

Linear Algebra

(mostly review) A. DOVER

Matrix

$$A = \left[\begin{array}{c|c|c|c} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \dots & \vec{a}_n \end{array} \right]$$

$\vec{a}_k \in \mathbb{R}^m$, n of them

$$A \in \mathbb{R}^{m,n}$$

Matrix - vector product

$$\vec{b} = A \vec{x} = \underbrace{x_1}_{\text{column vector}} \vec{a}_1 + \underbrace{x_2}_{=} \vec{a}_2 + \dots + \underbrace{x_n}_{=} \vec{a}_n$$

$$[m \times n] [n \times 1] = [m \times 1]$$

$$\vec{b} \in \mathbb{R}^m$$

Image, column space, or
range of A :

$$\vec{b} \in \text{im}(A)$$

Vector subspace of \mathbb{R}^n
spanned by columns of A

Columns of A are linearly
independent if

$$Ax = 0 \Leftrightarrow x = 0$$

$$r = \dim(\mathcal{V} \subset \mathbb{R}^n)$$

max number of lin. ind.
vectors in \mathcal{V} .

of vector in any basis
for \mathbb{R}^n

$$\dim (\mathbb{R}^n) = n \text{ . is finite}$$

~~~~~  
Inner or Dot product

$$\vec{x} \cdot \vec{y} = (x, y) = \langle x, y \rangle$$

$$= \begin{matrix} \vec{x} \\ \uparrow \end{matrix}^T y = \sum_{i=1}^n x_i y_i$$

$$[1 \times n] [n \times 1] = [1 \times 1] = \text{scalar}$$

$$\vec{x}, \vec{y} \in \mathbb{R}^n$$

If  $\mathbb{C}^n$  (complex number)

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^n \bar{x}_i y_i$$

$\vec{x} \perp \vec{y}$  (orthogonal) if

$$\vec{x} \cdot \vec{y} = 0$$

$$r = \text{rank}(A) = \dim(\text{im}(A))$$

# of lin. ind. rows or  
columns

$$\text{rank}(A) = \text{rank}(A^T)$$

$$r \leq \min(m, n)$$

If  $A\vec{x} = \vec{0} \Rightarrow$

$$\vec{x} \in \text{null}(A)$$

null space or kernel  
of  $A$

$$\dim(\text{null}(A)) = \underline{\text{nullity}}$$

If columns are lin. nd  $\Rightarrow$   
 $\text{null}(A) = 0$

Fund theorem of LA:

$$\text{rank} + \text{nullity} = n$$



$L: \mathcal{V} \rightarrow \mathcal{W}$   
vector spaces

$L$  is linear if

$$\left\{ \begin{array}{l} L(\vec{v}_1 + \vec{v}_2) = L\vec{v}_1 + L\vec{v}_2 \\ L(\alpha \vec{v}) = \alpha(L\vec{v}) \end{array} \right.$$

All linear mappings can  
be represented by a  
matrix

(finite-dimensional  $V$  and  $W$ )

$$L(\vec{\vartheta}) = \underbrace{L}_{\substack{\text{matrix-vector} \\ \text{product}}} \vec{\vartheta}$$

$$\vec{w} = \underbrace{L}_{n} \vec{\vartheta}$$

$$w_i = \sum_{j=1}^n L_{ij} \vartheta_j$$

contraction

$$w_i = (L_{i,:}, \vec{\vartheta})$$

i<sup>th</sup> row of matrix

Composition of linear  
mappings

$$\vec{z} = \underbrace{A(B\vec{x})}_{\text{associative}} = (\vec{AB})\vec{x} = C\vec{x}$$

$$C = AB \quad \begin{matrix} \text{matrix-} \\ \text{matrix} \\ \text{product} \end{matrix}$$

$$C_{ij} = \sum_{k=1}^p A_{ik} B_{kj} \quad \begin{matrix} k & k \\ \curvearrowright & \curvearrowright \\ \text{contract} & \end{matrix}$$

$$A = [m \times p]$$

$$B = [p \times n]$$

$$C = [m \times p] \underbrace{[p \times n]}_{k} = [m \times n]$$

In Matlab  $A * B$

