

Square Linear Systems

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E.g.

$$\left\{ \begin{array}{l} 3x_1 + 2x_2 = 2 \\ x_1 - x_2 + x_3 = 1 \\ 2x_1 + 3x_3 = 5 \end{array} \right.$$

$$A \xrightarrow{\text{matrix}} \xrightarrow{\text{solution}} = b \xleftarrow{\text{r.h.s.}}$$

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

$$A = [m \times n] \quad x = [n \times 1]$$

$$b = [m \times 1]$$

m eqs., n variables

For a
while

$$m = n$$

$$Ax = b \quad m=n$$

How many solutions?

- 0 \rightarrow A not invertible, $b \notin \text{Im}(A)$

- 1 \rightarrow A is invertible

$$x = A^{-1}b$$

- ∞ \rightarrow A not invertible, $b \in \text{Im}(A)$

If A is not invertible.

$Ax=0$? How many solutions?

\uparrow
Infinitely many solutions

$x, \lambda x$ is also a solution

Back to : $Ax = b$ for non-invertible

$$\left\{ \begin{array}{l} Ax_1 = b \quad \text{has 1 solution} \\ Ax_2 = 0 \quad \text{- infinitely} \\ \hline Ax_1 + Ax_2 = b \\ A(x_1 + x_2) = b \end{array} \right.$$

$$Ax = b, b \in \text{im}(A)$$

If $b \notin \text{im}(A) \Rightarrow$ no solution

If $b \in \text{im}(A) \Rightarrow$ mlin.
many solutions

$$Ax = b, \quad A \text{ is invertible}$$

$$\sum_{j=1}^n a_{ij} x_j = b_i \quad \forall i = 1, \dots, n$$

$$x = A^{-1} b$$

NOT how we compute it
 numerically

Never do $x = \text{inv}(A) * b$

$$\rightarrow \left[\begin{array}{ccc|ccc} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array} \right] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\frac{-a_{21}}{a_{11}}$

Reduced row echelon form

Step 1: Eliminate x_1

a) Multiply 1st equation by

$$l_{21} = \frac{a_{21}}{a_{11}}$$

and subtract it from second equations

b) Multiply 1st equation by

$$l_{31} = \frac{a_{31}}{a_{11}}$$

eq. #

and subtract from 3rd eq

$$\left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & - \\ \hline - & \overline{a_{22}} = & a_{23} & - \\ a_{21} - l_{21} a_{11} & \overline{a_{22} - l_{21} a_{12}} & a_{23} - l_{21} \cdot a_{13} & \\ \hline \textcolor{yellow}{= \emptyset} & \textcolor{red}{a_{32}^{(2)} =} & \textcolor{red}{a_{33}^{(2)} =} & \\ \hline \textcolor{yellow}{\emptyset} & \overline{a_{32}^{(1)} - l_{31} a_{12}^{(1)}} & \overline{a_{33}^{(1)} - l_{31} a_{13}^{(1)}} & \end{array} \right]$$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - l_{ik} \frac{a_{kj}^{(k)}}{a_{kk}^{(k)}}$$

$$l_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}, \quad i, j > k$$

eq. \uparrow \uparrow
var

$$\begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix} \rightarrow \begin{bmatrix} b_1^{(2)} = b_1^{(1)} \\ b_2^{(2)} = b_2^{(1)} - l_{21} b_1^{(1)} \\ b_3^{(2)} = b_3^{(1)} - l_{31} b_1^{(1)} \end{bmatrix}$$

$$b_i^{(k+1)} = b_i^{(k)} - l_{ik} b_k^{(k)}$$

Step 2

$$\begin{array}{c|ccc|c|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & x_1 & b_1^{(2)} \\ \hline l_{21} \cancel{\varnothing} & a_{22}^{(2)} & a_{23}^{(2)} & x_2 & b_2^{(2)} \\ l_{31} \cancel{\varnothing} & a_{32}^{(2)} & a_{33}^{(2)} & x_3 & b_3^{(2)} \end{array}$$

Multiply 2nd eq. by

$$l_{32} = \frac{a_{32}}{a_{22}^{(2)}}$$

eq.
var

and subtract from 3rd eq.

$$\left[\begin{array}{ccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & x_1 \\ -l_{21}^{(1)} a_{22}^{(2)} & a_{22}^{(2)} & a_{23}^{(2)} & x_2 \\ -l_{31}^{(1)} a_{32}^{(3)} & a_{32}^{(3)} & a_{33}^{(3)} & x_3 \end{array} \right] = \left[\begin{array}{c} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \end{array} \right]$$

