and $\displaystyle \bar{u}_j = \frac{1}{h}\int_{\mathcal{V}_j} u(x,t)\,dx$

So now do Taylor series with these integrals (see symbolic algebra Maple code) to see

$$\frac{d\bar{u}_j}{dt} + \frac{1}{h}\Big((au)(x_{j+1/2}) - (au)(x_{j-1/2})\Big) = O(h^3)$$

This shows that for higher than 2nd order there is a difference FD vs. FV

interpretation (scheme is the same, it's how we interpret the output / input)

For $a = \text{const}$, modified equation for 3rd order upwind biased spatial discretization is:

$$\tilde{u}_t + a\tilde{u}_x = \underbrace{-\frac{|a|}{12} h^3 \tilde{u}_{xxxx}}_{\text{stabilizing artificial dissipation (good!)}} \leftarrow O(h^3)$$

which is higher-order & less dissipation than upwind

This was so far just MOL & focused on <u>spatial</u> discretization..

To get an actual 3rd order upwind biased method, we need a temporal integrator. Which one to use? $\boxed{\text{Discuss in class}}$

The temporal integrator in MOL schemes will itself add some artificial dispersion / dissipation. We can analyze this <u>formally</u> by considering a "perfect" spatial discretization. Let's do this for linear equations (we will consider implicit methods in homework).

(from book on "Advection - Diffusion - Reaction" by Hundsdorfer / Verwer)

Since we have linear systems of ODEs, consider *generically*:

$$u' = \frac{du}{dt} = A u \quad , \quad A \text{ matrix}$$

$$u^{n+1} = u^n + (1-\theta)\tau A u^n + \theta \tau A u^{n+1}$$

$$\begin{cases} \theta = 0 : & \text{forward Euler} \\ \theta = 1/2 : & \text{Implicit midpoint (2nd order)} \\ \theta = 1 : & \text{Backward Euler (implicit)} \end{cases}$$

Local truncation error:

$$\delta_n = \left(\frac{u(t_{n+1}) - u(t_n)}{\tau} \right) - \left(\frac{u^{n+1} - u^n}{\tau} \right)$$

$$= \left(\frac{1}{2} - \theta\right) \tau u''(t_n) + \left(\frac{1}{6} - \frac{\theta}{2}\right) \tau^2 u'''(t_n)$$

$$\delta_n = \left[\left(\frac{1}{2} - \theta\right)\tau A^2 + \left(\frac{1}{6} - \frac{\theta}{2}\right)\tau^2 A^3\right] u(t_n)$$

Therefore, the modified equation is

$$\tilde{u}' = \tilde{A}\tilde{u}$$

$$\tilde{A} = A + \left(\theta - \frac{1}{2}\right)\tau A^2 + \left(\frac{\theta}{2} - \frac{1}{6}\right)\tau^2 A^3$$

$$\begin{cases} FE\ (\theta = 0): & \tilde{A} \simeq A - \frac{\tau A^2}{2} \\ BE\ (\theta = 1): & \tilde{A} \simeq A + \frac{\tau A^2}{2} \\ Mid\ (\theta = 1/2): & \tilde{A} \simeq A - \frac{\tau^2}{12} A^3 \end{cases}$$

For advection

$$A = -a\partial_x \Rightarrow$$

$$A^2 = a^2 \partial_{xx} \qquad A^3 = -a^3 \partial_{xxx}$$

So we get:

$$\tilde{u}_t + a\tilde{u}_x = \begin{cases} -\dfrac{\tau a^2}{2}\tilde{u}_{xx} & \text{FE (anti-diffusion)} \\ \dfrac{\tau a^2}{2}\tilde{u}_{xx} & \text{BE artificial dissipation} \\ -\dfrac{\tau^2}{12} a^3 \tilde{u}_{xxx} & \text{implicit midpoint} \end{cases}$$

artificial dispersion

This artificial dispersion / dissip. adds with the one from the spatial discretization.

