Linear Algebra

(mostly review) A. Donev

Matrix $A = \left[\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \dots \,|\, \vec{a}_n \right]$

$\vec{a}_k \in \mathbb{R}^m$, n of them

$A \in \mathbb{R}^{m,n}$

Matrix-vector product

$$\vec{b} = A\vec{x} = \underline{\underline{x_1}}\,\vec{a}_1 + \underline{\underline{x_2}}\,\vec{a}_2 + \dots + \underline{\underline{x_n}}\,\vec{a}_n$$

column vector

$$[m \times n]\,[n \times 1] = [m \times 1]$$

$b \in \mathbb{R}^m$

Image, column space, or range of A:

$\vec{b} \in \text{im}(A)$

Vector subspace of $\mathbb{R}^m$ spanned by columns of A

Columns of A are linearly independent if

$$Ax = 0 \iff x = 0$$

$$r = \dim(\mathcal{V} \subset \mathbb{R}^n)$$

max number of lin. ind. vectors in $\mathcal{V}$.

# of vector in $\underline{\text{any}}$ basis for V

$\dim(\mathbb{R}^n) = n$. is finite

Inner or dot product

$$\vec{x} \cdot \vec{y} = (x, y) = \langle x, y \rangle$$

$$= x^T y = \sum_{i=1}^{n} x_i y_i$$

$$[1 \times n][n \times 1] = [1 \times 1] = \text{scalar}$$

$\vec{x}, \vec{y} \in \mathbb{R}^n$

If $\mathbb{C}^n$ (complex number)

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} \overline{x_i}\, y_i$$

$\vec{x} \perp \vec{y}$ (orthogonal) if

$\vec{x} \cdot \vec{y} = 0$

$r = \text{rank}(A) =$

$\dim(\text{im}(A))$

# of lin. ind. rows or columns

$\text{rank}(A) = \text{rank}(A^T)$

$r \leq \min(m, n)$

If $A\vec{x} = \vec{0} \Rightarrow$

$\vec{x} \in \text{null}(A)$

null space or kernel of A

$\dim(\text{null}(A)) = \underline{\underline{\text{nullity}}}$

If column are lin. ind $\Rightarrow$

$\text{null}(A) = 0$

<u>Fund</u> theorem of LA:

rank + nullity = n

$L : V \longrightarrow W$

vector spaces

L is linear iff

$$\begin{cases} L(\vec{v}_1 + \vec{v}_2) = L\vec{v}_1 + L\vec{v}_2 \\ L(\alpha \vec{v}) = \alpha(L\vec{v}) \end{cases}$$

All linear mappings can be represented by a matrix
(finite-dimensional $\mathcal{V}$ and $\mathcal{W}$)

$$L(\vec{v}) = \underbrace{\overleftrightarrow{L}\,\vec{v}}_{\text{matrix-vector product}}$$

$$\vec{w} = \overleftrightarrow{L}\,\vec{v}$$

$$w_i = \sum_{j=1}^{n} L_{ij}\, v_j$$

contraction

$$w_i = (L_{i,:}\,,\ \vec{v})$$

i^{th} row of matrix

Composition of linear mappings

$$\vec{z} = \underbrace{A(B\vec{x})}_{\text{associative}} = (AB)\vec{x}$$

$$= C\vec{x}$$

$$C = AB$$ matrix-matrix product

$$C_{ij} = \sum_{k=1}^{p} A_{\underline{\underline{i}}\,k} B_{k\,\underline{\underline{j}}}$$

contract

$$A = [m \times p]$$
$$B = [p \times n]$$
$$C = [m \times p][p \times n] = [m \times n]$$

In Matlab $A * B$

Matrix multiplication is not commutative

$$AB \neq BA \quad \text{(in general)}$$

$$A : \mathbb{R}^n \underset{A^{-1}}{\overset{\longleftarrow}{\longrightarrow}} \mathbb{R}^n$$

$$A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

$$AA^{-1} = A^{-1}A = I$$

I = identity matrix

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

If A^{-1} exists, matrix is invertible (square)

A is invertible iff:
(all of these are equivalent)

1) A is full rank
$\text{rank}(A) = n$

2) columns & rows are lin. ind.

