

Linear Algebra

(mostly review) A. DOVER

Matrix

$$A = \left[\begin{array}{c|c|c|c} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \dots & \vec{a}_n \end{array} \right]$$

$\vec{a}_k \in \mathbb{R}^m$, n of them

$$A \in \mathbb{R}^{m,n}$$

Matrix - vector product

$$\vec{b} = A \vec{x} = \underbrace{x_1}_{\text{column vector}} \vec{a}_1 + \underbrace{x_2}_{=} \vec{a}_2 + \dots + \underbrace{x_n}_{=} \vec{a}_n$$

$$[m \times n] [n \times 1] = [m \times 1]$$

$$\vec{b} \in \mathbb{R}^m$$

Image, column space, or
range of A :

$$\vec{b} \in \text{im}(A)$$

Vector subspace of \mathbb{R}^n
spanned by columns of A

Columns of A are linearly
independent if

$$Ax = 0 \Leftrightarrow x = 0$$

$$r = \dim(\mathcal{V} \subset \mathbb{R}^n)$$

max number of lin. ind.
vectors in \mathcal{V} .

of vector in any basis
for \mathbb{R}^n

$$\dim (\mathbb{R}^n) = n \cdot \text{is finite}$$

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Inner or Dot product

$$\vec{x} \cdot \vec{y} = (x, y) = \langle x, y \rangle$$

$$= \begin{matrix} \vec{x} \\ \uparrow \end{matrix}^T y = \sum_{i=1}^n x_i y_i$$

$$[1 \times n] [n \times 1] = [1 \times 1] = \text{scalar}$$

$$\vec{x}, \vec{y} \in \mathbb{R}^n$$

If  $\mathbb{C}^n$  (complex number)

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^n \bar{x}_i y_i$$

$\vec{x} \perp \vec{y}$  (orthogonal) if

$$\vec{x} \cdot \vec{y} = 0$$

$$r = \text{rank}(A) = \dim(\text{im}(A))$$

# of lin. ind. rows or  
columns

$$\text{rank}(A) = \text{rank}(A^T)$$

$$r \leq \min(m, n)$$

If  $A\vec{x} = \vec{0} \Rightarrow$

$$\vec{x} \in \text{null}(A)$$

null space or kernel  
of  $A$

$$\dim(\text{null}(A)) = \underline{\text{nullity}}$$

If columns are lin. nd  $\Rightarrow$   
 $\text{null}(A) = 0$

Fund theorem of LA:

$$\text{rank} + \text{nullity} = n$$



$$L: \mathcal{V} \rightarrow \mathcal{W}$$

↗ ↘  
vector spaces

$L$  is linear if

$$\left\{ \begin{array}{l} L(\vec{v}_1 + \vec{v}_2) = L\vec{v}_1 + L\vec{v}_2 \\ L(\alpha \vec{v}) = \alpha(L\vec{v}) \end{array} \right.$$

All linear mappings can  
be represented by a  
matrix

(finite-dimensional  $V$  and  $W$ )

$$L(\vec{v}) = \underbrace{\begin{array}{c} \leftrightarrow \\ L \end{array}}_{\substack{\text{matrix-vector} \\ \text{product}}} \vec{v}$$

$$\vec{w} = \underbrace{\begin{array}{c} \leftrightarrow \\ L \end{array}}_n \vec{v}$$

$$w_i = \sum_{j=1}^n L_{ij} v_j$$

$\nearrow$   
contraction

$$w_i = \left( \underbrace{L_{i,:}}_{\substack{i^{\text{th}} \\ \text{row}}} , \vec{v} \right)$$

Composition of linear  
mappings

$$\vec{z} = \underbrace{A(B\vec{x})}_{\text{associative}} = (\vec{AB})\vec{x} = C\vec{x}$$

$$C = AB \quad \begin{matrix} \text{matrix-} \\ \text{matrix} \\ \text{product} \end{matrix}$$

$$C_{ij} = \sum_{k=1}^p A_{ik} B_{kj} \quad \begin{matrix} k & k \\ \curvearrowright & \curvearrowright \\ \text{contract} & \end{matrix}$$

$$A = [m \times p]$$

$$B = [p \times n]$$

$$C = [m \times p] \underbrace{[p \times n]}_{k} = [m \times n]$$

In Matlab  $A * B$

Matrix multiplication is  
not commutative

$$AB \neq BA \quad (\text{in general})$$



$$A : \mathbb{R}^n \xrightarrow{A^{-1}} \mathbb{R}^n$$

$$A^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$AA^{-1} = A^{-1}A = I$$

$I$  = identity matrix

$$I = \begin{bmatrix} 1 & & & \\ & \ddots & & \emptyset \\ \emptyset & & \ddots & \\ & & & 1 \end{bmatrix}$$

If  $A^{-1}$  exists, matrix  
is invertible (square)

$A$  is invertible iff:  
(all of these are equivalent)

