Math 496-004 - Multivariable Analysis **Fall 2008**

Complete Course Outline

This is a summary of the main points of each section, together with references to pages in the course texts. For the sake of brevity, statements here are compressed and are only intended to remind you things written out carefully in your notes and the texts. (Don't sue me if you misunderstand something after looking only at this outline).

I use S for *Schaum's outline of Advanced Calculus* by Speigel and E for *Advanced Calculus of Several Variables* by Edwards.

(1) Introduction
- **Goal** careful study of calculus of functions $f : \mathbb{R}^n \to \mathbb{R}^m$
- δ–ϵ definition of continuity for $f : \mathbb{R} \to \mathbb{R}$ (S, 45-47)

(2) Euclidean space, $\mathbb{R}^n$
- vector space structure—vector addition, scalar multiplication, properties, linear independence, basis, dimension of a subspace, properties of dimension (S, 150-153, 155-156; E, 1–9)
- norm and dot product—Euclidean norm, properties, dot product, properties, Cauchy's inequality, proof of triangle inequality, geometric significance of norm and dot product, circular and rectangular neighborhoods of a point in $\mathbb{R}^n$ (S, 153-154; E, 10–19)
- open and closed sets—limit point/cluster point/point of accumulation, closed set, interior point, open set, boundary point, basic properties, S is open if and only if S' is closed (S, 116-119; E, 49-50)

(3) Limits in Euclidean space
- Limits for $F : \mathbb{R}^n \to \mathbb{R}^m$—definition, equivalent formulations, example proof, 'limits and components' theorem, iterated limits, limits along lines, existence of limit implies limits along lines are all equal, $xy/(x^2 + y^2)$ example (S 118–119; E, 41–45)
- Continuity for $F : \mathbb{R}^n \to \mathbb{R}^m$—definition, 'continuity and components' corollary, 'continuity and limits' lemma (S, 119; E, 45–48), all polynomial functions are continuous

(4) The Derivative for $F : \mathbb{R}^n \to \mathbb{R}^m$
- Definition and basic properties—motivation from one-variable case, definition of differential and derivative(E, 67; S, 120-121), differentiable implies continuous, (review of matrix of a linear transformation), definition of directional derivative and partial derivative (E, 65-66; S, 186), 'differentials and directional derivative' theorem (E, 68), the matrix of df_a (the Jacobian matrix), uniqueness of differentials and derivatives, how to compute a differential, 'derivatives and components' theorem, differentiability theorem (E, 71-73).
- Curves, a special case—derivative of a curve, properties of derivatives (S, 156-157; E, 57-61), regular curves and reparametrization, length of a curve, unit tangent, unit normal and unit binormal vectors (S, 177), Frenet-Serret formulae (S, 181), osculating plane & circle
- Chain Rule & Equality of Mixed Partials—Chain Rule in terms of differentials, in terms of matrices (E, 76-78), and in terms of partials (S, 122), applications to the dot product, polar-rectangular coordinates (E, 80-81) and Laplace's equation (E, 80-82; S, 142).

(5) Higher Order Derivatives and Taylor's Theorem
- Mixed Partials—Meaning of $D_v(D_w f)$, Example of function with different mixed partials (S, 141), Equality of Mixed Partials (S, 130; E, 86-87),

- Extrema,I—First Derivative Test (S, 187; E, 102), Motivation for Second Derivative as a Quadratic Form, definition of Quadratic form
- Taylor's Theorem in one variable—Taylor Polynomial of order n, characterization of Taylor polynomial, Taylor's Theorem, Computing limits using Taylor's Theorem and big O notation, Corollary: $\lim_{x\to a}(f(x)-P_n(x))/(x-a)^k$ and P_n is the only polynomial of degree at most n with this property, 2nd derivative test from Taylor's theorem
- Taylor's Theorem in several variables—statement from calculus text, k-tuple notation for partial derivatives and vectors, kth (total) differential and its various formulations, kth differentials and kth directional derivatives theorem, lemma about kth directional derivatives and $f \circ \gamma$, kth order Taylor polynomial of $f : S \to \mathbb{R}$, Taylor's Theorem (E, 131), approximation property (E, 135).
- Extrema,II—definition of Hessian; positive/negative definite, positive/negative semidefinite, and indefinite for quadratic forms; equivalence of convexity, positive definite $d^{(2)}f$, and $f(x)$ above tangent hyperplane; for semidefiniteness, no info about local max/min, structure of quadratic forms (E, 146) ; characterizations of positive/negative definite in terms of eigenvalues and determinants (E, 149).

(6) Inverse and Implicit Function Theorems
- One-variable—one-variable inverse function theorem, proof using tangent line approximation
- Background—Lipschitz function, linear transformations are Lipschitz, continuous functions on
- Inverse Function Theorem—definition of local invertibility, local invertibility theorem, technical lemma, inverse function theorem, inverse function is C^1 (handout)
- Motivation for Implicit Function Theorem—how it generalizes the Inverse Function Theorem, Schaum's calculations,
- Implicit Function Theorem— definition of a local solution of $G(x, y) = 0$ for y in terms of x, derivative of a local solution, definition of $d_{(1)}G$ and $d_{(2)}G$ (E, 189), Implicit Function Theorem, computing derivatives

(7) Change of Variables Formula
- Applications—simplifying integrals by changing variables to either simplify either the region or the integrand
- Motivation—computing change of variables in terms of differentials, why result is true for linear transformations

(8) Lagrange Multipliers and Manifolds
- tangent vectors—definition of tangent vectors to a set M at a point p, T_pM, properties of T_pM, if $f : M \to \mathbb{R}$ has an extremum at p then $df_p(T_pM) = 0$ (E, 102),
- $n - 1$ dimensional manifolds—$T_pG(f)$ is an $n - 1$ dim. subspace of $\mathbb{R}^n$ (E, 106 [proof of Thm 5.3]), definition of $n-1$ dim. coordinate patch and of $n-1$ dim. manifold (E, 104), generalizing Implicit Function Theorem, one constraint Lagrange multipliers (E, 107)
- k dimensional manifolds—definition of k dim. coordinate patch and of k dim. manifold (E, 110), tangent space at a point of a k dim. manifold (E, 111), multiple constraint Lagrange multipliers (E, 113).