Consider the simplest FD/FV 1st order upwind:

$$u_j^{n+1} = u_j^n - \frac{a\tau}{h}\left(u_j^n - u_{j-1}^n\right)$$

(or add bars for FV)

$$\tilde{u}_t + a\tilde{u}_x = \underbrace{\frac{ah}{2}\tilde{u}_{xx}}_{\text{spatial artificial}} - \underbrace{\frac{a^2\tau}{2}\tilde{u}_{xx}}_{\text{temporal diffusion}}$$

Define CFL number $\nu = \frac{a\tau}{h}$

$0 \le \nu \le 1$ for stability

(from Num. Meth. II)

$$\tilde{u}_t + a\tilde{u}_x = \underbrace{\frac{ah}{2}(1-\nu)\tilde{u}_{xx}}_{\text{artificial dissipation}}$$

Observe that for $a =$ const and $\nu = 1$ the scheme is an exact $u_j^{n+1} = u_{j-1}^n$ $(a > 0)$

In 10.4 in FD book of LeVeque,
this is derived directly:

$$\tilde{u}(x, t+\tau) - \tilde{u}(x,t) =$$

$$\underbrace{- \frac{a\tau}{h} \left(\tilde{u}(x,t) - \tilde{u}(x-h,t) \right)}$$

This uses FD interpretation but
also OK to 2^{nd} order for FV
and is simpler so we use it

Now do Taylor series $\Rightarrow$

$$\tilde{u}_t + a\tilde{u}_x = \frac{ah}{2}\tilde{u}_{xx} - \frac{\tau}{2}\tilde{u}_{tt}$$

Since $\tilde{u} \approx u$ and

$$u_{tt} = -(au_x)_t = -a(u_t)_x = a^2 u_{xx}$$

$u_{tt} = a^2 u_{xx}$ – will be used multiple times later

$$\Rightarrow \quad \tilde{u}_t + a\tilde{u}_x = \frac{ah}{2}\left(1 - \frac{a\tau}{h}\right)\tilde{u}_{xx}$$

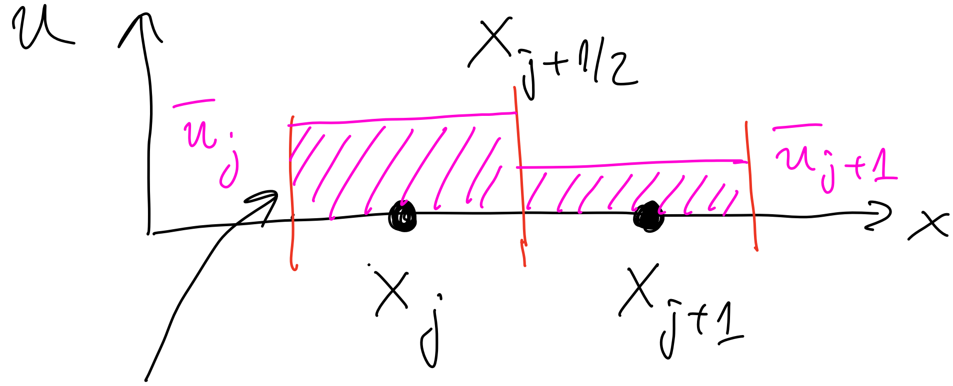
as derived already

Second-order FV in 1D

How can we construct other (better) methods for advection-diffusion that are at least 2nd order accurate?

The book of Le Veque does this in chapter 10 very nicely from an FD perspective. The same works for FV also to second order. But here I will focus on other approaches that generalize to more cases & fit FV interpretation better.

