

Finite Element Methods

A. DÖNER, Fall 2021

These lecture notes are based on lecture notes by Georg Stadler.

I consider the background on piecewise smooth local basis functions to be part of interpolation in 2D/3D so see those lecture notes first.

Like pseudo spectral methods, FEM is a series method, meaning that the discrete solution is a function that is a sum of basis functions and the discrete unknowns are

①

the series coefficients :

$$u_h(x) = \sum_{i=1}^N u_i \varphi_i(x) \approx u(x)$$

↑
"grid size"
(discrete) ↑
unknown
coefficients ↗
basis
functions

A key difference is that now the basis functions $\varphi_i(x)$ are piecewise polynomials with localized support — this will be key for efficiency as it will lead to sparse matrices not dense like for orthogonal polynomials.

But the heart of FEM methods is their relation to weak & variational formulation of elliptic (parabolic) PDEs

②

Consider PDE on bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$:

$$\begin{aligned}
 & - \sum_{i,j=1}^n \partial_j (a_{ij}(x) \partial_i u) = -\nabla \cdot (A(x) \nabla u) \\
 & + \sum_{i=1}^n b_i(x) \partial_i u = f(x) \\
 & + c(x) u = g(x)
 \end{aligned}$$

Importantly, $A(x)$ is uniformly (symmetric) positive definite, i.e., $\nabla \cdot (A \nabla u)$ is an elliptic operator.

BCs can be Dirichlet (u), Neumann ($\partial u / \partial n$), or Robin/mixed.

If $b = 0$ (no "advection"), we have a **variational formulation** of PDE. Take for simplicity

$$\left\{ \begin{array}{l} -\nabla \cdot (A \nabla u) + u = f \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial \Omega_1 \quad (\text{essential BC}) \\ a \frac{\partial u}{\partial n} = g \quad \text{on } \partial \Omega_2 \quad (\text{natural BC}) \end{array} \right.$$

Take a test function $\vartheta \in C(\bar{\Omega})$ with $\vartheta|_{\partial \Omega_1} = 0$ (essential BCs must be incorporated into FEM spaces / enforced explicitly in the strong sense), multiply PDE and integrate by parts to lower smoothness requirements

(4)

$$-\int_{\Omega} \nabla \cdot (a \nabla u) \varphi \, dx + \int_{\Omega} u \varphi \, dx =$$

$$= \int_{\Omega} a \nabla u \cdot \nabla \varphi \, dx - \int_{\partial \Omega} a \frac{\partial u}{\partial n} \varphi \, ds$$

$$+ \int_{\Omega} u \varphi \, dx = \int_{\Omega} f \varphi \, dx$$

Using B.Cs we get

$$\int_{\Omega} a \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} u \varphi \, dx =$$

$$\int_{\Omega} f \varphi \, dx + \int_{\partial \Omega} g \varphi \, dx \dots (*)$$

Weak formulation: $(*)$ is true
for all suitable $\varphi(x)$

(5)

The right function space is the same for u and ϑ (for self-adjoint problems, but in Petrov-Galerkin methods u and ϑ belong to different spaces) is

$$H^1_{0, \partial\Omega_1}(\Omega) = \left\{ u \in L^2(\Omega) : \right.$$

$\frac{\partial u}{\partial x_i} \in L^2(\Omega) \quad \forall i = 1, \dots, n,$

$u = 0 \quad \text{on} \quad \partial\Omega_1 \right\}$

Sobolev space $H^1 \equiv W^{1,2}$

Denote **bilinear form**

$$a(u, \vartheta) = \int_{\Omega} a(x) \nabla u \cdot \nabla \vartheta \, dx$$

$$+ \int_{\Omega} u \vartheta \, dx$$

(6)

and linear form

$$l(\varphi) = \int_{\Omega} f\varphi \, dx + \int_{\partial\Omega_2} g\varphi \, ds$$

Variational / weak form of PDE:

$$\begin{cases} a(u, \varphi) = l(\varphi), u \in H_0^1, \\ \text{if } \varphi \in H_0^1 \end{cases}$$

$$\text{If } f \in L^2(\Omega), g \in L^2(\partial\Omega_2)$$

then Lax-Milgram lemma says u is a unique solution.

Key condition is coercivity/ellipticity:

$$a(\varphi, \varphi) \geq c_0 \|\varphi\|_{H^1}$$

$$(\varphi, w)_{H^1} = \int_{\Omega} (\nabla \varphi \cdot \nabla w + \varphi \cdot w) \, dx$$

(7)

If $A(x)$ is SPD for all x ,

$$a(u, v) = a(v, u)$$

we have also equivalent

energy / variational formulation

$$u = \arg \min_{v \in H^1_0, \partial\Omega_1} J(v)$$

$$J(v) = \frac{1}{2} a(v, v) - l(v)$$

Steps in FEM

1) Write weak form of PDE

2) Choose finite dimensional spaces
for all function spaces

3) Solve resulting system of
equations

So instead of

Find $u \in V$ s.t. $a(u, \varphi) = l(\varphi)$ for

choose finite-dimensional $V_h \subset V$

made of piecewise polynomial
functions and solve

Find $u_h \in V_h$ s.t.

$$a(u_h, \varphi_h) = l(\varphi_h) \quad \forall \varphi_h \in V_h$$

by solving a system of equations.

