1

Fall 2022

PDE ~~SPRING 2018~~

LECTURE 2

Initial & Boundary Conditions

Consider the PDE

$$\boxed{u_{tx} - 4xt = 0} \Rightarrow$$

$$\partial_t (u_x - 2xt^2) = 0 \Rightarrow$$

$$u_x - 2xt^2 = \int \emptyset \cdot dt = \underline{\underline{f(x)}}$$

Integration "constant" for ODEs now becomes an <u>arbitrary function</u> of x!

In some sense a PDE is like a system of infinitely many ODEs (uncountably many), one for each x in example above.

NOTE: Infinite dimensional linear spaces and linear algebra are <u>different</u>!

(2)

We need one initial condition for each ODE, i.e. for each x in the example.

ODE: $u_x = f(x) + 2xt^2$

$$u = \int \left[f(x) + 2xt^2 \right] dx$$

$$= g(x) + x^2 t^2 + h(t)$$

integration "constant"

where

$$g(x) = \int f(x)\,dx$$

is one indefinite integral of $f(x)$. Since $f(x)$ was arbitrary so is $g(x)$, and we can forget about $f(x)$

Solution:

$$\boxed{u(x,t) = g(x) + h(t) + x^2 t^2}$$

(3)

To make the solution unique we need a way to fix/determine $g(x)$ and $h(t)$

Let's say $t \equiv$ time
$x \equiv$ space

The condition on time will be an initial condition, meaning at $t=0$, just like ODEs:

$$u(x, t=0) = g(x) + h(0) = u_1(x)$$

Initial Condition (IC)

The condition in space will be called a boundary condition since it has to do with the physical/spatial domain of the PDE:

$$u(x=0, t) = g(0) + h(t) = u_2(t)$$

Boundary Condition (BC)

4

Note:

$$u(0,0) = u_1(0) = u_2(0)$$

So if $u_1(0) \neq u_2(0)$ there will be a <u>discontinuity</u> at $(0,0)$

Whether that is permissible depends on the type of the PDE

Another "better" way to write solution:

$$g(x) = u_1(x) - h(0)$$
$$h(t) = u_2(t) - g(0) \quad \Rightarrow$$

$$u(x,t) = u(x,0) + u(0,t) + x^2t^2 - (h(0) + g(0))$$

But

$$u(0,0) = h(0) + g(0) \quad \Rightarrow$$

$$\boxed{u(x,t) = x^2t^2 + u(x,0) + u(0,t) - u(0,0)}$$

(5) Let's consider now the advection equation

$$\boxed{u_t + c u_x = 0}$$

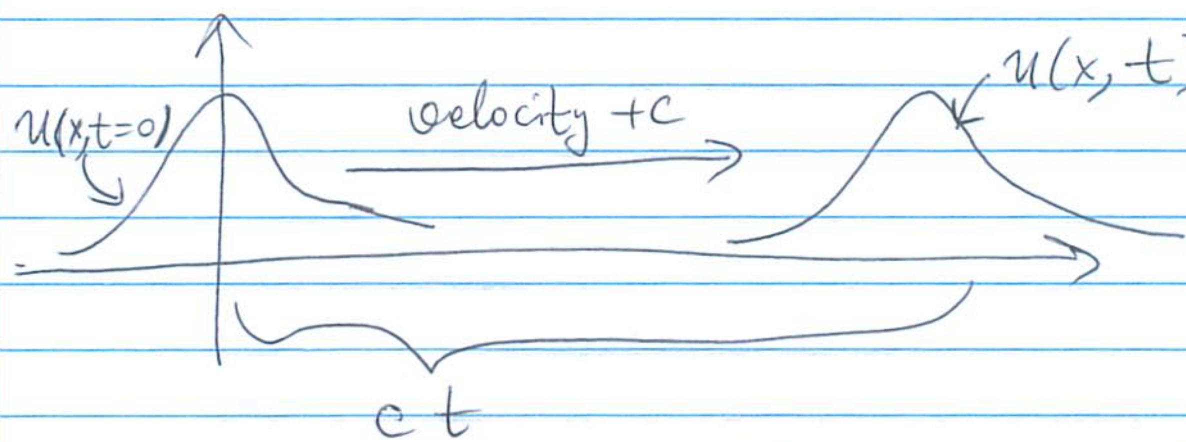
Let's show that

$$u = f(x - ct)$$

is a solution:

$$\left\{\begin{array}{l} u_t = -c\, f'(x-ct) \\ c u_x = c\, f'(x-ct) \end{array}\right. \Rightarrow u_t + c u_x = 0$$

$$\boxed{f(x) = u(x,0) \quad \text{IC}}$$



c has units $\left[\frac{m}{s}\right] \equiv$ [velocity]

or speed of propagation of information (same applies to wave equation)

