INTRO TO FINITE ELEMENT METHODS

A. DONEV, COURANT 1

We will consider 1D elliptic BVPs (Sturm-Louville problems)

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]+b(x)u=f$$

$$x\in[0,1]$$

$$a(x)>0\quad,\quad b(x)\geq 0$$

And $\left\{\begin{array}{l} u(0)=\alpha \\ u(1)=\beta \end{array}\right.$ } essential BCs

The key idea of **series methods** is to approximate u with a linear combination of **basis functions**: (2)

$$\begin{cases} u_m = \varphi_0(x) + \sum_{\ell=1}^{m} \gamma_\ell \, \varphi_\ell(x) \\ \varphi_0 \text{ satisfies inhomogeneous BCs} \\ \varphi_\ell \text{ satisfies homogeneous BCs}, \ \ell \geq 1 \end{cases}$$

We want $u_m(x)$ to **approximate** $u(x)$

Plug u_m into PDE to get the residual or defect:

$$d_m(x) = -\frac{d}{dx}\left(a \frac{du_m}{dx}\right) + b\,u_m - f$$

We can minimize $d_m(x)$ in a few ways

1. Collocation: Impose $d_m(x_j) = 0$ at a grid of points x_j.
   E.g. use Chebyshev grid and polynomials
2. Minimize some norm of $d_m(x)$ over the γ's

Imagine that

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$$d_m(x) \in \overset{\circ}{H}_m = \mathrm{Span}\{\varphi_1, \dots, \varphi_m\}$$

linear subspace

Then $d_m(x)$ being zero is equivalent to it being orthogonal to all non-zero elements of $\overset{\circ}{H}_m$, i.e.

$$\boxed{\langle d_m, \varphi_k \rangle = 0 \quad , \quad k = 1, \dots, m}$$

FOR SOME INNER PRODUCT

This is the $\boxed{\text{Galerkin equations}}$

We focus now on Gallerkin methods for second-order BVPs (5)

$$d_m = -\left[(a\varphi_0)' + \sum \gamma_\ell (a\varphi_\ell')' \right] + b\left[\varphi_0 + \sum \gamma_\ell \varphi_\ell \right] - f$$

Now compute $\langle d_m, \varphi_k \rangle$ to get a linear system of equations for the γ_ℓ's.

$$\sum_{\ell} \gamma_\ell \Big[\overbrace{\langle -(a\varphi_\ell')', \varphi_k \rangle}^{\text{Matrix}} + \underbrace{\langle b\varphi_\ell, \varphi_k \rangle}_{\text{vector}} \Big]$$

$$= \boxed{\overleftrightarrow{A}\,\vec{\gamma} = \vec{\delta}} =$$

$$\langle f, \varphi_k \rangle - \Big[\langle -(a\varphi_0')', \varphi_k \rangle + \langle b\varphi_0, \varphi_k \rangle \Big]$$

From now on assume homogeneous BCs so that $\gamma_0 = 0$, $\varphi_0 = 0$

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$$A^{(a)}_{kl} = \langle -(a\varphi_l')', \varphi_k \rangle$$

Let's <u>integrate by parts</u>

$$\langle v, w \rangle = \int_0^1 v(\tau)\, w(\tau)\, d\tau$$

in L_2

$$-\int_0^1 \left(a(\tau)\varphi_l'(\tau)\right)' \varphi_k(\tau)\, d\tau =$$

$$\underset{\substack{\parallel \\ 0}}{\text{boundary terms}} + \int a(\tau)\, \varphi_l'(\tau)\, \varphi_k'(\tau)\, d\tau$$

Finally we get

$$\overleftrightarrow{A}\,\vec{\rho} = \vec{\delta}$$

$$\boxed{A_{kl} = \int_0^1 \left[a(\tau)\varphi_l'(\tau)\varphi_k'(\tau) + b(\tau)\varphi_l(\tau)\varphi_k(\tau)\right] d\tau}$$

symmetric matrix

$$\delta_k = \int_0^1 \varphi(\tau)\varphi_k(\tau)\, d\tau + \begin{array}{c}\text{boundary}\\ \text{condition}\\ \text{terms}\\ \text{from } \varphi_0\end{array}$$

In finite element circles the matrix $\overleftrightarrow{A}$ is called the stiffness matrix. It is SPD for $a > 0$

