Fourier Bases on the "Skewed Sierpinski Gasket"

Calvin Hotchkiss

Iowa State University

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Part 1:

A Fast Fourier Transform for Fractal Approximations

The Fractal Approximation $\mathcal{S}_N$

Begin with an iterated function system generated by contractions $\{\psi_0, \psi_1, \ldots, \psi_{K-1}\}$ on $\mathbb{R}^d$ of the following form:

$$\psi_j(x) = A(x + \vec{b}_j)$$

where A is a $d \times d$ invertible matrix with $\|A\| < 1$. We require A^{-1} to have integer entries, the vectors $\vec{b}_j \in \mathbb{Z}^d$, and $\vec{b}_0 = \vec{0}$. Define the finite orbit:

$$\mathcal{S}_N(\vec{0}) := \{\psi_{j_{N-1}} \circ \psi_{j_{N-2}} \circ \cdots \circ \psi_{j_1} \circ \psi_{j_0}(\vec{0}) : j_k \in \{0, 1, \ldots, K-1\}\}.$$

This orbit is our Nth fractal approximation.

A Fourier Basis on $\mathcal{S}_N$

We then choose a second iterated function system generated by $\{\rho_0, \rho_1, \ldots, \rho_{K-1}\}$ of the form

$$\rho_j(x) = Bx + \vec{c}_j$$

where $B = (A^T)^{-1}$, with $\vec{c}_j \in \mathbb{Z}^d$, and $\vec{c}_0 = \vec{0}$. Define the finite orbit:

$$\mathcal{T}_N(\vec{0}) := \{\rho_{j_{N-1}} \circ \rho_{j_{N-2}} \circ \cdots \circ \rho_{j_1} \circ \rho_{j_0}(\vec{0}) : j_k \in \{0, 1, \ldots, K-1\}\}.$$

These are the frequencies for an exponential basis on $L^2(\mu_n)$, where $\mu_n = \frac{1}{K^N} \sum_{s \in \mathcal{S}_n} \delta_s$.

Example: The "Skewed Sierpinski Gasket"

Let:

$$A = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \qquad b_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad c_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

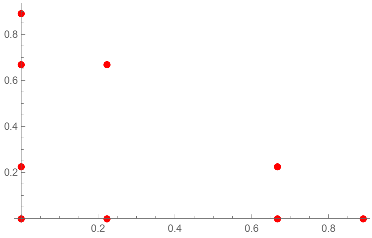
So that

$$\psi_0 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \rho_0 \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}$$

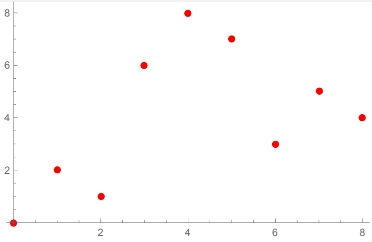
$$\psi_1 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x+2 \\ y \end{pmatrix} \qquad \rho_1 \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\psi_2 \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x \\ y+2 \end{pmatrix} \qquad \rho_2 \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

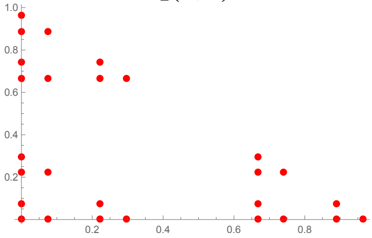
Approximations and Frequencies for Small N



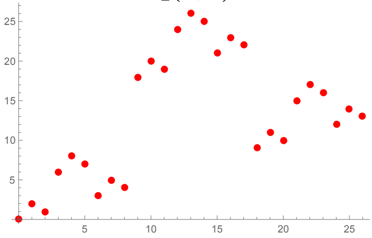
$\mathcal{S}_2(0,0)$



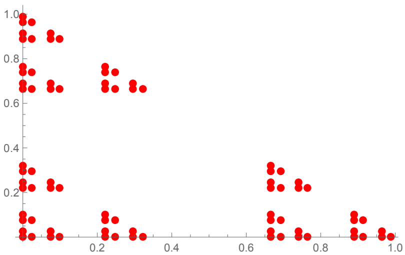
$\mathcal{T}_2(0,0)$



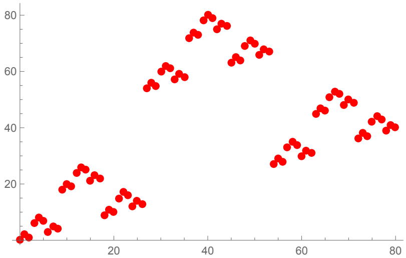
$\mathcal{S}_3(0,0)$



$\mathcal{T}_3(0,0)$



$\mathcal{S}_4(0,0)$



$\mathcal{T}_4(0,0)$

Matrices

The $\{\vec{c}_j\}$ must be chosen so that the matrix

$$M_1 = \left(e^{-2\pi i \vec{c}_j \cdot A\vec{b}_k}\right)_{j,k}$$

is invertible (or Hadamard).

