Math 107, Sect. 250/350 — Answers to Test # 1 — 14 Feb 05

. 24 1. Showing all steps, evaluate each of the following integrals:

$$\text{(a)} \int \frac{\sqrt{x^2-25}}{x}\,dx, \quad x>5; \qquad \text{(b)} \int_{-1}^{2} \frac{1}{x^{1/3}}\,dx.$$

Solution. These problems are similar to 7.3, # 11, and 7.7 # 31, both assigned homework problems. For (a),

$$\begin{aligned}
\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{5\tan\theta}{5\sec\theta} 5\sec\theta\tan\theta\,d\theta \begin{cases} x = 5\sec\theta, \\ dx = 5\sec\theta\tan\theta\,d\theta \end{cases} \\
&= 5\int \tan^2\theta\,d\theta \\
&= 5\int (\sec^2\theta - 1)\,d\theta \\
&= 5\tan\theta - 5\theta + C \\
&= 5\sqrt{x^2-25} - 5\text{arcsec}(x/5) + C
\end{aligned}$$

Alternatively, one could solve $\sqrt{x^2-25} = 5\tan\theta$ for θ, to get

$$= 5\sqrt{x^2-25} - 5\arctan(\sqrt{x^2-25}/5) + C$$

For (b), the function has an asymptote at $x=0$, so we split the integral into two limits

$$\begin{aligned}
\int_{-1}^{2} \frac{1}{x^{1/3}}\,dx &= \lim_{b\to 0^-} \int_{-1}^{b} x^{-1/3}\,dx + \lim_{a\to 0^+} \int_{a}^{2} x^{-1/3}\,dx \\
&= \lim_{b\to 0^-} \frac{3}{2}x^{2/3}\Big|_{-1}^{b} + \lim_{a\to 0^+} \frac{3}{2}x^{2/3}\Big|_{a}^{2} \\
&= \lim_{b\to 0^-} \frac{3}{2}(b^{2/3}-1) + \lim_{a\to 0^+} \frac{3}{2}(2^{2/3}-a^{2/3}) \\
&= \frac{3(2^{2/3}-1)}{2}.
\end{aligned}$$

. 12 2. Does $\displaystyle\int_{1}^{\infty} \frac{1}{(1+x)\sqrt{x}}\,dx$ converge or diverge? Justify your answer.

Solution. This is 7.7, # 17, an assigned homework problem. Here are two solutions. First, evaluating the integral:

$$\begin{aligned}
\int_{1}^{\infty} \frac{1}{(1+x)\sqrt{x}}\,dx &= \lim_{b\to\infty} \int_{1}^{b} \frac{1}{(1+x)\sqrt{x}}\,dx \\
&= \lim_{b\to\infty} \int_{1}^{\sqrt{b}} \frac{1}{1+u^2}\,du \begin{cases} u = x^{1/2} \\ du = \frac{1}{2\sqrt{x}}\,dx \end{cases} \\
&= \lim_{b\to\infty} 2\arctan(u)|_{1}^{\sqrt{b}} \\
&= \lim_{b\to\infty} 2\arctan(\sqrt{b}) - 2\arctan(1) = \pi - \frac{\pi}{2} = \frac{\pi}{2}.
\end{aligned}$$

So the integral converges.

Second, we use the Comparison Test. Notice that

$$\frac{1}{(1+x)\sqrt{x}} = \frac{1}{x + x^{3/2}} \leq \frac{1}{x^{3/2}}.$$

By our basic example with $p = 1.5$, we know $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges, By the Comparison Test, $\displaystyle\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx$ converges, too.

. 12 3. An experiment gives the following data for a function $f(x)$:

x	0	1/3	2/3	1	4/3	5/3	2
$f(x)$	0.2	0.4	0.7	1.0	0.8	0.6	0.1

(a) Estimate $\int_0^2 f(x)\,dx$ using the Trapezoid Rule.

(b) Give an upper bound for the error of your estimate, assuming that $\max\{|f''(x)| : x \in [0,2]\} \leq 5$,

Solution. Part (a) is similar to Quiz 4 (in the 10:30 class) and Part (b) is similar to Homework 4, except for using the Trapezoid rule instead of the Simpson's rule. There are several homework problems like this in 7.6.