Note that because we integrated by parts now φ's need to only be piecewise once differentiable and don't have to be twice differentiable. This gives us more flexibility in choosing the basis functions. 9

$\overleftrightarrow{A}$ is symmetric positive semidefinite

(so we can use Cholesky)

Proof: Define

$$y = x^T A x = \sum_{k,\ell} x_k A_{k\ell} x_\ell$$

$$\underbrace{x_k A_{kl} x_l}_{\substack{\text{Implied} \\ \text{summation} \\ \text{over } k,l}} = \int a\,(x_l \varphi_l)'(x_k \varphi_k)'\,d\tau$$

$$= \int a\left(\tilde{\varphi}'\right)^2 d\tau \geq 0$$

where $\tilde{\varphi} = \sum_l x_l \varphi_l$

Note: If we had $\mathring{H}$ be an infinite dimensional space (say L_2) then a solution of the Galerkin equations is a $\boxed{\text{weak solution}}$ of the PDE

Finite Element Basis functions

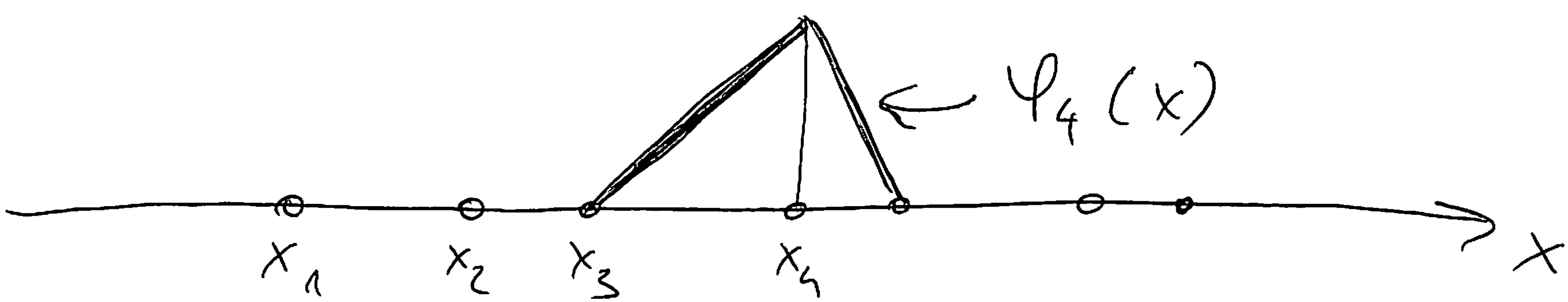
Up to now everything was general and really came from analysis. The basis functions could be Fourier basis or Chebyshev polynomials or other functions.

The key idea of the Finite Element Method

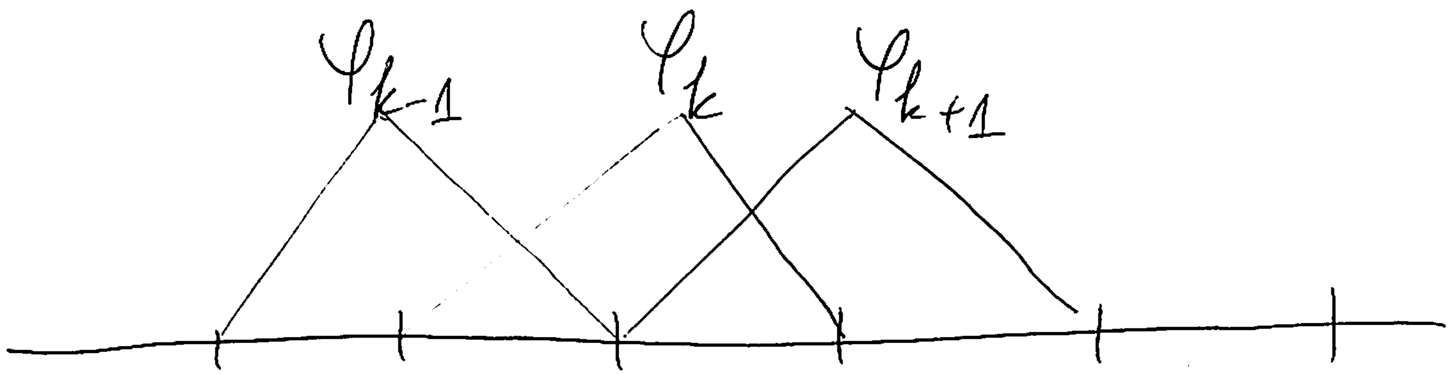
is to choose basis functions that are compactly supported over elements that tile the domain that is, the basis functions are <u>local</u> rather than global. (12)

this makes the stiffness matrix <u>sparse</u>

We need <u>piecewise differentiable</u> functions for our BVP, so choose <u>linear basis "hat" functions</u>:



$$x_{k+1} - x_k = h_k$$



Since the supports of φ_{k-1} and φ_{k+1} do <u>not</u> overlap, the stiffness matrix is <u>tridiagonal</u>

