Fall 2022

PDE ~~Spring 2018~~

Lecture 1

Why are Partial Differential Equations (PDEs) important?

They are the cornerstone of macroscopic (large/human scale) physics!

Examples of PDEs in action:

1. Weather prediction or climate modeling (Navier-Stokes equations)
2. Essentially all engineering, especially mechanical & aerospace
   - elasticity (bridges)
   - fluid flow (airplanes, rockets, ships)
3. Electromagnetism (Maxwell's equations) (cell phones, radio communication)

Everything around us that we can see & interact with can be described by a PDE

(2)

There are different kinds of PDEs related to different physics, e.g.

(1) Diffusion
(heat conduction, polutant dispersal, viscosity of liquids

(2) Advection
(hurricanes, wind dispersal / flow, airplane design, traffic flow)

(3) Deformation & Elasticity
(electric fields, elastic materials, magnetism)

and others.

Course webpage:

https: adonev.github.io/PDE2022

~~http://cims.nyu.edu/~donev/Teaching/PDE~~

$\boxed{\text{Visit the webpage ASAP}}$

Two ~~Three~~ textbooks ~~(PDE, Applied PDE, Essential PDE)~~

Ordinary Differential Equation (ODE):

Explicit: $$\frac{du(t)}{dt} = f(t, u(t))$$

Implicit: $$F\left(t, u(t), \frac{du}{dt}, \frac{d^2u}{dt^2}, \ldots, t\right) = 0$$

For PDEs there is more than one dependent variable:

$$u(\underbrace{x, y, z}_{\text{space}}, \ldots, \underbrace{t}_{\text{time}})$$

Notation:

$$\frac{\partial u}{\partial x} \equiv u_x \equiv \partial_x u$$

$$\frac{\partial^2 u}{\partial t \partial x} \equiv u_{tx} = u_{xt} \equiv \frac{\partial^2 u}{\partial x \partial t}$$

twice differentiable

$$\frac{\partial^2 u}{\partial x^2} \equiv u_{xx} \equiv \partial_x^2 u$$

Order of PDE

the order of a PDE is the highest order of derivative that appears in equation.

General PDE:

$F(u, x, y, z, t, \ldots$

first order $\to u_x, u_y, u_z, u_t, \ldots,$

second $\to u_{xx}, u_{xy}, \ldots, u_{tt}, u_{tx}, \ldots,$

third $\to u_{xxx}, u_{xyz}, \ldots) = 0$

Examples:

$u_t = u_{xx}$ is second order

What does it mean to solve a PDE?

Try $u(x,t) = e^{-t}\sin(x)$

(a) $u_t = -e^{-t}\sin x = -u$

(b) $u_x = e^{-t}\cos x \Rightarrow$

$u_{xx} = -e^{-t}\sin x = -u$

(c) $u_t = u_{xx}$ as needed

So $u(x,t) = e^{-t}\sin(x)$
is a solution of the PDE.

Is it the only? How do we choose among many (physical systems choose one after all)?

All of these are topics of the field of PDEs:
Mathematical Analysis

Practice

(1) Show that $u = \ln(r)$, where $r = \sqrt{x^2+y^2}$, solves

$$\nabla^2 u = 0 = \partial_{xx} u + \partial_{yy} u$$

(2) Show that $u = 1/r$ solves

$$\nabla^2 u(x,y,z) = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u = 0$$

these are the Laplace equation in two and three dimensions

Examples of PDEs

A First order:

1. Advection equation - (LINEAR)

$$\boxed{u_t + c\,u_x = 0},\quad c>0 \text{ is a constant (speed of flow)}$$

wind, flow in ocean, traffic (simple)

2. Inviscid Burgers - (NON-LINEAR)

$$u_t + u\,u_x = 0$$

(traffic flow)

Second order

1. Laplace's equation (LINEAR)

$u_{xx} + u_{yy} = 0$ in 2D (two dimensions)

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Short hand notation for Laplacian

physics

$$\boxed{\nabla^2 u = \Delta u = 0}$$

math

Second order contd.

2 Poisson's Equation (e.g. electrostatics)
(variant of Laplace)

$$\nabla^2 u = f(x, y, z, \dots) \quad \text{LINEAR}$$

2* Non-linear Poisson:

$$\nabla^2 u = f(x, y, z, u) \quad \text{NON-LINEAR}$$

3 Heat or diffusion eq (LINEAR)

$$u_t = k\, u_{xx}, \quad k > 0 \text{ is constant}$$

or generally $\boxed{u_t = k \nabla^2 u \equiv k \Delta u}$

4 Wave equation (LINEAR)
(e.g. sound or electrostatic waves)

$$u_{tt} = c^2 u_{xx}, \quad c = \text{sound speed, speed of light}$$

or more general

$$\boxed{u_{tt} = c^2 \nabla^2 u}$$

(5) Advection - diffusion - reaction
(pollutant transport) (LINEAR)

$$u_t + c u_x = k u_{xx} + r u$$

$\begin{cases} k > 0 \\ r \\ c \end{cases}$ constants (diffusion coefficient) (reaction rate) (speed of flow)

Or $ru \to f(u)$ for nonlinear

(6) Black - Scholes eq. (LINEAR)
(stock prices)

$$u_t + \frac{\sigma^2}{2} x^2 u_{xx} + r x u_x = r u$$

option price — volatility — stock price — return rate

Third order

(1) Korteveg de Vries (KdV) eq.

$$u_t + 6u u_x + u_{xxx} = 0$$

describes solitons
(e.g. lossless communication)

Excercise:

Show that $u = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$

solves the heat equation

$$u_t = u_{xx}$$

Dimensional Analysis
(PHYSICS)

t = time = [s] = [seconds]

units

$x \equiv$ space = [m] = [meters]

$\frac{\partial}{\partial x}$ has units of $[m^{-1}]$

$\frac{\partial}{\partial t}$ has units of $[s^{-1}]$

$\frac{\partial}{\partial x \partial x}$ has units of $[m^{-2}]$

u has some units $[u]$

e.g. kg, K (temperature)

mass

(10)

$u_t = k u_{xx}$

$$u_t = \left[\frac{u}{s}\right] = k u_{xx} = [k]\left[\frac{u}{m^2}\right]$$

$$\Rightarrow [k] = \left[\frac{m^2}{s}\right] \quad \text{(diffusion coefficient)}$$

Practice

What units does c have in

$$u_t + c u_x = 0$$

What about

$$u_t = c^2 u_{xx}$$

Principle (physics) The answer must not depend on the choice of the units!

This means that special functions must have dimensionless arguments (no unit)

$$u = \frac{1}{\sqrt{t}} e^{-x^2/4t} \quad \text{is NOT good}$$

$$\left[\frac{x^2}{t}\right] = \left[\frac{m^2}{s}\right]$$

Later we will construct a solution of the form

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} u(s,0)\, e^{-(x-s)^2/4kt}\, ds$$

Check units make sense

$$\left[\frac{1}{\sqrt{kt}}\right] = \frac{1}{\sqrt{\frac{m^2}{\cancel{s}}\cdot \cancel{s}}} = \frac{1}{m}$$

Unit of integral

$$\left[\int ds\right] = [ds] = m$$

So the unit of the whole expression is

$$\frac{[u]\cdot m}{m} = [u]$$

as it has to be.

[Showing that this is a solution requires calculus only (move derivatives inside the integral) → TRY IT