Implicit-Explicit (IMEX) temporal integrators

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Here are some standard 2nd order IMEX schemes for solving ODEs of the form:

$$u'(t) = \underbrace{f(u,t)}_{\substack{\underline{\text{not}}\text{ stiff} \\ \text{(explicit)} \\ \text{advection}}} + \underbrace{g(u,t)}_{\substack{\text{stiff} \\ \text{(implicit)} \\ \text{diffusion}}}$$

Often $g(u,t) = Lu$ is linear to avoid non-linear equations in implicit solve

Multistep

AB2 + CN

Adams - Bashforth + Crank- Nicolson (implicit midpoint)

See Section 11.5 in LeVeque

$$U^{n+1} = U^n + \frac{\tau}{2}\left(3f(U^n,t^n) - f(u^{n-1},t^{n-1})\right)$$

not L-stable

$$+ \frac{\tau}{2}\left(g(U^n,t^n) + g(U^{n+1},t^{n+1})\right)$$

implicit

(11.26) in Le Veque

Backwards Differentiation

Semi-implicit BDF = SBDF2

$$U^{n+1} = \frac{4}{3}U^n - \frac{1}{3}U^{n-1} + \frac{2}{3}\tau g(U^{n+1},t^{n+1})$$

$$+ \frac{2}{3}\tau\left[2f(U^n,t^n) - f(U^{n-1},t^{n-1})\right]$$

BDF2 is L-stable (8.3.2 in Le Veque)

(good for diffusion)

Runge-Kutta IMEX L-stable

$$U^{n+1/2,*} = U^n + \tau\left(\left(1+\frac{\sqrt{2}}{2}\right) g\left(U^{n+1/2,*}, t^n + \frac{\tau}{2}\right) - \left(\frac{1+\sqrt{2}}{2}\right) g\left(U^n, t^n\right) \right) + \frac{\tau}{2} f\left(U^n, t^n\right)$$

(Predictor to midpoint)

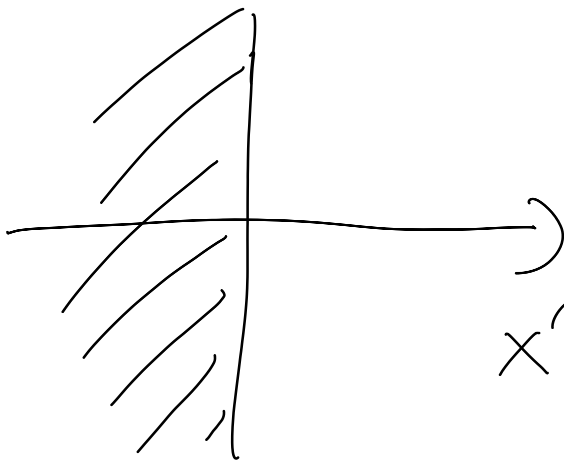
$$U^{n+1} = U^n + \left(\left(1+\frac{\sqrt{2}}{2}\right) g\left(U^{n+1}, t^{n+1}\right) - \left(1+\sqrt{2}\right) g\left(U^{n+1/2,*}, t^n + \frac{\tau}{2}\right) + \left(1+\frac{\sqrt{2}}{2}\right) g\left(U^n, t^n\right) \right)$$

(corrector)

explicit midpoint $\left\{ + \tau f\left(U^{n+1/2,*}, t^n + \tau/2\right) \right.$

L-stable scheme

(A-stable)



$$x'(t) = -\underbrace{\lambda}_{\text{large}} x(t)$$

$$X^n = \overset{R}{\overset{\parallel}{\underbrace{\left(\frac{1+\lambda\Delta t/2}{1-\lambda\Delta t/2}\right)}_{?}}}^{\,n} X^0$$

$$X^n = \underbrace{e^{-\lambda n \Delta t}}_{\text{decays rapidly}} X^0$$

$$\lambda \Delta t >> 1 \rightarrow X^n \rightarrow \underbrace{(-1)}_{\text{does}}^{n} X^0$$

L - stable :

Stability $\left| R(\lambda \Delta t = z) \right| \leq 1$

L-stable = A-stable +

$R(z \to \infty) \to 0$

Backward Euler is L-stable

An L-stable scheme applied to a linear parabolic PDE with $\Delta t \to \infty$ will converge to the steady-state solution (elliptic)

Good scheme:

- Stability $\nu = \frac{a\tau}{h} < C = O(1)$

- Diffusion only $\Rightarrow$ strictly dissipative (L-stable)

- 2^{nd} order