The role of “positivity” in moment and polynomial optimization problems

Lawrence Fialkow

Truncated K-Moment Problem

Given an n-dimensional multisequence of degree m,

$$\beta \equiv \beta^{(m)} = \{\beta_i : i \in \mathbb{Z}_+^n,\ |i| \leq m\},$$

and a closed set $K \subseteq \mathbb{R}^n$, find conditions on β so that there exists a positive Borel measure μ on $\mathbb{R}^n$ such that *supp* $\mu \subseteq K$ and

$$\beta_i = \int_{\mathbb{R}^n} x^i \, d\mu(x) \ (|i| \leq m)$$

$(x \equiv (x_1, \ldots, x_n),\ i \equiv (i_1, \ldots, i_n) \in \mathbb{Z}_+^n,\ x^i := x_1^{i_1} \cdots x_n^{i_n})$.
By analogy, in the *Full* K-Moment Problem, we are given $\beta^{(\infty)}$, with moment data of all degrees.

Multisequence notation and moment matrices

Let β denote an n-dimensional real multisequence of degree m,

$$\beta \equiv \beta^{(m)} = \{\beta_i : i \in \mathbb{Z}_+^n, \ |i| \leq m\},$$

Example. For $n = 1$, $m = 4$, $\beta^{(4)} : \beta_0, \ldots, \beta_4$, we associate $\beta^{(4)}$ to the moment matrix M_2, with rows and columns indexed by $1, \ x, \ x^2$, defined by

$$M_2(\beta) \equiv \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_3 & \beta_4 \end{pmatrix}.$$

Example. For $n = 2$, $m = 4$, consider $\beta^{(4)}$:
$\beta_{00}, \beta_{10}, \beta_{01}, \beta_{20}, \beta_{11}, \beta_{02}, \beta_{30}, \beta_{21}, \beta_{12}, \beta_{03}, \beta_{40}, \beta_{31}, \beta_{22}, \beta_{13}, \beta_{04}.$

We associate $\beta^{(4)}$ to the moment matrix M_2, with rows and columns indexed by $1,\ x,\ y,\ x^2,\ xy,\ y^2$, defined by

$$M_2(\beta) \equiv \begin{pmatrix} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix}.$$

The Polynomial Optimization Problem

Let $\mathcal{P} := \mathbb{R}[x_1, \ldots, x_n]$. For $q_0, \ldots, q_k \in \mathcal{P}$, $q_0 \equiv 1$, let K_Q denote the basic closed semialgebraic set

$$K_Q = \{x \in \mathbb{R}^n : \; q_i(x) \geq 0 \; (0 \leq i \leq k)\}.$$

For $p \in \mathcal{P}$, we seek to compute (or estimate)

$$p_* := \inf_{x \in K_Q} p(x).$$

Later, we will discuss an algorithm of J.-B. Lasserre [2000] which estimates p_* based on "moment relaxations", and whose stopping criterion (when p_* is computed exactly) is based on the theory of the truncated K-moment problem.

Representing measures

Suppose we have a measure μ as above:

$$\beta_i = \int_{\mathbb{R}^n} x^i d\mu(x) \ (|i| \leq m), \ supp\ \mu \subseteq K$$

μ is a *K-representing measure* for β.

For $K = \mathbb{R}^n$, μ is a *representing measure*.

μ is a *finitely atomic* K-representing measure if

$$\mu = \sum_{i=1}^{k} \rho_i \delta_{w_i} \ (\rho_i > 0, w_i \in K).$$

Question If there is a K-representing measure, is there a finitely atomic K-representing measure?

A theorem of V. Tchakaloff [1957] provides an affirmative answer for K compact. The complete answer was found 50 years later:

Theorem [C. Bayer and J. Teichmann, 2006]

If $\beta^{(m)}$ has a K-representing measure, then β has a finitely-atomic K-representing measure μ, with *card supp* $\mu \leq$ *dim* $\mathbb{R}_m[x_1, \ldots, x_n]$

The Full K-Moment Problem

$\beta \equiv \beta^{(\infty)} = \{\beta_i : i \in \mathbb{Z}_+^n\}$

Stieltjes [1894] $K = [0, +\infty)$

Hamburger [1920] $K = \mathbb{R}$

Hausdorff [1923] $K = [a, b]$

Results for FMP suggest results for TMP.