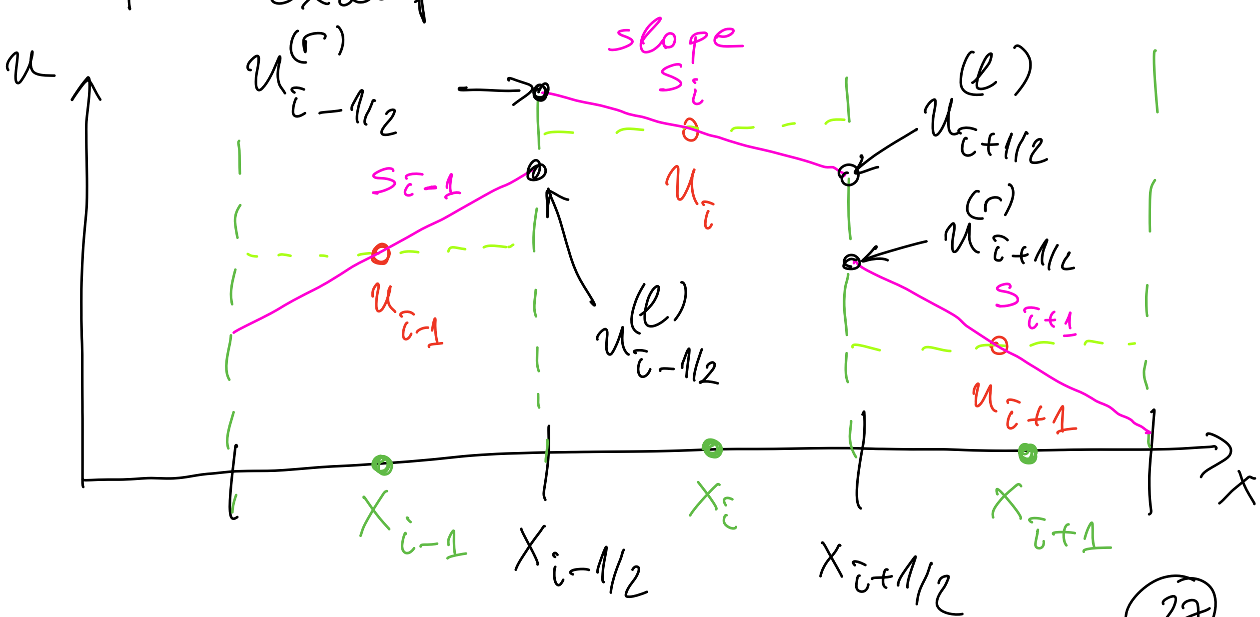
For upwind we used a



Piecewise constant reconstruction

$$u(x_{j-1/2} < x < x_{j+1/2}) = \bar{u}_j + O(h^2)$$

We can do better by using a higher-order reconstruction, for example linear:



In each cell, we first need to estimate slopes s_i of the linear reconstruction.

$$u(x_{i-1/2} < x < x_{i+1/2}) = \bar{u}_i \leftarrow \text{conservation}$$

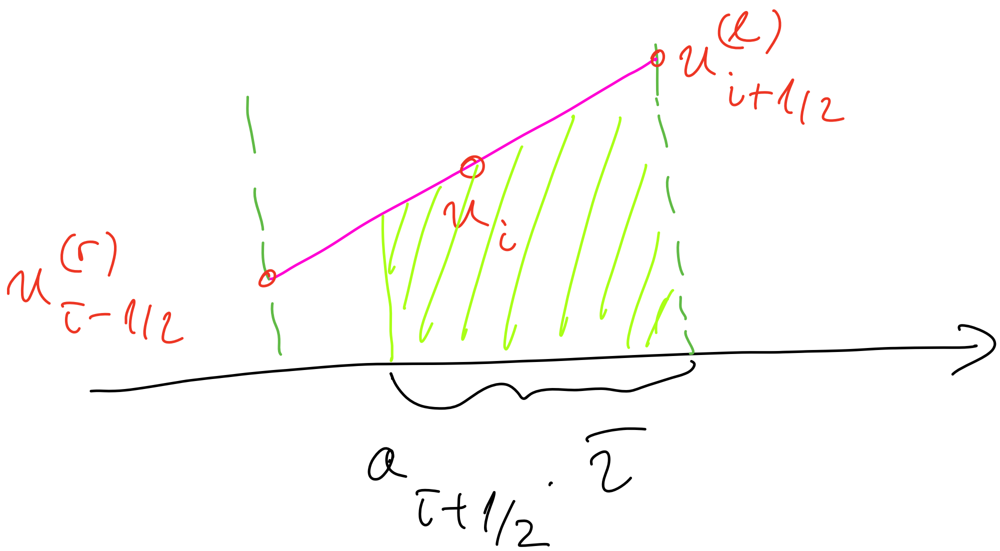
$$+ s_i (x - x_i) \quad \underbrace{+ O(h^2)}_{\text{still same order}}$$

