```
> restart:
> with(LinearAlgebra):
> N:=2: # Number of unknowns
> Digits:=6: # Number of digits to use for floating-point numbers (the numerical ODE solver will use double precision)
> with(plots):
```

Implicit part of ODE

Let's construct a linear term A*x that is stiff, with one eigenvalue of A being 0.2 and the other being lambda (an input parameter). Explicit methods will require a time step size dt~1/lambda, so for large lambda we need an implicit method

```
> eigvecs:=Matrix(N,N,[[1/3,sqrt(1-(1/3)^2)] , [sqrt(1-(1/3)^2),-1/3]]); # Normalized eigenvectors of matrix
```

$$eigvecs := \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ \frac{2\sqrt{2}}{3} & K\ \frac{1}{3} \end{bmatrix} \quad (1)$$

```
> simplify(MatrixInverse(eigvecs)-Transpose(eigvecs)); # Confirm eigenvecs is unitary
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (2)$$

```
> eigvals:=lambda->Vector(N,[0.2,lambda]); # lambda is assumed to be a large eigenvalue, thus a stiff system
```

$$eigvals := \lambda \mapsto Vector(N, [0.2, \lambda]) \quad (3)$$

```
> A:=unapply(evalf(eigvecs.DiagonalMatrix(eigvals(lambda)).Transpose(eigvecs)), lambda):
> A(lambda);
```

$$\begin{bmatrix} 0.0222222 + 0.888889\,\lambda & 0.0628539 K\ 0.314269\,\lambda \\ 0.0628536 K\ 0.314269\,\lambda & 0.177777 + 0.111111\,\lambda \end{bmatrix} \quad (4)$$

```
> Eigenvectors(A(10.0)); # Confirm the eigenvecs and eigenvals are correct:
```

$$\begin{bmatrix} 9.99999733241764 + 0.\,I \\ 0.200002667582359 + 0.\,I \end{bmatrix}, \begin{bmatrix} 0.942809093731676 + 0.\,I & 0.333333185831919 + 0.\,I \\ K\ 0.333333185831919 + 0.\,I & 0.942809093731676 + 0.\,I \end{bmatrix} \quad (5)$$

Explicit part of ODE

We will make the explicit terms be a simple quadratic function with a non-stiff Jacobian

```
> B:=x->Vector([0.2*x[1]*x[2], 0.3*x[1]^2-0.1*x[2]^2]);
```

$$B := x \mapsto Vector\left(\left[0.2 \cdot x_1 \cdot x_2,\ 0.3 \cdot x_1^2 - 0.1 \cdot x_2^2\right]\right) \quad (6)$$

```
> x_:=Vector([x1(t),x2(t)]);
```

$$x\_ := \begin{bmatrix} x1(t) \\ x2(t) \end{bmatrix} \quad (7)$$

Right hand side of ODE is A*x+B(x)

```
> rhs_homog:=(-A(lambda).x_+B(x_)); # Right hand side of ODEs
```

$$rhs\_homog := \left[\left[-(0.0222222 + 0.888889\,\lambda)\, x1(t) - (0.0628539 - 0.314269\,\lambda)\, x2(t) + 0.2\, x1(t)\, x2(t)\right], \left[-(0.0628536 - 0.314269\,\lambda)\, x1(t) - (0.177777 + 0.111111\,\lambda)\, x2(t) + 0.3\, x1(t)^2 - 0.1\, x2(t)^2\right]\right] \quad (8)$$

```
> dx_dt:=map(diff,x_,t);
```

$$dx\_dt := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\, x1(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\, x2(t) \end{bmatrix} \quad (9)$$

```
> ODEs:=dx_dt-rhs_homog;
```

$$ODEs := \left[\left[\frac{\mathrm{d}}{\mathrm{d}t}\, x1(t) + (0.0222222 + 0.888889\,\lambda)\, x1(t) + (0.0628539 - 0.314269\,\lambda)\, x2(t) - 0.2\, x1(t)\, x2(t)\right], \left[\frac{\mathrm{d}}{\mathrm{d}t}\, x2(t) + (0.0628536 - 0.314269\,\lambda)\, x1(t) + (0.177777 + 0.111111\,\lambda)\, x2(t) - 0.3\, x1(t)^2 + 0.1\, x2(t)^2\right]\right] \quad (10)$$

```
> #infolevel[dsolve]:=4:
> #dsolve(convert(ODEs,set) union {x1(0)=alpha,x2(0)=beta}, {x1(t),
  x2(t)});
```

Matlab's analytical solver does not return a solution. Therefore, we will use the **method of manufactured solutions**, constructing a very simple oscillatory solution:

```
> x1_sol:=t->sin(t); x2_sol:=t->cos(t);
```

$$x1\_sol := t \mapsto \sin(t)$$
$$x2\_sol := t \mapsto \cos(t) \quad (11)$$

```
> x_sol:=Vector([x1_sol(t),x2_sol(t)]);
```

$$x\_sol := \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \quad (12)$$

Now compute the inhomogeneous term required to get this solution:

```
> rhs_inhomog:=eval(ODEs, {x1=x1_sol, x2=x2_sol});
```

$$rhs\_inhomog := [[\cos(t) + (0.0222222 + 0.888889\,\lambda)\sin(t) + (0.0628539 \; \mathrm{K}\; 0.314269\,\lambda)\cos(t)\;\mathrm{K}\; 0.2\sin(t)\cos(t)], [\mathrm{K}\sin(t) + (0.0628536\,\mathrm{K}\; 0.314269\,\lambda)\sin(t) + (0.177777 + 0.111111\,\lambda)\cos(t)\;\mathrm{K}\; 0.3\sin(t)^2 + 0.1\cos(t)^2]] \tag{13}$$

Final ODE is

```
> ODE_rhs:=rhs_homog+rhs_inhomog: # rhs of ODE
> ODEs_manuf:=dx_dt-ODE_rhs:
> eval(convert(ODEs_manuf, set), {x1=x1_sol, x2=x2_sol}); # Confirm the manufactured solution works
```

$$\{0.\} \tag{14}$$

Numerical solution in Maple

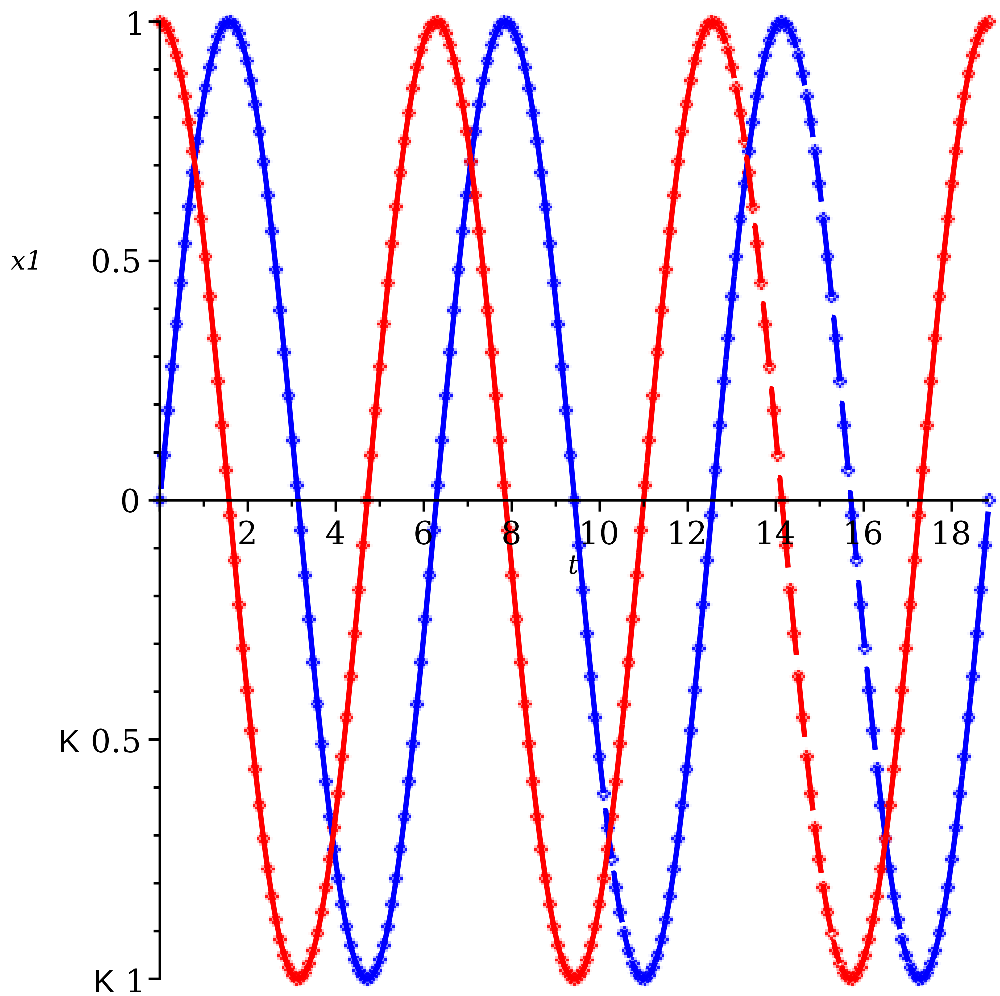
```
> lambda:=1:
> T:=evalf(6*Pi):
> num_solution:=dsolve(convert(ODEs_manuf,set) union {x1(0)=x1_sol(0), x2(0)=x2_sol(0)} , {x1(t),x2(t)}, type='numeric', stiff=true, range=0..T, relerr=1e-3, optimize=true);
```

$$num\_solution := \mathbf{proc}(x\_rosenbrock)\ \ldots\ \mathbf{end\ proc} \tag{15}$$