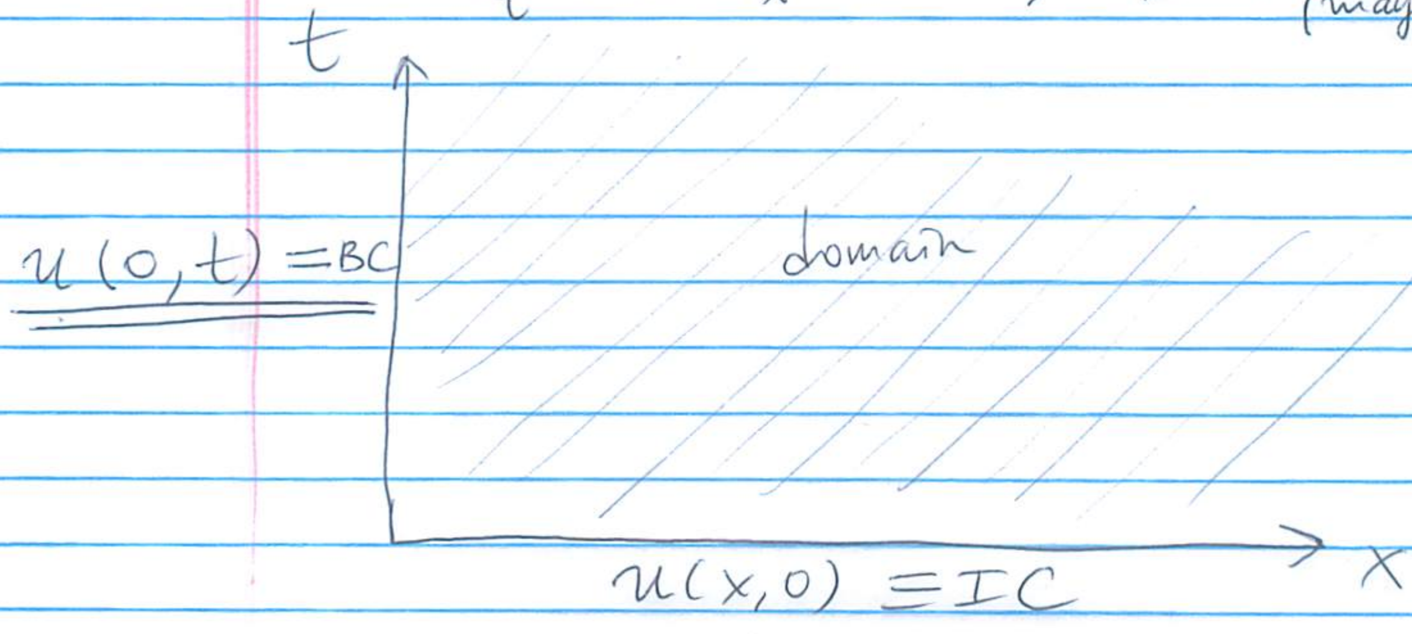
For this equation (advection eq)
If we have an initial condition
we can find the solution at
any point in time simply
by translating the IC

Conclusion: Advection equation posed
on the whole real line $\mathbb{R}$
is an
Initial Value Problem (IVP)

If we also need BCs it's a
Boundary Value Problem (BVP)

Consider advection equation on $\mathbb{R}^+$

$$u_t + c\,u_x = 0 \quad , x \geq 0 , t \geq 0$$
(maybe?)



