# Numerical Analysis
# (Review of) Linear Algebra

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# Outline

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# Linear Spaces

- A **vector space** $\mathcal{V}$ is a set of elements called **vectors** $\mathbf{x} \in \mathcal{V}$ that may be multiplied by a **scalar** $c$ and added, e.g.,

$$\mathbf{z} = \alpha \mathbf{x} + \beta \mathbf{y}$$

- I will denote scalars with lowercase letters and vectors with lowercase bold letters.
- Prominent examples of vector spaces are $\mathbb{R}^n$ (or more generally $\mathbb{C}^n$), but there are many others, for example, the set of polynomials in $x$.
- A **subspace** $\mathcal{V}' \subseteq \mathcal{V}$ of a vector space is a subset such that sums and multiples of elements of $\mathcal{V}'$ remain in $\mathcal{V}'$ (i.e., it is closed).
- An example is the set of vectors in $x \in \mathbb{R}^3$ such that $x_3 = 0$.

# Image Space

- Consider a set of $n$ vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n \in \mathbb{R}^m$ and form a **matrix** by putting these vectors as columns

$$\mathbf{A} = [\mathbf{a}_1 \,|\, \mathbf{a}_2 \,|\, \cdots \,|\, \mathbf{a}_m] \in \mathbb{R}^{m,n}.$$

- I will denote matrices with bold capital letters, and sometimes write $\mathbf{A} = [m, n]$ to indicate dimensions.
- The **matrix-vector product** is defined as a **linear combination** of the columns:

$$\mathbf{b} = \mathbf{A}\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \in \mathbb{R}^m.$$

- The **image** $\text{im}(\mathbf{A})$ or **range** $\text{range}(\mathbf{A})$ of a matrix is the subspace of all linear combinations of its columns, i.e., the set of all $\mathbf{b}'s$.
  It is also sometimes called the **column space** of the matrix.

# Dimension

- The set of vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are **linearly independent** or form a **basis** for $\mathbb{R}^m$ if $\mathbf{b} = \mathbf{A}\mathbf{x} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$.
- The **dimension** $r = \dim\mathcal{V}$ of a vector (sub)space $\mathcal{V}$ is the number of elements in a basis. This is a property of $\mathcal{V}$ itself and *not* of the basis, for example,

$$\dim \mathbb{R}^n = n$$

- Given a basis $\mathbf{A}$ for a vector space $\mathcal{V}$ of dimension $n$, every vector of $\mathbf{b} \in \mathcal{V}$ can be uniquely represented as the vector of coefficients $\mathbf{x}$ in that particular basis,

$$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

- A simple and common basis for $\mathbb{R}^n$ is $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}_k$ has all components zero except for a single 1 in position $k$.
  With this choice of basis the coefficients are simply the entries in the vector, $\mathbf{b} \equiv \mathbf{x}$.

# Kernel Space

- The dimension of the column space of a matrix is called the **rank** of the matrix $\mathbf{A} \in \mathbb{R}^{m,n}$,
$$r = \operatorname{rank} \mathbf{A} \leq \min(m, n).$$
- If $r = \min(m, n)$ then the matrix is of **full rank**.
- The **nullspace** $\operatorname{null}(\mathbf{A})$ or **kernel** $\ker(\mathbf{A})$ of a matrix $\mathbf{A}$ is the subspace of vectors $\mathbf{x}$ for which
$$\mathbf{A}\mathbf{x} = \mathbf{0}.$$
- The dimension of the nullspace is called the **nullity** of the matrix.
- For a basis $\mathbf{A}$ the nullspace is $\operatorname{null}(\mathbf{A}) = \{\mathbf{0}\}$ and the nullity is zero.

# Orthogonal Spaces

- An inner-product space is a vector space together with an **inner or dot product**, which must satisfy some properties.
- The standard dot-product in $\mathbb{R}^n$ is denoted with several different notations:

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

- For $\mathbb{C}^n$ we need to add complex conjugates (here $\star$ denotes a complex conjugate transpose, or **adjoint**),

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^\star \mathbf{y} = \sum_{i=1}^{n} \bar{x}_i y_i.$$

- Two vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if $\mathbf{x} \cdot \mathbf{y} = 0$.

# Part I of Fundamental Theorem

- One of the most important theorems in linear algebra is that the sum of rank and nullity is equal to the number of columns: For $\mathbf{A} \in \mathbb{R}^{m,n}$

$$\operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A} = n.$$

- In addition to the range and kernel spaces of a matrix, two more important vector subspaces for a given matrix $\mathbf{A}$ are the:
  - **Row space** or **coimage** of a matrix is the column (image) space of its transpose, $\operatorname{im} \mathbf{A}^T$.
    *Its dimension is also equal to the the rank*.
  - **Left nullspace** or **cokernel** of a matrix is the nullspace or kernel of its transpose, $\ker \mathbf{A}^T$.