$$V_h = \text{span} \{ \varphi_1, \varphi_2, \dots, \varphi_n \}$$

(linearly independent but not orthogonal)

$$u_h = \sum_{i=1}^n u_i \varphi_i(x)$$

Plug into weak form to get

⑨

$$\sum_{i=1}^n a(\varphi_i, \varphi_j) u_i = l(\varphi_j) \quad \forall j = 1, \dots, N$$

$$\left\{ \begin{array}{l} A U = L \\ \end{array} \right. - \text{system of } N \text{ equations}$$

$$A_{ij} = a(\varphi_i, \varphi_j) \quad \text{stiffness matrix}$$

$$L_i = l(\varphi_i)$$

Notes :

① Since computing A_{ij} requires integration, it may have to itself be approximated by **spectral quadrature** (e.g. Gauss quad). Always true for r.h.s. L

② By choosing piecewise basis wisely we can make A be sparse & SPD and thus solve system more efficiently

(10)

Parabolic problems (aside)

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + a u_x = \Delta u_{xx} + b u + f(x, t) \\ u(\partial \Omega) = 0 \end{array} \right. \quad \begin{array}{l} \Delta u_{xx} \\ b u \end{array} \text{constant coeff.}$$

Method of lines :

$$u(x, t) = (u(t))(x)$$

$$u: (0, T) \rightarrow H_0^1(\Omega)$$

Weak form:

$$\int_{\Omega} \vartheta u_t dx = \int_{\Omega} \vartheta (-au_x + \Delta u_{xx} + bu + f) \quad \forall \vartheta \in H_0^1(\Omega)$$

↑ integrate by parts

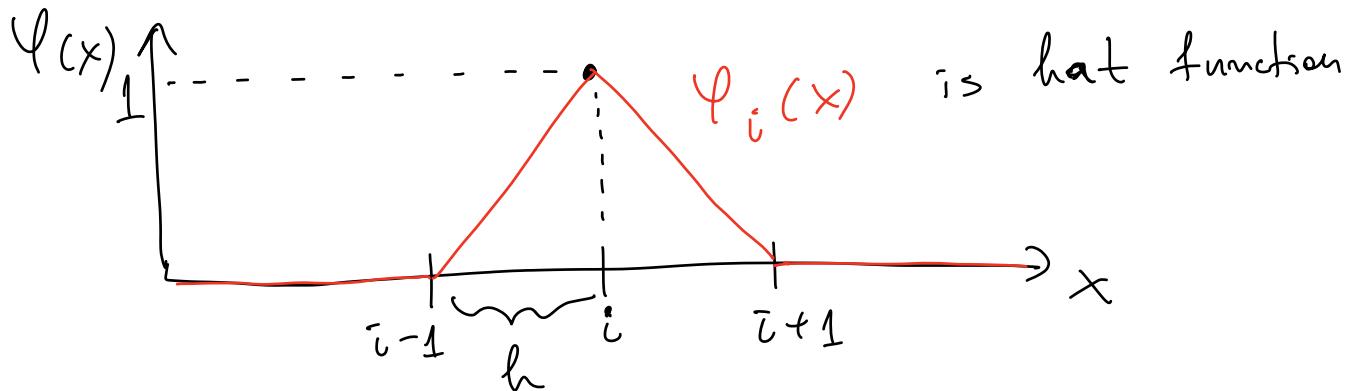
$$u(x, t) = \sum_{i=1}^N v_i(t) \varphi_i(x)$$

gives $\underset{\uparrow \text{mass matrix}}{M} \frac{dU}{dt} + AU = g \quad (\text{ODEs})$

$M_{ij} = (\varphi_j, \varphi_i)_{L^2(\Omega)}$

(11)

Take uniform grid in 1D



$$\Rightarrow \int \psi_j \psi_{j+1} dx = \frac{h}{6}$$

$$\int \psi_j^2 dx = \frac{2}{3} h$$

$$-\int \frac{d\psi_{j-1}}{dx} \psi_j dx = 1/2$$

$$\int \left(\frac{d\psi_j}{dx} \right)^2 dx = \frac{2}{h^2}$$

$$\int \left(\frac{d\psi_{j-1}}{dx} \right) \left(\frac{d\psi_j}{dx} \right) dx = -\frac{1}{h^2}$$

Gives discretization

$$M \frac{dU}{dt} + a \tilde{D} U = d D_2 U + \ell M U + F$$

where $M = \frac{1}{6} \begin{bmatrix} 4 & 1 & & & \\ 1 & -3 & 1 & & \\ & 1 & -3 & 1 & \\ & & 1 & -3 & 1 \\ & & & 1 & 4 \end{bmatrix}$ mass matrix

$$\tilde{D} = \frac{1}{2h} \begin{bmatrix} 0 & 1 & & & \\ -1 & -3 & 1 & & \\ & 1 & -3 & 1 & \\ & & 1 & -3 & 1 \\ & & & -1 & 0 \end{bmatrix} = \text{centered difference}$$

$$D_2 = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & \\ & 1 & -3 & 1 & \\ & & 1 & -3 & 1 \\ & & & 1 & -2 \end{bmatrix} = \text{standard Laplacian}$$

$$F_i = \frac{1}{h} \int_{x_{i-1}}^{x_{i+1}} f(x) \varphi_i(x) dx$$

Except for mass matrix, this is the same as the FD second order!

(13)

We know that centered difference is not good for advection (will require RK3+ to integrate).

But FEM can be higher order (with some conditioning issues) & use unstructured grids.

