MAC for Incompressible Navier-Stokes Flow

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The complete Navier Stokes equation, i.e., the momentum conservation equation, valid for inhomogeneous fluids as well, is:

$$\underline{\text{momentum density}} = \underset{\substack{\uparrow \\ \text{mass density}}}{\rho} \, \vec{u} \leftarrow \text{fluid velocity}$$

$$(\rho \vec{u})_t + \vec{\nabla} \cdot (\rho \vec{u} \vec{u}^T) + \nabla p = \nabla \cdot (\overleftrightarrow{\sigma}_v) + \rho \vec{g}$$

pressure: p; (e.g.) gravity: $\vec{g}$; viscous stress tensor: $\overleftrightarrow{\sigma}_v$

1

This needs to be supplemented with (at least) mass conservation:

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0$$

(continuity equation)

Equivalent form of momentum eq:

$$\rho \vec{u}_t + \rho \vec{u} \cdot \nabla \vec{u} = \nabla \cdot (\vec{\sigma}_v) + \rho \vec{g}$$

Where viscous stress tensor is:

$$\overleftrightarrow{\sigma}_v = \eta \left(\vec{\nabla} \vec{u} + \vec{\nabla}^T \vec{u} \right)$$

$$\sigma^{(v)}_{ij} = \eta \left(\partial_i u_j + \partial_j u_i \right)$$

Often one can make two key

assumptions: $\begin{cases} \rho = \text{const} \leftarrow \text{means flow is Low Mach number} \\ \eta = \text{const} \end{cases}$

constant viscosity

This allows for some key simplifications of the equations:

$$\nabla\cdot(\rho\vec{u}) = \rho\,\nabla\cdot\vec{u} = 0 \Rightarrow \nabla\cdot\vec{u} = 0$$

(incompressible flow)

This leads to more simplifications:

$$\nabla\cdot\overleftrightarrow{\sigma} = \eta\,\nabla^2\vec{u}$$

why: $\partial_j \sigma_{ji} = \eta\,\partial_j^2 u_i + \eta\,\partial_i \underbrace{(\partial_j u_j)}_{\text{zero}}$

$$\begin{cases} \rho(\vec{u}_t + \vec{u}\cdot\vec{\nabla}\vec{u}) + \nabla p = \eta\,\nabla^2\vec{u} \\ \nabla\cdot\vec{u} = 0 \end{cases}$$

pressure is the Lagrange multiplier for incompressibility

Challenge:

1) $\vec{u}\cdot\vec{\nabla}\vec{u}$ is <u>nonlinear</u>

2) Incompressibility <u>constraint</u>

Formally, NS equations are a differential-algebraic (DAE) system of equations of index 2.
This is hard even for ODEs!

In periodic / unbounded domains, we can eliminate pressure using a projection operator. Start with Hodge decomposition theorem:

$$\forall \vec{v} = -\vec{\nabla}\varphi + \vec{\nabla} \times \vec{A}$$
$$= -\vec{\nabla}\varphi + \vec{u}$$

s.t. $\vec{\nabla} \cdot \vec{u} = 0$

$$\vec{u} = \overleftrightarrow{\mathcal{P}}\, \vec{v} = \vec{v} + \vec{\nabla}\varphi$$

projection operator

How to compute φ ?

$$\vec{\nabla} \cdot \vec{u} = \nabla \cdot \vec{\vartheta} + \nabla^2 \varphi = 0$$

$$\Rightarrow \quad -\nabla^2 \varphi = \nabla \cdot \vec{\vartheta} \Leftarrow$$ Poisson equation for pressure

(Solving the incompressible equations is always at least as hard as solving a Poisson equation (elliptic), so generally much more expensive than simple advection-diffusion).

In periodic/unbounded domains we can write NS in unconstrained form (c.f. vorticity-stream):

$$u_t = \mathcal{P}\left[-u \cdot \nabla u + \nu \nabla^2 u + f \right]$$

$\nu = \eta / \rho$ gravity/wind/etc.

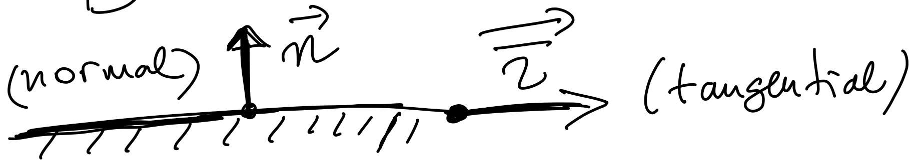
The problem is that this does NOT work with boundary conditions!
First methods for NS were based on the projection method of A. Chorin, but I will present a different version that deals correctly with BCs.