Matrix multiplication is  
not commutative

$$AB \neq BA \quad (\text{in general})$$



$$A : \mathbb{R}^n \xrightarrow{A^{-1}} \mathbb{R}^n$$

$$A^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$AA^{-1} = A^{-1}A = I$$

$I$  = identity matrix

$$I = \begin{bmatrix} 1 & & & \\ & \ddots & & \emptyset \\ \emptyset & & \ddots & \\ & & & 1 \end{bmatrix}$$

If  $A^{-1}$  exists, matrix  
is invertible (square)

$A$  is invertible iff:  
(all of these are equivalent)

- 1)  $A$  is full rank  
 $\text{rank}(A) = n$
- 2) columns & rows are <sup>lin.</sup><sub>ind.</sub>
- 3)  $\det(\overset{\leftrightarrow}{A}) \neq 0$
- 4)  $(\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n)$   
 $\lambda=0$  is not an eigenvalue
- 5)  $Ax=0 \Leftrightarrow x=0$

## Properties of matrix algebra

$$C(A+B) = CA + CB$$

$$\begin{aligned} ABC &= (AB)C \\ &= A(BC) \end{aligned}$$

E.g.

$$A \vec{x} \vec{x}^T A = A \underbrace{(\vec{x} \vec{x}^T)}_B A$$
$$(\vec{x}^T y) B$$

scalar

$$(AB)^+ = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

Matrix "division" ~~X~~

→ multiplication by inverse

$$(A^{-1}) A B = C (B^{-1})$$

$$\cancel{B} = \cancel{C} / A$$

$$\underbrace{(A^{-1} A)}_{\sim} B = A^{-1} C$$

$$\underbrace{I}_{\sim} B = A^{-1} C$$

$$B = A^{-1} C$$

$$A = C B^{-1}$$

# Vector norms

$p$ -norm or  $L_p$  norm

$$p \geq 1$$

$$\|\vec{x}\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$

1)  $L_1$  norm (Manhattan norm)

$$p=1, \|\vec{x}\|_1 = \sum |x_i|$$

2)  $L_2$  norm (Euclidean norm)

$$\|\vec{x}\|_2 = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\vec{x}^T \vec{x}}$$

$$\|\vec{x}\|_2 = \sqrt{\sum |x_i|^2}$$

3)  $L_\infty$  norm or max norm

$$\|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

All norms are related

to each other if  $n$

$(\vec{x} \in \mathbb{R}^n)$  is "small"

$\|\cdot\|$  matrix norm

$$\frac{\|\mathbf{A} \mathbf{x}\|_{1,2,\infty}}{\|\mathbf{x}\|_{1,2,\infty}} = \|\mathbf{A}\|_{1,2,\infty}$$

$\sup_{\mathbf{x} \neq 0}$

Matrix norm induced by  
the vector norm

$$\|Ax\| \leq \|A\| \|x\|$$

$$\|AB\| \leq \|A\| \|B\|$$

$$1) \quad \|A\|_1 = \max_j \|A_{:,j}\|_1$$

j<sup>th</sup> column  
 ↑

$$= \max_j \sum_{i=1}^n |a_{i,j}|$$

$$2) \|A\|_\infty = \max_i \|A_{i,:}\|_1$$

$$= \max_i \sum_j |a_{ij}|$$

$$3) \|A\|_2 = \max_i \lambda_i$$

$\lambda^2$  is an eigenvalue of  
 $A^T A$  or  $A A^T$   
symmetric

$$\|A\|_2 = \max_i \sqrt{\lambda_{A^T A}}$$

In Matlab

$$\text{norm}(A, p) \quad \begin{matrix} 1 \\ 2 \\ \inf \end{matrix} \quad \begin{matrix} p=1 \\ p=2 \\ p=\infty \end{matrix}$$

Conditioning of  
mappings / matrices

$$\vec{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\frac{\|\vec{f}(\vec{x} + \vec{\delta x}) - \vec{f}(\vec{x})\|}{\|\vec{\delta x}\|}$$

$$\sup_{\vec{\delta x} \neq 0}$$

$$= \underset{x}{\text{Cond}}(f)$$

(Local absolute condition  
