Square Linear Systems

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E.g.
$$\begin{cases} 3x_1 + 2x_2 = 2 \\ x_1 - x_2 + x_3 = 1 \\ 2x_1 + 3x_3 = 5 \end{cases}$$

$$\underset{\substack{\uparrow \\ \text{matrix}}}{\overleftrightarrow{A}} \; \underset{\substack{\nwarrow \\ \text{solution}}}{\vec{x}} = \vec{b} \leftarrow \text{r.h.s.}$$

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

$$A = [m \times n] \quad x = [n \times 1]$$
$$b = [m \times 1]$$

m eqs. , n variables

For a while $\quad m = n$

$$Ax = b \qquad m = n$$

How many solutions?

- 0 → A not invertible, $b \notin \operatorname{Im}(A)$
- 1 → A is invertible
  $x = A^{-1} b$
- ∞ → A not invertible, $b \in \operatorname{Im}(A)$

If A is <u>not</u> invertible.

$Ax = 0$? How many solutions?

Infinitely many solutions

x, αx is also a solution

Back to: $Ax = b$ for non-invertible

$$\begin{cases} Ax_1 = b & \text{has 1 solution} \\ Ax_2 = 0 & \text{- infinitely} \end{cases}$$

$$A(x_1 + x_2) = b$$

$Ax = b$, $b \in \operatorname{im}(A)$

If $b \notin \operatorname{im}(A) \Rightarrow$ no solution

If $b \in \operatorname{im}(A) \Rightarrow$ infin. many solutions

$Ax = b$, A is invertible

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \forall i \quad i=1,\dots,n$$

$x = A^{-1} b$

NOT how we compute it numerically

Never do $x = \text{inv}(A) * b$

$$\begin{array}{c c}
 & \\
\rightarrow & \\
-\frac{a_{21}}{a_{11}} & \\
 &
\end{array}
\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix}$$

Reduced row echelon form

Step 1: Climmate x_1

a) Multiply 1^{st} equation by

$$\ell_{21} = \frac{a_{21}}{a_{11}}$$

and subtract it from second equations

b) Multiply 1^{st} equation by

$$\ell_{31} = \frac{a_{31}}{a_{11}}$$

eq. # → 3, 1 ← var #

and subtract from 3^{rd} eq

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - l_{21} a_{11} = 0 & a_{22}^{(2)} = a_{22}^{(1)} - l_{21} \cdot a_{12}^{(1)} & a_{23}^{(2)} = a_{23} - l_{21} \cdot a_{13} \\ 0 & a_{32}^{(2)} = a_{32}^{(1)} - l_{31} a_{12}^{(1)} & a_{33}^{(2)} = a_{33}^{(1)} - l_{31} a_{13}^{(1)} \end{bmatrix}$$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - l_{ik} \, a_{kj}^{(k)}$$

$$\underset{\text{eq.}\ \ \text{var}}{l_{ik}} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}, \quad i, j > k$$

$$a_{kk}^{(k)} \neq 0$$

$$
\begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix} \to \begin{bmatrix} b_1^{(2)} = b_1^{(1)} \\ b_2^{(2)} = b_2^{(1)} - l_{21} b_1 \\ b_3^{(2)} = b_3^{(1)} - l_{31} b_1 \end{bmatrix}
$$

$$
b_i^{(k+1)} = b_i^{(k)} - l_{ik} b_k^{(k)}
$$

<u>Step 2</u>

$$
\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ l_{21}\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ l_{31}\ 0 & a_{32}^{(2)} & a_{33}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1^{(2)} \\ b_2^{(2)} \\ b_3^{(2)} \end{bmatrix}
$$

Multiply 2^{nd} eq. by

$$\ell_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}}$$

$\ell_{3\,2}$: 3 → eq., 2 → var

and subtract from 3^{rd} eq.

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ \ell_{21}\ \cancel{0} & a_{22}^{(2)} & a_{23}^{(2)} \\ \ell_{31}\ \cancel{0} & \ell_{32}\ \cancel{0} & a_{33}^{(3)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \end{bmatrix}$$

<u>Step 3</u>:

$$x_3 = \frac{b_3^{(3)}}{a_{33}^{(3)}}$$

Step 4:

$$x_2 = \frac{b_2^{(2)} - a_{23}^{(2)} x_3}{a_{22}^{(2)}}$$

Step 5: solve for x_1

Gaussian elimination

$$L = \begin{bmatrix} 1 & \emptyset & \emptyset \\ l_{21} & 1 & \emptyset \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$\det(L) = 1$ is invertible

unit lower triangular matrix

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ \emptyset & a_{22}^{(2)} & a_{23}^{(2)} \\ \emptyset & \emptyset & a_{33}^{(3)} \end{bmatrix}$$

upper triangular

<u>Theorem</u>: $A = LU$

LU factorization

$\Longleftrightarrow$ Gauss elimination

Code MyLU.m on webpage

$i > k$: $l_{ik} = \dfrac{a_{ik}}{a_{kk}} \Rightarrow a_{ik}$

$l_{kk} = 1$ (not stored)

<u>for</u> $k = 1 : (n-1)$

Eliminate x_k from

eqs. $k+1, \ldots, n$

$A((k+1):n, k) =$

$A((k+1):n, k) / A(k,k);$

Compute l_{ik}, $i > k$

$$a_{ij} \Leftarrow a_{ij} - l_{ik} a_{kj}$$

$$i, j > k$$

for $j = (k+1):n$

$A((k+1):n, j) = A((k+1):n, j)$

$- \underbrace{A((k+1):n, k)}_{\text{stores } l_{ik}} * A(k, j);$

"in-place" LU factorization

<u>end</u> [for j]

<u>end</u> [for k]

<u>Note</u> We could have done for $i = (k+1):n$