Key point to remember:
One cannot seperate pressure & velocity when there are physical boundaries: must solve for both at the same time.
In other words, we will need to solve something like Stokes equation not Poisson equation to get evolution of v/p.

BCs for incompressible NS

At a physical boundary, the following BCs are common.

$\vec{u} = u_n + \vec{u}_\tau$



Normal component (pick one)
(also for Euler eqs)

① No penetration (Dirichlet)

$u_n = \vec{u} \cdot \vec{n} = 0$ ← normal vector

② Specified normal stress :

(Neumann) $\vec{n} \cdot \overleftrightarrow{\sigma} \cdot \vec{n} = -p + 2\eta \frac{\partial}{\partial n} u_n$

Tangential component (pick one)
(only if $\eta > 0$)

① Specified slip (or no slip)

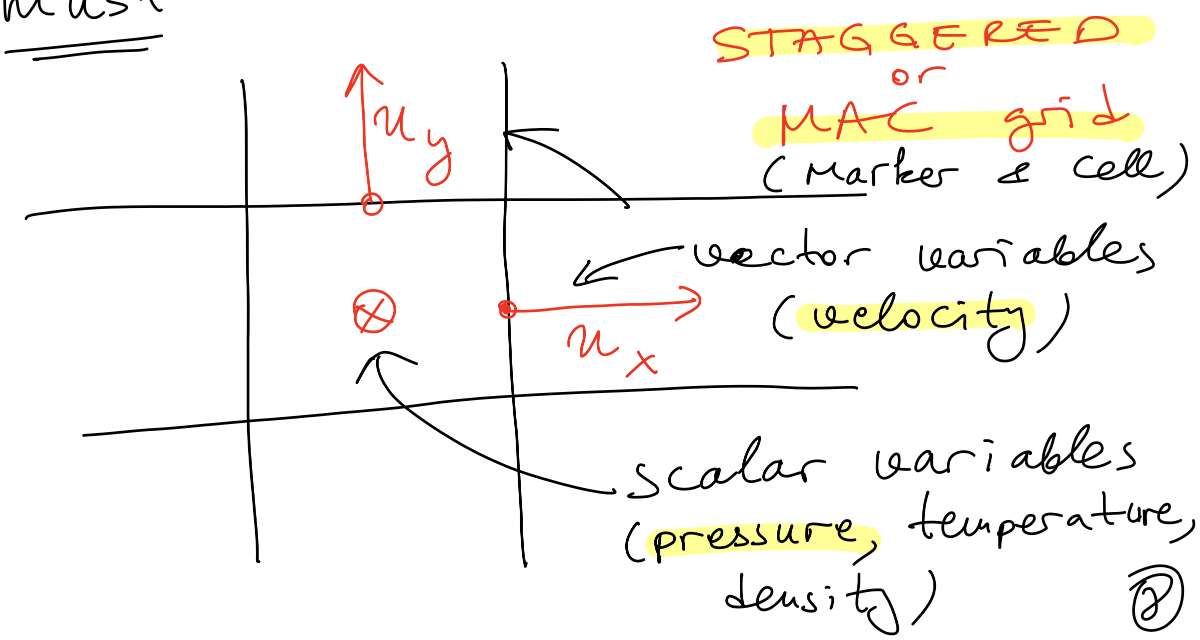
$\vec{u}_\tau = \vec{n} - \vec{u} \cdot \vec{n} = 0$ (Dirichlet)

② Specified traction

$\vec{\tau} \cdot \overleftrightarrow{\sigma} \cdot \vec{n} = \eta \left[\frac{\partial \vec{u}_\tau}{\partial n} + \frac{\partial u_n}{\partial \vec{\tau}} \right]$

Numerical Method: MAC

What I will describe can be viewed as a finite difference but also (discontinuous Galerkin) finite element method. It is the simplest 2nd order method but only works for regular grid. I will illustrate in 2D but also works in 3D. Domain must be an orthogonal cuboid.