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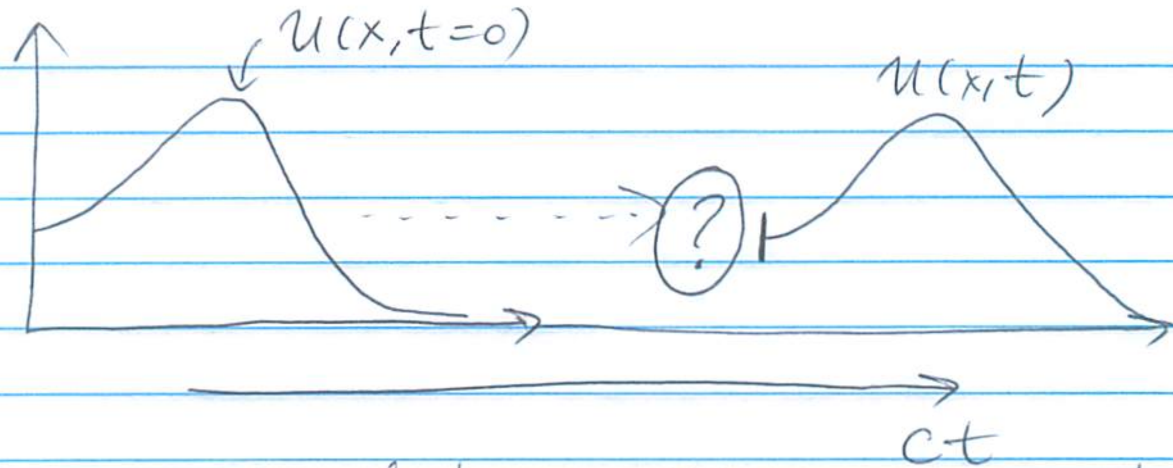


If we only had an IC we would **not** know $u(x < ct, t)$

We will show later that if we have the

Dirichlet BC : $u(0,t) = f(t)$

(Dirichlet means BC specifies u)

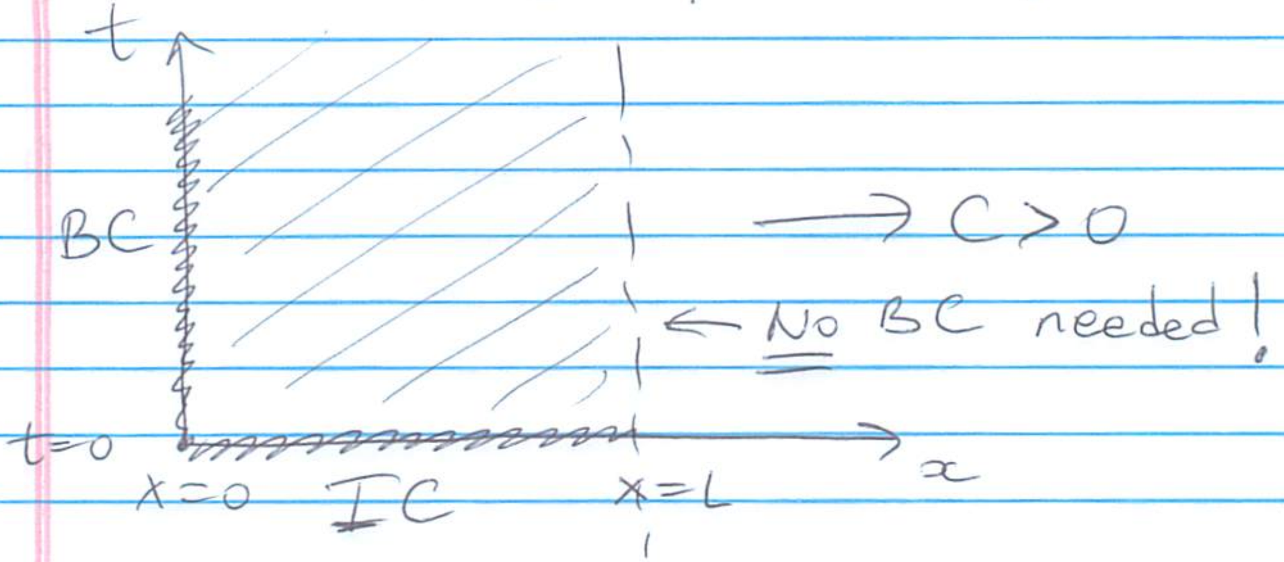
then we would know $u(x,t)$ over the whole space-time domain

BVP for advection

$$\begin{cases} u_t + c u_x = 0 , & x \geq 0, \ t \geq 0 \\ u(x,0) = g(x) = IC \\ u(0,t) = f(t) = BC \\ \boxed{c > 0} \end{cases}$$

(information is carried or flows to the right)

8 What if the domain were $0 \leq x \leq L$, $t \geq 0$?



Analogy: If the winds are carrying cold air from the Arctic to the Northeast, to forecast the weather over New England we need to know what's happening up north, but don't really need to know what is happening in Florida down south.