number in theory book)

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$\underset{x}{\text{Cond}}(f) = |f'(x)|$$

$$\|f(x+\delta x) - f(x)\| / \|f(x)\|$$

$$\text{cond}_x(f) = \sup_{\delta x \neq 0} \frac{\|\delta x\|}{\|f(x+\delta x) - f(x)\|}$$

If  $\text{cond}_x(f) = 10^4$

that means that

knowing 16 digits in  $x$

gives me  $16 - 4 = 12$  digits

in  $f(x)$

Loose four digits

$$\cancel{x} + \cancel{\delta x} - \cancel{x}$$

$$\text{cond}_x(Ax) = \sup_{\delta x \neq 0} \frac{\|A(x + \delta x) - Ax\| / \|Ax\|}{\|\delta x\| / \|x\|}$$

$$= \left( \sup_{\delta x \neq 0} \frac{\|A \delta x\|}{\|\delta x\|} \right) \frac{\|x\|}{\|Ax\|}$$

$$= \frac{\|x\|}{\|Ax\|} \|A\| \geq 1$$

$$\|Ax\| \leq \|A\| \|x\|$$

$$\text{cond}_x(Ax) = \frac{\|A^{-1}(A - x)\| \|A\|}{\|Ax\|}$$

$$\leq \frac{\|A^{-1}\| \|Ax\| \|A\|}{\|Ax\|} \\ \leq \|A\| \|A^{-1}\|$$

$$1 \leq \text{cond}_X(A) \leq \underline{\underline{\|A\| \|A^{-1}\|}}$$

Define cond. number of A

$$K(A) = \|A\| \|A^{-1}\|$$

↓  
 1, 2, ∞      ↓  
 1, 2, ∞      ↑  
 1, 2, ∞

## Eigenvalues

$$A \underset{\text{eigenvector}}{\underset{\uparrow}{X}} = \lambda \underset{\text{eigenvalue}}{\underset{\leftarrow}{X}}, \quad X \neq 0$$

$$A \in \mathbb{C}^{n \times n} \text{ or } \mathbb{R}^{n \times n}$$

$$Ax - \lambda x = 0$$

$$\underbrace{(A - \lambda I)}_{x \neq 0 ?} x = 0$$

$$\text{null } (A - \lambda I) \neq \{ \vec{0} \}$$

If  $A - \lambda I$  is invertible

$$X = (A - \lambda I)^{-1} 0 = 0$$

$A - \lambda I$  is not invertible

$$|A - \lambda I| = 0$$

Determinant

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \dots \\ \vdots & a_{22} - \lambda & \vdots \\ \vdots & \ddots & \ddots \\ & & a_{nn} - \lambda \end{bmatrix}$$

$$\det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} =$$

↑  
example

Characteristic  
polynomial

$$|A - \lambda I| = \text{poly}_n(\lambda) = 0$$

At most  $n$

If we allow  $\lambda \in \mathbb{C}$

At least one  $\lambda$  exists

At least one eigenvector  
for each distinct eigen.

$\times$  Not unique (can multiply  
by any constant)

Eigenvectors are directions  
(not vectors)

Matrix notation

$$\underline{\bar{X}} = [x_1 | x_2 | \dots | x_m]$$

↑ ↑ ↑

linearly independent

capital  $\lambda$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}$$

$$A \underline{x}_i = \lambda_i \underline{x}_i, i=1, \dots, m$$

$$A \underline{\bar{X}} = \underline{\bar{X}} \Lambda$$

$$1 \leq m \leq n$$

Is  $m = n$  ?  $\times$  not in general

Every  $\lambda$  has an algebraic multiplicity  $\alpha$  and a geometric multiplicity  $\beta$  (how many linearly independent eigenvectors)

$$1 \leq \beta \leq \alpha$$

If  $m = n$  we call that matrix non-selective or **diagonalizable matrix**

$$m = n \quad \sum \text{ is } [n \times n]$$

$\Rightarrow \sum$  is invertible  
Eigenvectors span all of  $\mathbb{R}^n$