Scalars "live" on a standard finite volume/difference grid:

$\nabla \cdot \vec{v}$ or pressure $\longrightarrow$ cell centered

but vector fields like velocity have different components "living" on distinct grids

velocity $\longrightarrow$ face centered

Velocities are on a <u>staggered grid</u>.

<u>Finite difference MAC</u> operators/matrices

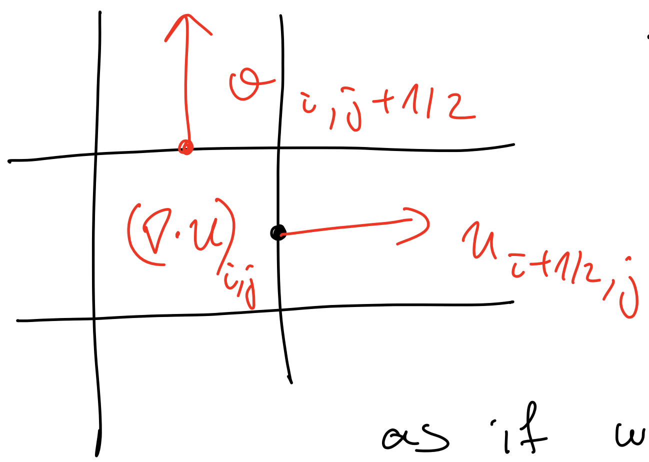
$\vec{u} = (u, v)$

1 $\quad \nabla \cdot u \longleftrightarrow D u$

discrete divergence

$$(Dv)_{i,j} = \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x}$$

$$- \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y}$$



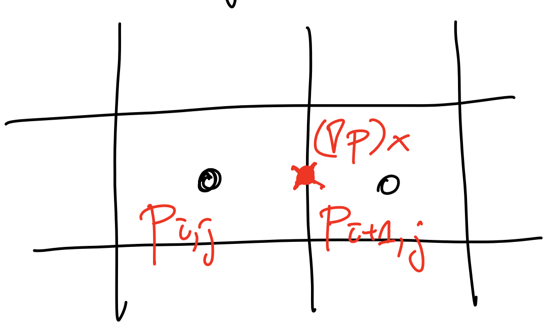
This really is a divergence of a flux through faces, exactly as if we were doing finite volume schemes for the scalar fields

② $\nabla p \longleftrightarrow Gp$

Discrete gradient

$$(Gp)^{x}_{i+1/2,j} = \frac{p_{i+1,j} - p_{i,j}}{\Delta x}$$

$$(Gp)^{y}_{i,j+1/2} = \frac{p_{i,j+1} - p_{i,j}}{\Delta y}$$



$D = -G^{T}$ *

adjoint

(3) Scalar / vector Laplacian is standard 5pt Laplacian, e.g.

$$(L_c p)_{i,j} = \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{\Delta x^2}$$

$$+ \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{\Delta y^2}$$

Important : $L_c = DG$

Similarly

$$(L^x u)_{i+1/2,j} = \frac{u_{i+3/2,j} - 2u_{i+1/2,j} + u_{i-1/2,j}}{\Delta x^2}$$

$$+ \frac{u_{i+1/2,j+1} - 2u_{i+1/2,j} + u_{i+1/2,j-1}}{\Delta y^2}$$

With these operators we can write a discrete $\underline{NS}$ equation

$$\begin{cases} \rho \dfrac{d\vec{u}}{dt} + G p = \eta \begin{pmatrix} L_x u \\ L_y v \end{pmatrix} - \vec{N}(\vec{u}) \\ D\vec{u} = 0 \end{cases}$$

where $\vec{N}(\vec{u}) \leftrightarrow \vec{u} \cdot \vec{\nabla} \vec{u}$

(DAE system of index 2)

<u>Things we need</u>:

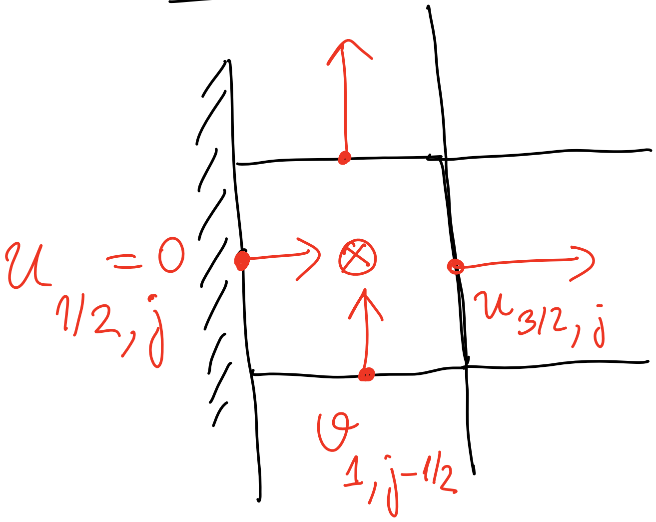
a) Temporal integrator

b) How to discretize $\vec{N}(\vec{u})$

c) Boundary conditions

The "easiest" of these is BCs, <u>especially</u> for Dirichlet BCs on $\vec{u}$ (e.g. no slip / no penetration)

No-slip BCs

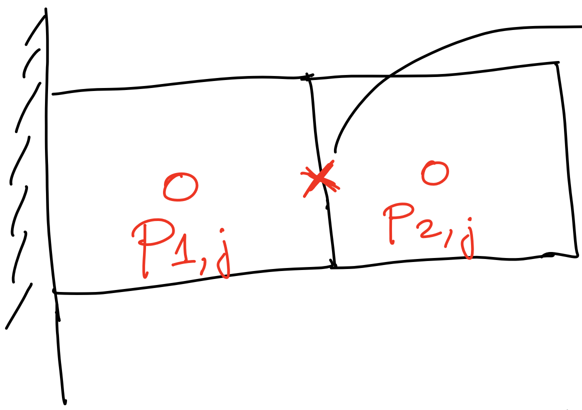


Discrete Divergence is easy to write:

$$(Dv)_{1,j} = \frac{u_{3/2,j} - \overset{\text{zero (no slip)}}{\cancel{u_{1/2,j}}}}{\Delta x} + \frac{v_{1,j+1/2} - v_{1,j-1/2}}{\Delta y}$$

Since normal velocity is given (zero for no-slip), the discrete divergence D maps from velocity on <u>interior</u> faces to cell centers.

The **discrete gradient** $G = -D^*$ therefore maps from cell centers to interior faces.



$$(Gp)_{3/2,j} = \frac{p_{2,j} - p_{1,j}}{\Delta x}$$

We do not need ghost cells to define either D or G – this is the real beauty of the staggered grid and why it is capturing the physics of incompressible flow.

$$L_c = DG$$

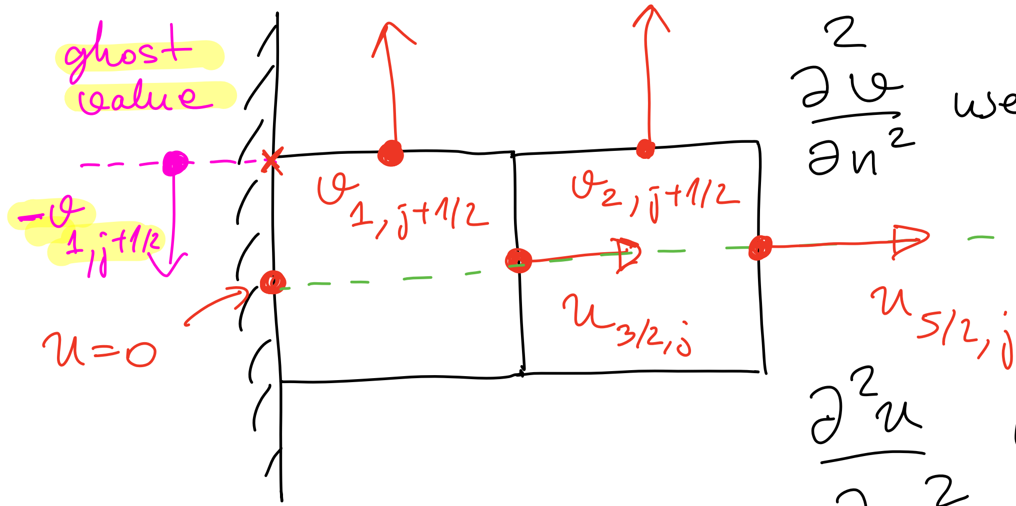
looks like a discrete Laplacian with **Neumann BCs for pressure**

$$\frac{\partial p}{\partial n} = 0 \quad \text{at impermeable boundary}$$

But this is misleading! Viscosity couples pressure and velocity at boundaries. The MAC grid shows that no BCs are needed for the pressure at no-slip walls.