For part (a), we use

$$\int_0^2 f(x)\,dx \approx \frac{1/3}{2}\,(0.2 + 2(0.4) + 2(0.7) + 2(1.0) + 2(0.8) + 2(0.6) + 0.1) = 1.217$$

and for part (b), we have

$$|E_T| \leq \frac{5 \cdot 2^3}{12 \cdot 6^2} = .0926.$$

Thus, the Trapezoid Rule estimates the integral at 1.217 and the error is at most .0926.

. 22 4. A conical tank (with pointy end down) has a depth of 20 feet and the width across the top is 10 feet. Find the work, including units, to fill the tank from a water source 10 feet below the bottom of the tank. (As usual, the pump is at the bottom of the tank.)

Remember that water weights 62.4 lbs/cu. ft.

Solution. This is similar to a problem on the old test.

Let x be the distance from the bottom of the tank and let r be the radius of a slice at distance x above the bottom of the tank. From similar triangles, $r = x/4$, so the weight of a slice is $62.4(x/4)^2\Delta x$. The distance we have to move a slice is $10+x$. Thus, we have

$$\begin{aligned}
\int_0^{20} 62.4\left(\frac{x}{4}\right)^2 (10+x)\,dx &= \frac{62.4}{16}\int_0^{20} 10x^2 + x^3\,dx \\
&= \frac{62.4}{16}\left(\frac{10x^3}{3} + \frac{x^4}{4}\Big|_0^{20}\right) \\
&= \frac{62.4}{16}\left(\frac{80,000}{3} + \frac{160,000}{4}\right) \\
&= 2.6\cdot 10^5 \text{ft-lbs}
\end{aligned}$$

. 18 5. Uranium-234 has a half-life of 246,000 years. If 10 g are put in a storage facility, how long before only 1 g is left?

Solution. This is similar to 6.5 # 31, another assigned homework problem. Let $U(t)$ be the amount of Uranium-234 in grams, where t is measured in thousands of years. We know $U(0) = 10$, $S(246) = 5$ and have to find t so that $U(t) = 1$. We know that

$$\frac{dU}{dt} = kU.$$

Separating variables,

$$\begin{aligned}
\frac{dU}{U} &= k\,dt \\
\ln|U| &= kt + C
\end{aligned}$$

Since $U \geq 0$,

$$\begin{aligned}
\ln U &= kt + C \\
U &= e^{kt+C} \\
U &= Ke^{kt} \text{ where } K = e^C
\end{aligned}$$

Since $10 = S(0) = Ke^0$, we have $K = 10$.
To find k, we use

$$5 = S(246) = 10e^{k246}$$

and so $k = \ln(1/2)/246 = -0.00282$.
Finally, we solve $1 = S(t) = 10e^{-0.00282t}$ to get $t = \ln(1/10)/(-0.00282) = 816.5$. So the quantity will be down to 1 gram in 817,000 years, more or less.

. 12 6. Let R be the region bounded by $y = \sqrt{\tan x}$, $x = 1$ and the x-axis. Find the volume of the solid formed by revolving R about the x-axis.

Solution. For the washer method, we'd be using vertical slices with endpoints $(x, 0)$ and $(x, \sqrt{\tan x})$. Since the corresponding point on the axis of revolution is $(x, 0)$, the inside radius is 0 and the outside radius is $\sqrt{\tan x}$. Since $\tan x \geq 0$ on $[0, 1]$, we get the following integral:

$$\begin{aligned}
\int_0^1 \pi \tan x \, dx &= \pi \int_0^1 \frac{\sin x}{\cos x} \, dx \\
&= -\pi \int_1^{\cos(1)} \frac{1}{u} \, du \begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases} \\
&= -\pi \left(\ln |u| \big|_1^{\cos(1)} \right) \\
&= -\pi(\ln(\cos(1)) - \ln(1)) = -\pi \ln(\cos(1)).
\end{aligned}$$