3) $\det(A) \neq 0$

4) $\left(\det(A) = \underline{\lambda_1} \cdot \underline{\lambda_2} \cdots \underline{\lambda_n}\right)$

$\lambda = 0$ is not an eigenvalue

5) $Ax = 0 \Leftrightarrow x = 0$

Properties of matrix algebra

$$C(A+B) = CA + CB$$

$$ABC = (AB)C$$
$$= A(BC)$$

E.x.

$$A\vec{x}\vec{x}^T A = A\underbrace{(\vec{x}\vec{x}^T)}_{B}A$$

$$\underbrace{(\vec{x}^T y)}_{\text{scalar}}B$$

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Matrix "division"

→ multiplication by inverse

$$^{(A^{-1})}A\,B^{(B^{-1})} = {}^{(A^{-1})}C^{(B^{-1})}$$

$$B = \cancel{C} / A$$

$$(\underbrace{A^{-1}A}) B = A^{-1} C$$

$$\underbrace{I} B = A^{-1} C$$

$$B = A^{-1} C$$

$$A = C B^{-1}$$

Vector norms

p- norm or L_p norm

$p \geq 1$

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

1) L_1 norm (Manhattan norm)

$p=1$, $\|\vec{x}\|_1 = \sum |x_i|$

2) L_2 norm (Euclidean norm)

$$\|x\|_2 = \sqrt{x \cdot x} = \sqrt{x^T x}$$

$$\|\vec{x}\|_2 = \sqrt{\sum |x_i|^2}$$

3) L_∞ norm or max norm

$$\|\vec{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

All norms are related to each other if n $(\vec{x} \in \mathbb{C}^n)$ is "small"

$\|A\|$ Matrix norm

$$\sup_{x \neq 0} \frac{\|Ax\|_{1,2,\infty}}{\|x\|_{1,2,\infty}} = \|A\|_{1,2,\infty}$$

Matrix norm <u>induced</u> by the vector norm

$$\|Ax\| \leq \|A\| \, \|x\|$$

$$\|AB\| \leq \|A\| \, \|B\|$$

1) $$\|A\|_1 = \max_j \|A_{:,j}\|_1$$

j^{th} column

$$= \max_j \sum_{i=1}^{n} |a_{ij}|$$

2) $$\|A\|_\infty = \max_i \|A_{i,:}\|_1$$

$$= \max_i \sum_j |a_{ij}|$$

3) $\|A\|_2 = \max_i \lambda_i$

λ^2 is an eigenvalue of

$\underbrace{A^T A}_{\text{symmetric}}$ or $A A^T$

$$\|A\|_2 = \max_i \sqrt{\lambda_{A^T A}}$$

In Matlab

$$\text{norm}\left(A, \begin{matrix} 1 \\ 2 \\ \text{inf} \end{matrix}\right) \begin{matrix} p=1 \\ p=2 \\ p=\infty \end{matrix}$$

Conditioning of mappings/matrices

$$\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$$

$$\sup_{\vec{\delta x} \neq 0} \frac{\| \vec{f}(\vec{x} + \vec{\delta x}) - f(\vec{x}) \|}{\| \vec{\delta x} \|} = \underline{\text{Cond}}_x (f)$$

(Local absolute condition number in theory book)

$$f : \mathbb{R} \to \mathbb{R}$$

$$\underline{\text{Cond}}(f) = | f'(x) |$$

$$\mathrm{cond}_x(f) = \sup_{\delta x \neq 0} \frac{\|f(x+\delta x) - f(x)\| / \|f(x)\|}{\|\delta x\| / \|x\|}$$

If $\mathrm{cond}_x(f) = 10^4$

that means that

knowing 16 digits in x

gives me $16-4=12$ digits

in $f(x)$

Loose four digits

$$\cancel{A}x + A\delta x - \cancel{A}x$$

$$\mathrm{cond}_x(Ax) = \sup_{\delta x \neq 0} \frac{\|A(x+\delta x) - Ax\| / \|Ax\|}{\|\delta x\| / \|x\|}$$

$$= \left(\sup_{\delta x \neq 0} \frac{\|A\delta x\|}{\|\delta x\|} \right) \frac{\|x\|}{\|Ax\|}$$

