SERIES-EXPANSION METHODS

CFD SPRING 2013, A. DONEV

This is based on ch. 4 in "Numerical methods for wave equations" by Durran.

We want to solve PDE

$$\frac{\partial \psi}{\partial t} + F(\psi) = 0$$

We have a numerical approximation

$$\varphi \simeq \psi$$

and want to expand/represent it in some finite basis for the functional space of interest.

(2)

$$\varphi(x,t) = \sum_{k=1}^{N} a_k(t)\, \varphi_k(x)$$

basis functions

Residual

$$R(\varphi) = \frac{\partial \varphi}{\partial t} + F(\varphi)$$

should somehow be minimized

(1) Equivalent: Minimize $R(\varphi)$ in L_2 norm

or require that $R(\varphi)$ be orthogonal to the finite-dimensional subspace spanned by $\{\varphi_k(x)\}$: GALERKIN

(2) Collocation: $R[\varphi(j\Delta x)] = 0, \; j = 1, \ldots, N$

We adopt the GALERKIN approach. 3

Minimize over $\dot{a}_k = \dfrac{d a_k(t)}{dt}$ s.t.

$$\int R[\varphi(x)]\, \varphi_k(x)\, dx = 0 =$$

$$\int \left[\sum_{n=1}^{N} \dot{a}_n \varphi_n + F\left(\sum_{n=1}^{N} a_n \varphi_n\right) \right] \varphi_k \, dx$$

$$\Rightarrow \boxed{\sum_n M_{nk}\, \dot{a}_n = - \int F\left(\sum_n a_n \varphi_n\right) \varphi_k \, dx}$$

for $k = 1, \dots, N$ LINEAR SYSTEM of eqs.

$$\boxed{M_{nk} = \int \varphi_n \varphi_k \, dx} \quad \text{(mass matrix)}$$

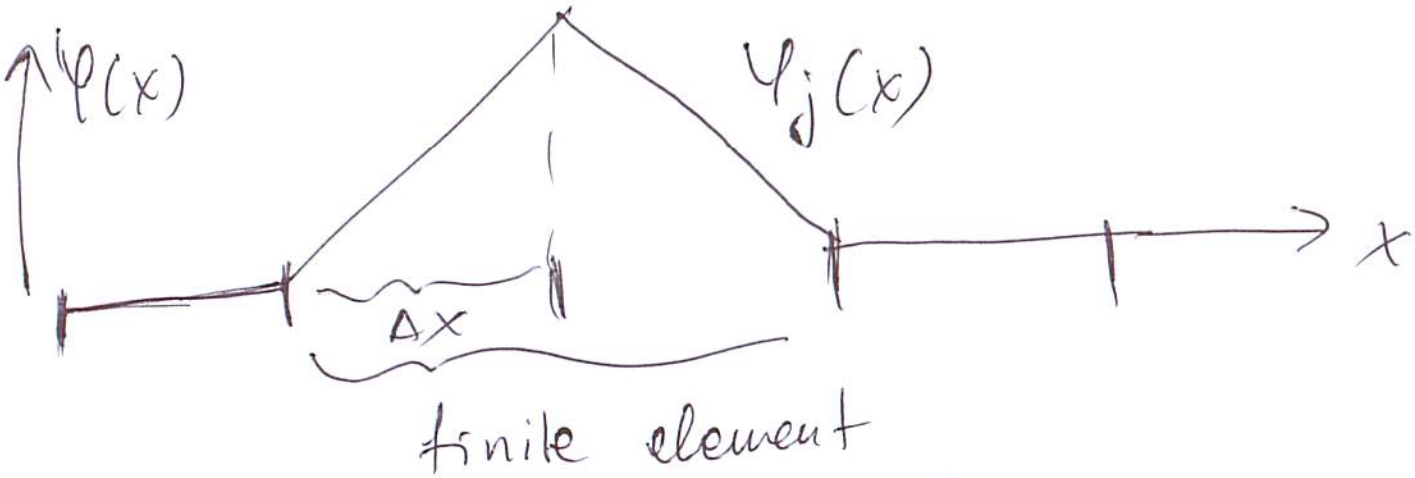
Note that if the basis functions are orthonormal then M is the identity matrix, $M = I$

We have now converted the PDE into a system of ODEs for $a_k(t)$
→ spatial discretization

Depending on the choice of basis functions, we can be doing spectral, finite-element, etc.

