CS 109: Probability for Computer Scientists
Problem Set#3

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1. Do not paste your code here. Submit your completed .py file on Gradescope under "PSet 3 - Coding".

2. Lyft line gets 2 requests every 5 minutes, on average, for a particular route. A user requests the route and Lyft commits a car to take her. All users who request the route in the next five minutes will be added to the car as long as the car has space. The car can fit up to three users. Lyft will make \$6 for each user in the car (the revenue) minus \$7 (the operating cost).

   a. How much does Lyft expect to make from this trip?

   b. Lyft has one space left in the car and wants to wait to get another user. What is the probability that another user will make a request in the next 30 seconds?

Answer.

a. Lyft has one user it to whom it has committed. Now, in the next 5 minutes, we know that Lyft can expect to receive 2 requests given its average request rate of 2 requests every 5 minutes. This means that Lyft can expect the car to be full and to make $\boxed{(6 \times 3) - 7 = \$11}$.

b. This situation can be modeled as an exponential distribution. Using the CDF of an exponential random variable (1) and taking the rate λ to be 0.4 requests/min (2), we can solve for the probability that a user makes a request in the next 30 seconds, converting seconds to minutes for unit consistency (3).

$$F(x) = 1 - e^{-\lambda x} \tag{1}$$

$$\lambda = 2 \text{ requests / 5 minutes} = 0.4 \text{ req/min} \tag{2}$$

$$\boxed{P(X \leq 0.5) = F(0.5) = 1 - e^{-0.4(0.5)} \approx 0.1813} \tag{3}$$

3. Suppose it takes at least 9 votes from a 12-member jury to convict a defendant. Suppose also that the probability that a juror votes that an actually guilty person is innocent is 0.25, whereas the probability that the juror votes that an actually innocent person is guilty is 0.15. If each juror acts independently and if 70% of defendants are actually guilty, find the probability that the jury renders a correct decision. Also determine the percentage of defendants found guilty by the jury.

Answer.

a. Let P(I) and P(G) represent the probability of innocence and guilt, respectively. Let (A) and (B) represent the probability of an innocent vote by the jury and a guilty (convicted) vote by the jury. We are given conditional probabilities that an individual juror will vote guilty given guilt and (indirectly) guilty given innocence. We now have to model the situation as a binomial distribution with 12 trials (jurors). For the sake of consistency, let's only count guilty votes. If we are calculating $P(B \mid G)$ and $P(B \mid I)$ for jury as a whole, we need there to be more than 9 guilty votes for it to pass, so a random variable, call it M, representing the number of guilty votes, must be greater than or equal to 9. From the law of total probability, we can express the probability of a correct decision from the jury, converting $P(B \mid I)$ to its complement $P(A \mid I)$.

$$P(B \mid G) = 1 - 0.25 = 0.75 \tag{1}$$

$$P(B \mid I) = 0.15 \tag{2}$$

$$\sum_{x=9}^{12} P(B \mid G) = \sum_{x=9}^{12} \binom{12}{x} (0.75)^x (1-0.75)^{12-x} \approx 0.6488 \tag{3}$$

$$\sum_{x=9}^{12} P(B \mid I) = \sum_{x=9}^{12} \binom{12}{x} (0.15)^x (1-0.15)^{12-x} \approx 5.478E-6 \tag{4}$$

$$P(\text{correct}) = P(B \mid G)P(G) + P(A \mid I)P(I) = P(B \mid G)P(G) + (1 - P(B \mid I))P(I) \tag{5}$$

$$\boxed{P(\text{correct}) \approx (0.6488)(0.7) + (1 - 5.478E-6)(0.3) \approx 0.754} \tag{6}$$

b. Since we only directly computed the conditional probabilities of guilty votes in the first part, we no longer have to convert $P(B \mid I)$ to its complement and can again use the law of total probability to compute our result.

$$\boxed{P(B) = P(B \mid G)P(G) + P(B \mid I)P(I) \approx (0.6488)(0.7) + (5.478E-6)(0.3) \approx 0.4541} \tag{1}$$

4. To determine whether they have measles, 60 people have their blood tested. However, rather than testing each individual separately, it is decided to first place the people into groups of 6. The blood samples of the 6 people in each group will be pooled and analyzed together. If the test is negative, one test will suffice for the 6 people, whereas if the test is positive, each of the 6 people will also be individually tested and, in all, 7 tests will be made on this group. Note that we assume that the pooled test will be positive if at least one person in the pool has measles. Assume that the probability that a person has measles is 5% for all people, independently of each other, and compute the expected number of tests necessary for each group of 6 people.