Connection between TMP and FMP

Theorem [Jan Stochel, 2001]

The full multisequence $\beta \equiv \beta^{(\infty)}$ has a K-representing measure if and only if $\beta^{(m)}$ has a K-representing measure for every $m \geq 1$.

In some cases (one of which is illustrated below), we can use solutions to TMP, together with Stochel's theorem, to solve FMP.

Riesz functional

$\mathcal{P} := \mathbb{R}[x_1, \ldots, x_n]$

$\beta \equiv \beta^{(\infty)} = \{\beta_i : i \in \mathbb{Z}_+^n\}$

Riesz functional: $L_\beta : \mathcal{P} \longmapsto \mathbb{R}$

$$p \equiv \sum a_i x^i \longmapsto L_\beta(\sum a_i x^i) = \sum a_i \beta_i \ (= \int_K p(x) d\mu(x))$$

Note: If β has a K-rep. measure μ, then L_β is K-*positive*, i.e.,

$$p|K \geq 0 \Longrightarrow L_\beta(p) \geq 0.$$

(For $K = \mathbb{R}^n$, we say L_β is *positive*.)

“Abstract” solution of the Full K-Moment Problem

Theorem [M. Riesz, 1923 ($n = 1$), E.K. Haviland, 1936 ($n \geq 2$)]

The full multisequence $\beta \equiv \beta^{(\infty)}$ has a K-representing measure if and only if L_β is *K-positive*, i.e.,

$$p \in \mathcal{P}, \;\; p|K \geq 0 \Longrightarrow L_\beta(p) \geq 0 \; .$$

A limitation of Riesz-Haviland:

For a general closed set K (even for $\mathbb{R}^2$), there is no concrete structure theorem for K-positive polynomials, so it is difficult to check that L_β is K-positive.

Moment matrices

Given $\beta \equiv \beta^{(\infty)}$, we define the moment matrix $M \equiv M_\infty(\beta)$:

$$M_\infty(\beta) = (\beta_{i+j})_{(i,j)\in \mathbb{Z}_+^n \times \mathbb{Z}_+^n}.$$

$M_\infty(\beta)$ is uniquely determined by

$$\langle M_\infty(\beta)\hat{p}, \hat{q}\rangle = L_\beta(pq) \ \forall p, q \in \mathcal{P},$$

where $\hat{s}$ denotes the coefficient vector of $s \in \mathcal{P}$ relative to the basis of monomials in degree-lexicographic order.
If L_β is positive (in particular, if β has a representing measure), then $M_\infty(\beta) \succeq 0$ (positive semidefinite):

$$\langle M_\infty(\beta)\hat{p}, \hat{p}\rangle = L_\beta(p^2) = \int p^2 d\mu \geq 0.$$

Summary so far: rep. meas. $\Longleftrightarrow$ L_β pos. $\Longrightarrow$ $M \succeq 0$.

Sums of squares

There is one situation where the "concrete" condition $M \succeq 0$ readily implies that L_β is positive. Consider the following property: $(H_{n,d})$ Every $p \in \mathcal{P}_d$ with $p|\mathbb{R}^n \geq 0$ can be expressed as $p = \sum p_i^2$. If $(H_{n,d})$ holds and $M \succeq 0$, then L_β is positive:

$$L_\beta(p) = L_\beta(\sum p_i^2) = \sum \langle M\hat{p}_i, \hat{p}_i \rangle \geq 0.$$

Hilbert's theorem on sums of squares [D. Hilbert, 1888]

$(H_{n,d})$ holds $\iff n = 1$, or $(n,d) = (2,4)$, or $d = 2$.

The moment problem can be solved concretely in the positive cases of Hilbert's theorem; we will discuss the first two cases in the sequel.

We consider FMP in the first case of Hilbert's theorem, $n = 1$.

Theorem [Hamburger, 1920]

Let $n = 1$, $K = \mathbb{R}$. The full multisequence $\beta \equiv \beta^{(\infty)}$ has a representing measure if and only if $M_\infty(\beta) \succeq 0$.

Proof.

