Numerical Methods II, Spring 2023
Assignment II: Pseudospectral semi-discretization of the KdV equation

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Due: **5pm on Tuesday 2/14 2023**

The optional parts of this assignment do not carry any points; only do optional things if you have time and interest for your own benefit.

1 [50 points] Spatial discretization of KdV

Consider the Korteweg de Vries PDE

$$\partial_t \phi = \mathcal{K}\left[\phi(\cdot, t)\right] = -\partial_{xxx}\phi - 3\partial_x\left(\phi^2\right),$$

where $\mathcal{K}\left[\phi(\cdot, t)\right]$ denotes the functional on the right hand side, which only involves derivatives of x.

On an unbounded line, this has traveling wave solutions in the forms of "solitons,"

$$\phi_{\text{sol}}\left(x, t\right) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}\left(x - ct\right)\right), \tag{1}$$

where c is the speed of the soliton moving to the right. While the soliton solution (1) is not periodic, it decays exponentially for large x so we can pretend it is periodic. Here we will consider the KdV equation on a periodic domain $x \in [-L/2, L/2)$ with $L = 60$ [*Hint: This is not on our standard interval of length 2π.*], and set $c = 1$. Note that direct differentiation and some algebra gives

$$\partial_t \phi_{\text{sol}}\left(x, t = 0\right) = 1/2 \, \frac{c^{5/2} \sinh\left(1/2 \, \sqrt{c}x\right)}{\left(\cosh\left(1/2 \, \sqrt{c}x\right)\right)^3}.$$

In a series of assignments related to KdV we will consider pseudo-spectral methods for solving this PDE, in which we represent the solution as a Fourier series [*Hint: Because we are not on the standard interval the wavenumber k is not an integer but rather has units of inverse length.*]

$$\phi\left(x, t\right) = \sum_k \hat{\phi}_k(t)e^{ikx}.$$

Recall from class that a pseudo-spectral method approximates the r.h.s of the KdV equation in Fourier space as

$$\widehat{\mathcal{K}\left[\phi(\cdot)\right]} = \boldsymbol{\mathcal{F}}\left(\mathcal{K}\left[\phi(\cdot)\right]\right) = \boldsymbol{F}\left(\hat{\boldsymbol{\phi}}\right) = ik^3 \boxdot \hat{\boldsymbol{\phi}} - 3i\boldsymbol{k} \boxdot \boldsymbol{\mathcal{F}}\left(\left(\boldsymbol{\mathcal{F}}^{-1}\hat{\boldsymbol{\phi}}\right)^{\boxed{2}}\right),$$

where $\boldsymbol{\mathcal{F}}$ denotes a Fourier transform. This allows us to convert the KdV PDE into the system of ODEs

$$\frac{d\hat{\boldsymbol{\phi}}\left(t\right)}{dt} = \boldsymbol{F}\left(\hat{\boldsymbol{\phi}}\left(t\right)\right),$$

which we will come back to later.

In this first series we focus on numerically approximating the right hand side $\boldsymbol{F}\left(\hat{\boldsymbol{\phi}}\right)$ for a given set of truncated Fourier coefficients $\hat{\boldsymbol{\phi}}$. For simplicity here we can just take $t = 0$ since the KdV equation is autonomous.

To explore how to smoothness of the solution in space affects the accuracy, in addition to the soliton wave you will consider a triangle wave

$$f(x) = \begin{cases} (10 + x) & x < 0 \text{ and } x \geq -10 \\ (10 - x) & x \geq 0 \text{ and } x \leq 10 \\ 0 & \text{otherwise} \end{cases},$$

and the sawtooth wave

$$g(x) = \begin{cases} (10 + x) & x < 0 \text{ and } x \geq -10 \\ 0 & \text{otherwise} \end{cases}.$$

Note: This homework works with functions and function norms, not with vectors and vector norms. In order to approximate function norms, you need to evaluate the solution on a finer grid, just like you did in Homework 1. So if you use N points for the Fourier transforms, evaluate the errors/functions on a grid of $M \gg N$ points using your interpft or the built-in interpft (for this homework it is fine to use the built-in function). Think about how to choose M in a way that your answers don't really depend on M, if that is possible, just like you want to plot a function with enough points so you cannot see the fact it is only sampled on a finite grid.

1.1 [15 pts] Error in initial condition

If we start with an initial condition $\phi(x, t = 0) = \phi_{\text{sol}}(x, t = 0)$, then the solution is given by the traveling soliton wave (1). However, we make an error already at $t = 0$ just by approximating this initial condition with a truncated Fourier series $\tilde{\phi}(x, t)$ with N Fourier modes. *Hint: You studied this error in Homework 1 so reuse your codes for this part.*

[optional] Try to make the code work also for even number of points this time around.

[5pts] Compute how the discretization error in the initial condition depends on the number of Fourier modes N for *different* function norms ($L_{1/2/\infty}$) when the initial condition is $\phi_{\text{sol}}(x, t = 0)$. Do you see a difference between the different norms? To make the comparison fair use *relative error*

$$\epsilon = \frac{\left\| \phi - \tilde{\phi} \right\|}{\| \phi \|}.$$

[5pts] Plot the error $\phi - \tilde{\phi}$ for the triangle and sawtooth waves, and also see whether you see a difference between the different norms for these non-smooth initial conditions.

1.2 [35 pts] Pseudo-spectral approximation of $\mathcal{K}[\phi(\cdot)]$

[10pts] Write code to evaluate $\boldsymbol{F}\left(\hat{\phi}\right)$ for a given grid size N.

[optional] Make sure the code works also for even N (explain what you did), and use even values for N for this assignment.

[15pts] Compute the error in the approximation of $\mathcal{K}[\phi_{\text{sol}}(., t = 0)]$ in different function norms (is there a difference and why?) and see how the error changes with the number of points N. Is this discretization spectrally accurate?

[10pts] How many points N do you need to obtain 9 digits of accuracy?

[optional] What if the constant c and/or the length L of the periodic domain $x \in [-L, L)$ changed – how do you expect N to change and why? *Hint: It is always a good idea to confirm your predictions numerically.*

1.3 [optional] Aliasing error

Implement anti-aliasing and compare the numerical approximations of $\mathcal{K}[\phi_{\text{sol}}(x, t = 0)]$ with and without anti-aliasing, and explain whether and when you see a noticeable difference and when you see no difference, and why.

Repeat for the triangle and square waves and comment on your observations. Note that for these functions the r.h.s. of the KdV equation does not have a classical interpretation so it is not really clear it makes sense to talk about an error in the approximation, however, we can certainly (naively) solve the system of ODEs and see what we get.