φ_k is called <u>nodal basis</u> function

$\boxed{\gamma_k = u_m(x_k)}$ is the nodal value

x_k's are <u>nodes</u>

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$$h_i = x_i - x_{i-1}$$

(Notation from section 0.5 in book by Susanne Brenner)

$$h = \max h_i$$

Take equation $-\dfrac{d^2u}{dx^2} = f$

$$\begin{cases} A_{i,i+1} = \displaystyle\int_{x_i}^{x_{i+1}} \varphi_i(x)\,\varphi_{i+1}(x)\,dx = -\frac{1}{h_{i+1}} \\ A_{i,i} = \displaystyle\int_{x_{i-1}}^{x_{i+1}} \varphi_i^2(x)\,dx = \frac{1}{h_i} + \frac{1}{h_{i+1}} \end{cases}$$

And

$$(f, \varphi_i) = \int_{x_{i-1}}^{x_{i+1}} f(x)\, \varphi_i(x)\, dx$$

Now we cannot compute this exactly for arbitrary $f(x)$.

In FEM one generally uses <u>Gauss quadrature</u> to approximate integrals. Here, to <u>first order</u>

$$(f, \varphi_i) = \frac{1}{2}(h_i + h_{i+1})\big(f(x_i) + O(h)\big)$$

So the FEM Galerkin approach with linear elements gives the linear system (16)

$$-\frac{2}{h_i+h_{i+1}}\left[\frac{U_{i+1}-U_i}{h_{i+1}}-\frac{U_i-U_{i-1}}{h_i}\right]=f(x_i)$$

$$+O(h)$$

LTE is only first order

But if $h_i = h =$ const we get our usual second-order

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}=f(x_i)$$

It turns out that even for unequal h_i the FEM discretization is second-order accurate, even in the L_∞ norm!

Proving this is greatly aided by the variational structure built into FEM

Theorem:

$$\| u - u_m \|_\infty \leq C h^2 \| u'' \|_\infty$$

Proof uses Greens functions similar to Finite Difference proof...

Some error estimates:

Denote by $S = \text{span}\{\varphi_1, \dots, \varphi_m\}$

$$S \subset V = \{ v \in L^2(0,1):$$

sufficiently smooth: $\int |v'|^2 \, dx < \infty$ and

(essential) BCs: $v(0) = v(1) = 0 \}$

Galerkin equations (Ritz-Galerkin)

Find

$$\boxed{u_S \in S \quad \text{s.t.} \quad a(u_S, v) = (f, v) \quad \forall v \in S}$$

$$\boxed{\text{where } a(u,v) = \int_0^1 u'(x)\, v'(x)\, dx \quad \underline{\text{bilinear form}}}$$

Defines a norm (<u>energy norm</u>):

$$\|v\|_E = \sqrt{a(v,v)}$$

<u>Schwarz' inequality</u>:

$$|a(u,v)| \leq \|u\|_E \, \|v\|_E$$

Approximation error result

(Brenner book)

$$\|u - u_S\|_E = \min \left\{ \|u - v\|_E : v \in S \right\}$$

i.e. the Galerkin approximation is the best one in the subspace S among all, in the energy norm

But what about the L_2 norm?

L_2 is weaker than energy norm since no derivatives.

