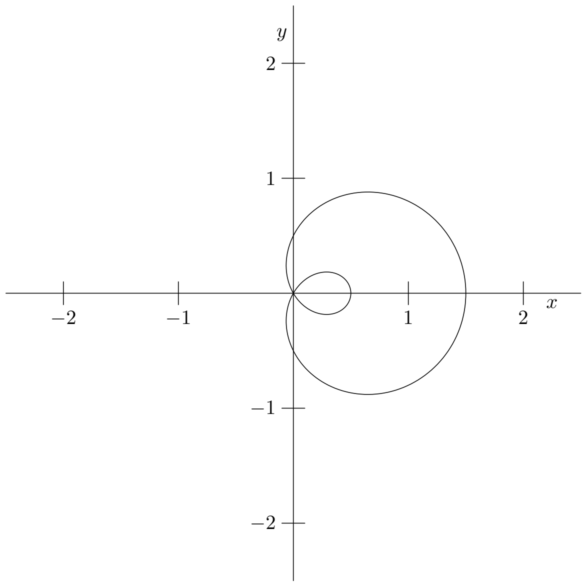
Math 107, Sect. 250/350 — Answers to Test # 3 — 18 Apr 07

. 20 1. Sketch the curve $r = 1/2 + \cos\theta$ on the axes below. Check for symmetries, giving algebraic justifications for any symmetries it might have. Finally, give the slope of the curve at $\theta = \pi/2$.

Solution. This problem is #21(a) from 9.2.



The graph suggests that curve is symmetric about the x-axis. To test this, we check if $(r, -\theta)$ satisfies the equation $r = 1/2 + \cos\theta$ when (r, θ) does. Since cos is an even function, we have

$$1/2 + \cos(-\theta) = 1/2 + \cos\theta = r,$$

and so $(r, -\theta)$ does satisfy the equation and the curve is symmetric about the x-axis.

To compute the slope, we use the formula for the slope of $r = f(\theta)$, namely

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Since $f(\theta) = 1/2 + \cos\theta$ and $f'(\theta) = -\sin(\theta)$, when $\theta = \pi/2$, $f(\pi/2) = 1/2$ and $f'(\theta) = -1$. Thus

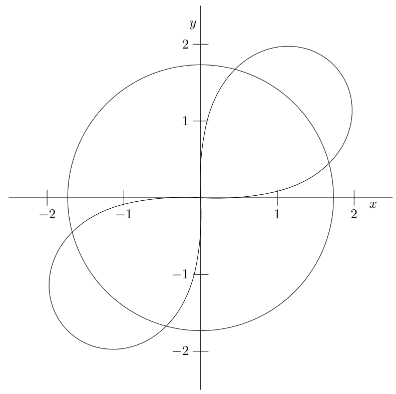
$$\frac{dy}{dx} = \frac{(-1)\cdot 1 + 1/2 \cdot 0}{(-1)\cdot 0 - (1/2)\cdot 1} = 2.$$

. 24 2. (a) Find the area inside $r^2 = 6\sin 2\theta$ that is outside $r = \sqrt{3}$.

(b) Find the length of the curve $r = \cos^3(\theta/3)$, $0 \le \theta \le \pi/2$.

Solution. The first problem is similar to # 11 and the second is # 23, both from 9.3.

For a), we sketch the graphs, as follows:



To find the points of intersection of $r^2 = 6 \sin 2\theta$ and $r = \sqrt{3}$, we substitute $r^2 = 3$ into the first equations and solve for $\sin 2\theta$ to get $\sin 2\theta = 1/2$, so $2\theta = \pi/6$ and $\theta = \pi/12$ (smallest solution). By the symmetry of the figure in $y = x$, the second intersection point is $\pi/2 - \pi/12 = 5\pi/12$. Using the symmetry of the figure about the x axis, the area we want is

