

Polynomial Approximation

in L_2

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So far we talked about interpolation as a way to approximate a non polynomial function $f(x)$ on some interval $[a, b]$. The advantages are:

- 1) All we needed were values of the function at the $(n+1)$ interpolation nodes, we didn't even need to know the function $f(x)$ [^{"black box"} mode]

①

2) Easy to evaluate interpolant using barycentric interpolation.

The main disadvantages are:

1) The choice of nodes really matters and equi-spaced is bad

2) Even if $P_n(x_k) = f(x_k)$ at the nodes, this does not guarantee anything about $|f(x) - p(x)|$ for x in-between the nodes (recall Runge function)

We improved on these issues

by using piecewise polynomial interpolation like splines, but that was not very accurate

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so to get 16 digits in $f(x)$
we would need thousands of
nodes/points.

Is there another way to
approximate $f(x)$ by $P_n(x)$
on $[a, b]$?

Yes!

$$P_n^* = \arg \min_{P_n \in \mathcal{P}^n} \|f - P_n\|_2$$

"Least squares" polynomial approx,

This is just like doing least
squares fitting to data, but
now we use the function
 L_2 norm not vector one.

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E.g. Approximate $f(x) = -2x^2$
 with a constant function ($n=0!$)
 on $[-1, 1]$:

$$\|f - p_0\|_2^2 = \int_{-1}^1 (-2x - a_0)^2 dx$$

$$\boxed{P_0(x) = a_0} = 2a_0^2 + \frac{8a_0}{3} + \frac{8}{5}$$

$$a_0^* = \arg \min_{a_0} \left(2a_0^2 + \frac{8a_0}{3} + \frac{8}{5} \right)$$

$$\left. \frac{d}{da_0} \left(2a_0^2 + \frac{8a_0}{3} + \frac{8}{5} \right) \right|_{a_0=a^*} = 0 \Rightarrow$$

$$a_0^* = -\frac{2}{3} = p_0^*(x)$$

Better & more instructive
 is to do this for a general
 $f(x)$ [OFTEN SIMPLER!!!] ④

$$a_0^* = \arg \min \int (f(x) - a_0)^2 dx$$

$$= \arg \min \int f(x)^2 dx - 2a_0 \int f(x) dx$$


 $+ \int a_0^2 dx$

Differentiate w.r.t a_0 :

$$2 a_0^* \int dx - 2 \int f(x) dx = 0$$

$$\Rightarrow a_0^* = \frac{\int f(x) dx}{\int dx} = \frac{\int f(x) dx}{b-a}$$

which is simply the mean
 (average) of $f(x)$ over $[a, b]$.

$$\text{If } f(x) = -2x^2, \text{ mean} = \frac{-2 \int_{-1}^1 x^2 dx}{2} = -\frac{2}{3} \quad (5)$$

Let's now repeat this for an arbitrary degree polynomial.

First, choose a basis for \mathcal{P}_n :

$$\text{basis} = \{p_0(x), p_1(x), p_2(x), \dots, p_n(x)\}$$

$$p_n^*(x) = \sum_{j=0}^n a_j p_j(x) = \begin{matrix} \text{best } L_2 \\ \text{approximation} \end{matrix}$$

How do we find the $(n+1)$ coefficients \vec{a} ? (drop *)

Standard approach in textbooks:

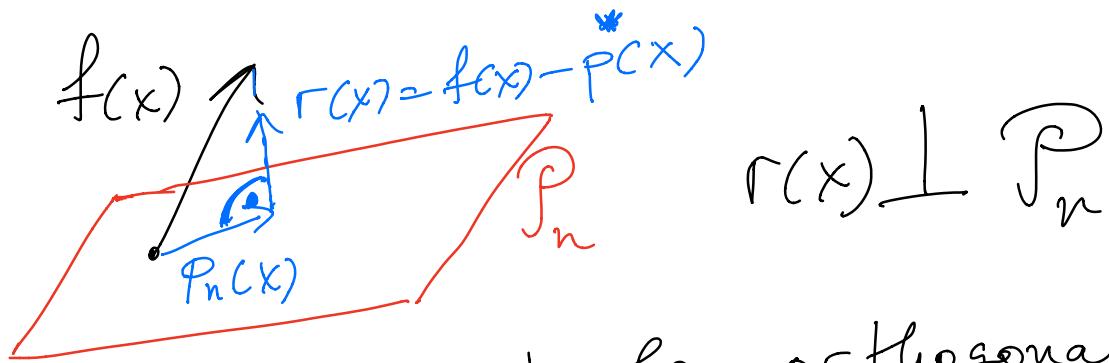
$$F(\vec{a}) = \int_a^b \left(f(x) - \sum c_j p_j(x) \right)^2 dx$$

$$\frac{\partial F}{\partial a_k} = 0, \quad k=0, \dots, n+1$$

$\nwarrow_{(n+1) \text{ equations}}$

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Easy to do by expanding the square. But let's instead follow a more linear algebra "geometric" approach:



$$r(x) \perp P_n$$

Residual must be orthogonal to all polynomials of degree n , i.e., it must be orthogonal to $P_k(x)$, $k=0, \dots, n$.

