Orthogonal Polynomials

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Interpolation works for a "black-box" function.
Equi-spaced nodes not a good choice in general

$$\underbrace{p_n(x_k) = f(x_k)}_{\text{nodes}}$$

? $\mathcal{E}(x) = \left| p_n(x) - f(x) \right|$

$x \neq x_k$

Piece-wise polynomial interp works for any nodes, but $\mathcal{E}(x)$ is not very small

Another way to approximate
$f(x) \simeq p_n(x) \in \mathcal{P}_n$ on $[a,b]$

$$p_n^* = \arg \min_{p_n \in \mathcal{P}_n} \| f(x) - p_n(x) \|$$

L_2- approximation

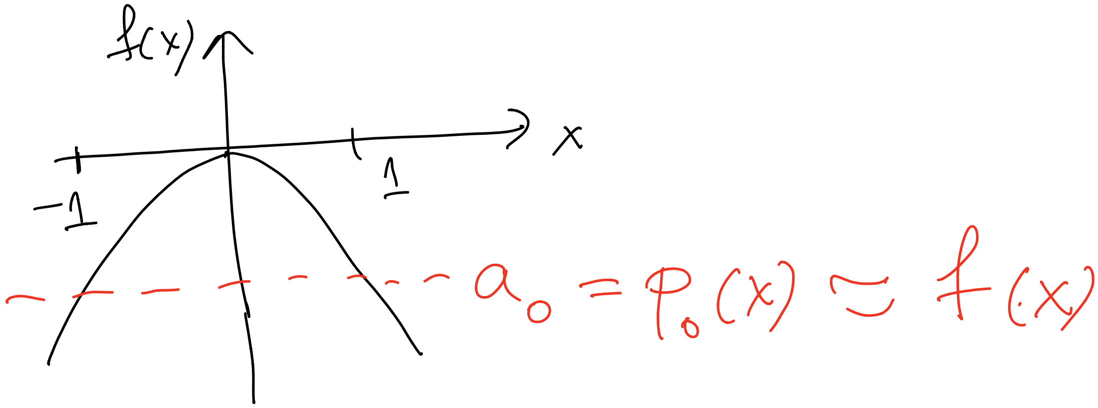
$$p_n^* = \arg \min_{p_n \in \mathcal{P}_n} \| f(x) - p_n(x) \|_2 = \left(\int |f(x) - p_n(x)|^2 dx \right)^{1/2}$$

Least-squares "fitting"

E.g. Approximate $f(x) = -2x^2$

in $P_{n=0}$ (space of constants)

on $[-1, 1]$



Idea: Average of function

$$a_0 = \frac{\int_{-1}^{1} f(x)\,dx}{2} \quad (?)$$

$$= \frac{\int_{-1}^{1} -(\cancel{2}x)^2\,dx}{\cancel{2}} = -\frac{x^3}{3}\Big|_{-1}^{1} = -\frac{2}{3}$$

$\underline{L_2}$:

$$\|f - a_0\|^2 = \int_{-1}^{1} \left(-2x^2 - a_0\right)^2 dx$$

$$a_0^* = \arg\min_{a_0} \; 2a_0^2 + \frac{8a_0}{3} + \frac{8}{5}$$

Derivative $= 0 \Rightarrow a_0^* = -\frac{2}{3}$

$$p_0^* \simeq a_0^* = -\frac{2}{3}$$

More generally:

$$a_0^* = \arg\min \int \left(f(x) - a_0\right)^2 dx$$

$$= \arg\min_{a_0} \left[\int f^2(x)\,dx - 2a_0 \int f(x)\,dx + a_0^2 \int dx \right]$$

$$\frac{\partial}{\partial a_0}\left[\cancel{\int f^2 dx} - 2a_0\int f\,dx + a_0^2\int dx\right]$$

$$-\cancel{2}\int f\,dx + \cancel{2}a_0\int dx = 0$$

$$\Rightarrow a_0 = \frac{\int f\,dx}{\int dx} = \frac{\int_a^b f(x)dx}{b-a}$$

average of function

Now make even more general

$$p_n^* \in \mathcal{P}_n$$

<u>Step 1</u>:

$$\text{basis} = \{\overbrace{p_0(x), p_1(x), \dots, p_n(x)}^{n+1}\} \subset \mathcal{P}_n$$

and linearly independent

$$P_n^*(x) = \sum_{j=0}^{n} a_j P_j(x)$$ Best L_2 approx

$$F(\vec{a}) = \int_a^b \Big(f(x) - \sum_{j=0}^{n} \underbrace{a_j}_{\text{unknown}} \overbrace{P_j(x)}^{\text{given}} \Big)^2 dx$$

"normal equations"

$$\frac{\partial F}{\partial a_k} = 0 \quad , \quad k = 0, \dots, n$$

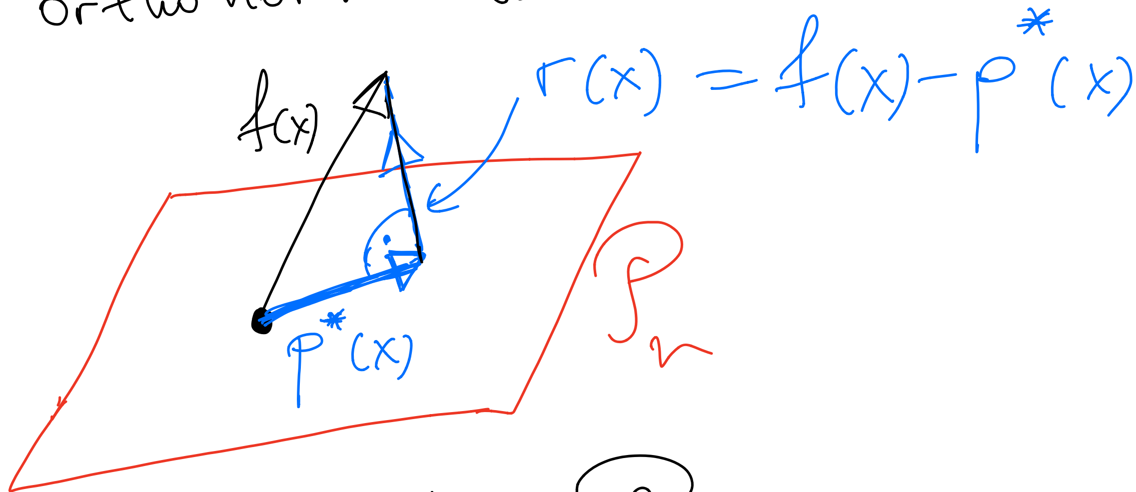
Linear System of $(n+1)$ equation for $(n+1)$ unknowns $\vec{a}$

