Implicit - Explicit (IMEX) temporal integrators

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Here are some standard 2nd order IMEX schemes for solving ODEs of the form:

$$u'(t) = \underbrace{f(u,t)}_{\substack{\text{not stiff} \\ \text{(explicit)} \\ \text{advection}}} + \underbrace{g(u,t)}_{\substack{\text{stiff} \\ \text{(implicit)} \\ \text{diffusion}}}$$

Often $g(u,t) = Lu$ is linear to avoid non-linear equations in implicit solve

Multistep AB2 + CN

Adams - Bashforth + Crank- Nicolson (implicit midpoint)

See Section 11.5 in LeVeque

$$U^{n+1} = U^n + \frac{\tau}{2}\left(3f(U^n,t^n) - f(U^{n-1},t^{n-1})\right)$$

not L-stable

$$+ \frac{\tau}{2}\left(g(U^n,t^n) + g(\underset{\uparrow \text{implicit}}{U^{n+1}}, t^{n+1})\right)$$

(11.26) in Le Veque

Backwards Differentiation

Semi-implicit BDF = SBDF2

$$U^{n+1} = \frac{4}{3}U^n - \frac{1}{3}U^{n-1} + \frac{2}{3}\tau g(U^{n+1}, t^{n+1})$$

$$+ \frac{2}{3}\tau\left[2f(U^n,t^n) - f(U^{n-1},t^{n-1})\right]$$

BDF2 is L-stable (good for diffusion) (8.3.2 in Le Veque)

Runge–Kutta IMEX L-stable

RK2–RK2

$$U^{n+1/2,*} = U^n + \tau\Big(\Big(1+\frac{\sqrt{2}}{2}\Big) g\big(U^{n+1/2,*}, t^n+\frac{\tau}{2}\big) - \Big(\frac{1+\sqrt{2}}{2}\Big) g\big(U^n, t^n\big)\Big) + \frac{\tau}{2} f\big(U^n, t^n\big)$$

(Predictor to midpoint)

$$U^{n+1} = U^n + \Big(\Big(1+\frac{\sqrt{2}}{2}\Big) g\big(U^{n+1}, t^{n+1}\big)$$

(corrector) $$- (1+\sqrt{2})\, g\big(U^{n+1/2,*}, t^n+\frac{\tau}{2}\big) + \Big(1+\frac{\sqrt{2}}{2}\Big) g\big(U^n, t^n\big)\Big)$$

explicit midpoint $\Big\{ + \tau f\big(U^{n+1/2,*}, t^n+\tau/2\big)$

One can use explicit trapezoidal for $f(u)$ instead of midpoint, which is recommended for advection, see paper in 2005 by Pareschi & Russo (see webpage).

For explicit RK3, the scheme recommended for advection is:

$$\begin{cases} u^{*} = u^{n} + \tau f^{n} \quad \text{(Euler step)} \\ u^{**} = \underbrace{\frac{3}{4} u^{n} + \frac{1}{4}\left[u^{*} + \tau f^{*} \right]}_{\text{convex combination of Euler steps}} \\ u^{n+1} = \frac{1}{3} u^{n} + \frac{2}{3} \left[\underbrace{u^{**} + \tau f^{**}}_{\text{Third Euler step}} \right] \end{cases}$$

It is possible to combine this explicit RK3 with the implicit RK2 from page 3 to obtain a great IMEX RK3-RK2 scheme for advection-diffusion MOL schemes. Here are the relevant pieces from the Pareschi + Russo 2005 paper:

2. IMEX RUNGE–KUTTA SCHEMES

An IMEX Runge–Kutta scheme consists of applying an implicit discretization to the source terms and an explicit one to the nonstiff term. When applied to system (1) it takes the form $\partial_t U = F(U) + R(U)$

$$U^{(i)} = U^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} \partial_x F(U^{(j)}) + \Delta t \sum_{j=1}^{\nu} a_{ij} R(U^{(j)}), \tag{3}$$

$$U^{n+1} = U^n - \Delta t \sum_{i=1}^{\nu} \tilde{w}_i \partial_x F(U^{(i)}) + \Delta t \sum_{i=1}^{\nu} w_i R(U^{(i)}). \tag{4}$$

The matrices $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = 0$ for $j \geqslant i$ and $A = (a_{ij})$ are $\nu \times \nu$ matrices such that the resulting scheme is explicit in F, and implicit in R. An IMEX Runge–Kutta scheme is characterized by these two matrices and the coefficient vectors $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_\nu)^T$, $w = (w_1, \ldots, w_\nu)^T$.

