Math 107-250/350 **Answers to Homework #7** **2 Mar '07**

Explain your reasoning. A correct answer poorly explained will not get full marks.

5 1. Do the following series converge or diverge?

$$\text{(a)} \sum_{n=1}^{\infty} \frac{2^n n!}{n^n}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{3^n n!}{n^n}.$$

HINT: Use the ratio test. It will help to remember that

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \approx 2.718.$$

Solution. For the first series, $a_n = \frac{2^n n!}{n^n}$, and so

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\dfrac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\dfrac{2^n n!}{n^n}} \\
&= \frac{2^{n+1}(n+1)!\, n^n}{2^n n!\, (n+1)^{n+1}} \\
&= \frac{2(n+1)n^n}{(n+1)^{n+1}} \\
&= \frac{2n^n}{(n+1)^n} = \frac{2}{\left(\frac{n+1}{n}\right)^n}.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{\left(\frac{n+1}{n}\right)^n} = \frac{2}{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n} = \frac{2}{e} < 1.$$

By the ratio test, the series converges.

For the second series, the calculations are similar, except that the final ratio is $3/e > 1$, so the series diverges by the ratio test. Here are the details,

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\dfrac{3^{n+1}(n+1)!}{(n+1)^{n+1}}}{\dfrac{3^n n!}{n^n}} \\
&= \frac{3^{n+1}(n+1)!\, n^n}{3^n n!\, (n+1)^{n+1}} \\
&= \frac{3(n+1)n^n}{(n+1)^{n+1}} \\
&= \frac{3n^n}{(n+1)^n} = \frac{3}{\left(\frac{n+1}{n}\right)^n}.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{\left(\frac{n+1}{n}\right)^n} = \frac{3}{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n} = \frac{3}{e} > 1.$$

By the ratio test, the series diverges.