$$(r, P_k) = 0 \quad \forall k$$

$$(f - P_n^*, P_k) = 0 \quad \forall k$$

\Rightarrow

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$$(f - p_n^*, \varphi_k) = 0 \quad \forall k$$

$$(f - \sum_{j=0}^n a_j \varphi_j, \varphi_k) = 0$$

$$\left(\sum_{j=0}^n a_j \varphi_j, \varphi_k \right) = (f, \varphi_k)$$

 (properties of inner product)

$$\sum_{j=0}^n a_j (\varphi_j, \varphi_k) = (f, \varphi_k)$$

$k = 0, \dots, n+1$

 n+1 equations for n+1 unknowns

⑧

In matrix form

$$\begin{matrix} \leftarrow \\ V \end{matrix} \vec{a} = \vec{f}$$

$$V_{ij} = (P_i, P_j) = \int_a^b P_i(x) P_j(x) dx$$

(A "Vandermonde" matrix)

$$\vec{f}_k = (f, P_k) = \int_a^b f(x) P_k(x) dx$$

Here we assumed a real-valued function.

Once again, like for interpolation, it boils down to solving a linear system!

Expensive? Ill-conditioned?

⑨

Let's take monomials as

basis :

$$P_k(x) = x^k \text{ on } [0, 1]$$

$$V_{ij} = \int_0^1 x^i x^j dx = \frac{x^{i+j+1}}{i+j+1} \Big|_0^1$$

$$V_{ij} = \frac{1}{i+j+1} \leftarrow \begin{matrix} \text{Hilbert} \\ \text{matrix} \end{matrix}$$

You learned in worksheets that the Hilbert matrix is very ill conditioned, just like the Vandermonde matrix.

So using a monomial basis is not a good idea.

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Instead, we need something akin to Lagrange polynomials for interpolation but for L_2 approximation.

What this means is that we want to choose basis such that $V_{ij} = \delta_{ij}$, i.e.,

\vec{V} is the identity matrix, because then we have

$$a_k = (f, p_k) = \int f(x) p_k(x) dx$$

if $V_{ij} = \int p_i(x) p_j(x) dx = \delta_{ij}$

$$\Rightarrow \int p_i(x) p_j(x) dx = 0$$

(11) if $i \neq j$ or 1 if $i=j$

This means that we want to use as basis a set of $(n+1)$ orthogonal polynomials.

How do we find an orthonormal basis for P_n ?

I.E. how do we find an orthonormal basis for the space spanned by $\{x^0, x^1, \dots, x^n\}$?

In the case of vectors, we did this using QR factorization, which was really simply doing the Gram-Schmidt orthogonalization process.

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Let's illustrate this on a very important example:

Find the first 3 orthogonal polynomials on $[-1, 1]$ in the standard L_2 inner product

$$(f, g)_2 = \int_{-1}^1 f(x) g(x) dx$$

Assume real-valued functions

Start with the 3 functions

$$\{1, x, x^2\}$$

(think of a matrix with 3 columns) and then find an orthogonal basis for P_2 (think of QR factorization of the matrix)

