

Computational PDEs, Fall 2021

Second Order Finite Volume Methods for Advection-Diffusion Equations

Aleksandar Donev

Courant Institute, NYU, donev@courant.nyu.edu

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Due Sunday **October 17th 2021**

All of the notes for the previous homework continue to apply, so read them again as needed.

1 Advection-Diffusion in Periodic Domains

Consider numerically solving the advection-diffusion equation

$$u_t + (a(x, t)u)_x = (d(x)u_x)_x,$$

on the periodic domain $0 \leq x < 1$, for a *smooth* initial condition

$$u(x, 0) = [\sin(\pi x)]^p,$$

on a uniform grid [Note: You can use the fact the initial condition is “smooth” if desired, but not its specific form, that is, in your code one should be able to change a line or two to change the initial condition.], where p is an exponent.

For $p = 2$ the initial condition is quite smooth even on coarse grids, but for $p = 100$ it is only smooth on finer grids. You should feel encouraged to try less smooth initial conditions such as square waves as well.

1.1 Variable coefficients

Make the advection speed and the diffusion coefficient non-constant, though quite “smooth”,

$$a(x, t) = a_0 \cos(t) \left(\frac{3}{4} - \frac{1}{4} \sin(2\pi x) \right).$$

$$d(x) = d_0 (2 + \cos(2\pi x)).$$

By default, use $a_0 = 1$ and $d_0 = 0.001$, but it is important to vary d_0 to vary the relative importance of advection versus diffusion. It is advised to validate/test the advection and diffusion pieces *separately*.

Develop a **finite volume** scheme to solve the PDE to time $T = 1$. Your goal is accomplish the following things, either with a MOL or a space-time scheme:

1. Treat advection explicitly but diffusion implicitly, with the goal that the time step is only limited by advection and not by diffusion, that is, the scheme is stable if $\tau = Ch / \|a\|_\infty$ where $C = O(1)$ *independent* of the value of the diffusion coefficient. Report whether you think you accomplished that goal and why.
2. Achieve at least 2nd order under space-time refinement and confirm that order unambiguously in all three norms for a smooth periodic solution.
3. Confirm that asymptotically for a refined grid the error in the solution $\rightarrow Kh^2 e(x)$ where $e(x)$ is some function, and provide an estimate of the actual error (difference between true solution and your approximation) for one of the grid sizes (usually one of the finer grids).
4. Validate that your scheme converges to the correct solution of the non-constant coefficient PDE at second-order in space-time, using the method of manufactured solutions, at least for a periodic domain.
5. Test the scheme on less smooth solutions and explain what sort of numerical artifacts arise in your method.

1.2 Boundary conditions

Now solve the PDE with boundary conditions

$$\begin{aligned}u(0, t) &= \sin^p(-\pi t) \\ u_x(1, t) &= 0 \quad \text{if } d_0 > 0.\end{aligned}$$

Observe that if $d = 0$ the exact solution here is $u(0, t) = \sin^p(\pi(x - t))$, which is the same as for periodic conditions.

Develop a finite-volume method (choose the advective/diffusive stencils, the boundary condition treatment, number of grid points, etc., and *explain your choices*) to solve the equation up to time $T = 1$.

1. For $p = 100$ and constant advection $a(x, t) = a_0$, show the solution at several times up to time T for $d_0 = 0.1$, $d_0 = 0.01$, $d_0 = 0.001$, and $d_0 = 0$, and comment on your observations. For your first submission you can restrict your code/results to $d_0 = 0$ if you run out of time, but the final submission should handle diffusion as well.
2. Discuss what happens as $d \rightarrow 0$ with the PDE and its true analytical solution, and compare to ~~your~~ what happens in your numerical method solution. Is your method *consistent* with the PDE for $d_0 = 0$? If it is, what is the order of convergence for $d_0 = 0$?
3. ~~Empirically~~ Determine the spatio-temporal order of convergence for your discretization for both zero and finite d_0 (explain what value you chose and why) in different norms, for a smooth solution. If you find it necessary, you can use constant advection speed for this part (certainly useful for initial testing). Does the observed order ~~it~~ agree with theoretical expectations (explain what your expectation is)?
4. (Optional) Repeat all the above steps with a Dirichlet condition on the right, $u(1, t) = 0$ and comment on the differences with the Neumann BC.