Note that

$$M^{-1} \tilde{D} U \approx \frac{\partial u}{\partial x} + O(h^4)$$

in the finite difference sense
 (called "compact finite difference")
 so in practice the method will be better than 2nd order FD for advection. But each timestep requires solving $Mx=b$!

Lumped mass approximation: Approx. M by a diagonal matrix

Back to time-independent problems

We will not go into the extensive & well-developed theory of FEM methods, which relies heavily on Sobolev function spaces. Some notes:

① Cea Lemma:

The FEM solution is nearly optimal in the approximation space:

$$\|u - u_h\|_{H^1} \leq \frac{c_1}{c_0} \min_{v_h \in V_h} \|u - v_h\|_{H^1}$$

As long as the constants c_1 and c_0 are well-behaved, and the approximation is suited to the PDE, we don't have to worry & have strong theoretical guarantees.

② The target in the FEM world
is to prove a priori error bound

$$\|u - u_h\|_{H^1} \leq C h^p$$

where p is the degree of the polynomial basis functions (so linear gives first order convergence in H^1 in general)

③ For purely elliptic PDEs,
define inner product

$$(\varphi, w)_a = a(\varphi, w)$$

From PDE $\left\{ \begin{array}{l} a(u, \varphi_h) = l(\varphi_h) \\ a(u_h, \varphi_h) = l(\varphi_h) \end{array} \right. + \forall \varphi_h \in V_h$

FEM $\left\{ \begin{array}{l} a(u_h, \varphi_h) = l(\varphi_h) \\ \Rightarrow a(u - u_h, \varphi_h) = 0 \end{array} \right. \quad \textcircled{16}$

\Rightarrow Error is orthogonal to V_h in
the new inner product, i.e.

$$\|u - u_h\|_a = \min_{\substack{\varphi \in V_h \\ \varphi \in V_h}} \|u - \varphi\|_a$$

FEM approximation is optimal in
the a -norm (improved Cea lemma)

For example, for

$$\begin{cases} -u'' + u = f & \text{on } (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

and a regular 1D grid using:

Cea's lemma + interpolation error

found + elliptic regularity one
 $(\|u\|_{H^2} \leq C \|f\|_{L_2})$

gets :

$$\|u - u_h\|_{H^1} \leq \frac{2h}{\pi} \left(1 + \frac{h^2}{\pi^2}\right)^{1/2} \|f\|_{L_2}$$

(17)

However, since we know that
 for regular grids + linear basis
 FEM is the same as FD ^{2nd}
 order, we expect that the solution
 is more accurate than just 1st
 order. For this one needs to
 switch to a different norm that
 does not test derivatives since
 those are indeed only first-order
 accurate. Specifically, one can
 show

$$\|u - u_h\|_{L_2} \leq 4h^2 \|u\|_{H^2}$$

i.e. solution is second-order
 accurate in L_2 norm.

However, FEM error bounds can become useless if the approximation space is not suited to the PDE. Notably, for advection-diffusion:

$$\begin{cases} -\delta \nabla^2 u + \vec{a} \cdot \nabla u = f \\ \nabla \cdot \vec{a} = 0 \end{cases}$$

the standard FEM discretization gives an error constant

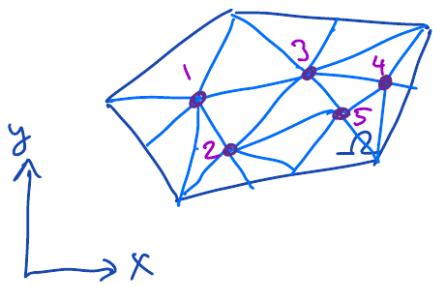
$$C \sim \sqrt{1 + Pe^2}$$

where Pe is the Peclet number. So for advection-dominated problems $C \gg 1$ and FEM does not work well without some "stabilization"

Some practicalities :

FEM Grids & Matrices

$\Omega \subset \mathbb{R}^2$ polygonal boundary, Cover Ω with triangles



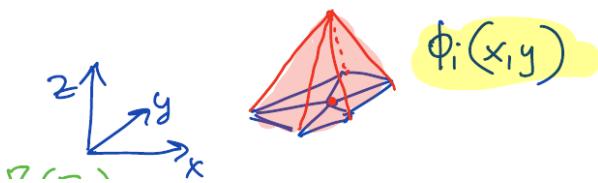
• interior points

V_h ... space of continuous functions that are linear on each triangle

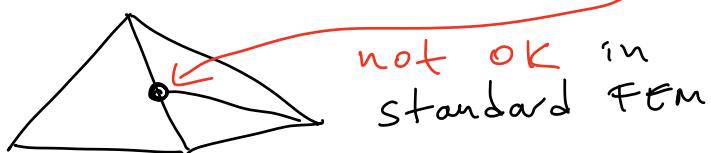
$$V_h \subset V$$

ϕ_i basis for each interior node, $i = 1, \dots, 5$

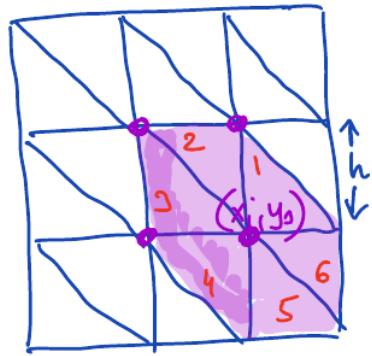
2-dim. hat functions:



In 2D, almost any domain of interest can be triangulated, so take FEM cells to be triangles, FEM nodes to be the vertices, no hanging nodes



