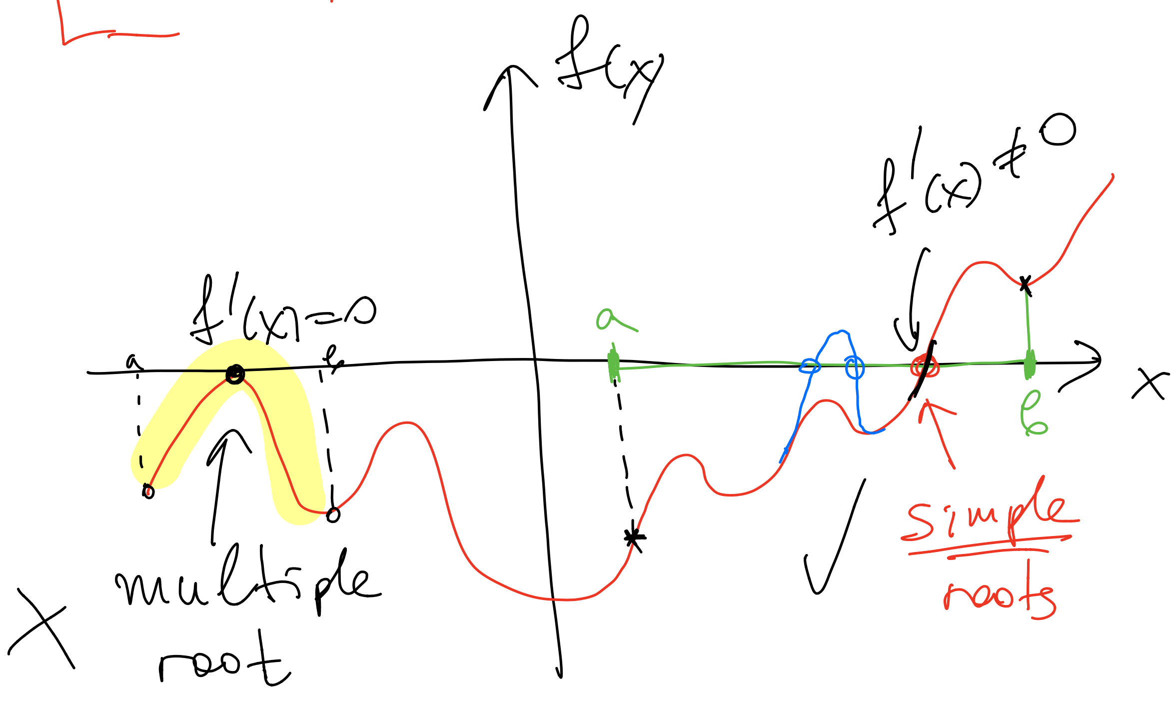
Nonlinear equations
(in one variable)