The IMEX Runge–Kutta schemes can be represented by a double *tableau* in the usual Butcher notation,

$$
\begin{array}{c|c}
\tilde{c} & \tilde{A} \\
\hline
 & \tilde{w}^T
\end{array}
\qquad\qquad
\begin{array}{c|c}
c & A \\
\hline
 & w^T
\end{array}\,,
$$

where the coefficients $\tilde{c}$ and c used for the treatment of non autonomous systems, are given by the usual relation

$$\tilde{c}_i = \sum_{j=1}^{i-1} \tilde{a}_{ij}, \qquad c_i = \sum_{j=1}^{i} a_{ij}. \tag{5}$$

The use of a DIRK scheme for R is a sufficient condition to guarantee that F is always evaluated explicitly.

Table II. Tableau for the Explicit (left) Implicit (right) IMEX-SSP2(2,2,2) L-Stable Scheme RK2-RK2

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1 & 0 \\
\hline
 & 1/2 & 1/2
\end{array}
\qquad
\begin{array}{c|cc}
\gamma & \gamma & 0 \\
1-\gamma & 1-2\gamma & \gamma \\
\hline
 & 1/2 & 1/2
\end{array}
\qquad
\gamma = 1 + \frac{1}{\sqrt{2}}
$$

Use instead of scheme on page 3

I prefer + sign

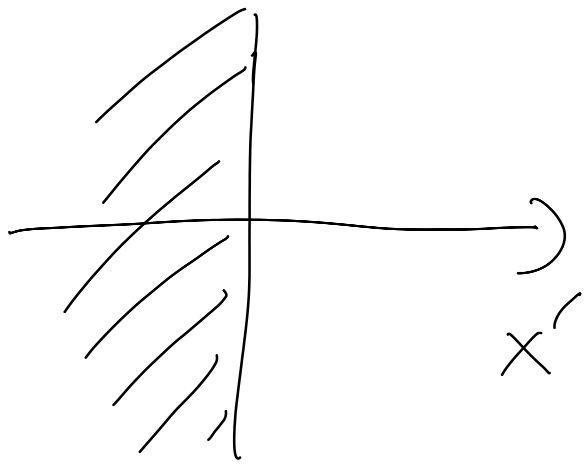
Table V. Tableau for the Explicit (left) Implicit (right) IMEX-SSP3(3,3,2) L-Stable scheme RK3-RK2

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1/2 & 1/4 & 1/4 & 0 \\
\hline
 & 1/6 & 1/6 & 2/3
\end{array}
\qquad
\begin{array}{c|ccc}
\gamma & \gamma & 0 & 0 \\
1-\gamma & 1-2\gamma & \gamma & 0 \\
1/2 & 1/2-\gamma & 0 & \gamma \\
\hline
 & 1/6 & 1/6 & 2/3
\end{array}
\qquad
\gamma = 1 - \frac{1}{\sqrt{2}}
$$

Aside

L-stable scheme

(A-stable)



$$x'(t) = -\underbrace{\lambda}_{\text{large}} x(t)$$

$$x^n = \underbrace{\left(\overset{R}{\overset{\shortparallel}{\frac{1+\lambda\Delta t/2}{1-\lambda\Delta t/2}}}\right)^n}_{1?} x^0$$

$$x^n = \underbrace{e^{-\lambda n \Delta t}}_{\text{decays rapidly}} x^0$$

$$\lambda \Delta t \gg 1 \rightarrow x^n \rightarrow (\underbrace{-1}_{\text{does}})^n x^0$$

L - stable :

Stability $\left| R(\lambda \Delta t = z) \right| \leq 1$

L-stable = A-stable +

$R(z \to \infty) \to 0$

Backward Euler is L-stable

An L-stable scheme applied to a linear parabolic PDE with $\Delta t \to \infty$ will converge to the steady-state solution (elliptic)

Good scheme:

- Stability $\nu = \frac{a\tau}{h} < C = O(1)$

- Diffusion only $\Rightarrow$ strictly dissipative (L-stable)

- 2nd order