Let $A \subseteq \mathbb{R}^n$. We will show $\overline{A}$ is closed.

Let $\mathbf{x}$ be a limit point of $\overline{A}$. We must show $\mathbf{x} \in \overline{A}$.

We have a sequence $(\mathbf{x}_m)$ converging to $\mathbf{x}$ with each $\mathbf{x}_m$ in $\overline{A}$.

Fixing m, as $\mathbf{x}_m$ is in the closure of A, there is a sequence $(\mathbf{z}_{m,k})$ converging to $\mathbf{x}_m$ as k goes to infinity.

The right thing to do is, for each m, to pick an integer k_m so that $\|\mathbf{z}_{m,k_m} - \mathbf{x}_m\|$ is at most $1/m$.

Define $\mathbf{y}_m$ to be $\mathbf{z}_{m,k_m}$. By definition, each $\mathbf{y}_m$ is in A.

To see that $(\mathbf{y}_m)$ converges to $\mathbf{x}$, let $\epsilon > 0$. Choose N so that for all $m \geq N$, $\|\mathbf{x}_m - \mathbf{x}\| < \epsilon/2$.

Let $M = \max\{N, 2/\epsilon\}$. Note that for $m \geq M$, $1/m \leq 1/M \leq \epsilon/2$. Thus, for all $m \geq M$, we have

$$\begin{aligned} \|\mathbf{y}_m - \mathbf{x}\| &\leq \|\mathbf{y}_m - \mathbf{x}_m\| + \|\mathbf{x}_m - \mathbf{x}\| \\ &\leq \|\mathbf{z}_{m,k_m} - \mathbf{x}_m\| + \frac{\epsilon}{2} \\ &\leq \frac{1}{m} + + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

By the definition, this shows $(\mathbf{y}_m)$ converges to $\mathbf{x}$. Thus, $\mathbf{x}$ is a limit point of A and so $\mathbf{x} \in \overline{A}$, as required.