If $\Omega = [0, 1]^2$ unit square
with uniform triangulation



Piecewise linear tent functions, give

$$U_i = u_h(x_i)$$

as in FD

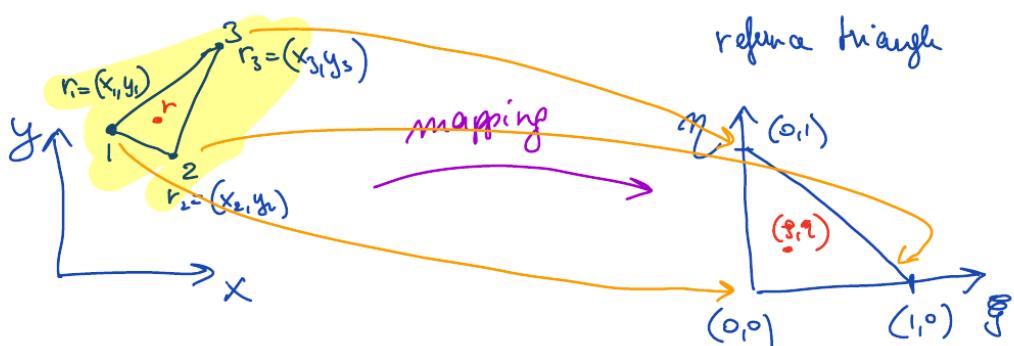
For the Laplacian,

$$A_{ij} = \int \nabla \varphi_i \cdot \nabla \varphi_j \, dx \, dy$$

is the standard FD 5th Laplacian

and so just as ill-conditioned
as for FD methods: Efficient
linear solvers are iterative &
based on geometric or
algebraic multigrid method (AMG)

In FEM, typically things are precomputed for a reference triangle, and results are mapped to each triangle of the grid using suitable Jacobians.



$$r = (x, y) = \underbrace{(1 - \xi - \eta)}_{\Psi_1(\xi, \eta)} r_1 + \underbrace{\xi}_{\Psi_2(\xi, \eta)} r_2 + \underbrace{\eta}_{\Psi_3(\xi, \eta)} r_3$$

Consider map: $(\xi, \eta) \mapsto r = (x, y)$

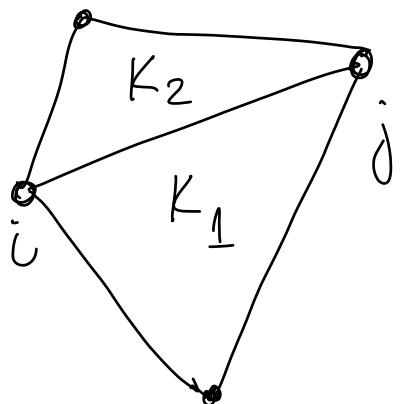
$$\mathbf{J} = \frac{\partial (x, y)}{\partial (\xi, \eta)} = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$$



$$|\mathbf{J}| = 2 A_{123} \leftarrow \text{area of triangle}$$

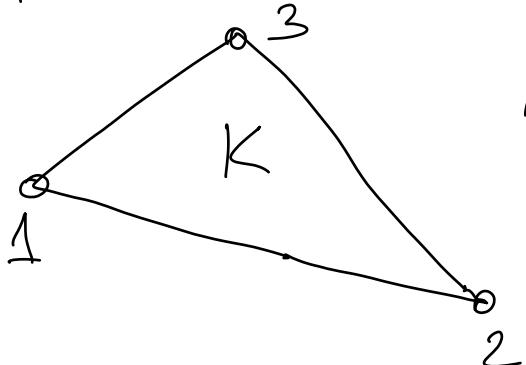
Recall that for Laplacian we need $\int \nabla \Psi_i \cdot \nabla \Psi_j \, dx$.

But the supports of φ_i and φ_j only overlap if nodes i and j are neighbors, and therefore



we get a nonzero contribution to the stiffness matrix from at most two triangles in 2D.

We therefore focus on a triangle K at a time, and assemble the stiffness matrix from triangle stiffness matrices



$$A_{ij}^K = \int_K \nabla \varphi_i \cdot \nabla \varphi_j \, dx$$

K

(3x3 matrix)

$$A^K = \frac{1}{4A_{123}} \begin{bmatrix} |r_2-r_3|^2 & (r_2-r_3) \cdot (r_3-r_1) & (r_2-r_3) \cdot (r_1-r_2) \\ (r_3-r_1)^2 & |r_3-r_1|^2 & (r_3-r_1) \cdot (r_1-r_2) \\ (r_1-r_2)^2 & (r_1-r_2)^2 & |r_1-r_2|^2 \end{bmatrix}$$

