Addendum: Fromm method with source & non-constant advection

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$$u_t = -(a(x)u)_x + s(x)$$

$\uparrow$ constant in time

We will follow the approach of extrapolating state to faces at midpoint in time:

assume $a_{j+1/2} > 0$

$$u^{n+1/2}_{j+1/2} = \overline{u}^n_j + \frac{\Delta x}{2}\left(u_x^n\right)_j + \frac{\Delta t}{2}\left(u_t^n\right)_j$$

And here we will use chain rule + PDE to estimate u_t^n:

$$u_t = -\underbrace{a u_x}_{\text{use } a \text{ at face}} - \underbrace{u a_x}_{\text{easy to get at cell centers}} + s$$

$$\left(u_t^n\right)_j = -a_{j+1/2} \underbrace{\left(\frac{u_{j+1} - u_{j-1}}{2h}\right)}_{\text{centered slopes}} - u_j\left(\frac{a_{j+1/2} - a_{j-1/2}}{h}\right) + S_j$$

Advective flux estimate:

$$F^{adv}_{j+1/2} = a_{j+1/2}\, u^{n+1/2}_{j+1/2}$$

Homework: Implement method and confirm second-order in space-time.

Remember that for advection-diff.

$$S_j^n = \frac{d}{h^2}\left(u^n_{j+1} - 2u_j + u^n_{j-1}\right)$$

comes from diffusion. For constant

$$a(x) = a = \text{const}$$

this approach gives an extra term:

$$u_j^{n+1} = u_j^n - \tau a \left(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2} \right)$$

$$= \text{usual Fromm scheme}$$

$$- \frac{a d \tau^2}{2} \underbrace{\left(\frac{u_{j+1} - 3u_j + 3u_{j-1} - u_{j-2}}{h^3} \right)}_{\text{Upwind difference for } (u_{xxx})_j}$$

to be compared to Lax-Wendroff scheme:

$$- \frac{a d \tau^2}{2} \left(\frac{u_{j+2} - 2u_{j+1} + 2u_{j-1} - u_{j-2}}{h^3} \right)$$

which is centered (no u_j!) and leads to spurious stability limit