Answer.

Since we are calculating the expected number of tests necessary for any group of 6 people, we can consider just one group of 6. We can model the probability of measles as a binomial distribution (1), and we are looking for the probability that at least one person has measles. This is just the complement to no one having measles, which is easier to calculate (2). Now, stepping back, the number of tests for a group can be either 1 or 7. This is similar to a Bernoulli random variable. The expectation of this random variable, call it Z, is the weighted average of the number of possible tests from the definition of expectation (3).

$$X \sim Bin(6, 0.05) \tag{1}$$

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{6}{0} 0.05^0 (1 - 0.05)^{6-0} \approx 0.2649 \tag{2}$$

$$\boxed{E[Z] = 1(1 - 0.2649) + 7(0.2649) \approx 2.589} \tag{3}$$

5. An urn contains 4 white balls and 4 black balls. Two balls are drawn randomly (without replacement) from the urn. If they are the same color, you win $2.00. If they are different colors, you lose $1.00 (i.e., you win -$1.00). Let X = the amount you win.

   a. What is $E[X]$?

   b. What is $\mathrm{Var}(X)$?

Answer.

a. The expectation of a random variable is defined as the sum of the values the variable can take on weighted by their respective probabilities. In this case, our random variable, call it X, can only take on the values of 2 and -1. The probability that X takes on value 2 is the probability that we pick two balls of the same color. The number of ways to choose two balls that are the same color can be thought of as first choosing 1 color from 2, then choosing 2 balls from 4 of those same-colored balls. We divide this by the total number of ways to choose 2 balls from 8 to obtain the probability that we choose 2 balls that are the same color (1). The probability of choosing two balls that are different colors is the complement. The expectation of X can now be computed (2).

$$P(X=2) = \frac{\binom{2}{1} \times \binom{4}{2}}{\binom{8}{2}} = \frac{12}{28} = \frac{3}{7} \tag{1}$$

$$\boxed{E[X] = 2\left(\frac{3}{7}\right) + (-1)\left(1 - \frac{3}{7}\right) = \frac{2}{7} \approx 0.2857} \tag{2}$$

b. The variance of X can be computed using the formula $Var(X) = E[X^2] - E[X]^2$.

$$E[X^2] = 2^2\left(\frac{3}{7}\right) + (-1)^2\left(1 - \frac{3}{7}\right) = \frac{16}{7}$$

$$E[X]^2 = \left(\frac{2}{7}\right)^2 = \frac{4}{49}$$

$$\boxed{Var(X) = \frac{16}{7} - \frac{4}{49} = \frac{108}{49} \approx 2.204}$$

6. Let X be a continuous random variable with probability density function:

$$f(x) = \begin{cases} c(2 - 2x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a. What is the value of c?
b. What is the cumulative distribution function (CDF) of X?
c. What is $E[X]$?

Answer.

a. The total integral of a PDF must be 1, so we must solve the definite integral and have c be the constant that normalizes it to 1.

$$\int_{-1}^{1} (2 - 2x^2)dx = \left(2x - \frac{2}{3}x^3\right)\Big|_{-1}^{1} = \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

$$\frac{8}{3}c = 1 \rightarrow \boxed{c = \frac{3}{8}}$$

b. The CDF is the integral from the the lowest bound of the random variable to a given value x.

$$\int_{-1}^{x} (2 - 2x^2)dx = \left(2x - \frac{2}{3}x^3\right)\Big|_{-1}^{x} = \left(2x - \frac{2}{3}x^3\right) - \left(-2 + \frac{2}{3}\right) = \boxed{-\frac{2}{3}x^3 + 2x + \frac{4}{3}}$$

c. The expectation of a continuous random variable X is the total integral of the PDF of X times its value. In this case, the expectation is $X = 0$ given the shape of the PDF.

$$\int_{-1}^{1} x(2 - 2x^2)dx = \int_{-1}^{1} x(2x - 2x^3)dx = \left(x^2 - \frac{1}{2}x^4\right)\Big|_{-1}^{1} = \left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{2}\right) = \boxed{0}$$

7. Scores on the SAT math (out of 800) are normally distributed with a mean of 500 and a standard deviation of 100.

   a. What fraction of students receive a score within 1.5 standard deviations of the mean?

