Partial Differential Equations, Fall 2022
Homework I: Introduction to PDEs

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Total number of points is 45. Show all steps leading to the answer with suitable justification/explanation.

1. (5pts) Show that
$$u(x,y) = h\left(y^2+2x\right) + g\left(y^2-2x\right)$$
satisfies the PDE
$$y^2 u_{xx} + \frac{1}{y} u_y - u_{yy} = 0,$$
for arbitrary sufficiently differentiable functions h and g.
2. (5pts) Show that $u(x,y) = f(x)g(t)h(y)$ is a solution of the PDE
$$u\, u_{xy} = u_x\, u_y$$
for arbitrary sufficiently differentiable functions f and h and an arbitrary function g.
3. (10pts) Verify that
$$u(x,t) = \frac{1}{2v}\int_{x-vt}^{x+vt} f(s)\, ds$$
is a solution of the wave equation $u_{tt} = v^2 u_{xx}$, where $v > 0$ is a constant and f is an arbitrary differentiable function.
Hint: *Lookup "Leibniz integral rule" on Wikipedia.*
4. (5pts) Find the general solution of the equation $u_{xt}(x,t) = 0$ in terms of arbitrary functions.
5. (7.5) Find a function $u(x,y)$ that satisfies the PDE
$$u_{xx} = 0, \quad 0 < x < 1,\ t > 0$$
subject to the boundary conditions
$$u(0,t) = t$$
$$u(1,t) = 1.$$
6. (5pts) For what values of α and β is $u(x,t) = u_0 e^{-\alpha t}\cos(\beta x)$ a solution of the heat equation $u_t = D u_{xx}$. What units do α and β have if the unit of u is $[U]$, the units of x are $[m]$ and the units of t are $[s]$? From this, what is the unit of D [*Hint: We answered this in class*]?
7. (7.5pts, 2.5pts per equation) What is the dispersion relation $\omega(k)$ between the frequency ω and wavenumber k if the so-called plane wave
$$u(x,t) = \exp\left(-i\left(kx + \omega t\right) + \phi\right),$$
is a solution of:
    (a) The wave equation $u_{tt} = v^2 u_{xx}$.
    (b) The advection equation $u_t + v u_x = 0$.
    (c) The diffusion equation $u_t = \kappa u_{xx}$.