Finite - Element Method

The main difference with (pseudo) spectral is that now we choose a localized basis function set, e.g.



$$\int \varphi_j \varphi_{j+1} \, dx = \int_0^{\Delta x} \frac{x}{\Delta x} \frac{\Delta x - x}{\Delta x} \, dx = \frac{\Delta x}{6}$$

$$\int \varphi_j^2 \, dx = \frac{2\Delta x}{3}$$

and we also need

$$-\underbrace{\int \frac{\partial \varphi_{j-1}}{\partial x} \varphi_j \, dx}_{} = \frac{1}{2}$$ 9

can be done using integration by parts if φ is not differentiable

So for $\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$ this gives the Galerkin finite-element discretization

$$\underbrace{\frac{\dot{a}_{j+1} + 4\dot{a}_j + \dot{a}_{j-1}}{6}}_{\text{sparse matrix}} + c\left(\frac{a_{j+1} - a_{j-1}}{2\Delta x}\right) = 0$$

Note that $a_j \equiv \psi_j$ for this basis

Denote $M = \frac{1}{6}\begin{bmatrix} 4 & 1 & & \\ 1 & 4 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & 4 \end{bmatrix} \Rightarrow$ (10)

$$\dot{a} = \dot{\psi} = -M^{-1}(c\Delta a) = -c(M^{-1}\Delta)\psi$$

centered finite difference

$$\Rightarrow \boxed{\dot{\psi} = -c\,(M^{-1}\Delta)\,\psi}$$

Finite-difference Approximation of $\frac{\partial}{\partial x}$ $\rightarrow$ called "compact finite difference" in the literature.

It turns out $M^{-1}\Delta$ is a fourth-order approximation of $\frac{\partial}{\partial x}$!

So this finite-element method is fourth order in space. (11)

An alternative approach to minimizing the residual $R(\varphi) = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}$ is to replace the time derivative right-away with a difference:

$$\begin{cases} \dfrac{\varphi^{n+1} - \varphi^{n}}{\Delta t} + c\,\dfrac{\partial \varphi}{\partial x} = 0 \quad \ldots\ldots (*) \\ \text{where} \quad \varphi^{n} = \sum a_k^{n}\,\varphi_k(x) \end{cases}$$

Now minimize the residual in (*) to get $\varphi^{n+1} \leftarrow \varphi^{n}$, i.e., $a^{n+1} \leftarrow a^{n}$

Take the advection equation

$$\frac{\partial \psi}{\partial t} + c(x) \frac{\partial \psi}{\partial x} = 0$$

$$\Rightarrow \quad \frac{da_k}{dt} = -\frac{i}{2\pi} \sum_{n=-N}^{N} n\, a_n \int_{-\pi}^{\pi} c(x,t)\, e^{-i(n-k)x}\, dx$$

if one uses the finite (truncated) Fourier basis.

Now, assume $c(x,t)$ is also approximated (represented) in the finite Fourier basis

$$c(x,t) = \sum_{m=-N}^{N} c_m(t)\, e^{imx}$$

$$\Rightarrow \frac{da_k}{dt} = - \sum_{\substack{m+n=k \\ |m|,|n| \le N}} i n c_m a_n$$

convolution

$$\frac{\partial \hat{\psi}}{\partial t} + \widehat{\left(c(x) \frac{\partial \psi}{\partial x} \right)} = \partial_t \hat{\psi} + \hat{c} \circledast (ik \hat{\psi})$$