   b. Irina scores 750. What percent of students scored lower than 750? (Irina's percentile)

Answer.

a. Let X be the normal random variable where $X \sim \mathcal{N}(500, 100^2)$. Since the standard deviation is 100 and $100 \times 1.5 = 150$, we are looking for is $P(350 < X < 750)$. We can find this integral by transforming X to the standard normal and subtracting the lower bound from the higher bound.

$$P(350 < X < 650) = F(650) - F(350) = \Phi\left(\frac{650-500}{100}\right) - \Phi\left(\frac{350-500}{100}\right) = \Phi\left(\frac{3}{2}\right) - \Phi\left(-\frac{3}{2}\right)$$

$$\boxed{P(350 < X < 650) = \Phi\left(\frac{3}{2}\right) - \left(1 - \Phi\left(\frac{3}{2}\right)\right) = 0.9332 - (1 - 0.9332) = 0.8664 = \frac{1083}{1250}}$$

b. We actually already computed this in the last part. This time we are looking for $P(X < 750)$, which can be computed once transforming X into the standard normal.

$$\boxed{P(X < 750) = F(750) = \Phi\left(\frac{750-500}{100}\right) = \Phi\left(\frac{5}{2}\right) = 0.9938 = 99.38\%}$$

8. The number of times a person's computer crashes in a month is a Poisson random variable with $\lambda = 7$. Suppose that a new operating system patch is released that reduces the Poisson parameter to $\lambda = 2$ for 80% of computers, and for the other 20% of computers the patch has no effect on the rate of crashes. If a person installs the patch, and has their computer crash 4 times in the month thereafter, how likely is it that the patch has had an effect on the user's computer (i.e., it is one of the 80% of computers that the patch reduces crashes on)?

Answer.

Here we are looking for the conditional probability that the path is effective given the random variable X = 4. The probability that the path is effective $P(E)$ or not effective $P(N)$ are given (1). Since we are modeling this situation as a Poisson distribution, when the patch is effective $\lambda = 2$ and when the path is not effective $\lambda = 7$. We can compute the conditional probabilities $P(X = 4 \mid E)$ and $P(X = 4 \mid N)$ using the PMF of a Poisson random variable (2, 3). The general probability that a computer crashes 4 times in a month can be expanded using the law of total probability and computed (4, 5). Finally, using Bayes' theorem, we can calculate $P(E \mid X = 4)$ (6).

$$P(E) = 0.8 \quad P(N) = 0.2 \tag{1}$$

$$P(X = 4 \mid E) = \frac{2^4}{4!}e^{-2} \approx 0.09022 \tag{2}$$

$$P(X = 4 \mid N) = \frac{7^4}{4!}e^{-7} \approx 0.09123 \tag{3}$$

$$P(X = 4) = P(X = 4 \mid E)P(E) + P(X = 4 \mid N)P(N) \tag{4}$$

$$P(X = 4) = (0.09022)(0.8) + (0.09123)(0.20) \approx 0.09042 \tag{5}$$

$$\boxed{P(E \mid X = 4) = \frac{P(X = 4 \mid E)P(E)}{P(X = 4)} \approx \frac{(0.09022)(0.80)}{0.09042} \approx 0.7982} \tag{6}$$

9. Consider a hash table with n buckets. Now, m strings are hashed into the table (with equal probability of being hashed into any bucket).

   a. Let $n = 2{,}000$ and $m = 10{,}000$. What is the (Poisson approximated) probability that the first bucket has 0 strings hashed to it?

   b. Let $n = 2{,}000$ and $m = 10{,}000$. What is the (Poisson approximated) probability that the first bucket has 8 or fewer strings hashed to it?

   c. Let $m = 10{,}000$. What is largest integer value n such the Poisson approximated probability that an arbitrary bucket in the hash table will have no strings hashed to it is less than 0.5 (= 50%)?

   d. Let X be a Poisson random variable with parameter λ, that is: $X \sim \text{Poi}(\lambda)$. What value of λ maximizes $P(X = 3)$? Show formally (mathematically) how you derived this result. (Hint: at some point in your derivation you should be differentiating with respect to λ.)

   (Questions such as this allow us to compute appropriate sizes for hash tables in order to get good performance with high probability in applications where we have a ballpark idea of the number of elements that will be hashed into the table.)