So we expect L_2 to be even better

Assume the following approximation assumption

MUST CHECK! $\boxed{\|u - u_S\|_2 \le \varepsilon \|u - u_S\|_E}$

where ε is small, e.g., $\varepsilon \sim h$ or $\varepsilon \sim h^2$

then

$$\|u - u_S\|_2 \le \varepsilon \|u - u_S\|_E \le \varepsilon^2 \|u''\|$$

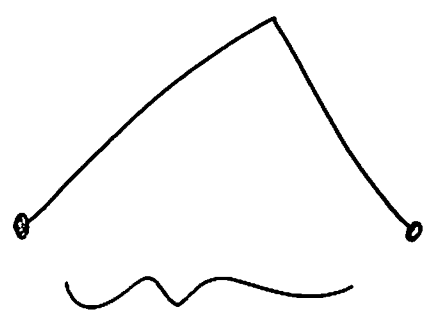
$$\Rightarrow \boxed{\|u - u_S\|_2 \le \varepsilon^2 \|f\|}$$

For FEM linear elements $\varepsilon \sim h$

What if we had Neumann conditions? (22)

$$-u'' = f \quad \text{in } (0,1)$$
$$u(0)=0 \quad \text{and} \quad u'(1)=0 \qquad (*)$$

It does not seem possible to impose $u'(1)=0$ with the linear hat functions!?!



slope not zero anywhere

But, turns out we don't need to impose Neumann BC explicitly

<u>theorem</u> (see 1D proof in theorem 0.1.4 in Brenner ch. $\emptyset$):

If $u \in V$ and

$$a(u,v) = f(u,v) \quad \forall v \in V$$

where

$$V = \{ v \in L^2(0,1) : a(v,v) < \infty \text{ and } v(0) = 0 \}$$

<u>Only</u> Dirichlet BC

then u solves the PDE (*)

the Dirichlet BC is <u>essential</u>

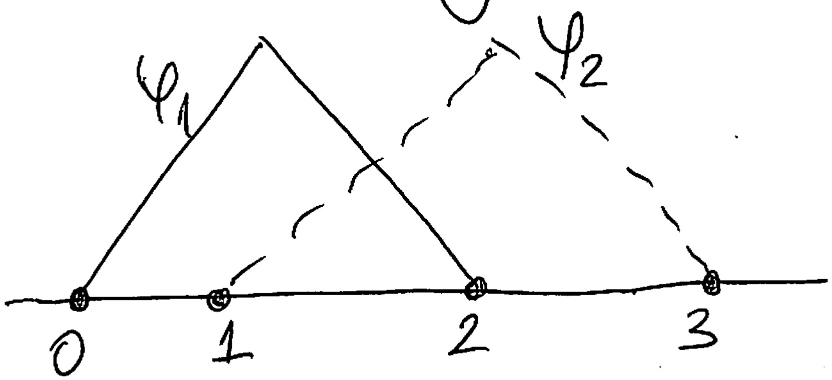
the Neumann BC is <u>natural</u>

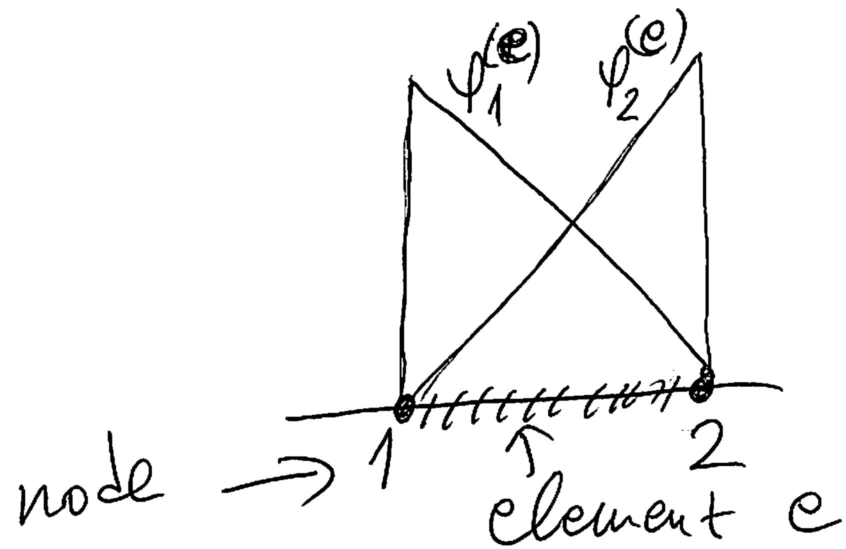
Summary of FEM philosophy:

1. Approximate solution in a finite-dimensional **subspace** $\varphi_0 + \mathring{H}_m \subset H$ ($\varphi_0 + S \subset V$)
2. Retain only **essential** BCs (variational formulation)
3. Make residual orthogonal to $\mathring{H}_m$ (Galerkin eqs.)
4. Integrate by parts to remove derivatives
5. Use **compactly-supported** basis (FEM)