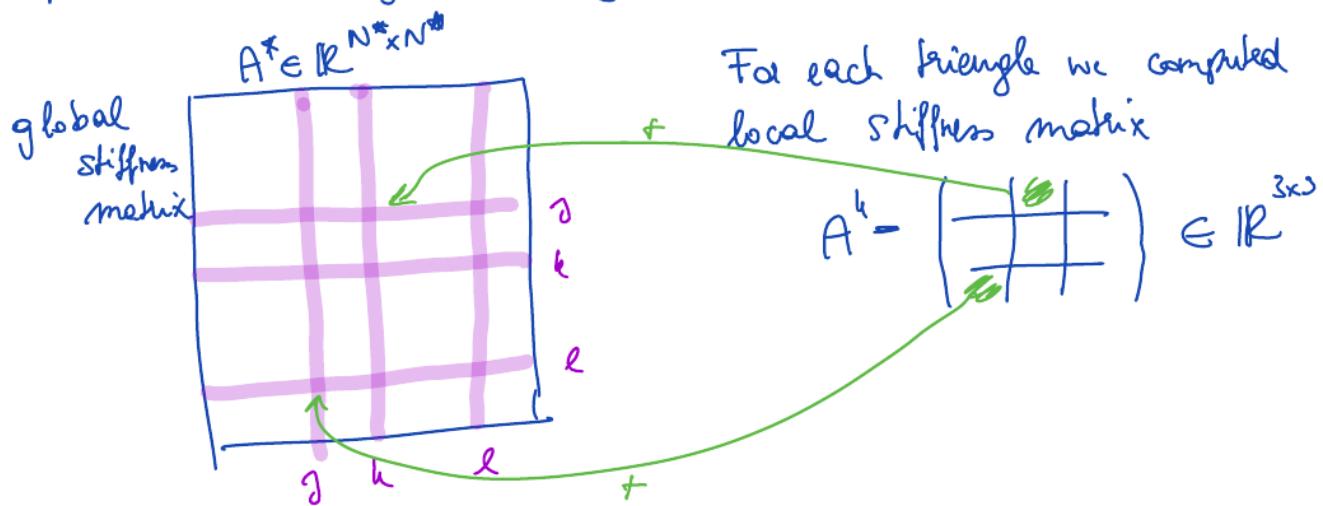
Symmetric

local stiffness matrix, corresponding to triangle

K

$$A = \sum_{K=1}^M A^K$$

Matrix Assembly summary:

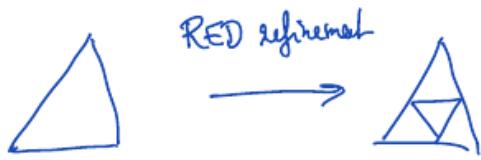


All you need is a loop over triangles and an $M \times 3$ ($3 \times M$) matrix mapping local DOFs to global DOFs

(24)

Refining grid based on a posteriori error estimate (from G. Stadler):

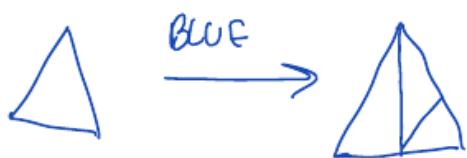
In 2D: How to refine a triangle



Split into 4 triangles,
that are shape-regular.
Problem: What to do with neighboring
element?
"hanging node"

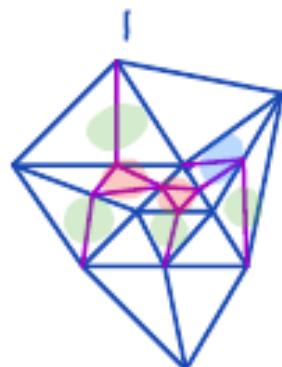


Splits into 2 triangles. Problem:
iterative GREEN refinement can result
in poor shape regularity:



Splits one triangle into three, helps with
shape regularity.

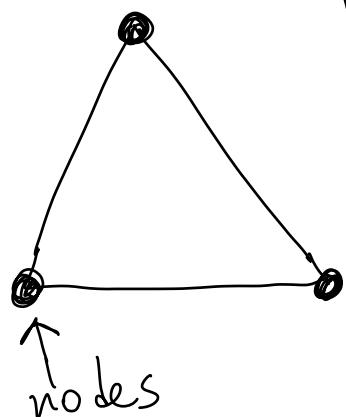
In practice, combine them:



24+1/2

Recall from interpolation lecture notes different elements & nodes in 2D:

① Linear triangles:



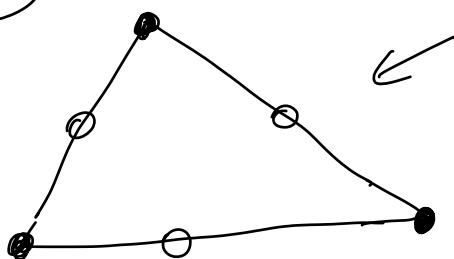
V_h = Space of piecewise linear functions over triangle

Basis functions are tent functions

$$U_i \equiv u(x_i)$$

Functions in discrete space are continuous across edges, i.e., they are continuous on \mathcal{S} .

② Lagrange quadratic triangle

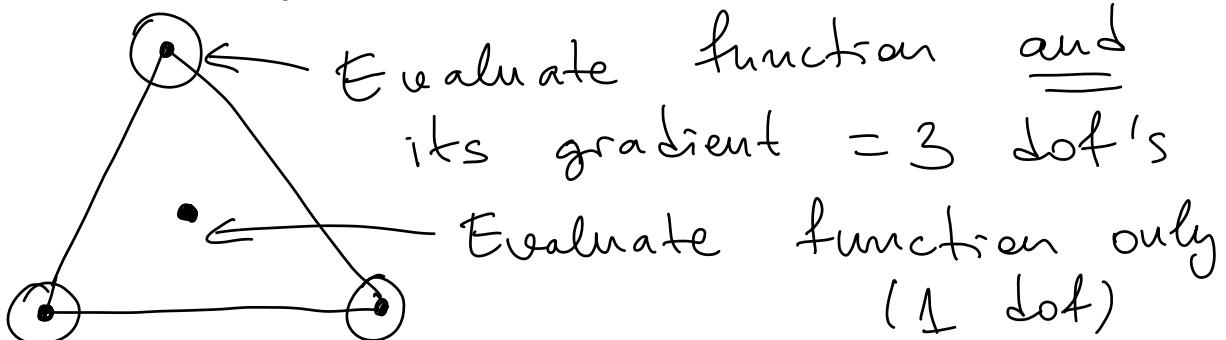


6 nodes, and now functions have continuous tangential derivatives along edges as well