Answer.

a. To calculate λ, we need the number of trials (strings) multiplied by the probability of success at each trial (probability of an individual bucket). This works out to be $\lambda = 10000 \times \dfrac{1}{2000} = 5$. Now let X be a random variable where $X \sim Poi(5)$.

$$\boxed{P(X = 0) = \frac{5^0}{0!}e^{-5} \approx 0.006738}$$

b. This amounts to the probability that X is less than or equal to 8 $P(X \leq 8)$, which is a sum from $P(X = 0)$ to $P(X = 8)$.

$$\boxed{P(X \leq 8) = \sum_{x=0}^{8} \frac{5^x}{x!}e^{-5} = e^{-5}\sum_{x=0}^{8} \frac{5^x}{x!} \approx 0.9319}$$

c. We need to solve an inequality to find this integer.

$$e^{-\frac{10000}{n}} \leq 0.5 \rightarrow -\frac{10000}{n} \leq \ln(0.5) \rightarrow \frac{1}{n} \geq -\frac{\ln(0.5)}{10000} \rightarrow n \leq -\frac{10000}{\ln(0.5)}$$

$$n \leq 14426.95 \rightarrow \boxed{n = 14426}$$

d. We can maximize $P(X = 3)$ using a calculus optimization.

$$P(X = 3) = \frac{\lambda^3}{3!}e^{-\lambda} \rightarrow \frac{d}{d\lambda}\left(\frac{\lambda^3}{3!}e^{-\lambda}\right) = \frac{1}{6}\frac{d}{d\lambda}\left(\lambda^3 e^{-\lambda}\right) = \frac{1}{6}\lambda^2 e^{-}\lambda(3 - \lambda) = 0 \rightarrow 3 - \lambda = 0 \rightarrow \boxed{\lambda = 3}$$

10. Say there are k buckets in a hash table. Each new string added to the table is hashed to bucket i with probability p_i, where $\sum_{i=1}^{k} p_i = 1$. If n strings are hashed into the table, find the expected number of buckets that have at least one string hashed to them. (Hint: Let X_i be a binary variable that has the value 1 when there is at least one string hashed to bucket i after the n strings are added to the table (and 0 otherwise). Compute $E\left[\sum_{i=1}^{k} X_i\right]$.)

Answer.

If X_i is a binary variable that has the value 1 when there is at least one string hashed to bucket i after n strings are added to the table and 0 otherwise, we need to find the probability that there is at least one string hashed to bucket i after n strings are added. This can be thought of immediately as p_i^n, but more rigorously this probability can be expressed as a binary variable where the number of trials is n and the probability of success in a given trial is p_i. We end up with the expectation of a sum which is equivalently a sum of expectations.

$$Y_i \sim Bin(n, p_i)$$

$$P(Y_i \geq 1) = 1 - P(Y_i = 0) = 1 - \binom{n}{0} p_i^0 (1 - p_i)^{n-0} = 1 - (1 - p_i)^n = p_i^n$$

$$X_i \sim Ber(P(Y \geq 1)) \rightarrow X_i \sim Ber(p_i^n)$$

$$\boxed{E\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} E\left[X_i\right] = \sum_{i=1}^{k} p_i^n}$$

11. A Bloom filter is a probabilistic implementation of the *set* data structure, an unordered collection of unique objects. In this problem we are going to look at it theoretically. Our Bloom filter uses 3 different independent hash functions H_1, H_2, H_3 that each take any string as input and each return an index into a bit-array of length n. Each index is equally likely for each hash function.

To add a string into the set, feed it to each of the 3 hash functions to get 3 array positions. Set the bits at all these positions to 1. For example, initially all values in the bit-array are zero. In this example $n = 10$:

Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	0	0	0	0	0	0	0	0	0

After adding a string "pie", where H_1("pie") = 4, H_2("pie") = 7, and H_3("pie") = 8:

Index:	0	1	2	3	4	5	6	7	8	9
Value:	0	0	0	0	**1**	0	0	**1**	**1**	0

Bits are never switched back to 0. Consider a Bloom filter with $n = 9,000$ buckets. You have added m = 1,000 strings to the Bloom filter. Provide a **numerical answer** for all questions.

a. What is the (approximated) probability that the first bucket has 0 strings hashed to it?

