Math 825 **Answers to Test # 1** **13 Oct 03**

20 1. State and prove the Bolzano-Weierstrass Theorem. Explain clearly your use of any lemmas.

Solution. This is Theorem 2.6.4 in the text.

20 2. For each of the following statements, determine if it is true or false and provide either a proof or a counterexample, as appropriate.

(a) For $n \in \mathbb{N}$ and $A \subseteq \mathbb{R}$, define A^n to be $\{a^n : a \in A\}$. If A is a bounded-above nonempty set of nonnegative real numbers. then, for $n \in \mathbb{N}$, $\sup A^n = (\sup A)^n$.

(b) If (x_n) has the property that every subsequence of (x_n) has a convergent subsubsequence, then (x_n) converges.

Solution. For part (a), the statement is true. Observe that if $0 \leq x \leq L$, then $x^n \leq L^n$. Since $\sup A$ is an upper bound for A, it follows that $(\sup A)^n$ is an upper bound for A^n.

On the other hand, if $M \geq 0$ is an upper bound for A^n, then $x^n \leq M$ for all $x \in A$. Taking nth roots, we have $x \leq M^{1/n}$ for all $x \in A$ and so $M^{1/n}$ is an upper bound for A. By the definition of $\sup A$, $\sup A \leq M^{1/n}$. Using the observation above, we have $\sup A^n \leq M$. Thus, $(\sup A)^n$ is the least upper bound for A^n.

For part (b), the statement is false. Consider the sequence (a_n) given by $a_n = (-1)^n$ for all n. Fix a subsequence (a_{n_k}).

If infinitely many n_k are odd, then we define a subsubsequence by choosing n_k so that n_k are odd. Since this subsubsequence has all terms equal to -1, it clearly converges.

If only finitely many n_k are odd, then there must be infinitely many n_k that are even. We define a subsubsequence by choosing n_k so that n_k are even. Since this subsubsequence has all terms equal to 1, it converges.

20 3. If $\lim\limits_{n\to\infty} a_n = a$ and there are infinitely many terms of (a_n) which are greater than a, then there is an decreasing subsequence of a_n which converges to a.

Solution. Since every subsequence of (a_n) converges to a, it suffices to construct a decreasing subsequence. We do this recursively.

Let n_1 be the smallest k so that $a_k > a$. Let $\epsilon_1 = a_{n_1} - a > 0$. Since a_n converges to a, there is N_1 so that for all $n \geq N_1$, $|a_n - a| < \epsilon_1$. In particular,

$$a_n - a < \epsilon_1 = a_{n_1} - a,$$

so $a_n < a_{n_1}$. Since there infinitely many n with $a_n > a$, we can pick $n > N_1$ so that $a_n > a$. Let n_2 be the least such n. Since $n_2 > N_1$, we have $a_{n_2} < a_{n_1}$.

In general, given n_l, we construct n_{l+1} as in the last paragraph. That is, let $\epsilon_l = a_{n_l} - a > 0$ and find N_l so that for all $n \geq N_l$, we have $|a_n - a| < \epsilon_l$. Letting n_{l+1} be the least $n > N_l$ so that $a_n > a$, we have

$$a_{n_{l+1}} < a + \epsilon_l = a_{n_l},$$

and so we have constructed a strictly decreasing sequence.

. 20 4. Suppose the sequence (a_n) is decreasing and $a_n - a_{n-1} > -1/n^2$ for all $n \in \mathbb{N}$. Prove that (a_n) converges.

Solution. By the Completeness Theorem, it is enough to show (a_n) is a Cauchy sequence.

First, we claim that for $m = n + l \geq n$, we have

$$a_m - a_n > -\sum_{k=n+1}^{m} \frac{1}{n^2}.$$

To prove this, we use induction on $l = 1, 2, \ldots$. For $l = 1$, this is exactly the statement of the question. Assume the result holds for some l. Then

$$a_{n+l+1} - a_n = (a_{n+l+1} - a_{n+l}) + (a_{n+l} - a_n) > -\frac{1}{(n+l+1)^2} - \sum_{k=n+1}^{n+l} \frac{1}{k^2} = -\sum_{k=n+1}^{n+l+1} \frac{1}{k^2},$$

so by induction, the claim is proved.

Now, $\sum\limits_{n=1}^{\infty} 1/n^2$ converges by the Integral Test, since

$$\lim_{k \to \infty} \int_1^{k+1} \frac{1}{x^2}\, dx = \lim_{k \to \infty} \frac{1}{x}\bigg|_{x=1}^{x=k+1} = \lim_{k \to \infty} \frac{k}{k+1} = 1.$$

Let $\epsilon > 0$. By the Cauchy criterion for series, there is N so that for all $m \geq n \geq N$, $\sum\limits_{k=n+1}^{m} \frac{1}{k^2} < \epsilon$. Then we have, for $m \geq n \geq N$,

$$a_n \geq a_m \geq a_n - \sum_{k=n+1}^{m} \frac{1}{k^2} \geq a_n - \epsilon.$$

That is, $|a_n - a_m| < \epsilon$, showing (a_n) is a Cauchy sequence.