$x = \sqrt{c} \iff x^2 - c = 0$

$\cos x + x^2 - 7 = 0$

Solve $f(x) = 0 \quad x \in [a, b]$

Find at least one solution if one exists



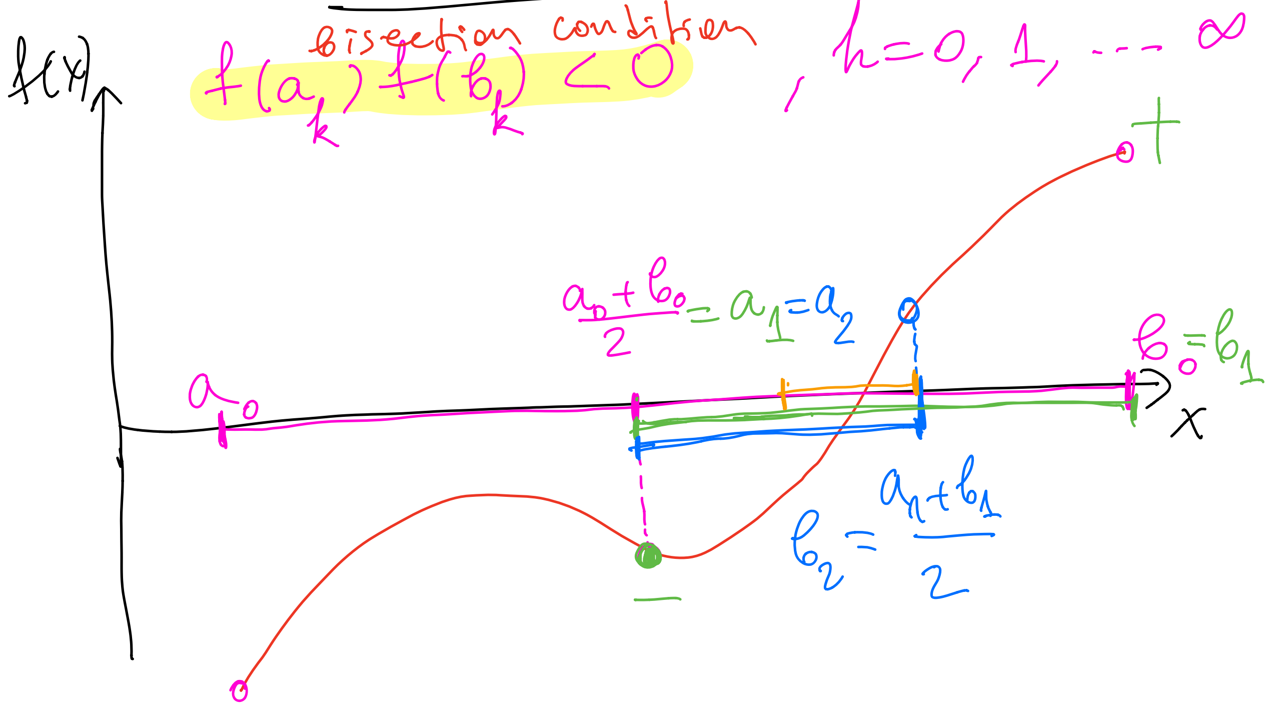
$f(x)$ is continuous on $[a,b]$

If $f(a)\cdot f(b) < 0$

$\Rightarrow \exists x \in [a,b]$ s.t. $f(x)=0$

(Q) Why don't I write $f(a) f(b) \leq 0$?

Bisection method



Algorithm

$$\begin{cases} \text{Input: } [a, b] \text{ s.t. } f(a)f(b)<0 \\ k_{max} \in \mathbb{Z}^+ \end{cases}$$

$$\text{Output: } \begin{cases} \tilde{x} \quad \text{s.t. } f(\tilde{x}) \simeq 0 \\ [a, b] \text{ s.t. true root is in } [a, b] \\ x \in [a, b],\ f(x)=0 \end{cases}$$

For $k = 0, 1, 2, \ldots, k_{max}$

$$x_k = \frac{a_k + b_k}{2}$$

$$f_k = f(x_k)$$

(actually, we only need sign of f_k)

If $f_k \cdot f(a_k) < 0$ then

left half $[a_k, x_k]$ is a bisection interval, $x \in [a_k, x_k]$

$$a_{k+1} = a_k \; ; \; b_{k+1} = x_k$$

else [if $f_k \cdot f(b_k) < 0$] (keep right half)

$$b_{k+1} = b_k \; ; \; a_{k+1} = x_k$$

[else if $f_k = 0$ then
return x_k

end if

End for

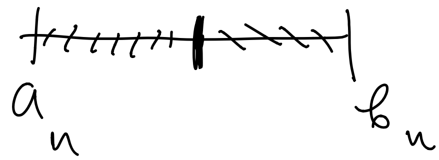
Output : $x \simeq x_{k_{max}} = \dfrac{a_{k_{max}} + b_{k_{max}}}{2}$

and $x \in [a_{k_{max}}, b_{k_{max}}]$

We know that [$n = k_{max}$]

$$|x - x_n| \le \frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2 \cdot 2^n}$$