# Part II of Fundamental Theorem

- The **orthogonal complement** $\mathcal{V}^{\perp}$ or orthogonal subspace of a subspace $\mathcal{V}$ is the set of all vectors that are orthogonal to every vector in $\mathcal{V}$.
- Let $\mathcal{V}$ be the set of vectors in $x \in \mathbb{R}^3$ such that $x_3 = 0$. Then $\mathcal{V}^{\perp}$ is the set of all vectors with $x_1 = x_2 = 0$.
- Second fundamental theorem in linear algebra:

$$\operatorname{im} \mathbf{A}^T = (\ker \mathbf{A})^{\perp}$$

$$\ker \mathbf{A}^T = (\operatorname{im} \mathbf{A})^{\perp}$$

# Outline

# Linear Transformation

- A function $L : \mathcal{V} \to \mathcal{W}$ mapping from a vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is a **linear function** or a **linear transformation** if

$$L(\alpha \mathbf{v}) = \alpha L(\mathbf{v}) \text{ and } L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2).$$

- Any linear transformation $L$ can be represented as a multiplication by a matrix $\mathbf{L}$

$$L(\mathbf{v}) = \mathbf{L}\mathbf{v}.$$

- For the common bases of $\mathcal{V} = \mathbb{R}^n$ and $\mathcal{W} = \mathbb{R}^m$, the product $\mathbf{w} = \mathbf{L}\mathbf{v}$ is simply the usual **matix-vector product**,

$$w_i = \sum_{k=1}^{n} L_{ik} v_k,$$

which is simply the dot-product between the $i$-th row of the matrix and the vector $\mathbf{v}$.

# Matrix algebra

$$w_i = (\mathbf{L}\mathbf{v})_i = \sum_{k=1}^{n} L_{ik} v_k$$

- The composition of two linear transformations $\mathbf{A} = [m, p]$ and $\mathbf{B} = [p, n]$ is a **matrix-matrix product** $\mathbf{C} = \mathbf{AB} = [m, n]$:

$$\mathbf{z} = \mathbf{A}(\mathbf{B}\mathbf{x}) = \mathbf{A}\mathbf{y} = (\mathbf{AB})\,\mathbf{x}$$

$$z_i = \sum_{k=1}^{n} A_{ik} y_k = \sum_{k=1}^{p} A_{ik} \sum_{j=1}^{n} B_{kj} x_j = \sum_{j=1}^{n} \left( \sum_{k=1}^{p} A_{ik} B_{kj} \right) x_j = \sum_{j=1}^{n} C_{ij} x_j$$

$$C_{ij} = \sum_{k=1}^{p} A_{Ik} B_{kj}$$

- Matrix-matrix multiplication is **not commutative**, $\mathbf{AB} \neq \mathbf{BA}$ in general.

# The Matrix Inverse

- A square matrix $\mathbf{A} = [n, n]$ is **invertible or nonsingular** if there exists a **matrix inverse** $\mathbf{A}^{-1} = \mathbf{B} = [n, n]$ such that:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I},$$

  where $\mathbf{I}$ is the identity matrix (ones along diagonal, all the rest zeros).
- The following statements are equivalent for $\mathbf{A} \in \mathbb{R}^{n,n}$:
  - $\mathbf{A}$ is **invertible**.
  - $\mathbf{A}$ is **full-rank**, $\operatorname{rank} \mathbf{A} = n$.
  - The columns and also the rows are linearly independent and form a **basis** for $\mathbb{R}^n$.
  - The **determinant** is nonzero, $\det \mathbf{A} \neq 0$.
  - Zero is not an eigenvalue of $\mathbf{A}$.

# Matrix Algebra

- Matrix-vector multiplication is just a special case of matrix-matrix multiplication. Note $\mathbf{x}^T\mathbf{y}$ is a scalar (dot product).

$$\mathbf{C}\left(\mathbf{A}+\mathbf{B}\right)=\mathbf{CA}+\mathbf{CB} \text{ and } \mathbf{ABC}=\left(\mathbf{AB}\right)\mathbf{C}=\mathbf{A}\left(\mathbf{BC}\right)$$

$$\left(\mathbf{A}^T\right)^T=\mathbf{A} \text{ and } \left(\mathbf{AB}\right)^T=\mathbf{B}^T\mathbf{A}^T$$

$$\left(\mathbf{A}^{-1}\right)^{-1}=\mathbf{A} \text{ and } \left(\mathbf{AB}\right)^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1} \text{ and } \left(\mathbf{A}^T\right)^{-1}=\left(\mathbf{A}^{-1}\right)^T$$