. 20 5. Prove that every conditionally convergent series has a rearrangement that diverges to $+\infty$, i.e., the sequence of partial sums diverges to $+\infty$.

Solution. Suppose $\sum_{k=1}^{\infty} a_k$ converges conditionally. Letting b_k be the kth positive term and c_k the kth negative term, we have $\sum_{k=1}^{\infty} b_k = +\infty$ and $\sum_{k=1}^{\infty} |c_k| = +\infty$. Define a sequence (m_k) by m_k is the least integer so that

$$u_k = \sum_{n=1}^{m_k} b_n + \sum_{n=1}^{k-1} c_n > k.$$

(This is always possible since the series for the b_k diverges to $+\infty$ and so has a partial sum greater than $k + \sum_{n=1}^{k} |c_n|$.)
Our rearrangement is the series

$$b_1 + b_2 + \cdots + b_{m_1} + c_1 + b_{m_1+1} + \cdots + b_{m_2} + c_2 + b_{m_2+1} + \cdots .$$

Let s_n be the nth partial sum of this series. Notice that for $n \le m_1$, we have $s_n > 0$. In general, for n with $k-1+m_{k-1} < n \le k+m_k$, we have $k-1 < u_{k-1} \le s_n \le u_k$. Thus, for all $n \ge k-1+m_{k-1}$, we have $s_n > k-1$. This shows that the partial sums of our series diverge to $+\infty$.

. 20 6. Suppose that (n_k) is a strictly increasing sequence of positive integers so that

$$\lim_{k\to\infty} \frac{n_k}{n_1 n_2 \cdots n_{k-1}} = +\infty.$$

Prove that $\sum_{i=1}^{\infty} \frac{1}{n_i}$ converges to an irrational number.

Solution. Notice that $\sum_{i=1}^{\infty} \frac{1}{n_i}$ converges to an irrational number if and only if $\sum_{i=2}^{\infty} \frac{1}{n_i}$ does. Also, if $n_1 = 1$, then $\lim_{k\to\infty} \frac{n_k}{n_2 \cdots n_{k-1}} = +\infty$. So we may throw away the first term of sequence if necessary, and assume that $n_1 \ge 2$.
To see that the series converges, first observe that there is K so that for all $k \ge K$, we have $\frac{n_k}{n_1 n_2 \cdots n_{k-1}} > 1$. For such k, $n_k > n_1 \cdots n_{k-1} > 2^{k-1}$. Thus, $n_k^{-1} < 2^{-k+1}$ for all $k \ge K$. By the comparison test, $\sum_{i=1}^{\infty} \frac{1}{n_i}$ converges.
Assume $\sum_{i=1}^{\infty} \frac{1}{n_i} = p/q$ for positive integers p and q.
Choose K so that for all $k \ge K$, we have $n_k > q+1$ and $n_k/(n_1 n_2 \cdots n_{k-1}) > q+1$.

Observe that

$$\begin{aligned}(n_1 \cdots n_{k-1})p &= q(n_1 \cdots n_{k-1})\frac{p}{q} \\ &= q(n_1 \cdots n_{k-1})\sum_{i=1}^{\infty}\frac{1}{n_i} \\ &= q\sum_{i=1}^{k-1}\frac{n_1 \cdots n_{k-1}}{n_i} + q\sum_{i=k}^{\infty}\frac{n_1 \cdots n_{k-1}}{n_i}\end{aligned}$$

Since the lefthand side and the first term of the righthand side are integers, we can conclude that the second term of the righthand side is also an integer. However, for $i \geq k$, we have

$$\frac{n_1 \cdots n_{k-1}}{n_i} = \frac{1}{n_k \cdots n_{i-1}}\frac{n_1 \cdots n_{i-1}}{n_i} < \frac{1}{(q+1)^{i-k}}\frac{1}{q+1} = \frac{1}{(q+1)^{i-k+1}}$$

Thus, we have

$$0 < q\sum_{i=k}^{\infty}\frac{n_1 \cdots n_{k-1}}{n_i} < q\sum_{i=k}^{\infty}\frac{1}{(q+1)^{i-k+1}} = q\frac{1/(q+1)}{1-1/(q+1)} = 1.$$

Since there are no integers between 0 and 1, this is a contradiction and shows that the series must converge to an irrational number.