For $p \in \mathbb{R}[x]$, $p|\mathbb{R} \geq 0 \Longrightarrow p = r^2 + s^2$ for some $r, s \in \mathbb{R}[x]$.
Then $L_\beta(p) = L_\beta(r^2) + L_\beta(s^2) = \langle M\hat{r}, \hat{r}\rangle + \langle M\hat{s}, \hat{s}\rangle \geq 0$.
Apply Riesz' Theorem. $\square$

Conditions for solving TMP (with R. Curto)

K-positivity in TMP

$\beta \equiv \beta^{(m)}$

$\mathcal{P}_k := \{p \in \mathcal{P} : deg\ p \leq k\}$

Riesz functional: $L_\beta : \mathcal{P}_m \longrightarrow \mathbb{R}$

$$p \equiv \sum a_i x^i \longmapsto L_\beta(\sum a_i x^i) = \sum a_i \beta_i \ (= \int_{\mathbb{R}^n} p(x) d\mu(x))$$

If β has a K-representing measure, then L_β is *K-positive*, i.e.,

$$p \in \mathcal{P}_m,\ \ p|K \geq 0 \Longrightarrow L_\beta(p) \geq 0\ .$$

If $\beta \equiv \beta^{(m)}$ has a K-representing measure, then L_β is K-*positive*.

Tchakaloff's Thm. implies that the converse is true for K *compact*. For the noncompact case, we introduce an example.

For $n = 1$, $K = \mathbb{R}$, $\beta \equiv \beta^{(4)}$, consider the moment problem with

$$\beta_0 = \beta_1 = \beta_2 = \beta_3 = 1, \;\; \beta_4 = 2.$$

We will show below that L_β is positive.

Is there a representing measure?

Question What is the analogue of R-H for TMP?

Moment matrices for TMP

For $\beta \equiv \beta^{(2d)}$, we define the d-th order moment matrix $M_d(\beta)$:

$$M_d(\beta) = (\beta_{i+j})_{(i,j)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n:|i|,|j|\le d}.$$

$M_d(\beta)$ is uniquely determined by

$$\langle M_d(y)\hat{p}, \hat{q}\rangle = L_\beta(pq)\ \forall p, q \in \mathcal{P}_d.$$

Positivity condition for TMP

Necessary condition 1: positivity

If β has a representing measure, then $M_d(\beta) \succeq 0$:

$$\langle M_d(\beta)\hat{p}, \hat{p}\rangle = L_\beta(p^2) = \int p^2 d\mu \geq 0$$

In the preceding example, we have

$$M_2(\beta) \equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

$M(2) \succeq 0$ and therefore (by sos) L_β is positive. Does β have a representing measure?

Recursiveness

Columns of $M_d(\beta)$: $1, X_1, \ldots, X_n, \ldots, X_1^d, \ldots, X_n^d$

A dependence relation in *Col* $M_d(\beta)$ can be denoted by $p(X) \equiv p(X_1, \ldots, X_n) = 0$ for some $p \in \mathcal{P}_d$.

Necessary condition 2: recursiveness

$M_d(\beta)$ is *recursively generated*:

$$p(X) = 0 \Longrightarrow (pq)(X) = 0 \text{ whenever } pq \in \mathcal{P}_d.$$

In the example, $X = 1$, but $X^2 \neq X$, so L_β is positive, but there is no measure. The direct analogue of R-H fails for TMP.

An analogue of the Riesz-Haviland theorem for TMP

Theorem [C-F, 2008]

Let $\beta \equiv \beta^{(2d)}$ or $\beta \equiv \beta^{(2d+1)}$. β has a K-representing measure $\Longleftrightarrow L_\beta$ admits a K-positive extension $L_{\tilde{\beta}} : \mathcal{P}_{2d+2} \longrightarrow \mathbb{R}$.

Issue: In general it is very difficult to establish K-positivity.

Example of a concrete truncated moment theorem

We return to the first case of Hilbert's theorem, $n = 1$.

Theorem [C-F, 1991]

Let $n = 1$. $\beta \equiv \beta^{(2d)}$ has a representing measure $\iff M_d(\beta) \succeq 0$ and $M_d(\beta)$ is recursively generated.

Using TMP to solve FMP

Another proof of Hamburger's Theorem

Theorem

Let $n = 1$. The full multisequence $\beta \equiv \beta^{(\infty)}$ has a representing measure if and only if $M_\infty(\beta) \succeq 0$.

Proof.

Suppose $M_\infty \succeq 0$. Then, for each d, $M_d(\beta)$ is positive semidefinite and recursively generated, so $\beta^{(2d)}$ has a representing measure. Stochel's theorem now implies that $\beta^{(\infty)}$ has a representing measure. $\square$

The variety of a moment matrix

The *variety* of $\beta \equiv \beta^{(2d)}$ (or of $M_d(\beta)$):

$$\mathcal{V}(\beta) = \bigcap_{p \in \mathcal{P}_d, p(X)=0} \mathcal{Z}(p),$$

where $\mathcal{Z}(p) = \{x \in \mathbb{R}^n : p(x) = 0\}$.