$$
\begin{aligned}
&= 2 \int_{\pi/12}^{5\pi/12} \left[\frac{1}{2}(6 \sin 2\theta) - \frac{1}{2} 3 \right] d\theta \\
&= 3 \int_{\pi/12}^{5\pi/12} 2 \sin 2\theta - 1 d\theta \\
&= 3 \left[- \cos 2\theta - \theta |_{\pi/12}^{5\pi/12} \right] \\
&= 3 \left[\left(- \cos \frac{5\pi}{6} - \frac{5\pi}{12} \right) - \left(- \cos \frac{\pi}{6} - \frac{\pi}{12} \right) \right] \\
&= 3 \left[\frac{\sqrt{3}}{2} - \frac{5\pi}{12} + \frac{\sqrt{3}}{2} + \frac{\pi}{12} \right]
\end{aligned}
$$

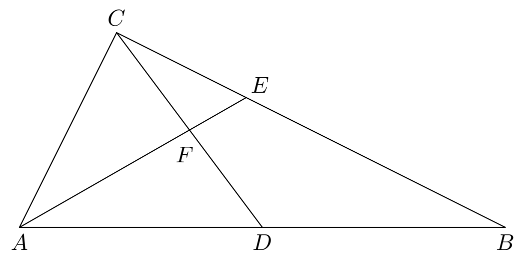
For b), the formula for the integral is

$$
L = \int_0^{\pi/2} \sqrt{r^2 + \frac{dr}{d\theta}^2} \, d\theta.
$$

In our case, $dr/d\theta = - \cos^2(\theta/3) \sin(\theta/3)$, and so

$$
\begin{aligned}
L &= \int_0^{\pi/2} \left(\cos^6(\theta/3) + \cos^4(\theta/3) \sin^2(\theta/3) \right)^{1/2} \, d\theta \\
&= \int_0^{\pi/2} \cos^2(\theta/3) \left(\cos^2(\theta/3) + \sin^2(\theta/3) \right)^{1/2} \, d\theta \\
&= \int_0^{\pi/2} \cos^2(\theta/3) \, d\theta \\
&= \int_0^{\pi/2} \frac{1 + \cos(2\theta/3)}{2} \, d\theta \\
&= \frac{\theta}{2} + \frac{3}{4} \sin(2\theta/3) \Big|_0^{\pi/2} = \frac{\pi}{4} + \frac{3\sqrt{3}}{8}
\end{aligned}
$$

. 12 3. In the triangle $\triangle ABC$ below, the D is the midpoint of $\overline{AB}$ and E is one third of the way from C to B. Use vectors to prove that F is the midpoint of $\overline{CD}$.



Solution. This is Homework 9.
We write the two vectors $\overrightarrow{AG}$ and $\overrightarrow{AE}$ in terms of $\overrightarrow{CA}$ and $\overrightarrow{CB}$. First, since D is the midpoint of AB, we have

$$\overrightarrow{CD} = \frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{CB}\right).$$

Since E is one third of thw way from C to B, $\overrightarrow{CE} = \frac{1}{3}\overrightarrow{CB}$.
Now, observe that

$$\begin{aligned}\overrightarrow{AE} &= -\overrightarrow{CA} + \overrightarrow{CE} \\ &= -\overrightarrow{CA} + \frac{1}{3}\overrightarrow{CB},\end{aligned}$$

while

$$\begin{aligned}\overrightarrow{AG} &= -\overrightarrow{CA} + \overrightarrow{CG} \\ &= -\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CD} \\ &= -\overrightarrow{CA} + \frac{1}{2}\left(\frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{CB}\right)\right) \\ &= -\frac{3}{4}\overrightarrow{CA} + \frac{1}{4}\overrightarrow{CB} \\ &= \frac{3}{4}\left(-\overrightarrow{CA} + \frac{1}{3}\overrightarrow{CB}\right) \\ &= \frac{3}{4}\overrightarrow{AE}.\end{aligned}$$

Since one vector is 3/4 of the other, they both point in the same direction. This means that G is on the straight line from A to E, i.e., the midpoint of $\overline{CD}$ is given by the intersection of $\overline{AE}$ and $\overline{CD}$, as required.

. 10 4. For each of the following statements, circle the correct status. You do not need to justify your answers.