convolution

Also

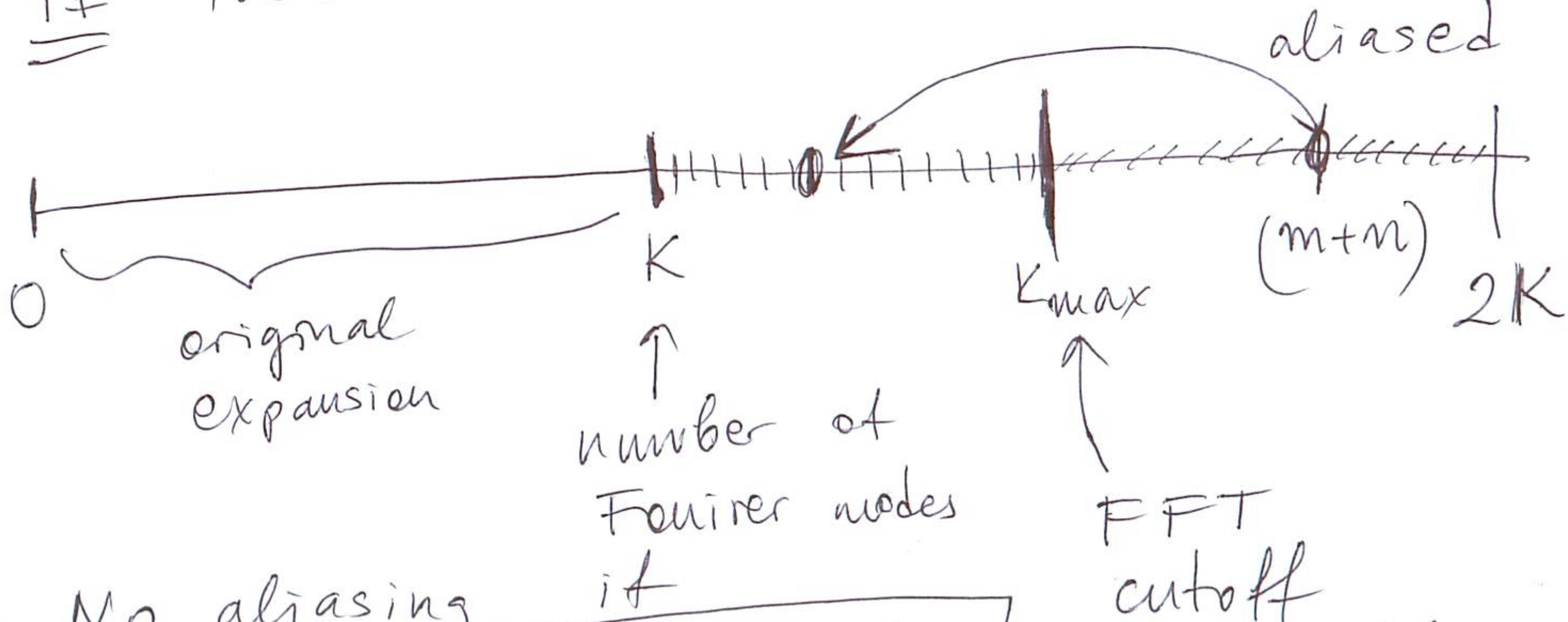
$$\frac{da_k}{dt} = - \sum_{\substack{n \\ (m=k-n) \\ |n|,|m| \le N}} i n a_n c_{k-n}$$

The convolution is expensive to calculate $\longrightarrow$ do it in real space (pseudospectral method)

The <u>pseudo spectral</u> approach:

$$\hat{c} \circledast (ik\hat{\psi}) = FFT\left\{ iFFT(\hat{c}) \cdot iFFT(ik\hat{\psi}) \right\}$$

is <u>equivalent</u> to the convolution sum <u>if</u> there are no aliasing errors



$\Rightarrow$ No aliasing if $\boxed{K_{max} = 3K/2}$ as we saw before