The variety condition

Proposition

If μ is a representing measure, then *supp* $\mu \subseteq \mathcal{V}(\beta)$ and

$$rank\ M_d(\beta) \le card\ supp\ \mu \le card\ \mathcal{V}(\beta).$$

Necessary condition 3: variety condition

$$r \equiv rank\ M_d(\beta) \le v \equiv card\ \mathcal{V}(\beta).$$

In the previous example, $r = 2$, $X = 1$, $V(\beta) = \{1\}$, $v = 1$.

The flat extension theorem

Recall: If μ is a representing measure, then

$$r \equiv rank\ M_d(\beta) \leq card\ supp\ \mu.$$

Theorem [C-F, 1996, 2005]

$\beta \equiv \beta^{(2n)}$ has an r-atomic representing measure $\Longleftrightarrow$ $M_d(\beta) \succeq 0$ and $M_d(\beta)$ has a *flat*, i.e., rank-preserving, moment matrix extension M_{d+1}. In this case, an r-atomic representing measure μ can be explicitly constructed with $supp\ \mu = \mathcal{V}(M_{d+1})$.

Solution to TMP based on moment matrix extensions

Theorem [C-F, 2005]

$\beta \equiv \beta^{(2d)}$ has a representing measure $\Longleftrightarrow M_d(\beta)$ admits a positive extension M_{d+k} (for some $k \geq 0$), and M_{d+k} has a flat extension M_{d+k+1}.

Note: When the strategy of this theorem can be implemented, this method circumvents the difficulty of positivity for $L_{\beta^{(2d+2)}}$ in the truncated R-H theorem. In this case, is there some way to recognize directly that for M_{d+1} (as above), $L_{\beta^{(2d+2)}}$ is positive?

TMP for K a planar curve of degree 2

Theorem [C-F, 2005]

Let $n = 2$ and suppose $p(X) = 0$ in $M_d(\beta)$ for some $p \in \mathcal{P}_2$. Then $\beta^{(2d)}$ has a representing measure (necessarily supported in $\mathcal{Z}_p$) $\Longleftrightarrow$ $M_d(\beta)$ is positive and recursively generated, and *rank* $M_d(\beta) \leq$ *card* $\mathcal{V}_\beta$. In this case, either $M_d(\beta)$ has a flat extension M_{d+1}, or $M_d(\beta)$ has a positive extension M_{d+1}, which in turn has a flat extension M_{d+2}.

Note

(i) This result solves the bivariate quartic moment problem ($n = 2$, $\beta \equiv \beta^{(4)}$) in the case when $M_2(\beta)$ is singular. For the case when $M_2(\beta) \succ 0$ we will use alternate methods based on approximation and convexity.

(ii) The above result does not extend to $y = x^3$ [F, 2008]

Approximation methods (with Jiawang Nie)

Let $\eta = dim\ \mathcal{P}_{2d}$, so $\beta^{(2d)} \in \mathbb{R}^\eta$.

$\mathcal{R}_{n,d} := \{\beta \in \mathbb{R}^\eta : \beta\ has\ a\ K - representing\ \ measure\}$, convex cone

$\mathcal{S}_{n,d} := \{\beta \in \mathbb{R}^\eta : L_\beta\ is\ K - positive\}$, convex cone

Theorem [F-Nie, 2009]

$\mathcal{S}_{n,d} = \overline{\mathcal{R}_{n,d}}$.

Strict K-positivity and representing measures

L_β is *strictly K-positive* if

$$p \in \mathcal{P}_{2d},\ \ p|K \geq 0,\ \ p|K \not\equiv 0 \Longrightarrow L_\beta(p) > 0.$$

K is *determining* if $p \in \mathcal{P}_{2d},\ p|K \equiv 0 \Longrightarrow p \equiv 0.$

Theorem [F-Nie, 2009]

If K is determining and L_β is strictly K-positive, then β has a K-representing measure.

Proof.

The hypotheses imply that
$\beta \in interior(\mathcal{S}_{n,d}) = interior(closure(\mathcal{R}_{n,d}))$
$= interior(\mathcal{R}_{n,d}) \subseteq \mathcal{R}_{n,d}$. $\square$

Bivariate quartic moment problem

Consider the second case of Hilbert's theorem, when $n = 2$, $d = 4$, and consider the corresponding moment problem for $\beta^{(4)}$. For $M_2(\beta)$ singular, the problem was solved by [Curto-F, 2005] (above).