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$	always true	**not** always true
$\|\mathbf{u}\| = \mathbf{u} \cdot \mathbf{u}$	always true	**not** always true
$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$	always true	**not** always true
$\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$	always true	**not** always true
$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$	always true	**not** always true

Solution. This problem is similar to problem 27 in Section 10.4; some parts are directly from this question.

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ is always true, since $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$.

$|\mathbf{u}| = \mathbf{u} \cdot \mathbf{u}$ is not always true; $\mathbf{u} \cdot \mathbf{u}$ is equal to $|\mathbf{u}|^2$ which can be differ from $|\mathbf{u}|$.

$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ is always true. Switching the order of the vectors switches the direction of the vector (by the righthand rule).

$\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$ is always true; $\mathbf{u}$ and $-\mathbf{u}$ make a parallelogram with no area, so their cross product has magnitude zero.

$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + \mathbf{w}$ is not always true. By the distributive law, $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$

. 14 5. Find the distance between the point $S(2, 1, 4)$ and the plane determined by $P(2, 3, -2)$, $Q(3, 4, 2)$, $R(1, -1, 0)$.

Solution. To find the distance from the point to the plane, we need the normal vector of the plane and a point in the plane. For the normal vector, observe that $\overrightarrow{PQ} = \vec{\imath} + \vec{\jmath} + 4\vec{k}$ and $\overrightarrow{PR} = -\vec{\imath} - 4\vec{\jmath} + 2\vec{k}$ are vectors in the plane, and so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 4 \\ -1 & -4 & 2 \end{vmatrix} = 18\vec{\imath} - 6\vec{\jmath} - 3\vec{k}$$

Choosing $\vec{n} = 6\vec{\imath} - 2\vec{\jmath} - \vec{k}$ as the normal vector and $R(1, -1, 0)$ as the point in the plane, the distance is

$$d = \frac{|\overrightarrow{RS} \cdot \vec{n}|}{|\vec{n}|} = \frac{\sqrt{21}}{\sqrt{41}}.$$

. 10 6. Find the velocity, speed, and acceleration functions for $\vec{r}(t) = (1+t^2)\vec{\imath} + (e^{2t})\vec{\jmath} + (\cos t)\vec{k}$.

Solution. We have

$$\begin{aligned} \vec{v}(t) = \vec{r}'(t) &= 2t\vec{\imath} + (2e^{2t})\vec{\jmath} + (-\sin t)\vec{k} \\ \vec{a}(t) = \vec{r}''(t) &= 2\vec{\imath} + (4e^{2t})\vec{\jmath} + (-\cos t)\vec{k} \end{aligned}$$

Finally, the speed is $s(t) = \sqrt{4t^2 + 4e^{4t} + \sin^2 t}$.

. 10 7. For the vector function $\vec{r}(t) = (\cos(\pi t))\vec{\imath} + (t + \sin(\pi t))\vec{\jmath} + (e^{t^2 - t})\vec{k}$, find a tangent line that is flat, i.e., does not intersect the x, y-plane.

Solution. For the tangent line to be flat, the velocity vector must have $\vec{k}$-component zero. The velocity vector is

$$\vec{v}(t) = (-\pi \sin(\pi t))\vec{\imath} + (1 + \pi \cos(\pi t))\vec{\jmath} + ((2t-1)e^{t^2-t})\vec{k},$$

So the $\vec{k}$-component vanishes when $(2t-1)e^{t^2-t} = 0$, i.e., $2t - 1 = 0$ or $t = 1/2$. At this time, $\vec{v}(1/2) = -\pi\vec{\imath} + \vec{\jmath}$. As $\vec{r}(1/2) = (1/2 + \sin(\pi/2)\vec{\jmath} + (e^{-1/4})\vec{k}$, the equation of the tangent line is

$$\vec{T}(t) = ((1/2 + \sin(\pi/2)\vec{\jmath} + (e^{-1/4})\vec{k}) + t(= -\pi\vec{\imath} + \vec{\jmath}).$$