(25)

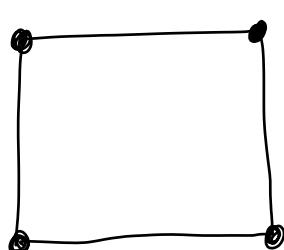
(3) Hermite triangle

$u(x)$ is now cubic on each element (dimension = 10 with basis $\{1, x, y, xy, x^2, y^2, x^3, y^3, x^2y, y^2x\}$)



$$= 3 \times 3 + 1 = 10 \text{ dof's total}$$

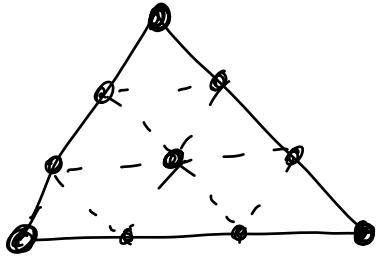
(4) Bilinear rectangle



$$P = \text{span} \{1, x, y, xy\} \\ (4 \text{ dof's})$$

In general use tensor product of polynomials in x and in y ; very simple but not all domains can be meshed with quadrilaterals.

⑤ Lagrange cubic triangle



Note that the global interpolant is still only C^0 since normal derivatives to an edge need not match

Sadly even Hermite triangles are not C^1 ! For higher order equations like biharmonic

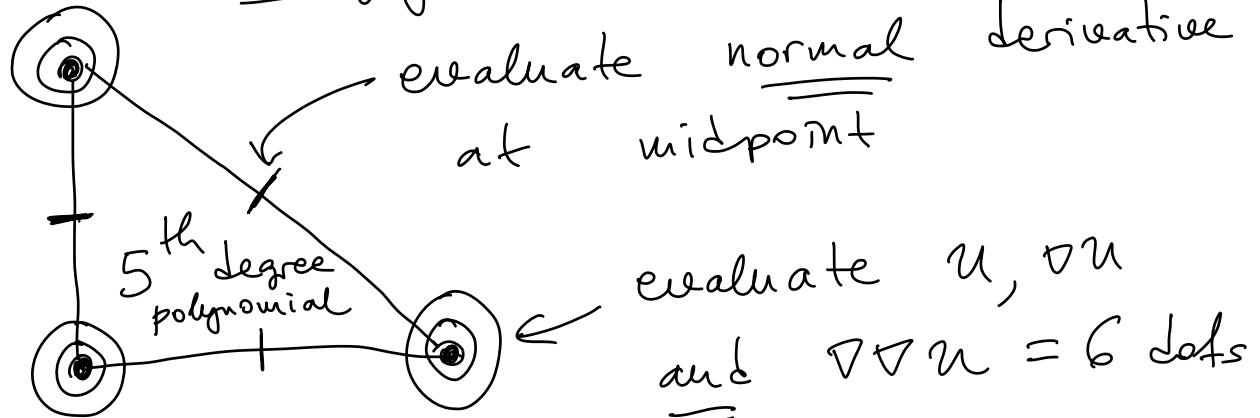
$$\nabla^4 u = f, \quad \nabla u \cdot n = 0 \text{ on } \partial\Omega \\ u = 0 \text{ on } \partial\Omega$$

the suitable space is H_0^2 .

Terminology: If $V_h \subset V$ (typical FEM), the FE approximation is **conforming** (otherwise non-conforming)

What element gives a conforming approximation to biharmonic eq.? ②7

The Argyris triangle



(interpolation error in $H^2(\Omega)$ is
 $\sim h^4 \|u\|_{H^6}$)

$$\# \text{DOFs} = 3 \times 6 + 3 = 21$$

The number of DOFs grows rapidly as one increases the order, and FEM methods can be expensive especially for vector equations.

Another, probably better, alternative is to introduce a new variable

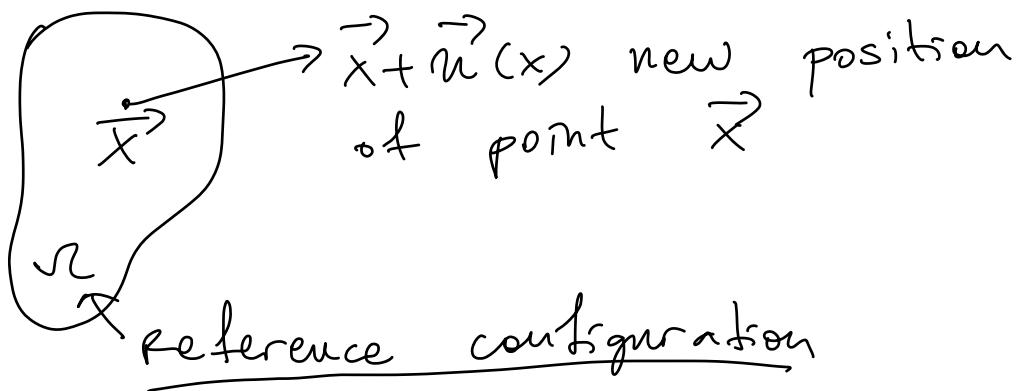
$$\begin{cases} \nabla^2 \psi = w \\ \nabla^2 w = f \end{cases} \rightarrow \text{mixed formulation}$$