$$X^{-1} \text{ exists } \boxed{AA^* = A^*A \Leftrightarrow A \text{ is non-defective}}$$

$$X^{-1}; AX = X\Lambda \quad | \quad X^{-1}$$

$\rightarrow$

$$\boxed{A = X\Lambda X^{-1}}$$

eigenvalue decomposition

If  $A$  is defective,  
Jordan form (decomposition)

$$\Lambda \rightarrow \begin{bmatrix} 1 & & & \\ - & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Not computable numerically

Assume  $A$  is non-defective

$$X^{-1} A X = \Lambda$$

Similarity transform

$A : \mathbb{R}^n \rightarrow \mathbb{R}^n$   
linear map

$$y = \sum_{i=1}^n a_i x_i$$

$$Ay = A \sum_i a_i x_i =$$

$$= \sum_i a_i (\underbrace{A x_i}_{\lambda_i x_i}) =$$

$$= \sum_i (a_i \lambda_i) x_i$$

{ In basis formed by eigenvectors  
 A is diagonal with  $\lambda_i$ 's  
 on the diagonal

{ If  $x_i$ 's are orthogonal  
 then matrix is called  
 unitarily diagonalizable

$$X \rightarrow U \text{ orthogonal matrix}$$

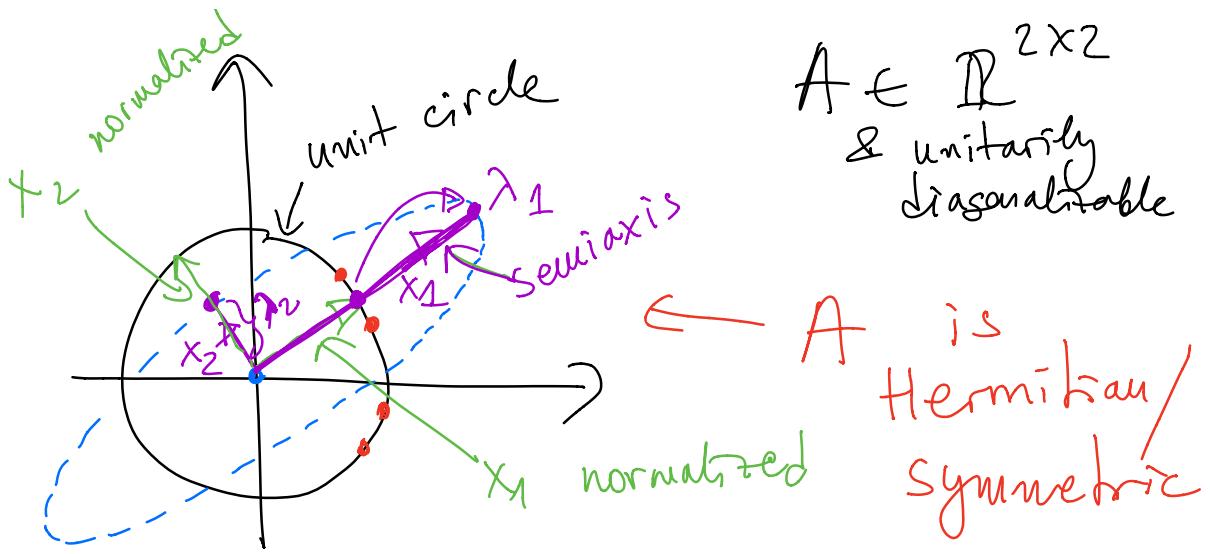
$$\|x_i\| = 1$$

$$U^{-1} = U^*$$

← complex conjugate transpose

$$A = U \Lambda U^*$$

Assume A can be factorized  
 like this



$$A = U \Lambda U^*$$

$$A^* = (U \Lambda U^*)^* =$$

$$= (U^*)^* \Lambda^* U^*$$

$$= U \Lambda^* U^*$$

If  $A^* = A$  Hermitian matrix  
or symmetric if in  $\mathbb{R}^{n \times n}$

$$\begin{aligned} & U \Lambda^* U^* = U \Lambda U^* \\ \Rightarrow & \Lambda^* = \Lambda \end{aligned}$$

eigenvalues are real

{ Theorem: If  $A^* = A$  then  
A is unitarily diagonalizable  
and eigenvalues are real

} From now on assume  
A is Hermitian }

If A is not unitarily  
diagonalizable, numerically  
best to use Schur  
decomposition

any matrix }  $A = U \underbrace{T}_{\text{upper triangular}} U^*$   
Eigenvalues on diagonal of T

Singular values of  
"tells you everything about A" matrix

"ultimate" decomposition

$$A = U \Sigma V^*$$

$\Sigma$  is Diagonal  
 $U$  is Unitary

works for any matrix  $A$ !

$$[m \times n] = [m \times m][m \times n][n \times n]$$

Singular value decomposition

Columns of  $U$  are left

Singular vectors

Columns of  $V$  are right

sing. vectors

Diag. elements of  $\Sigma$  are called singular values

$$\Sigma = m \begin{bmatrix} b_1 & \cdots & b_m \\ \vdash & \ddots & \vdash \\ b_m & \cdots & b_n \end{bmatrix} \quad \text{if } m < n$$

$$\Sigma = \begin{bmatrix} b_1 & \cdots & b_n \\ \vdash & \ddots & \vdash \\ b_m & \cdots & b_n \end{bmatrix}^n \quad \text{if } m > n$$

$$b_1, \dots, b_p \geq 0$$

$$b_{p+1}, \dots = 0$$

$$\phi = \min(m, n)$$

Observe

If  $A$  is Hermitian

$$\left\{ \begin{array}{l} A = U \cap U^* \\ A = U \sum V^* \end{array} \right. \quad \begin{array}{l} U = V \\ \Sigma = \Lambda \end{array}$$

For symmetric matrices  
no difference between eigenvalue  
 $\nwarrow$   
real  
& singular value decomposition.

{ SVD works for all  
matrices & can be  
computed numerically in  
a stable way