

Numerical Methods II, Spring 2019

Spectral Methods

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1 Spectral methods for periodic KdV

These are excerpts from student solutions to the first two homeworks.

1.1 interpft

Here N is number of nodes for Fourier approximation, and $M > N$ is number of fine nodes and we allow for arbitrary L .

We often consider the Fourier interpolation in the standard interval 2π , for now, we need to transfer to general length of interval L . The Fourier interpolation of $\phi(x)$ can be written as

$$\tilde{\phi}(x) = \begin{cases} \hat{\phi}_0 + \sum_{0 < k < \frac{N-1}{2}} \left(\hat{\phi}_k \exp(i \frac{2\pi}{L} kx) + \hat{\phi}_{N-1-k} \exp(-i \frac{2\pi}{L} kx) \right), & N \text{ odd}, \\ \hat{\phi}_0 + \sum_{0 < k < \frac{N}{2}} \left(\hat{\phi}_k \exp(i \frac{2\pi}{L} kx) + \hat{\phi}_{N-k} \exp(-i \frac{2\pi}{L} kx) \right) + \hat{\phi}_{N/2} \cos(\frac{2\pi}{L} \frac{N}{2} x), & N \text{ even}, \end{cases} \quad (1)$$

where N is the number of interpolation points and $\hat{\phi}_k$ are Fourier coefficients evaluated at interpolation grids $\frac{Lj}{N}$, $j = 0, \dots, N-1$.

$$\hat{\phi}_k = \frac{1}{N} \sum_{j=0}^{N-1} \phi(\frac{Lj}{N}) \exp(-i \frac{2\pi}{L} k \frac{Lj}{N}) = \frac{1}{N} \sum_{j=0}^{N-1} \phi_j \exp(-i \frac{2\pi jk}{N}), \quad (2)$$

which stays the same as in the standard 2π interval.

On finer grids, we want to estimate our interpolation at $x_j = \frac{Lj}{M}$ $j = 0, \dots, M-1$, $M \gg N$. The estimation can be written as

$$\tilde{\phi}(x_j) = \begin{cases} \hat{\phi}_0 + \sum_{0 < k < \frac{N-1}{2}} \left(\hat{\phi}_k \exp(i \frac{2\pi j}{M} k) + \hat{\phi}_{N-1-k} \exp(-i \frac{2\pi j}{M} k) \right), & N \text{ odd}, \\ \hat{\phi}_0 + \sum_{0 < k < \frac{N}{2}} \left(\hat{\phi}_k \exp(i \frac{2\pi j}{M} k) + \hat{\phi}_{N-k} \exp(-i \frac{2\pi j}{M} k) \right) + \hat{\phi}_{N/2} \cos(\frac{2\pi j}{M} \frac{N}{2}), & N \text{ even}, \end{cases} \quad (3)$$

which is still the same as in the standard 2π interval.

For the special $\frac{N}{2}$ mode when N is even, we can write

$$\hat{\phi}_{N/2} \cos(\frac{2\pi j}{M} \frac{N}{2} x) = \frac{\hat{\phi}_{N/2}}{2} \exp(i \frac{2\pi j}{M} \frac{N}{2}) + \frac{\hat{\phi}_{N/2}}{2} \exp(-i \frac{2\pi j}{M} \frac{N}{2}), \quad (4)$$

i.e separating the $\hat{\phi}_{N/2}$ into two halves and giving one half to the matched mode of $N/2$ in the finer grids.

So the routine of our Fourier Interpolations on a length- L interval is:

- If N is odd,

1. Use **fft** on the N interpolation points to obtain Fourier coefficients $\hat{\phi}_0, \dots, \hat{\phi}_{N-1}$;
 2. Apply oversampling to the Fourier coefficients
 $(\hat{c}_k)_{k=1}^N = (\hat{\phi}_0, \hat{\phi}_1, \dots, \hat{\phi}_{(N-1)/2}, 0, \dots, 0, \hat{\phi}_{(N-1)/2}, \dots, \hat{\phi}_{n-1})$;
 3. Apply **ifft** to $(\hat{c}_k)_{k=1}^N$, obtain the interpolation estimated on fine grids.
- If N is even,
 1. Use **fft** on the N interpolation points to obtain Fourier coefficients $\hat{\phi}_0, \dots, \hat{\phi}_{N-1}$;
 2. Apply oversampling to the Fourier coefficients
 $(\hat{c}_k)_{k=1}^N = (\hat{\phi}_0, \hat{\phi}_1, \dots, \hat{\phi}_{N/2-1}, \frac{\hat{\phi}_{N/2}}{2}, 0, \dots, 0, \frac{\hat{\phi}_{N/2}}{2}, \hat{\phi}_{N/2+1}, \dots, \hat{\phi}_{N-1})$;
 3. Apply **ifft** to $(\hat{c}_k)_{k=1}^N$, obtain the interpolation estimated on fine grids.

1.2 Spectral Differentiation

Here n is number of nodes for Fourier approximation, and $N > n$ is number of fine nodes, and we assume $L = 2\pi$.