$x_n = \frac{a_n + b_n}{2}$



Absolute error

$$e_n = |x - x_n| \le \frac{b - a}{2^{n+1}}$$

error estimate

Given an error tolerance

ε s.t. $|x - x_n| < \varepsilon$

$$\varepsilon \simeq \frac{b - a}{2^{n+1}}$$

$$2^{n+1} > \frac{b-a}{\varepsilon}$$

$$n+1 = \log_2\left(\frac{b-a}{\varepsilon}\right)$$

How many times I will evaluate $f(x)$

$$n = \left\lceil \log_2\left(\frac{b-a}{\varepsilon}\right) \right\rceil$$

round up $\lceil \ \rceil$ / ceil

in Matlab

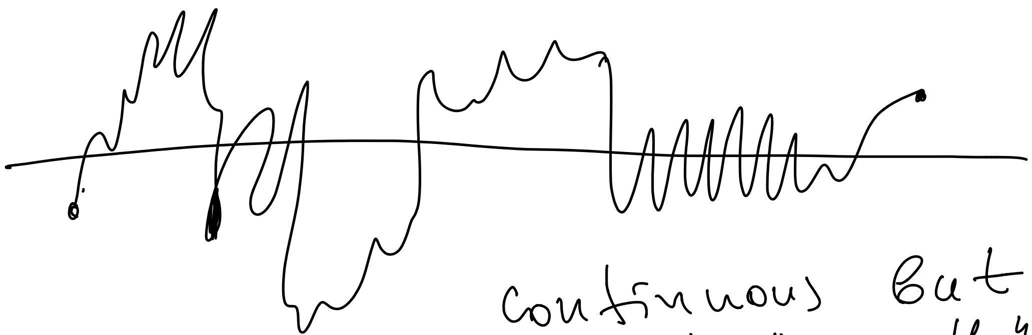
round down $\to \lfloor \ \rfloor$ / floor

In Matlab :

$$n_{est} = \text{ceil}(\log 2((b-a)/\varepsilon))$$

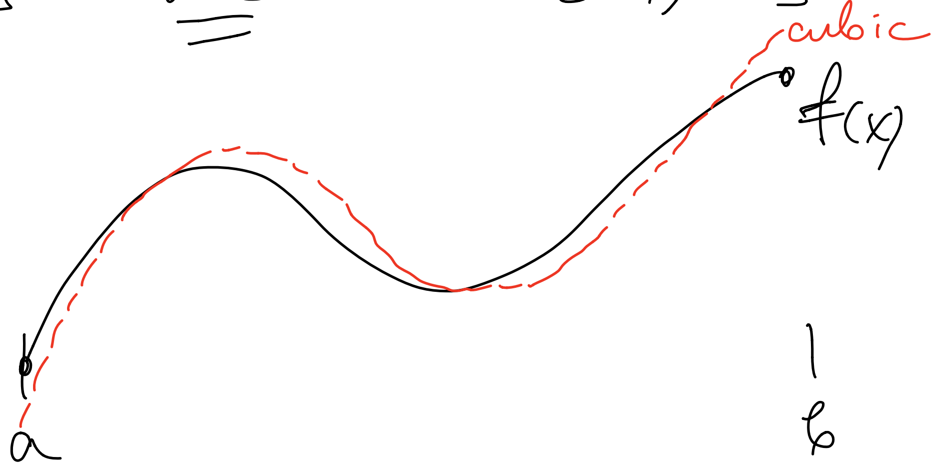
Our job is done

Can we do better?



continuous but not "smooth"

$f(x)$ is "smooth" on $[a,b]$ if it can be approximated "well" by a polynomial of low degree (linear, quadratic, cubic) over <u>all</u> of $[a,b]$