Step 3 :

$$x_3 = \frac{b_3^{(3)}}{a_{33}^{(3)}}$$

Step 4 :

$$x_2 = \frac{b_2^{(2)} - a_{23}^{(2)} x_3}{a_{22}^{(2)}}$$

Step 5 : solve for x_1

Gaussian elimination

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

det(L)=1
is invertible

unit lower triangular matrix

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{bmatrix}$$

Upper triangular

Theorem: $A = L U$

LU factorization

\Rightarrow Gauss elimination

Code MyLU.m on
webpage

$$i > k : l_{ik} = \frac{a_{ik}}{\overbrace{a_{kk}}} \Rightarrow a_{ik}$$

$$l_{kk} = 1 \text{ (not forced)}$$

for $k = 1 : (n-1)$
 Eliminate x_k from
 eqs. $k+1, \dots, n$

$$A((k+1):n, k) = A((k+1):n, k) / A(k, k);$$

Compute $l_{ik}, i > k$

$$a_{ij} \leftarrow a_{ij} - l_{ik} a_{kj}$$

$i, j > k$

for $j = (k+1) : n$

$$A((k+1) : n, j) = A((k+1) : n, j)$$

$$- A((k+1) : n, k) * A(k, j);$$

$\underbrace{}$ stores l_{ik} "in-place"

end [for j]

end [for k]

LU
factorization

Note We could have done
for $i = (k+1) : n$

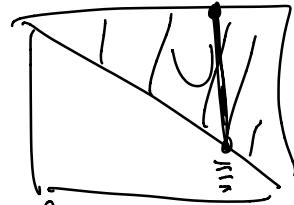
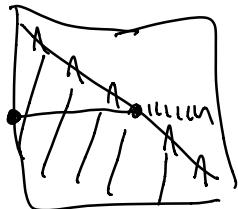
Last time: $A X = b$

GEM: $A = L + U$ diag.

"Proof":

$$a_{ij} = \sum_{k=1}^n l_{ik} u_{kj}$$

Dot product between i^{th}
row of L and j^{th} column of U



$$a_{ij} = \sum_{k=1}^{\min\{i,j\}} l_{ik} u_{kj}$$

If $i > j \Rightarrow \sum_{k=1}^j$

If $i \leq j \Rightarrow \sum_{k=1}^i$

Recall $\ell_{ii} = 1$

$$\textcircled{1} \quad a_{ij} = \sum_{k=1}^i \ell_{ik} u_{kj}$$

$$i = 1, \dots, n$$

$$j = i, \dots, n$$

$$= \sum_{k=1}^{i-1} \ell_{ik} u_{kj} + u_{ij}$$

$$\ell_{ii} = 1$$

$$\Rightarrow u_{ij} = a_{ij} - \sum_{k=1}^{i-1} \ell_{ik} u_{kj}$$

✓
GEM

We need $u_{(k < i), j}$
 $\ell_{ik}, (k < i)$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \ell_{ik} \cdot a_{kj}^{(k)}$$

②

$$a_{ij} = \sum_{h=1}^j l_{ih} u_{hj}$$

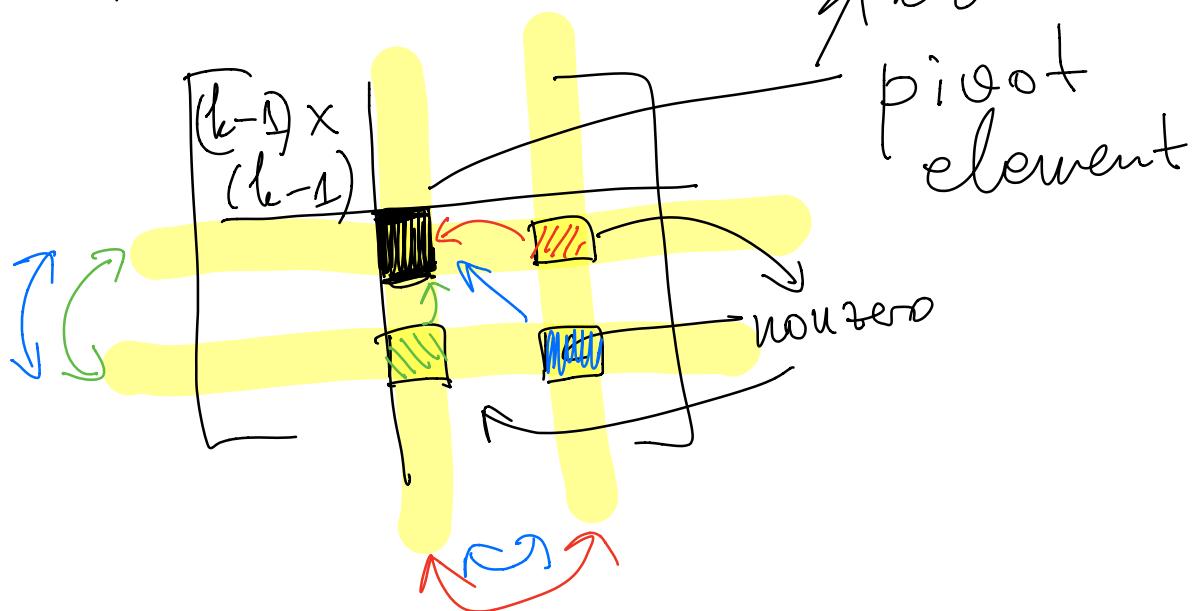
$$l_{ij} = \frac{(a_{ij} - \sum_{h=1}^{j-1} l_{ih} u_{hj})}{u_{jj}}$$
$$l_{ij} = \frac{a_{ij}^{(j)}}{a_{jj}^{(j)}}$$

which shows that GEM computed L & U s.t.