PHYSICAL INTUITION is KEY!

We only need information upwind/upstream to determine the solution downwind/downstream

Note: the naming IC/BC comes from physics but the distinction is arbitrary:

general: We need boundary conditions in the space-time domain on some of the boundaries

Let's consider the IVP for the wave equation

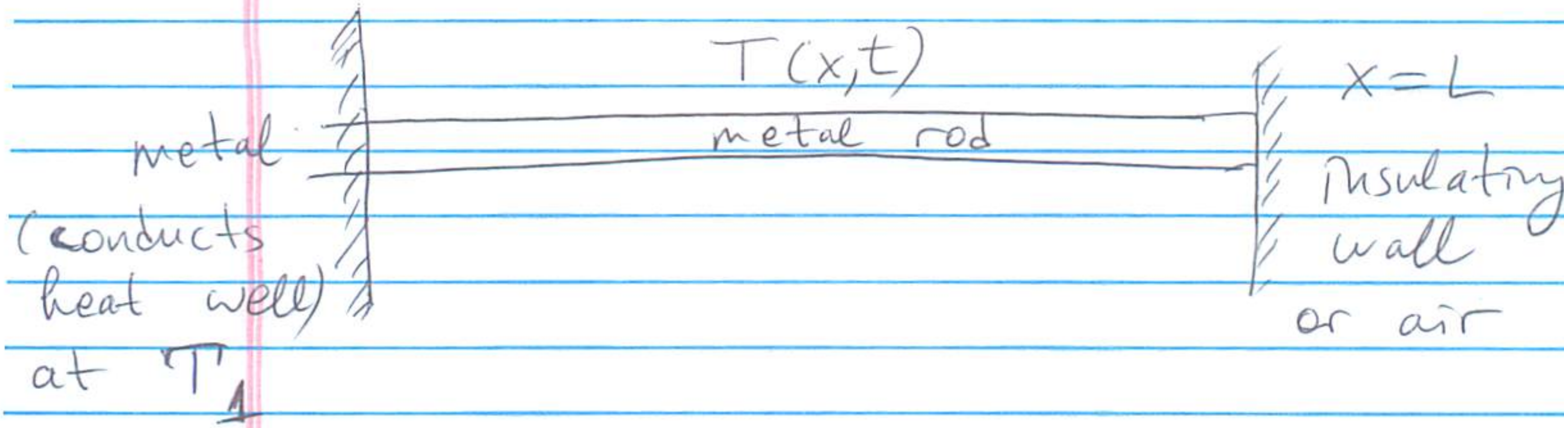
$$\boxed{u_{tt} = c^2 u_{xx}}$$

Here it will turn out later that we need two ICs, just like for second-order ODEs

ICs
$$\begin{cases} u(x, t=0) = f(x) \\ u_t(x, t=0) = g(x) \end{cases}$$

Example: If I strum a guitar string I need to know not only the initial position but also velocity of the string at $t=0$

(10) Consider now heat conduction



$u \equiv T$ (temperature or energy density)

$$\boxed{u_t = k\, u_{xx}}, \quad k>0 \qquad \text{heat equation}$$

IC: $u(x,0) = T_0 =$ constant

Dirichlet BC: on <u>left</u> wall:

$u(x=0,t) = T_1(t)$

<u>Neumann BC</u> on right wall

$\partial_x u(x=L,t) = 0$ (no heat flow)

(Neumann means BC specifies derivative of u, not u itself)

Note: It's OK if $T_1(0) \neq T_0$

11

The reason we need two BCs, both on left and on right, is because of the second derivative in u_{xx}

Example: Imagine there are two sources of smell in a room free of drafts (wind). What you smell will be a mix of information from both sources, left and right.

This is different from the case when wind brings only one of the smells to your nose.

Physics — Diffusion is different from advection (of mass, heat)

General rule of thumb — The BCs contain lower-order derivatives than the order of the PDE.
We always need an IC

Example: Setting $u_{xx}(x=0,t)=0$ is not allowed for the heat equation, as we will see later