For the discrete viscous Laplacian, we can reuse the things we did in 1D for advection - diffusion.

The idea is that perpendicular velocity is on a finite-difference (non-staggered) grid, but parallel velocity is on a finite-volume (staggered) grid.



$\frac{\partial^2 v}{\partial n^2}$ uses ghost cells

$\frac{\partial^2 u}{\partial n^2}$ uses BCs directly

This is done in the same way as if the x direction were the only dimension.

Corners are not that hard but they can lead to singularities (driven lid cavity test)

Stress-based BCs can also be handled but they are less natural than no slip – see article by Boyce Griffith linked on course webpage

Advection

First the "easy" case. Often the velocity advects other quantities like concentration or temperature, assumed here to be scalar (even if multi-component) fields:

$$\partial_t c + u \cdot \nabla c = D \nabla^2 c$$

This is exactly the finite volume setup we studied in 1D for advection - diffusion equations:

u is a face - centered velocity
and c is a cell - centered conserved quantity

One can generalize 2nd order Godunov methods to 2D/3D (good final project)

But what about $u \cdot \nabla u$?
One cannot do the extrapolation to midpoint in time for this term due to the staggered grid (it has been done for cell-centered velocities but this has other issues like BCs).
So we will do MOL, with second order in space finite-difference discretization.
This is not great for high Reynolds number flows, but really is that I know nothing of. So lets do FD MOL 2nd order:

$$u \cdot \nabla u = \begin{array}{cc} \begin{bmatrix} (u^2)_x + (uv)_y \\ \hline (uv)_x + (v^2)_y \end{bmatrix} & \begin{array}{l} \leftarrow u \\ \\ \leftarrow v \end{array} \end{array}$$

Idea: Place u^2 and v^2 on <u>cell centers</u> and uv on grid <u>corners</u> (In 3D, there are both corners/nodes & edges)

$$\left[(u^2)_x\right]_{i+1/2,j} = \frac{(u^2)_{i+1,j} - (u^2)_{i,j}}{\Delta x}$$

$$\left[(uv)_y\right]_{i+1/2,j} = \frac{(uv)^{(y)}_{i+1/2,j+1/2} - (uv)^{(y)}_{i+1/2,j-1/2}}{\Delta y}$$

To improve stability, we <u>upwind</u> to define u^2, v^2, and uv:

$$\left[u^2\right]_{i,j} = \bar{u}_{i,j} \cdot \underbrace{u_{i-1/2,j}}_{\text{upwind}}$$

if $\bar{u}_{i,j} \geq 0$

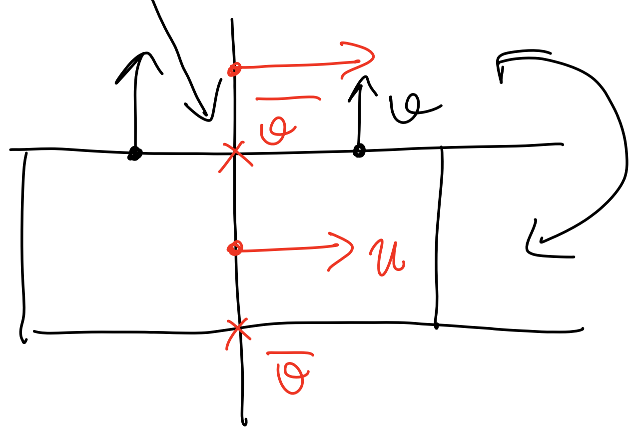
$$\bar{u}_{i,j} = \frac{u_{i+1/2,j} + u_{i-1/2,j}}{2}$$

The notion here is to <u>upwind</u> in the direction of the derivative:

$$(uv)^y_{i+1/2,\,j+1/2} = \overline{v}_{i+1/2,\,j+1/2} \cdot \underbrace{u_{i+1/2,\,j}}_{\text{upwind}}$$

$$\text{if} \quad \overline{v}_{i+1/2,\,j+1/2} \geq 0$$

$$\overline{v}_{i+1/2,\,j+1/2} = \frac{v_{i+1,\,j+1/2} + v_{i,\,j+1/2}}{2}$$



choose upwind value

For low Re numbers one may prefer centered discretizations, which are non dissipative but are dispersive.