Note $\int_{x_{i-1/2}}^{x_{i+1/2}} u(x)\,dx = \bar{u}_i \cdot h$ as it must in FV

This reconstruction is still only piecewise smooth, and at every face $i+1/2$ we still have two different states, one from the left $u^{(l)}_{i+1/2}$ and one from the right $u^{(r)}_{i+1/2}$

Now we need to decide if we want a MOL scheme (instantaneous fluxes) or a space-time scheme (time-average fluxes).

Let's start with Space-Time:



Upwind average flux:

$$f^{(l)}_{i+1/2} = \frac{\binom{\text{shaded}}{\text{area}}}{h} \cdot a_{i+1/2} = f_{i+1/2}$$

if $a_{i+1/2} > 0$

$$f_{i+1/2} = a_{i+1/2}\left(\bar{u}_i + \frac{h}{2} s_i \left(1-\nu_{i+1/2}\right)\right)$$

where <u>advective CFL number</u>

$$\nu_{i+1/2} = \frac{a_{i+1/2}\,\tau}{h}$$

Now let's assume $a = \text{const}$ for simplicity.

To complete the scheme we need to choose the <u>slopes</u>.

<u>Upwind slope</u>: $s_i = \dfrac{\bar{u}_{i+1} - \bar{u}_i}{h}$

leads to the Lax-Wendroff scheme:

$$f_{i+1/2} = \frac{a}{2}\left[(1+\nu)\bar{u}_i + (1-\nu)\bar{u}_{i+1}\right]$$

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \nu h \underbrace{\left(\frac{\bar{u}_{i-1} - \bar{u}_{i+1}}{2h} \right)}_{\text{centered advection}}$$

$$+ \frac{(\nu h)^2}{2} \underbrace{\left(\frac{\bar{u}_{i-1} - 2\bar{u}_i + \bar{u}_{i+1/2}}{h^2} \right)}_{\text{centered diffusive correction to dissipate}}$$

You may have encountered LW before and derived it from the second-order Taylor series <u>in time</u>:

$$u(t+\Delta t) = u(t) + \tau \frac{\partial u}{\partial t}(t) + \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2} + O(\tau^3)$$

Now from $u_t = -a u_x$ we already got using the PDE

$$u_{tt} = a^2 u_{xx}$$

Giving

$$u(t+\Delta t) \simeq u(t) - \nu h\, u_x(t) + \frac{(\nu h)^2}{2} u_{xx}(t)$$

which directly leads to LW if we use 2^{nd} order FD for all spatial derivatives, thus ensuring that the overall scheme is 2^{nd} order in space-time.

But this approach obsures the FV nature of the method & is hard to generalize to other hyperbolic equations

Note LW is <u>not</u> a MOL scheme – why?

Discuss in class

The modified equation for LW is:

$$\tilde{u}_t + a\tilde{u}_x + \underbrace{\frac{a\bar{\tau}^2}{6}(1-\nu^2)\tilde{u}_{xxx}}_{\text{dispersion}} = \underbrace{-\varepsilon\tilde{u}_{xxxx}}_{\text{dissipation}}$$

$$\varepsilon = O(\bar{\tau}^3, h^3)$$

Centered slopes

$$s_i = \frac{\bar{u}_{i+1} - \bar{u}_i}{2h}$$

with linear reconstruction gives the time-averaged upwind flux:

$$f_{i+1/2} = a\left(\bar{u}_i + \frac{(1-\nu)}{4}\left(\bar{u}_{i+1} - \bar{u}_{i-1}\right)\right)$$

Which leads to Fromm's method

$$\bar{u}_i^{n+1} = \bar{u}_i^n - (\nu h)\underbrace{\left(\frac{\bar{u}_i - \bar{u}_{i-1}}{h}\right)}_{\text{upwind}}$$

$$- \frac{\nu(1-\nu)h^2}{2}\underbrace{\left(\frac{\bar{u}_{i+1} - \bar{u}_i - \bar{u}_{i-1} + \bar{u}_{i-2}}{h^2}\right)}_{\text{upwinded difference } \approx u_{xx}}$$

This method is superior to Lax-Wendroff in practice.