Later we will mention
Discontinuous Galerkin (DGS)
as an alternative that avoids
increasing the number of global dots

Linear Elasticity in 2D

Equations are similar in structure
to fluids but variable is

displacement field $\vec{u}(x) \in \mathbb{R}^2$
(not velocity)



Strain tensor (similar to strain rate
for fluids)

$$\overleftrightarrow{\epsilon}(\vec{u}) = \frac{1}{2} (\vec{\nabla} \vec{u} + \vec{\nabla} \vec{u}^T) \in \mathbb{R}^{2 \times 2}$$

Linear elasticity (small deformation)

Stress tensor

$$\overset{\leftrightarrow}{\sigma} = L \overset{\leftrightarrow}{e}$$

$$\sigma_{ij} = L_{ijkl} e_{kl}$$

↓ implied summation

Isotropic material must have

$$L_{ijkl} = \lambda \delta_{ij} \delta_{kh} + \mu (\delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk})$$

$$\begin{cases} \mu > 0 \\ \lambda + 2\mu > 0 \end{cases}$$

↑
Lame' parameters
(like viscosity,
property of solid)

$$\Rightarrow \overset{\leftrightarrow}{\sigma} = 2\mu \overset{\leftrightarrow}{e} + \lambda \underset{\substack{\uparrow \\ \text{trace}}}{\text{Tr}}(\overset{\leftrightarrow}{e}) \overset{\leftrightarrow}{I}$$

Strong form of PDE

$$\vec{\nabla} \cdot \vec{\sigma} = \vec{f} \leftarrow \begin{array}{l} \text{body force} \\ (\text{applied force}) \end{array}$$

BCs are just like for

Navier-Stokes: Specify one

BC for normal direction

(either \vec{u} or $\vec{\sigma} \cdot \vec{n}$) and one

for tangential (either \vec{u} or $\vec{\sigma} \cdot \vec{z}$)

For essential BC (Dirichlet)

$\vec{u}(x) = \vec{0}$ we have an

Energy formulation:

$$\min_{u \in (H_0^1(\Omega))^2} \int_{\Omega} \left(\frac{1}{2} \vec{\sigma} : \vec{e} - \vec{f} \cdot \vec{u} \right) dx$$

\uparrow
 $\sigma_{ij} e_{ij}$ (double contraction)

(31)

Weak formulation

$$\begin{cases} \nabla \cdot \mathbf{v} = f & \text{in } \Omega \\ \mathbf{v} \cdot \mathbf{n} = g & \text{on } \partial\Omega_2 \\ \mathbf{v} = 0 & \text{on } \partial\Omega_1 \end{cases}$$

Find $\mathbf{v} \in \left(\bar{H}_0^1\right)^3$ s.t.

$$a(\mathbf{v}, \varphi) = l(\varphi) \quad \forall \varphi \in \left(\bar{H}_0^1\right)^3$$

$$\mathbf{v}(\partial\Omega_1) = \varphi(\partial\Omega_1) = 0$$

Where as before

$$a(\mathbf{v}, \varphi) = \frac{1}{2} \int_{\Omega} \mathbf{e}(\varphi) : \mathbf{L}(\mathbf{v}) \, dx$$

$$l(\varphi) = \int_{\Omega} f \cdot \varphi \, dx + \int_{\partial\Omega_2} g \cdot \varphi \, ds$$

Using a triangulation with linear hat basis functions

$$\{\gamma_1, \dots, \gamma_{2N}\} = \left\{ \begin{array}{l} \varphi_1 e_1, \varphi_1 e_2, \\ \varphi_2 e_1, \varphi_2 e_2, \dots \end{array} \right\}$$

$$\Rightarrow A_U = F$$

$$U = \left\{ U_1^x, U_1^y, \dots, U_N^x, U_N^y \right\}$$

$$= \left\{ U_1, U_2, \dots, U_{2N} \right\}$$

$$A_{kl} = \int_{\Omega} e(\gamma_k) : L e(\gamma_l) \, dx$$

$$F_k = \int_{\Omega} f \cdot \gamma_k \, dx + \int_{\partial \Omega} g \cdot \gamma_k \, ds$$

Explicit formulas can be obtained for linear triangles for the $(3 \cdot 2)^2 = 6 \times 6$ local stiffness matrix

(33)

Introduce

$$\boldsymbol{\varphi}(u) = \begin{pmatrix} e_{11}(u) \\ e_{22}(u) \\ 2e_{12}(u) \end{pmatrix}$$

$$\Rightarrow e(\boldsymbol{\varphi}) : L\boldsymbol{\varphi}(u) = \boldsymbol{\varphi}^T C \boldsymbol{\varphi} \text{ where}$$

$$C = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

On a triangle

$$\boldsymbol{\varphi}(u_h) = \begin{bmatrix} \varphi_{1x} & 0 & \varphi_{2x} & 0 & \varphi_{3x} & 0 \\ 0 & \varphi_{1y} & 0 & \varphi_{2y} & 0 & \varphi_{3y} \\ \varphi_{1y} & \varphi_{1x} & \varphi_{2y} & \varphi_{2x} & \varphi_{3y} & \varphi_{3x} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_6 \end{bmatrix}$$

$\underbrace{\hspace{10em}}$ R

Local DOFs
on triangle

$$A^K = |K| R^T C R \quad (\text{clearly SPD})$$

(34)

A more tricky example is

Stokes flow

$$\left. \begin{aligned} \nabla p &= -\nabla \cdot \sigma + f && \text{in } \Omega \\ \nabla \cdot u &= 0 \end{aligned} \right\}$$

$$\sigma = -\eta (\nabla u + \nabla u^T), \quad \eta = \text{const}$$

Now \vec{u} is velocity

What we say here also applies
to Navier-Stokes eqs.