To *check* whether a string is in the set, feed it to each of the 3 hash functions to get 3 array positions. If any of the bits at these positions is 0, the element is not in the set. If all bits at these positions are 1, the string *may* be in the set; but it could be that those bits are 1 because some of the other strings hashed to the same values. You may assume that the value of one bucket is independent of the value of all others.

b. What is the probability that a string which has *not* previously been added to the set will be misidentified as in the set? That is, what is the probability that the bits at all of its hash positions are already 1? Use approximations where appropriate.

c. Our Bloom filter uses three hash functions. Was that necessary? Repeat your calculation in (b) assuming that we only use a single hash function (not 3).

(Chrome uses a Bloom filter to keep track of malicious URLs. Questions such as this allow us to compute appropriate sizes for hash tables in order to get good performance with high probability in applications where we have a ballpark idea of the number of elements that will be hashed into the table.)

Answer.

a. We can model the probability of a bucket having a string hashed to if after one hash function as a Poisson distribution (Y). For all three hash functions, we can nest the Poisson distribution within a binomial distribution where the number of trials is 3 and the probability of "success" being the probability that $Y \geq 1$.

$$Y \sim Poi\left(1000 \times \frac{1}{9000}\right) \rightarrow Y \sim Poi\left(\frac{1}{9}\right)$$
$$X \sim Bin(3, P(Y \geq 0))$$
$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \frac{\frac{1}{9}^0}{0!}e^{-\frac{1}{9}} = e^{-\frac{1}{9}}$$
$$P(X = 0) = \binom{3}{0}(1 - e^{-\frac{1}{9}})^0(1 - (1 - e^{-\frac{1}{9}}))^{3-0} = (e^{-\frac{1}{9}})^3 = e^{-\frac{3}{9}} = \boxed{e^{-\frac{1}{3}} \approx 0.7165}$$

b. Since the value of one bucket is independent of all others, we can use the definiton of independence to solve this problem.

$$P(X = 1 \cap Y = 1 \cap Z = 1) = P(X = 1)P(Y = 1)P(Z = 1) = (1 - e^{-\frac{1}{3}})(1 - e^{-\frac{1}{3}})(1 - e^{-\frac{1}{3}})$$
$$\boxed{P(X = 1 \cap Y = 1 \cap Z = 1) = (1 - e^{-\frac{1}{3}})^3 \approx 0.02278}$$

c.

$$\boxed{P(X = 1) = (1 - e^{-\frac{1}{9}}) \approx 0.1052}$$

12. Last summer (May 2019) the concentration of CO_2 in the atmosphere was 414 parts per million (ppm) which is substantially higher than the pre-industrial concentration: 275 ppm. CO_2 is a greenhouse gas and as such increased CO_2 corresponds to a warmer planet.

    Absent some pretty significant policy changes, we will reach a point within the next 50 years (i.e., well within your lifetime) where the CO_2 in the atmosphere will be double the pre-industrial level. In this problem we are going to explore the following question: What will happen to the global temperature if atmospheric CO_2 doubles?

    The measure, in degrees Celsius, of how much the global average surface temperature will change (at the point of equilibrium) after a doubling of atmospheric CO_2 is called "Climate Sensitivity." Since the earth is a complicated ecosystem climate scientists model Climate Sensitivity as a random variable, S. The IPPC Fourth Assessment Report had a summary of 10 scientific studies that estimated the PDF of S:

    images/climateSensitivity.png

    In this problem we are going to treat S as part-discrete and part-continuous. For values of S less than 7.5, we are going to model sensitivity as a discrete random variable with PMF based on the average of estimates from the studies in the IPCC report. Here is the PMF for S in the range 0 through 7.5:

Sensitivity, S (degrees C)	0	1	2	3	4	5	6	7
Expert Probability	0.00	0.11	0.26	0.22	0.16	0.09	0.06	0.04

The IPCC fifth assessment report notes that there is a non-negligible chance of S being greater than 7.5 degrees but didn't go into detail about probabilities. In the paper "Fat-Tailed Uncertainty in the Economics of Catastrophic Climate Change" Martin Weitzman discusses how different models for the PDF of Climate Sensitivity (S) for large values of S have wildly different policy implications.