- Instead of **matrix division**, think of multiplication by an inverse:

$$\mathbf{AB}=\mathbf{C} \quad \Rightarrow \quad \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B}=\mathbf{A}^{-1}\mathbf{C} \quad \Rightarrow \quad \begin{cases} \mathbf{B} & =\mathbf{A}^{-1}\mathbf{C} \\ \mathbf{A} & =\mathbf{CB}^{-1} \end{cases}$$

# Outline

# Vector norms

- Norms are the abstraction for the notion of a length or **magnitude**.
- For a vector $\mathbf{x} \in \mathbb{R}^n$, the $p$-norm is

$$\|\mathbf{x}\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

and special cases of interest are:

1. The 1-norm ($L^1$ norm or Manhattan distance), $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$
2. The 2-norm ($L^2$ norm, **Euclidian distance**), $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$
3. The $\infty$-norm ($L^\infty$ or maximum norm), $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$

1. Note that all of these norms are inter-related in a finite-dimensional setting.

# Matrix norms

- Matrix norm **induced** by a given vector norm:

$$\|\mathbf{A}\| = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} \quad \Rightarrow \|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\| \, \|\mathbf{x}\|$$

- The last bound holds for matrices as well, $\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\| \, \|\mathbf{B}\|$.
- Special cases of interest are:
  1. The 1-norm or **column sum norm**, $\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$
  2. The $\infty$-norm or **row sum norm**, $\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$
  3. The 2-norm or **spectral norm**, $\|\mathbf{A}\|_2 = \sigma_1$ (largest singular value)
  4. The Euclidian or **Frobenius norm**, $\|\mathbf{A}\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$ (note this is not an induced norm)

# Outline

# Conditioning

- Consider a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, and perturb $\mathbf{x}$ to the **absolute condition number**

$$\text{Cond}_{\mathbf{x}}(f) = \sup_{\delta\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{f}(\mathbf{x}+\delta\mathbf{x}) - \mathbf{f}(\mathbf{x})\|}{\|\delta\mathbf{x}\|}$$

  where $\|\delta\mathbf{x}\| \ll \|\mathbf{x}\|$ is a small perturbation (assume $\mathbf{x} \neq \mathbf{0}$).
- This measures how **sensitive** the value of the function is to small errors in the input (e.g., roundoff or measurement).
- For differentiable scalar functions $f(x \in \mathbb{R}) \in \mathbb{R}$,

$$\text{Cond}_x(f) = \left|f'(x)\right|.$$

- More commonly used is the **relative condition number**

$$\text{cond}_{\mathbf{x}}(f) = \sup_{\delta\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{f}(\mathbf{x}+\delta\mathbf{x}) - \mathbf{f}(\mathbf{x})\| / \|\mathbf{f}(\mathbf{x})\|}{\|\delta\mathbf{x}\| / \|\mathbf{x}\|}$$

  which measures the maximum relative change in the output for a given small relative change in the input.

# Conditioning number

- Consider a linear mapping $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. What is the relative conditioning number?

$$\begin{aligned}\text{cond}_{\mathbf{x}}(f) &= \sup_{\delta\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}(\mathbf{x}+\delta\mathbf{x}) - \mathbf{A}\mathbf{x}\| / \|\mathbf{A}\mathbf{x}\|}{\|\delta\mathbf{x}\| / \|\mathbf{x}\|} \\ &= \frac{\|\mathbf{x}\|}{\|\mathbf{A}\mathbf{x}\|} \sup_{\delta\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\delta\mathbf{x}\|}{\|\delta\mathbf{x}\|} = \\ &= \frac{\|\mathbf{x}\|}{\|\mathbf{A}\mathbf{x}\|} \|\mathbf{A}\| \geq 1.\end{aligned}$$

- To get an upper bound, consider an invertible square $\mathbf{A}$,

$$\text{cond}_{\mathbf{x}}(f) = \frac{\left\|\mathbf{A}^{-1}(\mathbf{A}\mathbf{x})\right\|}{\|\mathbf{A}\mathbf{x}\|} \|\mathbf{A}\| \leq \left\|\mathbf{A}^{-1}\right\| \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{A}\mathbf{x}\|} \|\mathbf{A}\|$$

which leads us to define a **matrix condition number**

$$\kappa(\mathbf{A}) = \left\|\mathbf{A}^{-1}\right\| \|\mathbf{A}\| > 1.$$

# Outline

# Eigenvalue Decomposition

- For a square matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$, there exists at least one $\lambda$ such that

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \quad \Rightarrow \quad (\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0}$$

- Putting the **eigenvectors** $\mathbf{x}_j$ as columns in a matrix $\mathbf{X}$, and the **eigenvalues** $\lambda_j$ on the diagonal of a diagonal matrix $\boldsymbol{\Lambda}$, we get

$$\mathbf{A}\mathbf{X} = \mathbf{X}\boldsymbol{\Lambda}.$$

- A matrix is **non-defective** or **diagonalizable** if there exist $n$ **linearly independent eigenvectors**, i.e., if the matrix $\mathbf{X}$ is invertible:

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \boldsymbol{\Lambda}$$

leading to the **eigen-decomposition** of the matrix

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}.$$

# Unitarily Diagonalizable Matrices

- A **unitary** (complex) or **orthogonal** (real) matrix $\mathbf{U}$ has orthogonal colums each of which has unit $L_2$ norm:

$$\mathbf{U}^{-1} = \mathbf{U}^{\star}.$$

  Recall that star denotes **adjoint** (conjugate transpose).

