Spring 2021: Numerical Analysis
Assignment 6: Polynomial approximation and quadrature
Due May 10th 2pm EST

1. **[Connection between interpolation and 2-norm approximation, 10pts]** Using a set of disjoint points (nodes) $x_0, \ldots, x_n$ in $[a, b]$, we define an inner product for polynomials $p(x), q(x)$ as

$$\langle p, q \rangle := \sum_{i=0}^{n} p(x_i) q(x_i).$$

   (a) [3 pts] This is an inner product for each $\mathcal{P}_k$ with $k \leq n$, where $\mathcal{P}_k$ denotes the space of polynomials of degree k or less. Why is $\langle \cdot, \cdot \rangle$ not an inner product for $k > n$?

   (b) [3 pts] Show that the Lagrange polynomials $L_k(x)$ corresponding to the nodes $x_0, \ldots, x_n$ are orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$.

   (c) [4 pts] For a continuous function $f : [a, b] \to \mathbb{R}$, compute its optimal approximation in $\mathcal{P}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle$ and compare with the polynomial interpolant of f on the same set of nodes.

2. Compute explicitly the optimal L_2 approximation of $\sin(x)$ on $[0, \pi]$ for the standard L_2 inner product. Compare it on the same plot to the sine function, as well as the quadratic interpolant of $\sin(x)$ with nodes $x_0 = 0$, $x_1 = \pi/2$, $x_2 = \pi$ (see Worksheet 7 and be encouraged to also add the Hermite interpolant from that worksheet).

3. Obtain explicit formulas for the first three Chebyshev polynomials via the Gram-Schmidt orthogonalization process starting with the monomials $\{1, x, x^2\}$. Note that unless you normalize these polynomials they are only defined up to a constant.