Theorem [F-Jiawang Nie, 2009]

Let $n = 2$. If $M_2(\beta) \succ 0$, then β has a representing measure.

Proof.

Let $K = \mathbb{R}^2$, determining. Since $M_2(\beta) \succ 0$, Hilbert's theorem implies that L_β is strictly K-positive: If $p|\mathbb{R}^2 \geq 0$, $p \not\equiv 0$, then $p = \sum p_i^2$ (with some $p_i \not\equiv 0$), so $L_\beta(p) = \sum \langle M_2(\beta)\hat{p}_i, \hat{p}_i \rangle > 0$. Apply the previous theorem. $\square$

Lasserre's method for polynomial optimization

For simplicity, we consider the polynomial optimization problem for $K_Q = \mathbb{R}^n$, i.e., $Q = \{q_0 \equiv 1\}$. Let $p \in \mathbb{R}[x_1, \ldots, x_n]$. For $2t \geq deg\ p$, the t-th Lasserre "moment relaxation" for $p_* \equiv \inf_{x \in \mathbb{R}^n} p(x)$ is defined by

$$p_t := inf\{L_\beta(p) : \beta \equiv \beta^{(2t)},\ \beta_0 = 1,\ M_t(\beta) \succeq 0\}.$$

Then $p_t \leq p_*$, and for $t' \geq t$, $p_{t'} \geq p_t$; thus,$\{p_t\}$ is convergent, and $p^{mom} \equiv \lim_{t \to \infty} p_t \leq p_*$. In general, for fixed t, p_t is not necessarily attained at any β. Assuming that the infimum is attained, at some optimal sequence $\beta \equiv \beta^{\{t\}}$, we seek criteria so that $L_\beta(p) = p_*$, so that we have finite convergence of $\{p_s\}$ at stage t.

Lasserre's stopping criterion

Assume at stage t that $\beta \equiv \beta^{\{t\}}$ has a representing measure μ. Then

$$p_* = p_* \beta_0 = p_* \int 1 d\mu \leq \int p d\mu = L_\beta(p) = p_t \leq p_*,$$

so we have convergence at stage t. Although the existence of a representing measure for $\beta^{\{t\}}$ is difficult to ascertain in general, Lasserre focuses on the easy-to-check case when $M_t(\beta)$ is flat, i.e., $rank\ M_t(\beta) = rank\ M_{t-1}(\beta)$. In this case, β has a $rank\ M_t$-atomic representing measure, and the atoms are the global minimizers for p.
Can we find a more general, but still concrete, stopping criterion?

A more general stopping criterion

Theorem [LF-Jiawang Nie, 2010]

Let $\beta \equiv \beta^{\{t\}}$. If L_β is positive, then $p_t = p_*$.

Of course, in general, positivity for L_β is very difficult to check. In current work we are studying a class for which positivity is clear. Let $\mathcal{F}_d := \{\beta \equiv \beta^{(2d)} : M_d(y) \succeq 0 \textit{ is flat}\}$ (a subset of $\mathbb{R}^\rho$, where $\rho \equiv \rho_{2d} = dim\ \mathcal{P}_{2d}$). Consider $\overline{\mathcal{F}_d}$, the closure. If $\beta = \lim_{k\to\infty} \beta^{[k]}$, with each $M_d(\beta^{[k]})$ positive and flat, then each $L_{\beta^{[k]}}$ is positive, so L_β is positive. Thus, $M_d(\beta) \succeq 0$, and

$$rank\ M_d(\beta) \leq \liminf_{k\to\infty} rank\ M_d(\beta^{[k]} = \liminf_{k\to\infty} rank\ M_{d-1}(\beta^{[k]} \leq \rho_{d-1}.$$

If $M_d(\beta) \succeq 0$ and $rank\ M_d(\beta) \leq \rho_{d-1}$, does β belong to $\overline{\mathcal{F}_d}$?

On limits of positive flat moment matrices

Theorem [F-Nie, 2010]

Let $n = 1$, or $d = 1$, or $n = d = 2$. If $M_d(\beta) \succeq 0$ and *rank* $M_d(\beta) \leq \rho_{d-1}$, then $\beta \in \overline{\mathcal{F}_d}$.