Worksheet 7 (April 14th, 2021)

1 More on Polynomial Interpolation

In this worksheet you will explore some topics in polynomial interpolation. Each student should work on their own for a bit and then you can discuss and compare results, and come up with testing/validation strategies to see whose answer is correct as a group.

1.1 Approximating sine

Consider approximating $f(x) = \sin(\pi x)$ on $x \in [0, 1]$.

1.1.1 Lagrange Interpolation

Write down the interpolating polynomial of degree 2 (parabola) with equi-spaced nodes.

1.1.2 Hermite Interpolation

We want to find a polynomial $p(x)$ of as low a degree as possible such that the polynomial matches the value of the function *and* of the derivative at the end-points, $p(0) = f(0)$, $p(1) = f(1)$, $p'(0) = f'(0)$, $p'(1) = f'(1)$. Find such a Hermite polynomial approximant and then plot it on the same graph with the approximant from part 1.1, along with the true function.

1.2 Approximating derivatives and integrals

Consider a function $f(x)$ such that you know it at three equi-spaced nodes, for example, $f(-h) = f^-$, $f(h) = f^+$ and $f(0) = f^0$. If we approximate the function with an interpolating polynomial, then we can differentiate that polynomial to approximate the derivative of the function. Similarly, to integrate the function we can integrate the polynomial instead.

1. Use this approach to approximate $f'(0)$ and $f''(0)$. Find some way to validate your answer (as a group).
2. Use this approach to approximate $I = \int_{-h}^{h} f(x)dx$. Find some way to validate your answer (as a group).
3. Use the formula from part 1 to approximate $\sin''(\pi/4)$ for $h = 2^{-p}$ with $p = 3, 4, 5$,6 and report the error you get. Do you see any patterns?

4. (Optional) Another approach to approximating derivatives is to first approximate $f(x)$ with its Taylor series,
$$f(x) \approx \tilde{f}(x) = f^0 + f'(0)x + \frac{1}{2}f''(0)x^2,$$
and then obtain $f'(0)$ and $f''(0)$ from the equations $\tilde{f}(-h) = f^-$, $\tilde{f}(h) = f^+$. What does this approach give?

1.3 Interpolating derivatives

Consider n nodes $x_k \in [a, b]$, and let $L_k(x)$ be the Lagrange polynomial associated with node k. Show that the polynomial
$$K_k(x) = (L_k(x))^2 (x - x_k)$$
satisfies
$$\begin{aligned} K_k(x_i) &= 0 \quad \forall i \\ K_k'(x_i) &= \delta_{ik}, \end{aligned}$$
where δ_{ik} is the Kronecker delta symbol (one if $i = k$, zero otherwise).

From this result, find a polynomial $p(x)$ that has $x = 0, 1, 2$ as roots (it may have more roots), and satisfies
$$p'(0) = 1, p'(1) = -1, p'(2) = 2.$$
[*Hint: Think back to Lagrange interpolation and use the same idea.*] Make sure you test/validate your result.