Math 104-004 Answers to Test # 1a 21 Sept 07

. 18 1. Find $f'(x)$ for the following functions. (You need not simplify your answers.)

(a) $f(x) = 10x^4 - 3x^2 + \dfrac{5}{x}$

(b) $f(x) = \sqrt{x} + 5 - \dfrac{4}{\sqrt{x}}$

(c) $f(x) = \dfrac{x^3 + 2x + 11}{x}$

Solution. For part (a), we use the power rule with $n = 4$, $n = 2$, and $n = -1$.

$$f'(x) = 10 \cdot 4x^3 - 3 \cdot 2x + 5(-1)x^{-2} = 40x^3 - 6x - \frac{5}{x^2}.$$

For part (b), notice $f(x) = x^{1/2} + 5 - 4x^{-1/2}$ and so, using the power rule with $n = 1/2$, $n = 0$, and $n = -1/2$, we have

$$f'(x) = \frac{1}{2}x^{-1/2} + 0 - 4 \cdot \left(\frac{-1}{2}\right) x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{2}{x^{3/2}}.$$

For part (c), we rewrite $f(x)$ using

$$f(x) = \frac{x^3}{x} + \frac{2x}{x} + \frac{11}{x} = x^2 + 2 + 11x^{-1}$$

and then use the power rule to get

$$f'(x) = 2x + 0 - 11x^{-2}.$$

. 18 2. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 4} \frac{\sqrt{x} - 2}{x - 4}$.

(b) $\displaystyle\lim_{x\to 2^+} \frac{2x - 5}{x^2 - 3x + 2}$.

(c) $\displaystyle\lim_{x\to +\infty} 5 + \frac{x^3 - x}{5x^3 - 125}$.

Solution. For part (a),

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{\sqrt{x}-2}{(\sqrt{x}+2)(\sqrt{x}-2)}\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}.$$

For part (b), substituting $x=2$ into the numerator and denominator gives -1 and 0, respectively. Since we're dividing a nonzero number by zero, there is a vertical asymptote and the limit is either $+\infty$ or $-\infty$. Factoring the denominator, we have

$$\lim_{x\to 2^+}\frac{2x-5}{x^2-3x+2}=\lim_{x\to 2^+}\frac{2x-5}{(x-1)(x-2)}$$

As x approaches 2 from above, $2x-5$ approaches -1, $x-1$ approaches 1 and $x-2$ approaches 0 from above (is positive). Since we have two positive quantities and one negative, the limit is $-\infty$.
For part (c),

$$\begin{aligned}\lim_{x\to+\infty} 5+\frac{x^3-x}{5x^3-125} &= \lim_{x\to+\infty} 5+\lim_{x\to+\infty}\frac{x^3-x}{5x^3-125}\\ &= 5+\lim_{x\to+\infty}\frac{\frac{x^3}{x^3}-\frac{x}{x^3}}{\frac{5x^3}{x^3}-\frac{125}{x^3}}\\ &= 5+\lim_{x\to+\infty}\frac{1-\frac{1}{x^2}}{5}-\frac{125}{x^3}=5+\frac{1}{5},\end{aligned}$$

and so the limit is $26/5=5.2$.

. 10 3. Find the horizontal and vertical asymptotes for

$$f(x)=\frac{2x^2+8}{x^2+6x+8}.$$

As always, explain your reasoning.

Solution. To find the vertical asymptotes, we look for values of x where the denominator is zero. If the numerator is not zero, then there is a vertical asymptote and if the numerator is zero, we need to take limits to figure out what is happening.
So, we solve $x^2+6x+8=0$, which factors as $(x+2)(x+4)=0$, giving $x=-2$ and $x=-4$ as possible asymptotes. Since $2x^2+8$ is not zero for $x=-2$ or $x=-4$, these are both vertical asymptotes.

To find the horizontal asymptotes, we compute

$$\lim_{x\to\infty} \frac{2x^2+8}{x^2+6x+8} = \lim_{x\to\infty} \frac{\frac{2x^2}{x^2}+\frac{8}{x^2}}{\frac{x^2}{x^2}+\frac{6x}{x^2}+\frac{8}{x^2}} = \lim_{x\to\infty} \frac{2+\frac{8}{x^2}}{1+\frac{6}{x}+\frac{8}{x^2}} = 2$$

So there is a horizontal asymptote, $y = 2$.