We then define

$$M_N = \left(e^{-2\pi i \vec{s}_j \cdot \vec{t}_k}\right)_{\vec{t}_k \in \mathcal{T}_N, \vec{s}_j \in \mathcal{S}_N}$$

and show that M_N is a discrete Fourier Transform matrix for $\mathcal{S}_N$.

Diţă's Construction

If A is a $K \times K$ Hadamard matrix, B is an $M \times M$ Hadamard matrix, and $E_1, \ldots, E_{K-1}$ are $M \times M$ unitary diagonal matrices, then the $KM \times KM$ block matrix H defined by:

$$\begin{pmatrix} a_{00}B & a_{01}E_1B & \ldots & a_{0(K-1)}E_{K-1}B \\ a_{10}B & a_{11}E_1B & \ldots & a_{1(K-1)}E_{K-1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{(K-1)0}B & a_{(K-1)1}E_1B & \ldots & a_{(K-1)(K-1)}E_{K-1}B \end{pmatrix}$$

is a Hadamard matrix.

Similarly, if A, B, $E_1, \ldots, E_{K-1}$ invertible, H will also be invertible.

For $C = A^{-1}$, H^{-1} is:

$$\begin{pmatrix} c_{00}B^{-1} & c_{01}B^{-1} & \cdots & c_{0(K-1)}B^{-1} \\ c_{10}B^{-1}E_1^{-1} & c_{11}B^{-1}E_1^{-1} & \cdots & c_{1(K-1)}B^{-1}E_1^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(K-1)0}B^{-1}E_{K-1}^{-1} & c_{1(K-1)}B^{-1}E_{K-1}^{-1} & \cdots & c_{(K-1)(K-1)}B^{-1}E_{K-1}^{-1} \end{pmatrix}.$$

Complexity of Matrix Multiplication

Let $\vec{v}$ be a vector of length KM. Consider $H\vec{v}$ where H is the block matrix as in Equation (9). Utilizing the block form of the matrix H, we obtain that the computational complexity of $H\vec{v}$ is

$$O(M^2K + MK^2),$$

whereas for a generic $KM \times KM$ matrix, the computational complexity is $O(K^2M^2)$. Thus, the block form of H reduces the computational complexity of the matrix multiplication.

A Fast Fourier Transform on $\mathcal{S}_N$

The matrix M_N representing the exponentials with frequencies given by $\mathcal{T}_N(\vec{0})$ on the fractal approximation $\mathcal{S}_N(\vec{0})$, both ordered in a particular manner, has the form:

$$\begin{pmatrix} m_{00}M_{N-1} & m_{01}D_{N,1}M_{N-1} & \cdots & m_{0(K-1)}D_{N,K-1}M_{N-1} \\ m_{10}M_{N-1} & m_{11}D_{N,1}M_{N-1} & \cdots & m_{1(K-1)}D_{N,K-1}M_{N-1} \\ \vdots & \vdots & \vdots\,\vdots & \vdots \\ m_{(K-1)0}M_{N-1} & m_{(K-1)1}D_{N,1}M_{N-1} & \cdots & m_{(K-1)(K-1)}D_{N,K-1}M_{N-1} \end{pmatrix}.$$

Where $m_{jk} = [M_1]_{jk}$ and the $D_{N,m}$ are unitary diagonal matrices.

Therefore, by Diţă's construction, M_N is invertible, and if M_1 is Hadamard, then M_N is also Hadamard.

Block form for Inverse

The matrix $\widetilde{M}_N$ representing the exponentials with frequencies given by $\mathcal{T}_N$ on the fractal approximation $\mathcal{S}_N$, both ordered in a different manner, has the form:

$$\begin{pmatrix} m_{00}\widetilde{M}_{N-1} & m_{01}\widetilde{M}_{N-1} & \cdots & m_{0(K-1)}\widetilde{M}_{N-1} \\ m_{10}\widetilde{M}_{N-1}\widetilde{D}_{N,1} & m_{11}\widetilde{M}_{N-1}\widetilde{D}_{N,1} & \cdots & m_{1(K-1)}\widetilde{M}_{N-1}\widetilde{D}_{N,1} \\ \vdots & \vdots & \vdots\vdots & \vdots \\ m_{(K-1)0}\widetilde{M}_{N-1}\widetilde{D}_{N,K-1} & m_{(K-1)1}\widetilde{M}_{N-1}\widetilde{D}_{N,K-1} & \cdots & m_{(K-1)(K-1)}\widetilde{M}_{N-1}\widetilde{D}_{N,K-1} \end{pmatrix}.$$

Where $\widetilde{D}_{N,\ell}$ are also unitary diagonal matrices.