One can combine in code:

$$(u^2)_{i,j} = \underbrace{\frac{u_{i+1/2,j} + u_{i-1/2,j}}{2}}_{\bar{u}_{i,j}} \left(\frac{1 \mp \gamma}{2} u_{i+1/2,j} + \frac{1 \pm \gamma}{2} u_{i-1/2,j} \right)$$

Choose + sign if $\bar{u}_{i,j} \geq 0$

$\gamma = 0$ for centered

$\gamma = 1$ for upwinding

Adaptive γ would be a "poor person's" limiter.

There are better ways but none are very natural on the staggered grid.

Time Stepping

The last piece is how to integrate in time. The standard/old approach has been to use **projection methods**, but I suggest a better alternative that is equally efficient (I learned from Boyce Griffith).

We want:

$$\begin{cases} \rho \dfrac{u^{n+1} - u^{n}}{\Delta t} + \nabla p^{n+1/2} = - \left[\rho u \cdot \nabla u \right]^{n+1/2} \\ \text{Crank-Nicolson} \longrightarrow + \eta \nabla^2 \left(\dfrac{u^{n} + u^{n+1}}{2} \right) \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

This is a so-called saddle-point linear system:

$$
\left[\begin{array}{c|c} \frac{\rho}{\Delta t} - \frac{\eta}{2} L & G \\ \hline -D & 0 \end{array}\right]
\left[\begin{array}{c} u^{n+1} \\ \hline p^{n+1/2} \end{array}\right]
=
\left[\begin{array}{c} \left(\frac{\rho}{\Delta t} + \frac{\nu}{2} L_v\right) u^{n} - \left[\rho u \cdot \nabla u\right]^{n+1/2} \\ \hline 0 \end{array}\right]
$$

saddle-point

For Stokes flow, $Re \to 0$, use L-stable backward Euler:

$$
\left[\begin{array}{c|c} -\eta L & G \\ \hline -D & 0 \end{array}\right]
\left[\begin{array}{c} u^{n+1} \\ \hline p^{n+1} \end{array}\right]
=
\left[\begin{array}{c} \text{rhs} \\ \hline 0 \end{array}\right]
$$

How do we solve this saddle-point linear system ?

(Harder than solving a Poisson equation for pressure but similar)

Use GMRES with a smart preconditioner (see webpage)

(MULTIGRID method, more on this once we cover FEM)

One can prove (mostly in FEM literature) that the # of GMRES iterations is roughly independent of grid size if the preconditioner is chosen smartly.

For high Re numbers one can approximate the GMRES solution to second-order accuracy using a

Projection Method

(1) Time lag pressure and solve for u^{n+1} only:

$$\begin{cases} \rho \dfrac{u^{n+1,*} - u^n}{\Delta t} + G p^{n-1/2} = \eta L_0 \left(\dfrac{u^n + u^{n+1,*}}{2} \right) - \left[\rho u \cdot \nabla u \right]^{n+1/2} \\ u^{n+1,*}_{\partial\Omega} = 0 \quad (\text{no slip}) \end{cases}$$

This is just advection-diffusion for velocity

② Solve Poisson equation for pressure:

$$L_c \varphi^{n+1} = (DG)\varphi^{n+1} = \frac{Du^{n+1,*}}{\Delta t}$$

Recall: No BCs required on staggered grid

③ Project velocity onto space of discretely divergence-free fields:

$$u^{n+1} = u^{*} - \Delta t \, G \varphi^{n+1}$$

④ Correct pressure (key!)

$$p^{n+1/2} = p^{n-1/2} + \varphi^{n+1} - \frac{\eta \Delta t}{2\rho}\left(L_c \varphi^{n+1}\right)$$

Notes:

1) For periodic domain, this method is equivalent to the full GMRES solve approach

2) For non-periodic domains the projection method is a great preconditioner for the GMRES solver (use inexact sub-solvers), see paper by Boyce Griffith.

Only missing piece is how to get 2^{nd} order (midpoint) estimate for $[u \cdot \nabla u]^{n+1/2}$.

Let's show one approach on a relevant problem.