$$= \frac{\|x\|}{\|Ax\|} \qquad \|A\| \geq 1$$

$$\|Ax\| \leq \|A\| \, \|x\|$$

$$\mathrm{cond}_x(Ax) = \frac{\|A^{-1}(Ax)\| \, \|A\|}{\|Ax\|}$$

$$\leq \frac{\|A^{-1}\| \, \|\cancel{A}x\| \, \|A\|}{\|\cancel{A}x\|}$$

$$\leq \|A\| \|A^{-1}\|$$

$$1 \leq \text{cond}_x(A) \leq \underline{\underline{\|A\| \|A^{-1}\|}}$$

Define cond. number of A

$$K(A) = \|A\| \, \|A^{-1}\|$$

1, 2, ∞ 1, 2, ∞ 1, 2, ∞

Eigenvalues

$$\underset{\substack{\uparrow \\ \text{eigenvector}}}{Ax} = \underset{\substack{\nwarrow \\ \text{eigenvalue}}}{\lambda} x, \quad x \neq 0$$

$A \in \mathbb{C}^{n \times n}$ or $\mathbb{R}^{n \times n}$

$$Ax - \lambda x = 0$$

$$\underbrace{(A - \lambda I)}_{} x = 0$$

$x \neq 0$?

$$\text{null}(A - \lambda I) \neq \{\vec{0}\}$$

If $A - \lambda I$ is invertible

$$x = (A - \lambda I)^{-1} 0 = 0$$

$A - \lambda I$ is not invertible

$$|A - \lambda I| = 0$$

determinant

$$\begin{bmatrix} a_{11}-\lambda & & & \\ & a_{22}-\lambda & & \\ & & \ddots & \\ & & & a_{nn}-\lambda \end{bmatrix}$$

$$\det \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} =$$

example

characteristic polynomial

$$|A - \lambda I| = \mathrm{poly}_n(\lambda) = 0$$

At most n

If we allow $\lambda \in \mathbb{C}$

At least one λ exists

At least one eigenvector for each distinct eigenv.

$\times$ **Not** unique (can multiply by any constant)

Eigenvectors are <u>directions</u>
(not vectors)

<u>Matrix notation</u>

$$\overline{\underline{X}} = [\, x_1 \mid x_2 \mid \dots \mid x_m \,]$$

linearly independent

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \emptyset \\ & \lambda_2 & & \\ & & \ddots & \\ \emptyset & & & \lambda_m \end{bmatrix}$$

capital λ

$$A x_i = \lambda_i x_i \, , \quad i = 1, \dots, m$$

$$A \overline{\underline{X}} = \overline{\underline{X}} \Lambda$$

$1 \leq m \leq n$

Is $m = n$? $\times$ not in general

Every λ has an algebraic multiplicity α and a geometric multiplicity β (how many linearly independent eigenvectors)

$1 \leq \beta \leq \alpha$

If $m = n$ we call that matrix non-defective or diagonalizable matrix

$m = n$ $\quad X$ is $[n \times n]$

$\Rightarrow X$ is invertible

Eigenvectors span all of $\mathbb{R}^n$

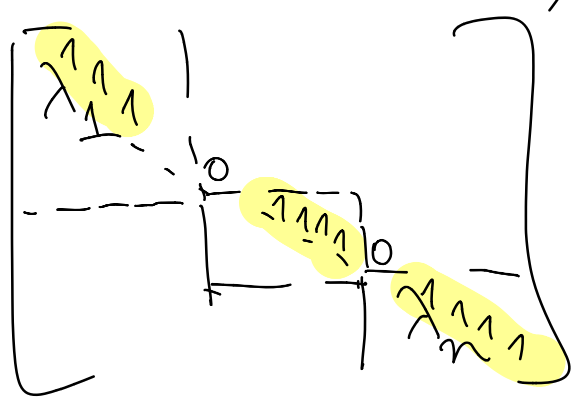
X^{-1} exists

$$X^{-1} \cdot \Big| \quad AX = X\Lambda \quad \Big| \cdot X^{-1}$$

$$\longrightarrow X^{-1}AX = \Lambda$$

$$\underbrace{\boxed{A = X\Lambda X^{-1}}}_{\text{eigenvalue decomposition}}$$

If A is defective,
Jordan form (decomposition)