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Gram-Schmidt process

$$P_0(x) = 1$$

$$P_2(x) = x^2$$

$$P_1(x) = x$$

~~~~~  
We want  $\varphi_0(x), \varphi_1(x), \varphi_2(x)$   
that are orthogonal to each  
other

$$\varphi_0(x) = P_0(x) = 1$$

$$\varphi_1 = P_1 - \text{Proj}_{\{\varphi_0\}} P_1$$

$$= P_1 - \frac{(\varphi_0, P_1)}{(\varphi_0, \varphi_0)} \varphi_0 =$$

$$= x - \frac{\int_{-1}^1 x \cdot 1 \cdot dx}{\int_{-1}^1 1 \cdot 1 \cdot dx} \cdot 1 = x = P_1$$

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Since  $p_1$  and  $p_0$  are already orthogonal on  $[-1, 1]$ .

Now

$$\varphi_2 = p_2 - \text{Proj}_{\{\varphi_0, \varphi_1\}} p_2 =$$

$$= p_2 - \frac{(\varphi_0, p_2)}{(\varphi_0, \varphi_0)} \varphi_0 - \frac{(\varphi_1, p_2)}{(\varphi_1, \varphi_1)} \varphi_1$$

$$= x^2 - \frac{\int_{-1}^1 x^2 dx}{\int_{-1}^1 1 \cdot dx} - \frac{\int_{-1}^1 x^2 \cdot x \cdot dx}{\int_{-1}^1 x^2 dx} \cdot x$$

(But  $\int_{-1}^1 x^2 dx = \frac{x^3}{3} \Big|_{-1}^1 = \frac{2}{3}$ ,  $\int_{-1}^1 1 dx = 2$   
 $\int_{-1}^1 x^3 dx = \frac{x^4}{4} \Big|_{-1}^1 = 0$ )

$$= x^2 - \frac{1}{3} - 0 = x^2 - \frac{1}{3}$$
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So we obtain that

$$\varphi_0(x) = 1$$

$$\varphi_1(x) = x$$

$$\varphi_2(x) = x^2 - \frac{1}{3}$$

$$\varphi_n(x) = x^n - \sum_{j=0}^{n-1} \left( \frac{\int_{-1}^1 x^n \varphi_j(x) dx}{\int_{-1}^1 \varphi_j^2(x) dx} \right) \varphi_j(x)$$

are orthogonal polynomials in  $L_2$   
on  $[-1, 1]$

They are known as Legendre  
polynomials and are very  
important for polynomial  
approximation

It is easy to see by change  
of coordinate

$$t = \left(\frac{x+1}{2}\right)(b-a) + a \in [a, b]$$

that if  $j \neq k$

$$0 = \int_a^b \varphi_j(t) \varphi_k(t) dt = \int_{-1}^1 \varphi_j(x) \varphi_k(x) dx$$

which means  $\{\varphi_k(t)\}$  are

orthogonal on  $[a, b]$  in  $L_2$ .

So we can easily generalize  
to another interval  $[a, b]$ ,  
though  $[-1, 1]$  is standard.

Now go back to our goal  
of approximating functions:

$$P_n^* = \arg \min_{P_n \in \mathcal{P}^n} \|f - P_n\|_2$$

$$P_n^*(x) = \sum_{j=0}^n a_j P_j(x) = \text{best } L_2 \text{ approximation}$$

$$a_k = (f, P_k) = \int f(x) P_k(x) dx$$

$$\text{if } V_{ij} = \int P_i(x) P_j(x) dx = \delta_{ij}$$

Now we have polynomials s.t.

$$V_{ij} = \int_{-1}^1 \varphi_i(x) \varphi_j(x) dx = 0 \text{ if } i \neq j$$

(orthogonal Legendre polynomials)

They are not orthonormal  
because we did not normalize  
them, i.e., in general

$$\int \varphi_j^2(x) dx \neq 1$$

(we could have normalized them!)

So now we have

$$f(x) \approx P_n^*(x) = \sum_{k=0}^n a_k x^k \quad (\text{best } L_2 \text{ approx. of } f \text{ in } P_n)$$

$$a_k = \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)} \leftarrow \begin{array}{l} \text{explicit} \\ \text{formula} \\ \text{by integral} \end{array}$$

This works for any choice  
of the inner product, interval  
[a, b] - what changes are the  
 $\varphi_k(x)$ .

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E.g. Find the best quadratic approximation to  $\sin(x)$  on  $[0, \pi]$  in standard  $L_2$  norm.

Solution:  $t = \frac{x+1}{2}\pi \Rightarrow x = \frac{2t}{\pi} - 1$

$$\varphi_0(t) = 1$$

$$\varphi_1(t) = x = \frac{2t}{\pi} - 1$$

$$\varphi_2(t) = x^2 - \frac{1}{3} = \left(\frac{2t}{\pi} - 1\right)^2 - \frac{1}{3}$$