Boussinesq approximation:

$$\begin{cases} \rho \partial_t u + \nabla \Pi + \rho u \cdot \nabla u = \eta \nabla^2 v + \vec{g} c \\ \partial_t c + u \cdot \nabla c = D \nabla^2 c \end{cases}$$

gravity / buoyancy

concentration / temperature / salinity

We have two options for the advected scalar c:

1) Discretize $u \cdot \nabla c$ at a specific point in time, as in MOL schemes (e.g. something like third order upwind biased in 2d)

2) Solve the advection-diffusion equation using a second-order high-resolution Godunov method like Fromm + limiters + diffusion as you did in homework

Notation:

$$\nabla \cdot \mathbf{F}^{(adv)} = \text{SpaceTime}(c^n, u, s, \Delta t)$$

computes advective fluxes over one timestep of duration Δt:

$$\partial_t c + u \cdot \nabla c = s$$

source

kept fixed in time

$$\underbrace{\left[u \cdot \nabla c\right]^{n+1/2}}_{\text{accurate to } O(\Delta t)} \equiv \nabla \cdot F^{(adv)}$$

Recall that this works even if $D=0$ and is only limited in stability by advective CFL.

By contrast, an RK2 MOL approach is unstable without sufficient diffusion.

Algorithm: Boussinesq predictor-corrector

1. Solve using GMRES for $u^{n+1,*}$ and $p^{n+1/2,*}$ (predictor):

$$
\begin{cases}
\rho \dfrac{u^{n+1,*} - u^n}{\Delta t} + \nabla p^{n+1/2,*} = -\rho\left[u\cdot\nabla u\right]^n \\
\qquad + \eta \nabla^2\left(\dfrac{u^n + u^{n+1,*}}{2}\right) + c^n \vec{g} \\
\nabla\cdot u^{n+1,*} = 0
\end{cases}
$$

and define

$$\boxed{u^{n+1/2,*} = \frac{u^n + u^{n+1,*}}{2}}$$

2. Update concentration

$$
\frac{c^{n+1} - c^n}{\Delta t} = D\nabla^2\left(\frac{c^n + c^{n+1}}{2}\right)
- \begin{cases}
\left[u^{n+1/2,*}\cdot\nabla c\right]^n \text{ (MOL approach)} \\
\text{SpaceTime}\left(c^n, u^{n+1/2,*}, D\nabla^2 c^n, \Delta t\right)
\end{cases}
$$

③ Solve using GMRES for u^{n+1} and $p^{n+1/2}$ (corrector)

$$\begin{cases} \rho \dfrac{u^{n+1} - u^n}{\Delta t} + \nabla p^{n+1/2} = -\rho \left[u \cdot \nabla u \right]^{n+\frac{1}{2},*} \\ \quad + \eta \nabla^2 \left(\dfrac{u^n + u^{n+1}}{2} \right) + \left(\dfrac{c^n + c^{n+1}}{2} \right) g \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

This is a <u>one-step</u> RK2 type method but requires <u>two</u> Stokes solves per step, which is the most expensive step by far. We can do cheaper by using Adams–Bashforth for the $u \cdot \nabla u$ term:

Algorithm Boussinesq AB2

① Predict concentration:

$$\frac{c^{n+1,*} - c^{n}}{\Delta t} = D\nabla^2\left(\frac{c^{n} + c^{n+1,*}}{2}\right) - \begin{cases} [u^{n} \cdot \nabla c]^{n} & (\text{MOL approach}) \\ \text{SpaceTime}\left(c^{n}, u^{n}, D\nabla^2 c^{n}, \Delta t\right) \end{cases}$$

② Solve using GMRES

$$\begin{cases} \rho\dfrac{u^{n+1} - u^{n}}{\Delta t} + \nabla p^{n+1/2} = \left(\dfrac{c^{n} + c^{n+1,*}}{2}\right) g \\ \quad + \eta\nabla^2\left(\dfrac{u^{n} + u^{n+1}}{2}\right) + \\ \quad \left(\underbrace{\dfrac{3}{2}[u \cdot \nabla u]^{n} - \dfrac{1}{2}[u \cdot \nabla u]^{n-1}}_{AB2}\right) \end{cases}$$

(3) Define $u^{n+1/2} = \frac{u^n + u^{n+1}}{2}$ and correct concentration:

$$\frac{c^{n+1} - c^n}{\Delta t} = D \nabla^2 \left(\frac{c^n + c^{n+1}}{2} \right) - \begin{cases} \left[u^{n+1/2} \cdot \nabla c \right]^n & (\text{MOL approach}) \\ \text{SpaceTime}\left(c^n, u^{n+1/2}, D \nabla^2 c^n, \Delta t \right) & \end{cases}$$