Therefore, $\widetilde{M}_N^{-1}$ has the form of Diţă's construction.

Complexity is $\mathcal{O}(n \log n)$

By utilizing the block forms above, the computational complexity of multiplying a vector $\vec{v}$ by either M_N or M_N^{-1} is reduced to $\mathcal{O}(N \cdot K^N)$, or, in terms of the matrix size $n = K^N$, $\mathcal{O}(n \log n)$.

This is the same computation time as the standard Fast Fourier Transform.

Part 2:

Fourier Bases on the Skewed Sierpinski Gasket

The full Skewed Sierpinski Gasket!

For any iterated function system on $\mathbb{R}^d$ generated by contractions $\{\psi_0, \psi_1, \dots, \psi_{K-1}\}$ of the form:

$$\psi_j(x) = A(x + \vec{b}_j)$$

where A is a $d \times d$ invertible matrix with $\|A\| < 1$, J.E. Hutchinson (1981, [5]) proved there is a unique closed bounded set S with

$$S = \bigcup_{j=0}^{K-1} \psi_j(S)$$

That is, the "compact attractor set" S is invariant under $\{\psi_0, \psi_1, \dots, \psi_{K-1}\}$ and their compositions.

In the same paper, Hutchinson also showed that there is a Borel probability measure ν on S with the property that, for all continuous f:

$$\int_S f(\vec{x})d\nu(x) = \frac{1}{K}\left(\sum_{j=0}^{K-1}\int f(\psi_j(\vec{x}))d\nu(x)\right).$$

For our example of $\{\psi_0, \psi_1, \psi_2\}$ above, the invariant set

$$\mathcal{S} = \{(x,y)|x \in C_3, y \in C_3, x+y \in C_3\}$$

where C_3 is the standard middle-third Cantor set. We call the corresponding probability measure ν_3.

Is $\mathcal{T} = \bigcup_N \mathcal{T}_N(0,0)$ a complete set of frequencies for $\mathcal{S}$?

To (try to) show that

$$\mathcal{E} = \{e^{2\pi i(t\cdot x)} \mid t \in \bigcup_N \mathcal{T}_N(0,0)\}$$

is a Fourier basis for $L^2(\nu_3)$, we reconstruct it using a representation of the Cuntz algebra $\mathcal{O}_3$ on $L^2(\nu_3)$.

First, choose filters for $L^2(\nu_3)$:

$$m_0(x,y) = \frac{1}{\sqrt{3}} e^{2\pi i(x,y)\cdot(0,0)} = \frac{1}{\sqrt{3}}$$
$$m_1(x,y) = \frac{1}{\sqrt{3}} e^{2\pi i(x,y)\cdot(2,1)}$$
$$m_2(x,y) = \frac{1}{\sqrt{3}} e^{2\pi i(x,y)\cdot(1,2)}$$

An Orthonormal Set of Exponentials on $\mathcal{S}$

Then for $j = 0, 1, 2$, $R(x, y) = 3(x, y) \mod 1$:

$$S_j f(x, y) = M = m_j(x, y) f(R(x, y))$$

satisfy the Cuntz relations: $S_i^* S_j = \delta_{i,j} I$ and $\sum_{j=0}^{2} S_j S_j^* = I$.

$\mathcal{E}$ is the orbit of the constant function $\mathbb{1}$ under specific powers of these $\{S_j\}$; thus $\mathcal{E}$ is an orthonormal set for $L^2(\nu_3)$.

We tried to show that $\mathcal{E}$ was complete; however ...

Answer: No!

Lemma (Jorgensen and Pedersen, 1998)

Let $Q_1(t) := \sum_{\lambda \in P} |\hat{\mu}(t-\lambda)|^2$*, for* $t \in \mathbb{R}^d$ *and* $\hat{\mu}$ *the inverse Fourier transform* $\hat{\mu}(t) = \int e^{-2\pi i(t\cdot x)}\, d\mu(x)$*.*
Then $\{e^{2\pi i(\lambda\cdot x)} : \lambda \in P\}$ *is an orthonormal basis for* $L^2(\mu)$ *if and only if* $Q_1 \equiv 1$ *on* $\mathbb{R}^d$*.*

In our case, for $P = \mathcal{T} = \bigcup_N \mathcal{T}_N(0,0)$,
$\widehat{\nu}_3((-1/2,-1)-(a,b)) = 0$ for all $(a,b) \in \mathcal{T}$. Therefore,
$Q_1(-1/2,-1) = 0$ and $\mathcal{E}$ is NOT a Fourier basis for $L^2(\nu_3)$!