Energy formulation:

$$\left. \begin{aligned} \min_{v \in V} \quad & \frac{\eta}{2} \int_{\Omega} \nabla v : \nabla v \, dx + \int_{\Omega} f \cdot v \, dx \\ \text{s.t.} \quad & \nabla \cdot v = 0 \end{aligned} \right\} \quad \begin{aligned} p &\text{ is Lagrange} \\ &\text{multiplier} \\ &\text{for constraint} \end{aligned}$$

Weak form involves different
 Spaces for φ and p : mixed FE.
 Find $u \in (H_0^1(\Omega))^d$, $p \in L^2(\Omega)$
 s.t. \uparrow
 up to a constant

$$\left\{ \begin{array}{l} a(u, \varphi) + b(\varphi, p) = F(\varphi) \\ b(u, q) = 0 \\ \uparrow \\ \text{mixed bilinear form} \\ f(\varphi, q) \in V \times Q \end{array} \right.$$

saddle-point system

$$\begin{pmatrix} A & B^T \\ B & \emptyset \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

Where $BU = 0$ defines the
 kernel space K (discretely 36)
 divergence-free velocity fields

$$a(u, \varphi) = \eta \int_{\Omega} (\nabla u + \nabla^T u) : \nabla \varphi \, dx$$

$$b(\varphi, p) = \int_{\Omega} p (\nabla \cdot \varphi) \, dx$$

Saddle-point system

$$\begin{cases} \varphi \\ \text{---} \end{cases} \quad A u + B^T p = f$$

$$\Rightarrow \varphi^T A u + (B/\varphi)^T p = \varphi^T f$$

zero

$$\Rightarrow \int \varphi^T A u = \varphi^T f$$

$\forall \varphi \in K$

(variational problem on K)

A key feature of Stokes flow
is that the saddle-point system
must be solvable & well-conditioned
as $h \rightarrow 0$

(37)

Mathematically, this is expressed as the inf-sup condition

(also called LBB = Lady Szczesnaya, Brezzi, Babuska):

$$\inf_{q_h \in Q_h} \sup_{\begin{subarray}{l} v_h \in V_h \\ v_h \neq 0 \end{subarray}} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} > \gamma > 0$$

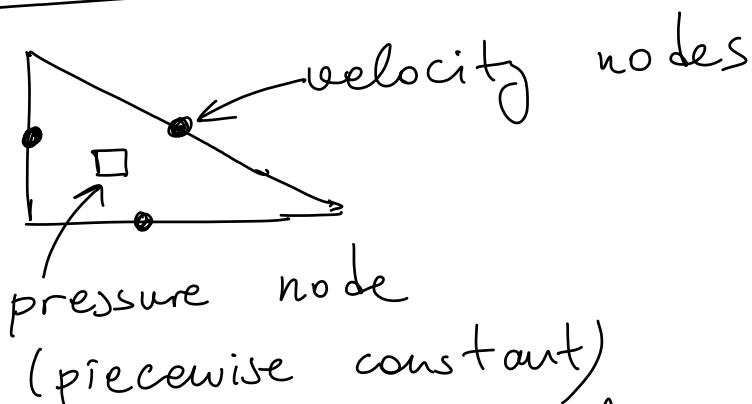
This condition means that pressure space Q_h cannot be "too large", since otherwise there will be some $q_h \in Q_h$ (a spurious or "parasitic mode") that will make the sup be zero.

V_h and Q_h must be chosen together not independently

Using linear triangles for both pressure & velocity is NOT inf-sup stable and does not converge for Stokes. Heuristically, the polynomial degree for pressure should be one lower than velocity.

Examples of stable elements:

Crouzeix - Raviart element

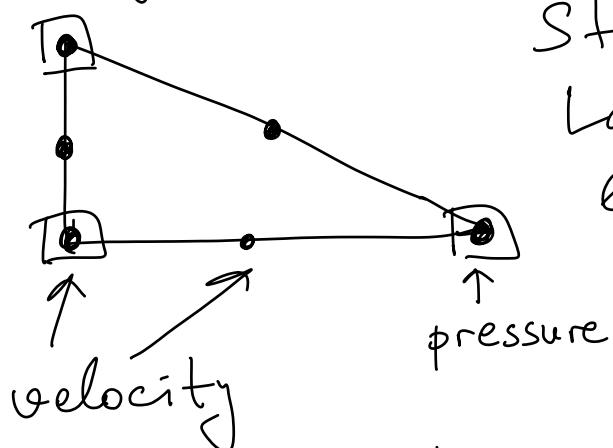


This is a non-conforming element since velocity is not continuous across edges so $V_h \notin H^1$

It is in some sense the equivalent of the MAC or staggered grid for triangles (but first order for velocity)

A more standard stable element is the

Taylor-Hood element



Standard quadratic for velocity,
Lagrange linear for pressure

Already quite a bit more expensive than MAC!

Also stable is V_h order $k \geq 2$ polynomial for cells, pressure degree $k-2$ discontinuous