$$A = LU$$

Pivoting

Assumed that $a_{kk}^{(h)} \neq 0$



Pivoting: Row, Column,
Complete = Row + column

Let's re-order equations

$$2x \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 13 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -4 \\ 0 & 3 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ -4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LU = PA$$

↑ permutation matrix

Theorem :

If A is non-singular
then pivoted LU factorization
will succeed

$$P A = L U$$

for some permutation
matrix P

What is the best P

Example :

$$\left[\begin{array}{c|cc} 10^{-20} & 1 & \\ \hline 10 & & \\ 1 & 1 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_1 \approx x_2 \approx 1$$

||

$$10^{-20} \times \begin{array}{c|c} 10 & 1 \\ \hline \emptyset & 1 - 10^{-20} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 - 10^{-20} \end{bmatrix}$$

↑ GEM

10^{-20} is circled in red.

Due to round off

$$\begin{array}{c|c} 10^{-20} & 1 \\ \hline & -10^{-20} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \\ -10 \end{bmatrix}$$

-10^{-20} is circled in yellow.

$$x_2 = \frac{-10^{-20}}{-10^{-20}} = 1 \quad \checkmark$$

$$10^{-20} x_1 = 1 - 1 = 0$$

$$\Rightarrow x_1 = 0 \neq 1 \quad \times$$

Not the correct solution

$$\begin{bmatrix} 1 & 1 \\ 10^{-20} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x_1 = x_2 = 1 \quad \checkmark$$

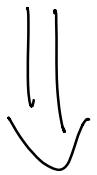
Or change order of
variables

$$\begin{bmatrix} 1 & 10^{-20} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

"Small" pivot elements are a problem in floating-point arithmetic

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc} 7 & 8 & 0 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$



$$\left[\begin{array}{ccc} 7 & 8 & 0 \\ 0 & 3/7 & 6/7 \\ 0 & 0 & 3 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -8/7 \\ 5/7 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 3/7 & 6 \end{array} \right] \sim \left[\begin{array}{c} 2 \\ 5/7 \\ -8/7 \end{array} \right]$$

↓

$$\left[\begin{array}{ccc} 7 & 8 & 0 \\ 0 & 6/7 & 3 \\ 0 & 0 & 9/2 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} 2 \\ 5/7 \\ -3/2 \end{array} \right]$$

$$P^{-1} \quad | \quad PA = LU = P \ell$$

$$P^{-1} = P^T$$

$$A = P^T LU = \ell$$

$$A = (P^T L) U \underset{\nearrow}{\sim} \tilde{L} U$$

permuted lower
triangular

$1 \begin{bmatrix} D \\ D & D \\ D & D & D \end{bmatrix}$	$3 \begin{bmatrix} D & D & D \\ D & D \\ D \end{bmatrix}$
2	2
3	1

In MATLAB:

$$Ax = b$$

$$L(U) x = b$$

(1) $L y = b$ forw ard subst. to get y

(2) $U x = y$ backward substitution for x

$$\left[\begin{array}{|c|c|c|} \hline l_{11} & & \\ \hline & l_{22} & \\ \hline 0 & \ddots & 0 \\ \hline \end{array} \right] \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \right] = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]$$

$$l_{11} y_1 = b_1 \Rightarrow$$

$$y_1 = b_1 / l_{11} = 1$$

$$y_2 = \frac{b_2 - l_{11} y_1}{l_{22}}$$

$$Ly = b$$

$$y_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} y_j}{l_{ii}} = 1$$

$$i = 1, \dots, n$$

Matlab code

for $i = 1 : n$ *Forward Substitution*
 $y(i) = b(i) - \sum (L(i, 1:i-1) \cdot y(1:i-1))$
end
=

For $Ux = y$ (Backward)

for $i = n : -1 : 1$

Homework

Matlab syntax

$\{ [L, U] = \text{lu}(A)$
is really L
 $y = L \backslash b; % \text{ Solve } Ly = b$
 $x = U \backslash y; % \text{ Solve } Ux = y$

alright

(equiva)

To solve $A * x = b$ in Matlab

$$\{ x = A \backslash b; \text{ "mldivide"}$$

Different from $x = \text{inv}(A) * b$

never do this