Math 325-001 **Assignment # 4** **4 Feb 09**

Due: Feb 11th

1. If $A = \{a + a^{-1} : a > 0\}$, then find $\inf A$ and $\min A$. Prove your answers (which may be that it doesn't exist) are correct.

2. Suppose that S and T are bounded nonempty subsets of $\mathbb{R}$.
   (a) If $S \subseteq T$, prove that $\inf T \leq \inf S \leq \sup S \leq \sup T$.
   (b) Prove that $\sup(S \cup T) = \max\{\sup S, \sup T\}$. (Do not assume S is a subset of T.)
   (c) Find a formula expressing $\inf(S \cup T)$ in terms of $\inf S$ and $\inf T$. You do **not** need to prove it is correct.

3. Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and all $t \in T$.
   (a) Observe that S is bounded above and T is bounded below.
   (b) Prove that $\sup S \leq \inf T$.
   (c) Give an example of such sets S and T where $S \cap T$ is nonempty.
   (d) Give an example of such sets S and T where $S \cap T = \varnothing$ and $\sup S = \inf T$.

4. Let $\mathbb{I}$ be the set of all irrational numbers, i.e., all real numbers that are not rational numbers. Prove that if $a, b \in \mathbb{R}$ and $a < b$, then there is $x \in \mathbb{I}$ with
$$a < x < b.$$
Hint: First show that $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$. You may assume $\sqrt{2} \in \mathbb{I}$.

5. Let A and B be nonempty bounded subsets of $\mathbb{R}$. If S is the set of all sums $a + b$ where $a \in A$ and $b \in B$, then prove the following:
   (a) $\sup S = \sup A + \sup B$, and
   (b) $\inf S = \inf A + \inf B$.

6. Extra Credit: If A is a nonempty subset of $\mathbb{R}$, we call b an **almost upper bound** for A if there are only **finitely** many numbers $a \in A$ with $a \geq b$.
   (a) Find all almost upper bounds for each of $\{n/(n+1) : n \in \mathbb{N}\}$ and $\{1/n : n \in \mathbb{N}\}$.
   (b) If A is a bounded infinite set, prove that the set of all almost upper bounds is nonempty and bounded below.
   (c) Find an infinite subset of $\mathbb{R}$ so that the set of all almost upper bounds is $\mathbb{R}$.