Is it second order?

We know that upwinding has a modified equation (to $O(h^2)$)

$$\tilde{u}_t + a\tilde{u}_x = \frac{ah}{2}\tilde{u}_{xx}$$

So for Fromm:

$$u(x, t+\Delta t) \simeq \underbrace{u - a\tau u_x + \cancel{\frac{ah\tau}{2}} u_{xx}}_{\text{upwind piece}}$$

$$- \cancel{\frac{\nu h^2}{2}} u_{xx} + \frac{(\nu h)^2}{2} u_{xx} + O(h^2)$$

which is the correct Taylor series just like LW, so Fromm is also <u>second order</u> in space-time.

Another way to get Fromm as a FV is to estimate fluxes at midpoint of time step to get 2^{nd} order in time:

$$f^n_{i+1/2} \simeq a u^{n+1/2}_{i+1/2} \simeq$$

$$a u\left(x_i + \frac{h}{2}, t^n + \frac{\tau}{2}\right)$$

This is still not a MOL scheme since we are not applying a midpoint temporal integrator to an ODE, but is similar & often simpler than computing time-averaged fluxes.

If $a>0$, use a Taylor series in the <u>upwind</u> cell:

$$u_{i+1/2}^{n+1/2} = u_i^n + \frac{\tau}{2}\left(u_t\right)_i^n + \frac{h}{2}\left(u_x\right)_i^n$$

$\downarrow$ PDE

$$= u_i^n - \frac{a\tau}{2}\left(u_x\right)_i^n + \frac{h}{2}\left(u_x\right)_i^n$$

$$= \bar{u}_i^n + \frac{h}{2}(1-\nu)\, S_i^n \leftarrow \text{slope}$$

which leads to Fromm once we use centered slopes (it is not hard to see this is identical to linear reconstruction + time averaged upwind)

Now let's consider higher-order reconstruction with **MOL schemes**

To get a 3^{rd} order MOL scheme, we need a quadratic reconstruction

$$u_i(x) = c_0 + c_1 x + c_2 x^2$$

With FV conditions:

$$\begin{cases} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} u_i(x)\,dx = \bar{u}_i \cdot h \\ \displaystyle\int_{x_{i-3/2}}^{x_{i-1/2}} u_i(x)\,dx = \bar{u}_{i-1} \cdot h \\ \displaystyle\int_{x_{i+1/2}}^{x_{i+3/2}} u_i(x)\,dx = \bar{u}_{i+1} \cdot h \end{cases}$$

Solving for c_1, c_2 & c_0 gives:

$$u_i(x;\bar{u}) = \bar{u}_i + \left(\frac{\bar{u}_{i+1} - \bar{u}_{i-1}}{2h}\right)(x - x_i)$$

$$+ \frac{1}{2}\left(\frac{\bar{u}_{i+1} - 2\bar{u}_i + \bar{u}_{i-1}}{h^2}\right)\left[(x-x_i)^2 - \frac{h^2}{12}\right]$$

If it were FD third-order we wouldn't have this piece!

Upwind instantaneous flux:

$$f_{i+1/2}(\bar{u}(t)) = a_{i+1/2}\, u_i(x_{i+1/2}; \bar{u}(t))$$

leads to the 3^{rd} order upwind-biased scheme:

$$\frac{d\bar{u}_j}{dt} = \frac{a}{h}\left[-\frac{1}{6}\bar{u}_{j-2} + \bar{u}_{j-1} - \frac{\bar{u}_j}{2} - \frac{\bar{u}_{j+1}}{3}\right]$$

if $a > 0$

that I mentioned before