```
> p1:=odeplot(num_solution, [t,x1(t)], color='blue', style='point'):
> p2:=odeplot(num_solution, [t,x2(t)], color='red', style='point'):
> p3:=plot(x1_sol(t), t=0..T, color='blue', style='line', thickness=2) :
> p4:=plot(x2_sol(t), t=0..T, color='red', style='line', thickness=2):
```

For some reason Maple's Rosenbrock solver has issues around T=15!?!

```
> display(p1,p2,p3,p4);
```



Matlab, Python, and Fortran code for A and B(x):

```
> with(CodeGeneration):
```

Fortran:

```
> explicit:=B(Vector([x[1],x[2]]))+rhs_inhomog;
```

$explicit :=$ **(16)**

$$\left[\left[0.2\,x_1\,x_2 + 0.748585\cos(t) + 0.911111\sin(t)\,\mathrm{K}\;0.2\sin(t)\cos(t)\right],\right.$$
$$\left[0.3\,x_1^2\,\mathrm{K}\;0.1\,x_2^2\,\mathrm{K}\;1.25142\sin(t) + 0.288888\cos(t)\,\mathrm{K}\;0.3\sin(t)^2\right.$$
$$\left.\left.+\,0.1\cos(t)^2\right]\right]$$

```
> Fortran(explicit);
      cg(1) = 0.2D0 * x(1) * x(2) + 0.748585D0 * cos(t) + 0.911111D0
* s
     #in(t) - 0.2D0 * sin(t) * cos(t)
      cg(2) = 0.3D0 * x(1) ** 2 - 0.1D0 * x(2) ** 2 - 0.125142D1 *
sin(t
     #) + 0.288888D0 * cos(t) - 0.3D0 * sin(t) ** 2 + 0.1D0 * cos(t)
**
      #2
```

```
> Fortran(A(eig));
      cg0(1,1) = 0.222222D-1 + 0.888889D0 * eig
      cg0(1,2) = 0.628539D-1 - 0.314269D0 * eig
      cg0(2,1) = 0.628536D-1 - 0.314269D0 * eig
      cg0(2,2) = 0.177777D0 + 0.111111D0 * eig
```

Matlab:

```
> Matlab(explicit);
cg1 = [0.2e0 * x(1) * x(2) + 0.748585e0 * cos(t) + 0.911111e0 * sin
(t) - 0.2e0 * sin(t) * cos(t) 0.3e0 * x(1) ^ 2 - 0.1e0 * x(2) ^ 2 -
0.125142e1 * sin(t) + 0.288888e0 * cos(t) - 0.3e0 * sin(t) ^ 2 +
0.1e0 * cos(t) ^ 2];
```

```
> Matlab(A(eig));
cg2 = [0.222222e-1 + 0.888889e0 * eig 0.628539e-1 - 0.314269e0 * eig;
0.628536e-1 - 0.314269e0 * eig 0.177777e0 + 0.111111e0 * eig;];
```

Python

```
> Python(explicit);
cg3 = numpy.mat([0.2e0 * x[0] * x[1] + 0.748585e0 * math.cos(t) +
0.911111e0 * math.sin(t) - 0.2e0 * math.sin(t) * math.cos(t),0.3e0 *
x[0] ** 2 - 0.1e0 * x[1] ** 2 - 0.125142e1 * math.sin(t) + 0.288888e0
* math.cos(t) - 0.3e0 * math.sin(t) ** 2 + 0.1e0 * math.cos(t) ** 2])
```

```
> Python(A(eig));
cg4 = numpy.mat([[0.222222e-1 + 0.888889e0 * eig,0.628539e-1 -
0.314269e0 * eig],[0.628536e-1 - 0.314269e0 * eig,0.177777e0 +
0.111111e0 * eig]])
```