- 1)  $A$  is full rank  
 $\text{rank}(A) = n$
- 2) columns & rows are <sup>lin.</sup><sub>ind.</sub>
- 3)  $\det(\overset{\leftrightarrow}{A}) \neq 0$
- 4)  $(\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n)$   
 $\lambda=0$  is not an eigenvalue
- 5)  $Ax=0 \Leftrightarrow x=0$

## Properties of matrix algebra

$$C(A+B) = CA + CB$$

$$\begin{aligned} ABC &= (AB)C \\ &= A(BC) \end{aligned}$$

E.g.

$$A \vec{x} \vec{x}^T A = A \underbrace{(\vec{x} \vec{x}^T)}_B A$$
$$(\vec{x}^T y) B$$

scalar

$$(AB)^+ = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

Matrix "division" ~~X~~

→ multiplication by inverse

$$(A^{-1}) A B = C (B^{-1})$$

$$\cancel{B} = \cancel{C} / A$$

$$\underbrace{(A^{-1} A)}_{\sim} B = A^{-1} C$$

$$\underbrace{I}_{\sim} B = A^{-1} C$$

$$B = A^{-1} C$$

$$A = C B^{-1}$$

# Vector norms

$p$ -norm or  $L_p$  norm

$$p \geq 1$$

$$\|\vec{x}\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$

1)  $L_1$  norm (Manhattan norm)

$$p=1, \|\vec{x}\|_1 = \sum |x_i|$$

2)  $L_2$  norm (Euclidean norm)

$$\|\vec{x}\|_2 = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\vec{x}^T \vec{x}}$$

$$\|\vec{x}\|_2 = \sqrt{\sum |x_i|^2}$$

3)  $L_\infty$  norm or max norm

$$\|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

All norms are related

to each other if  $n$

$(\vec{x} \in \mathbb{R}^n)$  is "small"

$\|\cdot\|$  matrix norm

$$\frac{\|\mathbf{A} \mathbf{x}\|_{1,2,\infty}}{\|\mathbf{x}\|_{1,2,\infty}} = \|\mathbf{A}\|_{1,2,\infty}$$

$\sup_{\mathbf{x} \neq 0}$

Matrix norm induced by  
the vector norm

$$\|Ax\| \leq \|A\| \|x\|$$

$$\|AB\| \leq \|A\| \|B\|$$

$$1) \quad \|A\|_1 = \max_j \|A_{:,j}\|_1$$

j<sup>th</sup> column  
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$$= \max_j \sum_{i=1}^n |a_{i,j}|$$

$$2) \|A\|_\infty = \max_i \|A_{i,:}\|_1$$

$$= \max_i \sum_j |a_{ij}|$$

$$3) \|A\|_2 = \max_i \lambda_i$$

$\lambda^2$  is an eigenvalue of  
 $A^T A$  or  $A A^T$   
symmetric

$$\|A\|_2 = \max_i \sqrt{\lambda_{A^T A}}$$

In Matlab

$$\text{norm}(A, p)$$
 where  $p=1, 2, \infty$

Conditioning of  
mappings / matrices

$$\vec{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\frac{\|\vec{f}(\vec{x} + \vec{\delta x}) - \vec{f}(\vec{x})\|}{\sup_{\vec{\delta x} \neq 0} \|\vec{\delta x}\|}$$

$$= \underset{x}{\text{Cond}}(f)$$

(Local absolute condition  
number in theory book)

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$\underset{x}{\text{Cond}}(f) = |f'(x)|$$

$$\|f(x+\delta x) - f(x)\| / \|f(x)\|$$

$$\text{cond}_x(f) = \sup_{\delta x \neq 0} \frac{\|\delta x\|}{\|f(x+\delta x) - f(x)\|}$$

If  $\text{cond}_x(f) = 10^4$

that means that

knowing 16 digits in  $x$

gives me  $16 - 4 = 12$  digits

in  $f(x)$

Loose four digits

$$\cancel{x} + \cancel{\delta x} - \cancel{x}$$

$$\text{cond}_x(Ax) = \sup_{\delta x \neq 0} \frac{\|A(x + \delta x) - Ax\| / \|Ax\|}{\|\delta x\| / \|x\|}$$

$$= \left( \sup_{\delta x \neq 0} \frac{\|A \delta x\|}{\|\delta x\|} \right) \frac{\|x\|}{\|Ax\|}$$

$$= \frac{\|x\|}{\|Ax\|} \|A\| \geq 1$$

$$\|Ax\| \leq \|A\| \|x\|$$

$$\text{cond}_x(Ax) = \frac{\|A^{-1}(A - x)\| \|A\|}{\|Ax\|}$$

$$\leq \frac{\|A^{-1}\| \|Ax\| \|A\|}{\|Ax\|} \\ \leq \|A\| \|A^{-1}\|$$

$$1 \leq \text{cond}_X(A) \leq \underline{\underline{\|A\| \|A^{-1}\|}}$$

Define cond. number of A

$$K(A) = \|A\| \|A^{-1}\|$$

↓  
 1, 2, ∞      ↓  
 1, 2, ∞      ↑  
 1, 2, ∞