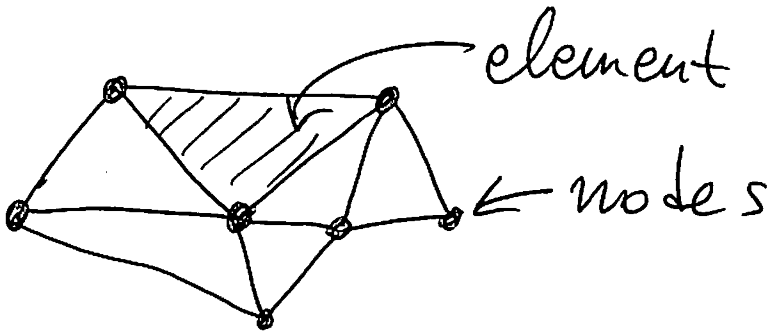
FEM in 2D

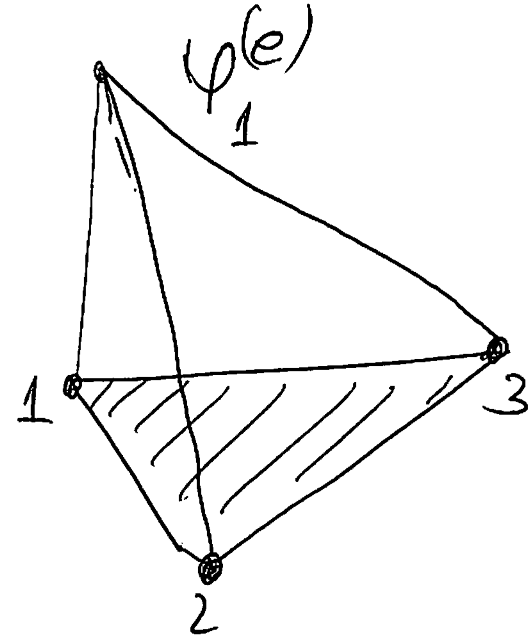
Instead of thinking of the basis functions being tent functions, associated with nodes of the grid, it is better to think of basis as being associated with elements

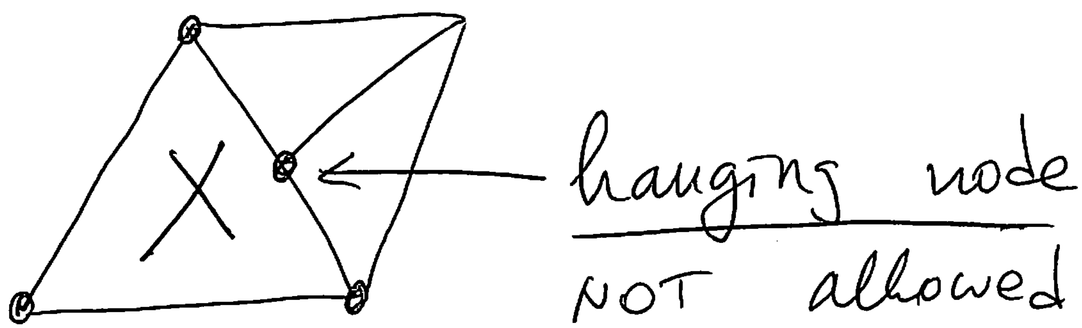




Now this is easy to transfer to <u>two dimensions</u> via <u>triangular elements</u>







Every triangle has 3 vertices/nodes

Basis functions are now
tent functions over each triangle

$$\varphi^{(e)\leftarrow \text{element}}_{\underset{\nwarrow \text{verex of element}}{i}} = \begin{cases} 1 & \text{at vertex } i \\ 0 & \text{at all other vertices} \end{cases}$$

$$\varphi^{(e)}_i = \underbrace{\alpha + \beta x + \gamma y}$$

three coefficients $\Leftrightarrow$
three values at vertices of triangle

Linear interpolant (piecewise linear)
is unique given vertex values.

The unknowns are now the values at the vertices

$$u = \sum_{e} \sum_{i \in e} u_i \, \varphi_i^{(e)}(x,y)$$

One vertex (degree of freedom) is shared among multiple triangles.

Consider the Poisson equation.