Reminder: $Ax = b$, overdetermined

$$x^* = \arg\min_x \|Ax - b\|^2$$

① Normal eqs: $(A^T A)x = A^T b$

② QR factorization, Gram-Schmidt orthonormalization.



$$r(x) \perp \mathcal{P}_n$$

Conclusion: $r(x)$ must be orthogonal to all polynomials of degree n $\iff$

$$r(x) \perp P_k(x), \quad k=0,\dots,n$$

$$(r(x), P_k(x)) = 0 \quad \forall k$$

$$\left(f - p^*, p_k\right)_2 = 0 \quad \forall k$$

$$\left(f - \sum_{j=0}^{n} a_j p_j, p_k\right) = 0$$

$$(f, p_k) = \sum_{j=0}^{n} a_j (p_j, p_k)$$

$$\sum_{j=0}^{n} (p_j, p_k)\, a_j = (f, p_k)$$

$$k = 0, \dots, n$$

$(n+1)$ equations for $(n+1)$ coeffs.

$$\overleftrightarrow{V}\, \vec{a} = \vec{f}$$

Vandermonde matrix

$$V_{jk} = (p_j, p_k) = \int_{x=a}^{b} p_j(x)\, p_k(x)\, dx$$

Can pre-compute exactly

$$f_k = (f, p_k) = \int_{x=a}^{b} f(x)\, p_k(x)\, dx$$

cannot compute exactly for any $f(x)$

E.g.

Take monomials as basis for

P_n : $\{1, x, x^2, \ldots, x^n\}$

$\{p_0(x), p_1(x), \ldots, p_n(x)\}$

$a = 0$, $b = 1$

$$V_{ij} = \int_0^1 P_i(x)\, P_j(x)\, dx =$$

$$= \int_0^1 x^i \, x^j \, dx = \int_0^1 x^{i+j} \, dx$$

$$= \frac{x^{i+j+1}}{i+j+1} \Big|_0^1 = \frac{1}{i+j+1}$$

$$V_{ij} = \frac{1}{i+j+1} \leftarrow \text{Hilbert matrix}$$

Very ill conditioned for $n \geq 10$
(Worksheets)

Using monomials as a basis is a bad idea (ill-conditioned)

$$\overleftrightarrow{V}\,\vec{a} = \vec{f}$$

What's the simplest $\mathbf{V}$?

$$V = I \Rightarrow \vec{a} = \vec{f}$$

$$V_{ij} = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\begin{cases} \int p_i(x)\, p_j(x)\, dx = 0 & \text{if } i \neq j \\ \int p_i^2(x)\, dx = 1 & \end{cases}$$

← skip this in practice

orthonormal polynomials

$$a_k = f_k = (f, p_k) = \int f(x)\, p_k(x)\, dx$$

$$P_n^* = \sum_{k=0}^{n} a_k\, p_k(x)$$

aside: Often polynomials are orthogonal but not normalized

$$a_k = \frac{(f, p_k)}{(p_k, p_k)}$$

Orthogonal polynomials

How to find an orthogonal basis for $\mathcal{P}_n$?

Recall:

How to find an orthogonal basis for $\operatorname{col} \vec{A}$?

$$A = QR$$

upper triangular

orthonormal

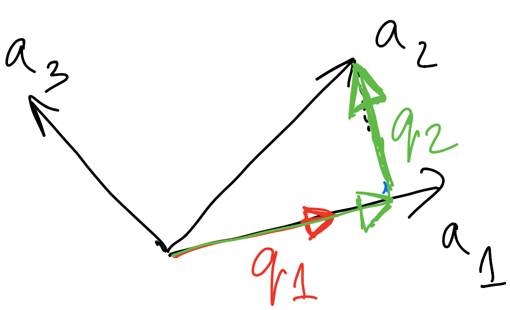
Compute by Gram-Schmidt process

"columns of A" $\stackrel{?}{\equiv} \{1, x, x^2, \dots, x^n\}$

$$\begin{cases} \mathrm{col}(A) \equiv P_n \\ \quad \Big\downarrow \text{GS} \\ Q \equiv \text{orthonormal basis for } P_n \end{cases}$$

Start with monomials, then do GS on them one by one to get orthogonal polynomials

Review:



q_1, q_2, q_3

$q_1 \perp q_2 \perp q_3$

$q_1 = a_1$

skip normalization

$$q_2 = a_2 - \mathrm{Proj}_{\{a_1\}} a_2$$

$$q_3 = a_3 - \mathrm{Proj}_{\{a_1, a_2\}} a_3 =$$

$$= a_3 - \operatorname{Proj}_{\{a_1\}} a_3 - \operatorname{Proj}_{\{a_2\}} a_3$$

$$\rightarrow \operatorname{Proj}_{\{a_j\}} a_k = \ ?$$