$$= \frac{4}{\pi} \left( \frac{t^2}{\pi} - t + \frac{\pi}{6} \right)$$

$$a_k = \frac{1}{\int_0^\pi \varphi_k^2(t) dt} \int_0^\pi \sin(t) \varphi_k(t) dt$$

$$k = 0, 1, 2$$

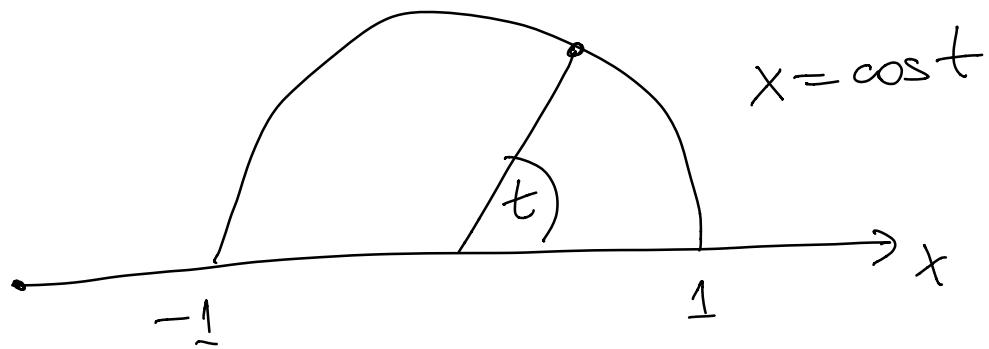
Homework: Compute  $\varphi_2^*(x)$  and plot it & compare to worksheet 7

## Chebyshev polynomials

It is not hard to show that

$$\int_0^{\pi} \cos(m t) \cos(n t) dt = 0$$

So  $\cos(k t)$  are orthogonal functions (called trigonometric polynomials) on  $(0, \pi)$



$$t = a \cos X \Rightarrow$$

$$dt = \frac{1}{\sqrt{1-x^2}} dx$$

$$\Rightarrow \int_{-1}^1 \cos(m \cdot \cos(x)) \cdot \cos(n \cdot \cos(x)) \cdot \frac{dx}{\sqrt{1-x^2}} = 0 \quad \dots \quad (\ast)$$

Define

$$T_k(x) = \cos(k \cdot \cos(x))$$

$$k = 0, 1, 2, \dots$$

It turns out that  $T_k(x)$  is actually a polynomial of  $x$  of degree  $k$ !

The Chebyshev polynomials  $T_k(x)$  are orthogonal on  $[-1, 1]$  with respect to the weighted

inner product

$$(f, g) = \int_{-1}^1 f g \frac{dx}{\sqrt{1-x^2}}$$

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Homework: Use Gram-Schmidt  
to obtain explicit formulas  
for

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_2(x) = 2x^2 - 1 \end{cases}$$

$$T_3(x) = 4x^3 - 3x$$

...

One can use either  
Legendre or Chebyshev  
polynomials, they are similar  
& which one is used depends  
on the context & choice by  
the numerical analyst.

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## Orthogonal polynomials & computation

We saw that to compute orthogonal polynomials or  $L_2$  function approximants, we need to be able to numerically compute integrals - we will study this next & see that orthogonal polynomials will make a re-appearance!

In practice, we rarely use optimal  $L_2$  approximation because it is hard to compute.

But, it turns out that orthonormal polynomials are very useful for polynomial interpolation, which is easy to do numerically.

The following two properties/theorems are key:

① The zeros of the orthogonal polynomial  $\varphi_{j \geq 1}$  are real & distinct and in  $(a, b)$

(see Theorem 9.4 in theory text book for proof)

② The roots of  $\varphi_j(x)$  in  $(a, b)$  can be used as nodes for polynomial interpolation that is accurate & stable, i.e., does not suffer from Runge's phenomenon. This is because the nodal polynomial is "well-behaved."

This is the key lesson. In practice, we use the roots of orthogonal polynomials as interpolation nodes for global polynomial interpolation.

while the resulting interpolating polynomial is not the best  $L_2$  approximant, in practice it is just as good & easy to compute.

Take the Chebyshev polynomials of first kind (difference is basically cos vs. sine)

$$T_k(x) = \cos(k \cdot \alpha \cos(x))$$

Let's find its roots :

$$T_k(x) = 0 \Rightarrow$$

$$k \cdot \alpha \cos(x) = (2n+1) \frac{\pi}{2}$$

$$n \in \mathbb{Z}$$

$$\Rightarrow t = \cos(x) = \frac{(2n+1)}{k} \frac{\pi}{2}$$

$$t \in [0, \pi]$$

$$x \in [-1, 1]$$

$$0 \leq \frac{2n+1}{k} \leq 2$$

$$\Rightarrow 0 \leq n \leq \frac{(2k-1)}{2}$$

$$\Rightarrow \begin{cases} x_n = \cos\left(\frac{n+1/2}{k}\pi\right) \\ 0 \leq n \leq k \end{cases}$$

which are called the Chebyshev nodes of the first kind  
 (second kind are  $x_n = \cos\left(\frac{n}{k+1}\pi\right)$ )  
 and you used them in Worksheet 6 for interpolation