Stiffness matrix is

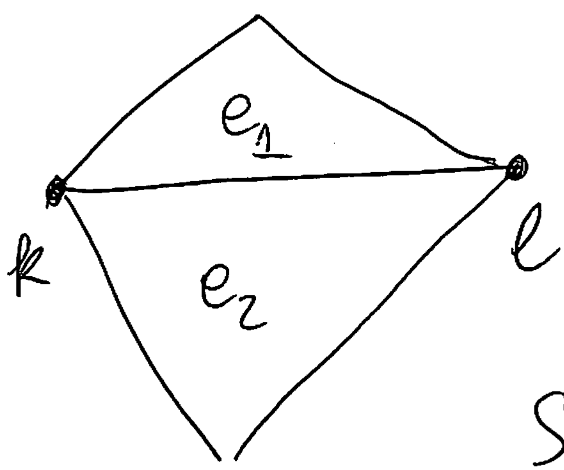
$$a_{k,l} = \int_{\Omega} (\vec{\nabla} \varphi_k) \cdot (\vec{\nabla} \varphi_l) \, dA$$

vertices

$$a_{k,\ell} = \sum_e \int_{\Omega_e} \left(\frac{\partial \varphi_k}{\partial x} \frac{\partial \varphi_\ell}{\partial x} + \frac{\partial \varphi_k}{\partial y} \frac{\partial \varphi_\ell}{\partial y} \right) dA$$

triangle

But integrand is only nonzero if vertices k and ℓ belong to triangle e, i.e., if they share triangle e.



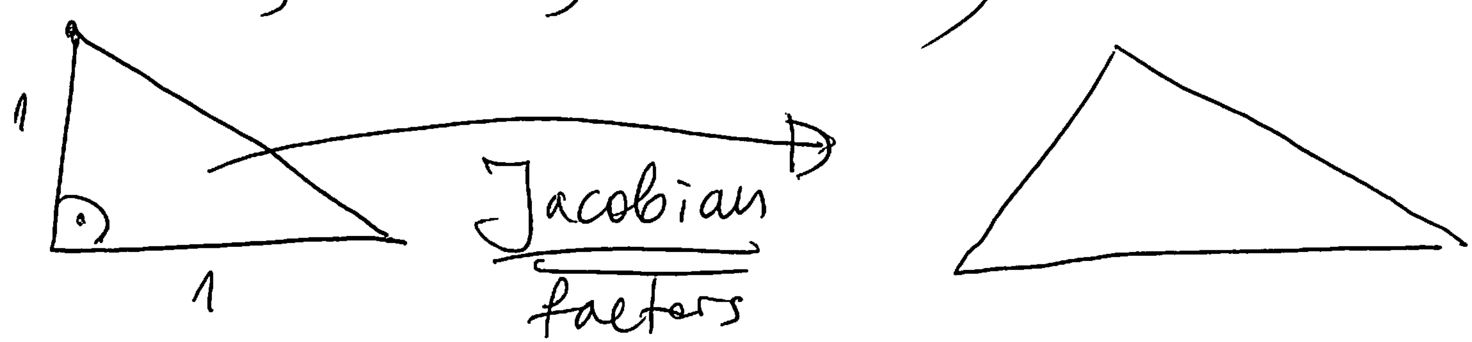
$$\boxed{a_{k,\ell} = \sum_{\substack{e \\ (k,\ell)\in e}} a^{(e)}_{k,\ell}}$$

Stiffness matrix assembly

We just need to worry about <u>one element</u> at a time.

Inside one element the functions are linear so the integrals are easy to compute analytically.

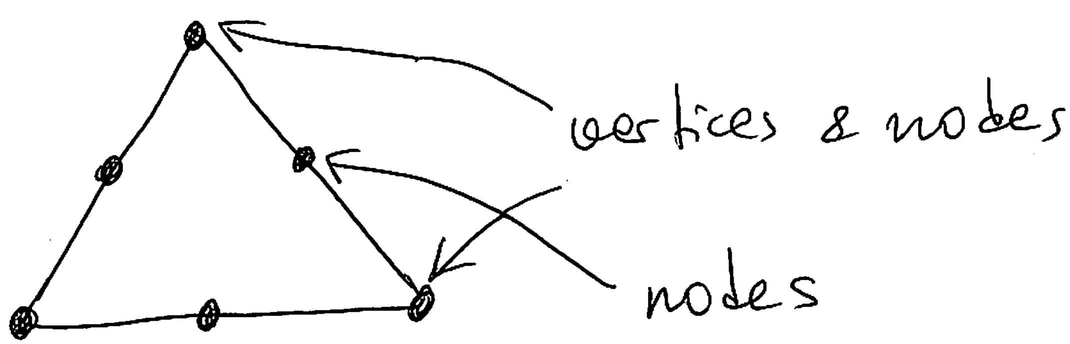
Usually this is precomputed on a <u>reference triangle</u> that is mapped by an affine transformation (translate, rotate, stretch)