Completing our spectrum

Theorem (Dutkay and Jorgensen, 2006)

Let $\Lambda \subset \mathbb{R}^d$ be the smallest set that contains $-C$ for every W_B-cycle C, and such that $S\Lambda + L \subset \Lambda$. Then $\{e^{2\pi i \lambda \cdot x} | \lambda \in \Lambda\}$ is an orthonormal basis for $L_2(\mu_B)$.

For the case of $\mu_B = \nu_3$, $S = 3I_2$ and $L = \{(0,0), (1,2), (2,1)\}$;
$S\Lambda + L = \{\rho_j(\Lambda) \mid j \in (0,1,2)\}$.
The W_B-cycles are $(0,0)$, $(1, 1/2)$, and $(1/2, 1)$.

A Complete Spectrum

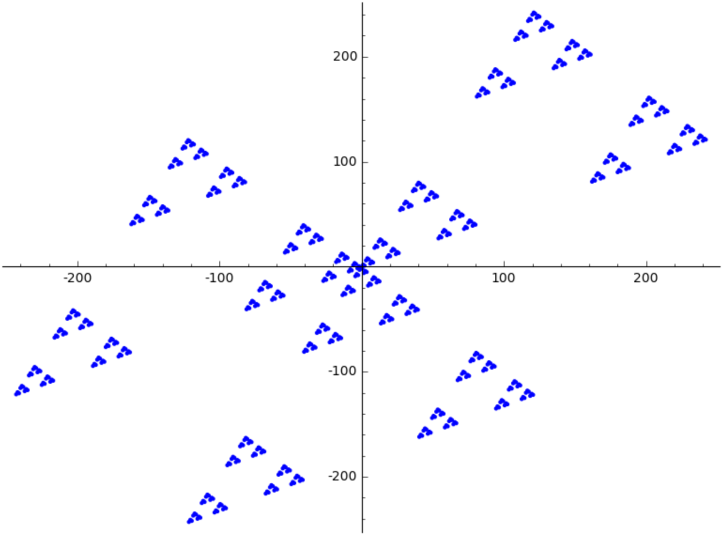
Therefore, for

$$\mathcal{T}_N(x,y) := \{\rho_{j_{N-1}} \circ \rho_{j_{N-2}} \circ \cdots \circ \rho_{j_1} \circ \rho_{j_0}(x,y) : j_k \in \{0,1,2\}\},$$

then

$$\overline{\mathcal{T}} = \bigcup_N \left(\mathcal{T}_N(0,0) \cup \mathcal{T}_N(-1/2,-1) \cup \mathcal{T}_N(-1,-1/2)\right)$$

forms a complete set of frequencies for $\mathcal{S}$.



$$\mathcal{T}_5(0,0) \cup \mathcal{T}_5(-1/2,-1) \cup \mathcal{T}_5(-1,-1/2)$$

Another spectrum

Theorem (H. and Weber, 2016)

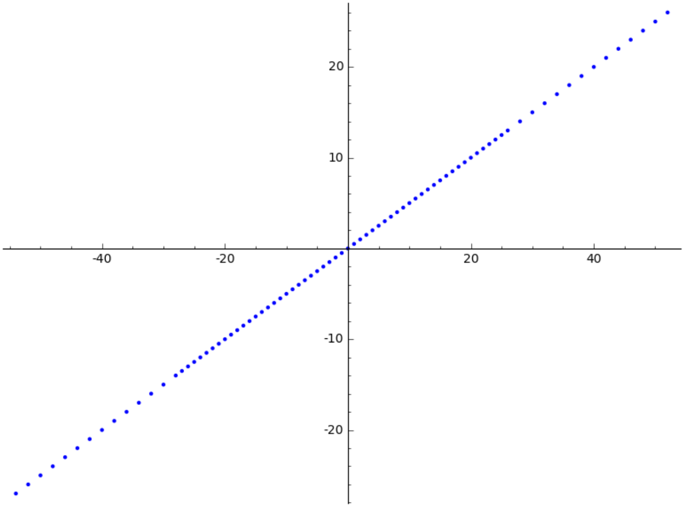
$$\{e^{2\pi i(u,u/2)\cdot(x,y)} | u \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\nu_3)$.

This spectrum comes from the dual iterated function system:

$$\eta_0 \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} \quad \eta_1 \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \eta_2 \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

which has W_B-cycles $(0,0)$, $(2,1)$, and $(1,1/2)$.



Third iteration of the dual function system $\{\eta_j\}$ applied to $(0,0)$, $(-2,-1)$, and $(-1,-1/2)$.

Thank you.

References

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