Computational PDEs, Fall 2021
Second Order Finite Volume Methods for Advection-Diffusion Equations

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All of the notes for the previous homework continue to apply, so read them again as needed.

1 Advection-Diffusion in Periodic Domains

Consider numerically solving the advection-diffusion equation

$$u_t + (a(x,t)u)_x = (d(x)u_x)_x \, ,$$

on the periodic domain $0 \leq x < 1$, for a *smooth* initial condition

$$u(x,0) = [\sin{(\pi x)}]^p \, ,$$

on a uniform grid, where p is an exponent.

For $p = 2$ the initial condition is quite smooth even on coarse grids, but for $p = 100$ it is only smooth on finer grids. You should feel encouraged to try less smooth initial conditions such as square waves as well.

1.1 Variable coefficients

Make the advection speed and the diffusion coefficient non-constant, though quite "smooth",

$$a(x,t) = a_0 \cos{(t)} \left(\frac{3}{4} - \frac{1}{4} \sin(2\pi x) \right).$$

$$d(x) = d_0 \left(2 + \cos(2\pi x)\right).$$

By default, use $a_0 = 1$ and $d_0 = 0.001$, but it is important to vary d_0 to vary the relative importance of advection versus diffusion. It is advised to validate/test the advection and diffusion pieces *separately*.

Develop a **finite volume** scheme to solve the PDE to time $T = 1$. Your goal is accomplish the following things, either with a MOL or a space-time scheme:

1. Treat advection explicitly but diffusion implicitly, with the goal that the time step is only limited by advection and not by diffusion, that is, the scheme is stable if $\tau = Ch/\left\|a\right\|_\infty$ where $C = O(1)$ *independent* of the value of the diffusion coefficient. Report whether you think you accomplished that goal and why.
2. Achieve at least 2nd order under space-time refinement and confirm that order unambiguously in all three norms for a smooth periodic solution.
3. Confirm that asymptotically for a refined grid the error in the solution $\to h^2 \, e(x)$ where $e(x)$ is some function, and provide an estimate of the actual error (difference between true solution and your approximation) for one of the grid sizes (usually one of the finer grids).
4. Validate that your scheme converges to the correct solution of the non-constant coefficient PDE at second-order in space-time, using the method of manufactured solutions, at least for a periodic domain. [*Hint: Make sure your manufactured solution is smooth and satisfies the BCs (for example, is a smooth periodic function).*]
5. Test the scheme on less smooth solutions and explain what sort of numerical artifacts arise in your method.

1.2 Boundary conditions

Now solve the PDE with boundary conditions

$$\begin{aligned} u(0,t) &= \sin^p(-\pi t) \\ u_x(1,t) &= 0 \quad \text{if} \quad d_0 > 0. \end{aligned}$$

Observe that if $d_0 = 0$ and $a = a_0$ is constant, the exact solution is $u(0,t) = \sin^p(\pi(x-t))$, which is the same as for periodic conditions.

If you find it necessary, for the first submission of this part, you can: (1) Use constant advection speed (certainly useful for initial testing), however, there is no real reason to do that except for testing, and the final submission should handle non-constant advection as well. (2) restrict your code/results to $d_0 = 0$ if you run out of time, but the final submission should handle diffusion as well.

Develop a finite-volume method (choose the advective/diffusive stencils, the boundary condition treatment, number of grid points, etc., and *explain your choices*) to solve the equation up to time $T = 1$.

1. For $p = 100$ and constant advection $a(x,t) = a_0$, show the solution at several times up to time T for $d_0 = 0.1$, $d_0 = 0.01$, $d_0 = 0.001$, and $d_0 = 0$, and comment on your observations (e.g., anything to note near the boundaries?). [*Note: Some of the times $t \le T$ you choose should have the peak overlap at least one of the two boundaries; if the function is essentially zero around the boundaries the problem is easier than a generic one.*]
2. Discuss what happens as $d \to 0$ with the PDE and its true analytical solution, and compare to what happens in your numerical method. Is your method *consistent* with the PDE for $d_0 = 0$? If it is, what is the order of convergence for $d_0 = 0$?
3. For non-constant coefficients, determine the spatio-temporal order of convergence for your discretization for both zero and finite d_0 (explain what value you chose and why) in different norms, for a smooth solution. [*Hint: "Order of **convergence**" assumes the method converges to the true solution to begin with, so you need to be convinced of that before measuring order. You can use either empirical order of accuracy or manufactured solution, or (best) both. If you use empirical order, you **must** also somehow confirm that the BCs are satisfied. For manufactured solution, make sure the solution satisfies the BCs by construction. For order of accuracy testing you want to use a smooth **easy** problem ($d_0 \to 0$ is a singular limit so not "easy").*] Does the observed order agree with theoretical expectations (explain what your expectation is)?
4. (Optional) Repeat all the above steps with a Dirichlet condition on the right, $u(1,t) = 0$ and comment on the differences with the Neumann BC.