And for even n , there is unmatched mode corresponding to frequency $n/2$, so we use minimal oscillation trigonometric interpolation,

$$\phi(x) = \hat{f}_0 + \sum_{0 < k < \frac{n}{2}} \left(\hat{f}_k e^{ikx} + \hat{f}_{n-k} e^{-ikx} \right) + \hat{f}_{n/2} \cos\left(\frac{nx}{2}\right), \quad (5)$$

where $\hat{f}_0, \dots, \hat{f}_{n-1}$ are n Fourier coefficients and can be obtained by **fft**. For the interpolation $\phi(x)$, we differentiate the form (5),

$$\begin{aligned} \phi'(x) &= \left(\hat{f}_0 + \sum_{0 < k < \frac{n}{2}} \left(\hat{f}_k e^{ikx} + \hat{f}_{n-k} e^{-ikx} \right) + \hat{f}_{n/2} \cos\left(\frac{nx}{2}\right) \right)' \\ &= \sum_{0 < k < \frac{n}{2}} \left(\hat{f}_k e^{ikx} \cdot ik + \hat{f}_{n-k} e^{-ikx} \cdot (-ik) \right) - \hat{f}_{n/2} \sin\left(\frac{nx}{2}\right) \cdot \frac{n}{2} \\ &= \sum_{0 < k < \frac{n}{2}} \left(\hat{f}_k e^{ikx} \cdot ik + \hat{f}_{n-k} e^{-ikx} \cdot (-ik) \right) + \hat{f}_{n/2} \frac{1}{2} \left(e^{i\frac{n}{2}x} \cdot i\frac{n}{2} + e^{i(-\frac{n}{2})x} \cdot (-i\frac{n}{2}) \right), \end{aligned} \quad (6)$$

which is very similar to our implementation of Fourier interpolation, just needs vector multiplication before doing the inverse Fourier transform.

The routine of approximating derivatives on a finer grid (upsampled) for an **even grid** is:

1. Use **fft** on the n interpolation points to obtain Fourier coefficients $\hat{f}_0, \dots, \hat{f}_{n-1}$;
2. Multiply the Fourier coefficients with the corresponding frequency (we do **not** zero out the special mode, but we would do this if $N = n!$)

$$\hat{\phi}'_k = \hat{f}_k \left(\frac{2\pi}{L} \right) \begin{cases} ik, & k \leq n/2, \\ -i(n-k), & k > n/2; \end{cases}$$

3. Apply oversampling to N Fourier coefficients

$$(\hat{c}_k)_{k=1}^N = (\hat{\phi}'_0, \hat{\phi}'_1, \dots, \hat{\phi}'_{n/2-1}, \frac{\hat{\phi}'_{n/2}}{2}, 0, \dots, 0, -\frac{\hat{\phi}'_{n/2}}{2}, \hat{\phi}'_{n/2+1}, \dots, \hat{\phi}'_{n-1})$$

4. Apply **ifft** to $(\hat{c}_k)_{k=1}^N$, obtain the derivative of interpolation estimated on fine grids.

Since we still use **fft**, **ifft**, just add an vector multiplication, so our cost is still $O(N \log(N))$.

1.3 KdV Equation

We now want to discretize in space:

$$\partial_t \phi = \mathcal{K}[\phi(\cdot, t)] = \frac{-\partial_{xxx} \phi - 3\partial_x(\phi^2)}{2},$$

where $\mathcal{K}[\phi(\cdot, t)]$ denotes the functional on the right hand side, which only involves derivatives of x . The r.h.s of the KdV equation in Fourier space as

$$\widehat{\mathcal{K}[\phi(\cdot)]} = \mathcal{F}(\mathcal{K}[\phi(\cdot)]) = \mathbf{F}(\hat{\phi}) = ik^3 \boxdot \hat{\phi} - 3i\mathbf{k} \boxdot \hat{\mathbf{w}},$$

where

$$\hat{\mathbf{w}} = \mathcal{F}\left(\left(\mathcal{F}^{-1}\hat{\phi}\right)^{\boxed{2}}\right).$$

Here N is number of nodes for Fourier approximation, and $M > N$ is number of fine nodes.

To compute $\mathcal{K}[\phi(\cdot, t)]$ when we are given either only $\hat{\phi}$ or both $\hat{\phi}$ and ϕ (code design to think about) for even-sized grids:

1. Set a larger size $N' = 2N$ (or more efficient but also works $N' = 3N/2$) if anti-aliasing, or $N' = N$ if not, and conduct the following step:

$$\hat{\mathbf{w}} = \frac{N'}{N} \text{truncate} \left[\mathcal{F} \left(\left(\mathcal{F}^{-1}(\text{zero-pad}(\hat{\phi})) \right)^{\boxed{2}} \right) \right]. \quad (7)$$

where **zero-pad** is pad the N Fourier modes to N' and **truncate** is truncate to save only the first N Fourier modes. Scaling factor depends on FFT conventions (to think about). *What to you do with the unmatched Fourier mode when you truncate?*

2. Multiply the Fourier coefficients with the corresponding frequency to obtain $\widehat{\mathcal{K}[\phi(\cdot)]}$ (note that again we do **not** zero out the $N/2$ mode unless $M = N$)

$$F_k = \begin{cases} i(\frac{2\pi}{L}k)^3 \hat{\phi}_k - 3i\frac{2\pi}{L}k \hat{\phi}_k^2, & k \leq N/2, \\ i(-\frac{2\pi}{L}(N-k))^3 \hat{\phi}_k - 3i\frac{2\pi}{L}(N-k) \hat{\mathbf{w}}_k, & k > N/2, \end{cases}$$

3. Apply oversampling to M points:

$$(\hat{c}_k)_{k=1}^N = (F_0, F_1, \dots, F_{N/2-1}, \frac{F_{N/2}}{2}, \dots, -\frac{F_{N/2}}{2}, F_{N/2+1}, \dots, F_{N-1});$$

4. Apply **ifft** to $(\hat{c}_k)_{k=1}^N$, obtain the approximation $\mathcal{K}[\phi(\cdot, t)]$ estimated on fine grids.