Math 325-001 **Assignment # 7** **4 Mar 09**

Due: March 11th

1. Consider a sequence (a_n). Show that if (b_k) is a subsequence of (a_n) and (c_m) is a subsequence of (b_k), then (c_m) is a subsequence of (a_n).

2. Let (s_n) be a sequence of real numbers.

   (a) Show that $\liminf s_n = -\limsup(-s_n)$.
   Hint: Recall that, for a nonempty subset $S \subseteq \mathbb{R}$, $\inf S = -\sup\{-s : s \in S\}$.

   (b) Show that if (t_k) is a monotonic subsequence of $(-s_n)$ converging to $\limsup(-s_n)$, then $(-t_k)$ is a monotonic subsequence of (s_n) converging to $\liminf s_n$.

   (c) Use part (b) to complete the proof of Corollary 11.4.

3. Don't turn this question in. Define sequences (s_n) and (t_n), both repeating in cycles of four, by

$$(s_n)_{n=1}^{\infty} = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots)$$
$$(t_n)_{n=1}^{\infty} = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, \ldots)$$

   Without proof, find each of the following:

   (a) $\liminf s_n + \liminf t_n$, (b) $\liminf(s_n + t_n)$
   (c) $\limsup s_n + \limsup t_n$, (d) $\limsup(s_n + t_n)$
   (e) $\liminf s_n + \limsup t_n$, (f) $\limsup(s_n t_n)$,
   (g) $\liminf(s_n t_n)$.

4. Suppose that (s_n) and (t_n) are bounded sequences. Show that

$$\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n.$$

   Hint: First show $\sup\{s_n + t_n : n \ge N\} \le \sup\{s_n : n \ge N\} + \sup\{t_n : n \ge N\}$ and then apply Problem 4(c) from Assignment 5.

5. Given a sequence (s_n) of nonnegative real numbers, define a new sequence (σ_n) by

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

   (a) Show that $\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n$.
   Hint: For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \le \frac{s_1 + s_2 + \cdots + s_N}{M} + \sup\{s_n : n > N\}.$$

   (b) Deduce that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and equals $\lim s_n$.