$$\Lambda \longrightarrow$$ 

Not computable numerically

Assume A is non-defective

$$X^{-1} A X = \Lambda$$

$\underbrace{X^{-1} A X}$ similarity transform

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

linear map

$$y = \sum_{i=1}^{n} a_i x_i$$

$$Ay = A \sum_i a_i x_i =$$

$$= \sum_i a_i \, (\underbrace{A x_i}_{\lambda_i x_i}) =$$

$$= \sum_i (a_i \lambda_i) \, x_i$$

In basis formed by eigenvectors A is diagonal with λ_i's on the diagonal

If x_i's are orthogonal then matrix is called unitarily diagonalizable

$X \rightarrow U$ orthogonal matrix

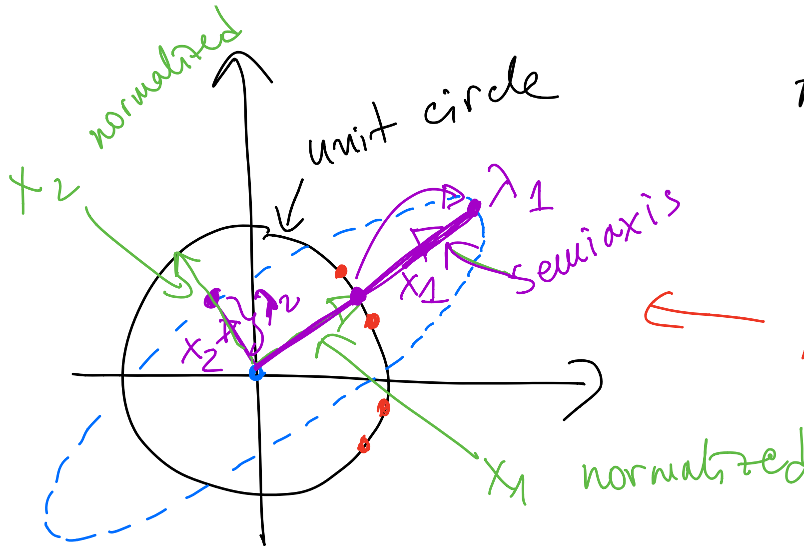
$\|x_i\| = 1$

$U^{-1} = U^{*}$ ← complex conjugate transpose

If and only if $AA^{*} = A^{*}A$

$$\boxed{A = U \Lambda U^{*}}$$

Assume A can be factorized like this



$A \in \mathbb{R}^{2\times 2}$
& unitarily diagonalizable

← A is Hermitian/symmetric

$$A = U \Lambda U^*$$

$$A^* = (U \Lambda U^*)^* =$$

$$= (U^*)^* \Lambda^* U^*$$

$$= U \Lambda^* U^*$$

If $A^* = A$ Hermitian matrix or symmetric if in $\mathbb{R}^{n\times n}$

$$\cancel{U} \Lambda^* \cancel{U^*} = \cancel{U} \Lambda \cancel{U^*}$$

$$\Rightarrow \Lambda^* = \Lambda$$

eigenvalues are <u>real</u>

Theorem: If $A^* = A$ then A is unitarily diagonalizable and eigenvalues are real

From now on assume A is **Hermitian**

If A is not unitarily diagonalizable, numerically best to use <u>Schur</u> decomposition

any matrix $\quad A = U T U^*$

T upper triangular

Eigenvalues on diagonal of T

Singular values of matrix

"tells you everything about A"

diagonal

$$A = U \Sigma V^*$$

works for any matrix A!

"ultimate" decomposition

unitary

$$[m \times n] = [m \times m][m \times n][n \times n]$$

Singular value decomposition

Columns of U are left singular vectors

Columns of V are right sing. vectors

Diag. elements of Σ are called singular values

$$\Sigma = m \begin{bmatrix} \sigma_1 & & 0 & \vdots & \\ & \ddots & & \vdots & 0 \\ 0 & & \sigma_m & \vdots & \end{bmatrix} \quad \text{If } m < n$$

$$\begin{array}{cc} m & n \end{array}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & \cdots & \sigma_n \\ \hline & 0 & \end{bmatrix} n \quad \text{If } m > n$$

$$\sigma_1, \dots\dots, \sigma_p > 0$$

$$\sigma_{p+1}, \dots\dots = 0$$

$$p = \min(m, n)$$

<u>Observe</u>

If A is Hermitian

$$\begin{cases} A = U \Lambda U^* & U = V \\ A = U \Sigma V^* & \Sigma = \Lambda \end{cases}$$

For symmetric matrices no real difference between eigenvalue & singular value decomposition.

SVD works for all matrices & can be computed numerically in a stable way