(See some examples / images on webpage)

If we want to do quadratic interpolation on triangles, then inside each triangle

$$\varphi(x,y) = \underbrace{\alpha + \beta x + \gamma y + \delta x^2 + \eta xy + \xi y^2}_{\text{six coefficients} \Rightarrow \text{six nodes}}$$

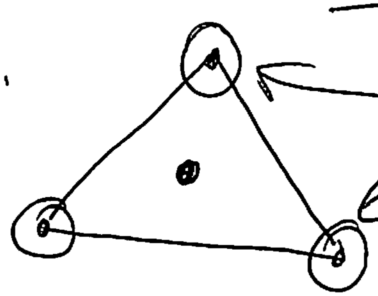


Observe now that along each edge the interpolant is a parabola, which is uniquely defined by the tree nodal values on each edge, so the interpolant is continuous accross triangle boundaries.

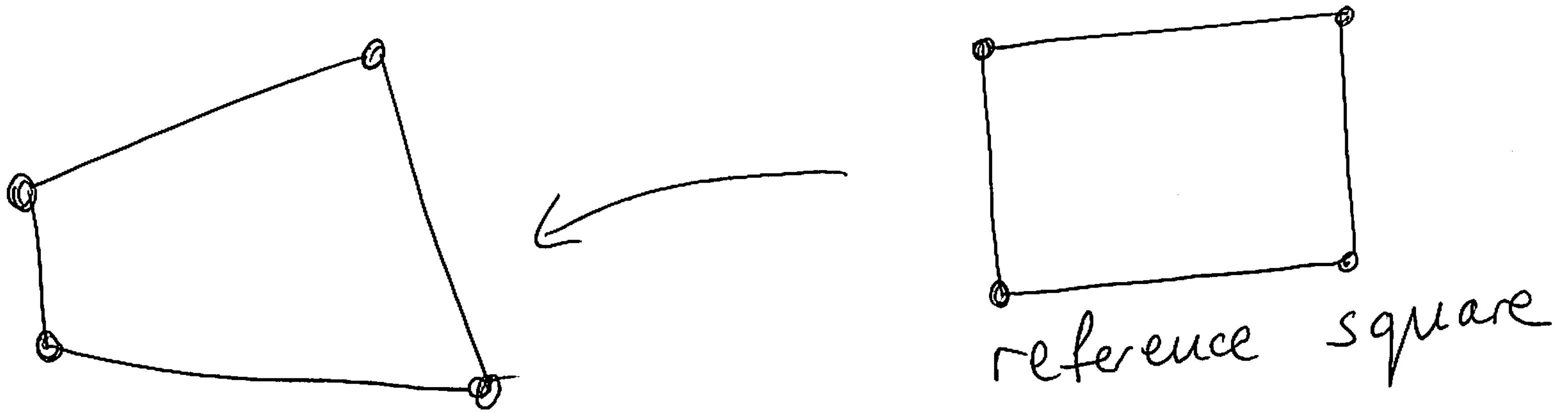
This is all we need for Poisson

But interpolant is not continuously differentiable.

Instead, do



It is possible sometimes to use quadrilateral meshes



Now we use bilinear interpolation, i.e., inside each element (reference element)

$$\varphi(x,y) = \varphi(x)\,\varphi(y) \leftarrow \text{continuous at edges}$$

or

$$\underbrace{\alpha + \beta x + \gamma y + \delta x y}_{\text{four coefficients}}$$

there is a zoo of FEM basis functions on trianglar, tetrahedral, rectangular, hexagonal and other grids.
Understanding which element smoothness is is required (how much smoothness is needed, conforming vs. non-conforming) and obtaining error estimates in various Sobolev norms is the topic of FEM textbooks and not trivial.