- A matrix is **unitarily diagonalizable** if there exist $n$ linearly independent **orthogonal eigenvectors**, $\mathbf{X} \equiv \mathbf{U}$,

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\star}.$$

  There is a **geometric interpretation** of this (sphere->ellipsoid).

- Theorem: **Hermitian matrices**, $\mathbf{A}^{\star} = \mathbf{A}$, are unitarily diagonalizable and have **real eigenvalues**.
  For real matrices we use the term **symmetric**.

# Non-diagonalizable Matrices

- For matrices that are not diagonalizable, one can use **Jordan form factorizations**, or, more relevant to numerical mathematics, the **Schur factorization** (decomposition):

$$\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^{\star},$$

  where **T** is **upper-triangular** (unlike **Jordan form** where only nonzeros are on super-diagonal).
- The eigenvalues are on the diagonal of **T**, and in fact if **A** is unitarily diagonalizable then $\mathbf{T} \equiv \mathbf{\Lambda}$.
- The Schur decomposition is **not unique** but it is the best generalization of the eigenvalue (spectral) decomposition to general matrices.

# Singular Value Decomposition (SVD)

Every matrix has a **singular value decomposition (SVD)**

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\star} = \sum_{i=1}^{p} \sigma_i \mathbf{u}_i \mathbf{v}_i^{\star}$$

$$[m \times n] = [m \times m]\,[m \times n]\,[n \times n]\,,$$

where **U** and **V** are **unitary matrices** whose columns are the left, $\mathbf{u}_i$, and the right, $\mathbf{v}_i$, **singular vectors**, and

$$\boldsymbol{\Sigma} = \mathrm{Diag}\left\{\sigma_1, \sigma_2, \ldots, \sigma_p\right\}$$

is a **diagonal matrix** with real positive diagonal entries called **singular values** of the matrix

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0,$$

and $p = \min(m, n)$ is the maximum possible rank of the matrix.

# Comparison to eigenvalue decomposition

- Recall the eigenvector decomposition for diagonalizable matrices

$$\mathbf{AX} = \mathbf{X\Lambda}.$$

- The singular value decomposition can be written similarly to the eigenvector one

$$\mathbf{AV} = \mathbf{U\Sigma}$$
$$\mathbf{A}^{\star}\mathbf{U} = \mathbf{V\Sigma}$$

and they both **diagonalize A**, but there are some important **differences**:

1. The SVD exists for any matrix, not just diagonalizable ones.
2. The SVD uses different vectors on the left and the right (different basis for the domain and image of the linear mapping represented by **A**).
3. The SVD always uses orthonormal basis (unitary matrices), not just for unitarily diagonalizable matrices.

# Relation to Eigenvalues

- For **Hermitian (symmetric) matrices**, there is **no fundamental difference** between the SVD and eigenvalue decompositions (and also the Schur decomposition).
- The squared singular values are **eigenvalues of the normal matrix**:

$$\sigma_i(\mathbf{A}) = \sqrt{\lambda_i(\mathbf{A}\mathbf{A}^\star)} = \sqrt{\lambda_i(\mathbf{A}^\star\mathbf{A})}$$

  since

$$\mathbf{A}^\star\mathbf{A} = (\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^\star)(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\star) = \mathbf{V}\boldsymbol{\Sigma}^2\mathbf{V}^\star$$

- Similarly, the singular vectors are eigenvectors of $\mathbf{A}^\star\mathbf{A}$ or $\mathbf{A}\mathbf{A}^\star$.

# Matrix norms

- Recall: Matrix norm **induced** by a given vector norm:

$$\|\mathbf{A}\| = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} \quad \Rightarrow \|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\| \, \|\mathbf{x}\|$$

- Special cases of interest are:
  1. The 2-norm or **spectral norm**, $\|\mathbf{A}\|_2 = \sigma_1$ (largest singular value)
  2. The Euclidian or **Frobenius norm**, $\|\mathbf{A}\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$ (note this is not an induced norm)
- Unitary matrices are important because they are **always well-conditioned**, $\kappa_2(\mathbf{U}) = 1$.