Math 496-004 **Assignment # 12** **19 Nov 08**

Due: Dec 3rd

1. Consider the equations

$$xu^2 + yv^2 + xy = 9,$$
$$xv^2 + yu^2 - xy = 7.$$

   Find conditions on $(x_0, y_0, u_0, v_0) \in \mathbb{R}^4$ so that there are real=valued C^1 functions $u(x, y)$ and $v(x, y)$ that solve these equations for u and v in terms of x and y around this point. Prove that the solutions satisfy $u^2 + v^2 = 16/(x + y)$.

2. In the Implicit Function Theorem, we take a function $G : S \to \mathbb{R}^n$, where $S \subset \mathbb{R}^{n+m}$ and separate the $n + m$ invariables into $\mathbf{y}$, which we are solving for, and $\mathbf{x}$, which we are solving in terms of. The theorem then gives a condition in terms of this choice.

   Suppose you don't care which variables are solved for or in terms of. Give a condition on the $n \times (m + n)$ matrix of $dG_{\mathbf{c}}$, where $\mathbf{c} \in \mathbb{R}^{n+m}$ that allows you to solve for some (unspecified) choice of n of the variables in terms of the other m. In terms of your condition, how do you choose the n variables to solve for?

   Prove that your answers are correct, of course.

   Hint: This is really more of a linear algebra exercise than an calculus one; your condition should be a familiar one from linear algebra.

3. For a C^1 function $G : \mathbb{R}^n \to \mathbb{R}$, let $S = \{\mathbf{x} \in \mathbb{R}^n : G(\mathbf{x}) = 0\}$. If $dG_{\mathbf{a}} \neq 0$ for some $\mathbf{a} \in \mathbb{R}^n$, show that there is an open set $N \subseteq \mathbb{R}^n$ so that $S \cap N$ is the graph of a C^1 function f from a suitable subset of $\mathbb{R}^n$ into $\mathbb{R}$.

4. Let $\mathbf{0} = (0, 0, 0)$ and define $T : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{R}^3 \backslash \{\mathbf{0}\}$ by $T(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|^2$, that is,

$$T(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right).$$

   (a) Show that T is globally invertible.

   (b) Find $JT_{\mathbf{x}}$. Hint: First use spherical coordinates.

   (c) Show that T maps $z = -1$ onto $\partial(B_{1/2}(0, 0, -1/2)) \backslash \{\mathbf{0}\}$.

   (d) (bonus question) Show that, for any plane, P, not containing $\mathbf{0}$, $T(P)$ is a sphere containing $\mathbf{0}$ whose tangent plane at $\mathbf{0}$ is parallel to P.
   Hint: This is really a geometry problem. Start by considering the map I from $\mathbb{R}^2 \backslash \{(0, 0)\}$ to itself that sends (x, y) to $(x/(x^2 + y^2), y/(x^2 + y^2))$ and show that a circle containing $(0, 0)$ is mapped to a line. Use this result to get the conclusion.