For values of S greater than or equal to 7.5 degrees Celsius, we are going to model S as a continuous random variable. Consider two different assumptions for S when it is at least 7.5 degrees Celsius: a fat tailed distribution (f_1) and a thin tailed distribution (f_2):

$$f_1(x) = \frac{K}{x} \text{ s.t. } 7.5 \leq x < 30$$

$$f_2(x) = \frac{K}{x^3} \text{ s.t. } 7.5 \leq x < 30$$

For this problem assume that the probability that S is greater than 30 degrees Celsius is 0.

a. Compute the probability that Climate Sensitivity is at least 7.5 degrees Celsius.
b. Calculate the value of K for both f_1 and f_2.
c. It is estimated that if temperatures rise more than 10 degrees Celsius, all the ice on Greenland will melt. Estimate the probability that S is greater than 10 under both the f_1 and f_2 assumptions.
d. Calculate the expectation of S under both the f_1 and f_2 assumptions.
e. Let $R = S^2$ be a crude approximation of the cost to society that results from S. Calculate $E[R]$ under both the f_1 and f_2 assumptions.

Notes: (1) Both f_1 and f_2 are "power law distributions". (2) Calculating expectations for a variable that is part discrete and part continuous is as simple as: use the discrete formula for the discrete part and the continuous formula for the continuous part.

Answer.

a. This can be computed a the complement of the probability that $S < 7.5$ since the discreet part of the function has a consistent summation.

$$P(S \geq 7.5) = 1 - P(S < 7.5) = 1 - \sum_{i=0}^{7} S_i = 1 - 0.94 = \boxed{0.06}$$

b. K must be a real-number satisfying the integral from 7.5 to 30 be equal to 0.06, so that the total probability in our models in equal to 1.

$$\int_{7.5}^{30} \frac{K}{x}dx = 0.06 \rightarrow K\ln(x)\Big|_{7.5}^{30} = 0.06 \rightarrow K\left[\ln(30) - \ln(7.5)\right] = 0.06$$

$$\boxed{K = \frac{0.06}{\ln(30) - \ln(7.5)} \approx 0.04328 \text{ (fat-tailed)}}$$

$$\int_{7.5}^{30} \frac{K}{x^3}dx = 0.06 \rightarrow K\left(-\frac{1}{2x^2}\right)\Big|_{7.5}^{30} = 0.06 \rightarrow K\left[-\frac{1}{2(30)^2} + \frac{1}{2(7.5)^2}\right] = 0.06$$

$$\boxed{K = \frac{0.06}{\frac{1}{120}} = \frac{36}{5} = 7.2 \text{ (thin-tailed)}}$$

c. Since we have the respective K values calculated, we can compute any integral $7.5 < S < 30$ using proper bounds, which will yield the probability.

$$\int_{10}^{30} \frac{K}{x}dx = K\ln(x)\Big|_{10}^{30} = K\left[\ln(30) - \ln(10)\right] \approx \boxed{0.04755 \text{ (fat-tailed)}}$$

$$\int_{10}^{30} \frac{K}{x^3}dx = K\left(-\frac{1}{2x^2}\right)\Big|_{10}^{30} = K\left[-\frac{1}{2(30)^2} + \frac{1}{2(10)^2}\right] = \boxed{\frac{4}{25} = 0.032 \text{ (thin-tailed)}}$$

d. The expectation of a sum is a sum of expectations so we can compute the expectation as a discreet sum over the discreet part of the function and as an integral sum over the continuous part of the functions. We multiply by x in the integral forms because we need the value of x at those infinitesimal parts.

$$E[S] = \sum_{i=0}^{7} S_i p_i + \int_{7.5}^{30} \frac{K}{x} x dx = 3.02 + K(30 - 7.5) \approx \boxed{3.994 \text{ (fat-tailed)}}$$

$$E[S] = \sum_{i=0}^{7} S_i p_i + \int_{7.5}^{30} \frac{K}{x^3} x dx = 3.02 + K\left[-\frac{1}{30} + \frac{1}{7.5}\right] = \boxed{3.74 \text{ (thin-tailed)}}$$

e. This is the same as the previous part, except now we must square the values of the discreet part of the function as well as square the PDFs for the continuous parts.

$$E[S] = \sum_{i=0}^{7} S_i^2 p_i + \int_{7.5}^{30} \frac{K^2}{x^2} x dx = 12.06 + K^2(\ln(30) - \ln(7.5)) \approx \boxed{12.0626 \text{ (fat-tailed)}}$$

$$E[S] = \sum_{i=0}^{7} S_i^2 p_i + \int_{7.5}^{30} \frac{K^2}{x^6} x dx = 12.06 + K^2\left[-\frac{1}{4(30)^4} + \frac{1}{4(7.5)^4}\right] = \boxed{12.0641 \text{ (thin-tailed)}}$$