Another (related) is that $f(x)$ is smooth if it is sufficiently differentiable

$e^{1/x}$ is a bad example

Idea: $p(x) \approx f(x)$ on $[a, b)$

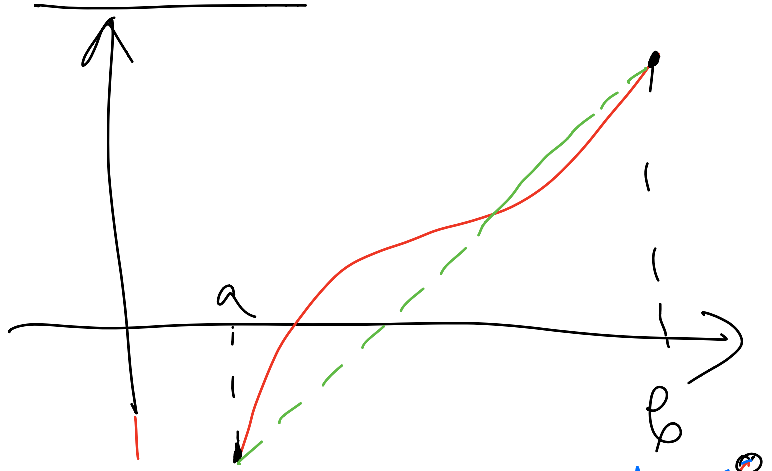
polynomial

Solve $p(x) = 0$ instead

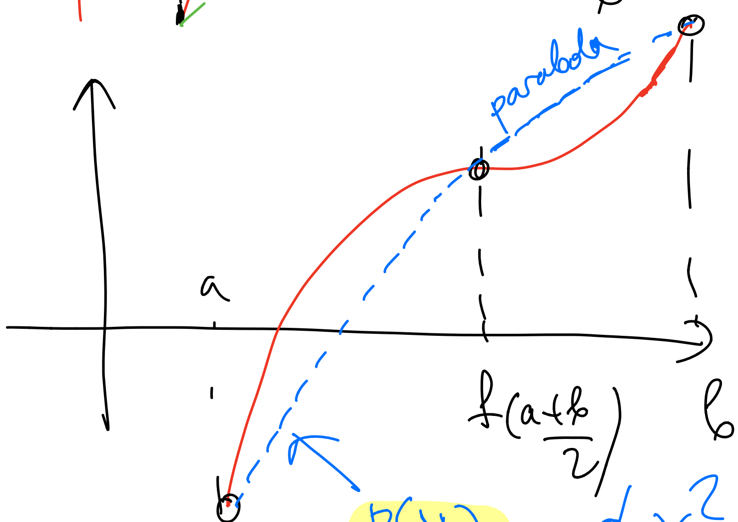
Typically $p(x)$ is linear or quadratic

How do we find $p(x)$?

Option 1

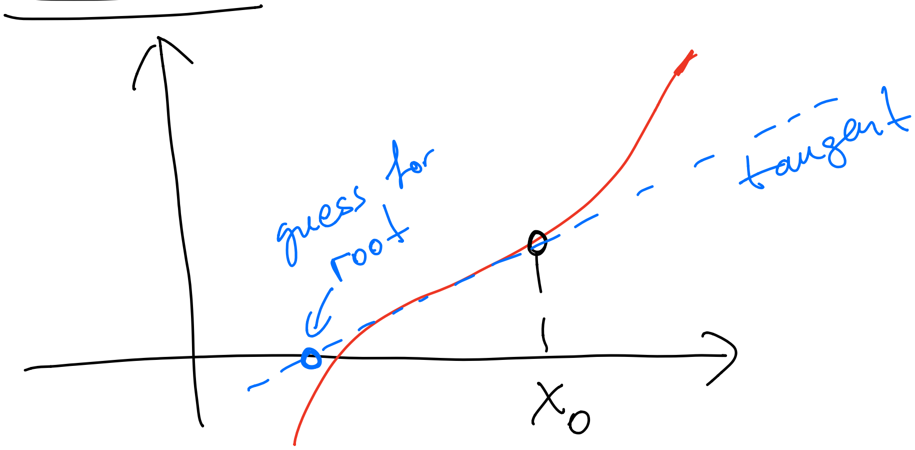


Interpolation



$P(x) = \alpha x^2 + \beta x + \gamma$

Option 2 :



Option 2 : Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \quad \ddot{\smile}$$

$$p(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \ldots$$

$$p(x) = \sum_{n=0}^{k} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Truncated Taylor series

$$\underbrace{f(x) - p(x)}_{\text{Remander}} = \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-x_0)^{k+1}$$

ξ is between x and x_0