. 10 4. Find the equation of the tangent line to the graph of $y = f(x) = 8/x - x^2/2$ when $x = 4$.

Solution. To find the equation of the tangent line, we need to find $f(4)$, as the line goes through the point $(4, f(4))$, and we need to find $f'(4)$, since this is the slope of the tangent line at $x = 4$.
First, $f(4) = 8/4 - 4^2/2 = 2 - 8 = -6$, and second, $f(x) = 8x^{-1} - x^2/2$, so

$$f'(x) = -8x^{-2} - (2x)/2 = -8/x^2 - x.$$

Thus, $f'(4) = -8/4^2 - 4 = -1/2 - 4 = -9/2$. So the equation of the line is

$$y - (-6) = \frac{-9}{2}(x-4).$$

. 12 5. Let $f(x) = \begin{cases} \dfrac{2x^2-7x+3}{x-3} & \text{if } x < 3, \\ 3x-4 & \text{if } x \geq 3. \end{cases}$.

(a) Find $\lim_{x\to 3^+} f(x)$ and $\lim_{x\to 3^-} f(x)$.

(b) Is $f(x)$ continuous at 3? Why or why not?

Solution. For part a), we have

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 3x - 4 = 3(3) - 4 = 5$$

and

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{2x^2-7x+3}{x-3} = \lim_{x\to 3^-} \frac{(2x-1)(x-3)}{x-3} = \lim_{x\to 3^-} 2x - 1 = 5.$$

For part b), since

$$\lim_{x\to 3^-} f(x) = 5 = \lim_{x\to 3^+} f(x),$$

we have $\lim_{x\to 3} f(x) = 5$ and since $f(3) = 9 - 4 = 5$ is the same as the limit, the function is continuous at $x = 3$.

. 20

6. Suppose the cost of producing x calculus textbooks is given by $C(x) = 100 + 2x - \frac{x^2}{200}$, in hundreds of dollars.

   (a) What is the **average cost** of a book if 100 books are made?

   (b) What is the **marginal cost** function?

   (c) What is the **marginal cost** when $x = 10$?

   (d) What is the exact cost of the 11th book?

   (e) If the book sells for \$110 each, what are the **revenue** function $R(x)$ and the **profit** function $P(x)$, in hundreds of dollars?

Solution. For part a), there are two acceptable answers. Using the usual meaning of average cost, it is

$$\frac{C(100)}{100} = \frac{100 + 2 \cdot 100 - \frac{100^2}{200}}{100} = \frac{250}{100} = 2.5$$

which means the cost per book is \$250 (since are units are hundreds of dollars). Using the average rate of change from $x = 0$ to $x = 100$, we have

$$\frac{C(100) - C(0)}{100 - 0} = \frac{250 - 100}{100} = 1.5$$

which means the average rate of change of the cost in making the first hundred books is \$150 per book.

For part b), the marginal cost function is

$$C'(x) = 2 - \frac{2x}{200} = 2 - \frac{x}{100}.$$

For part c), we have

$$C'(10) = 2 - \frac{10}{100} = 1.9$$

which, with units of hundreds of dollars, is \$190.

For part d), the exact cost is the cost of making 11 books minus the cost of making the first 10, i.e.,

$$C(11) - C(10) = \left(100 + 2 \cdot 11 - \frac{11^2}{200}\right) - \left(100 + 2 \cdot 10 - \frac{10^2}{200}\right) = 2 - \frac{21}{200} = 1.895$$

which, with units of hundreds of dollars, is \$189.50.

For part e), the revenue is $R(x) = 1.1x$ and the profit is

$$P(x) = R(x) - C(x) = .95x - \left(100 + 2x - \frac{x^2}{200}\right) = -100 - .9x + \frac{x^2}{200}.$$

12 7. **Using the definition**, find the derivative of the function $f(x) = 3x^2 - 5x + 1$ when $x = 2$.

Solution. By the definition,

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

Now,

$$\begin{aligned} f(2+h) &= 3(2+h)^2 - 5(2+h) + 1 \\ &= 3(4 + 4h + h^2) - 10 - 5h + 1 \\ &= 12 + 12h + 3h^2 - 10 - 5h + 1 = 3 + 7h + 3h^2 \end{aligned}$$

while $f(2) = 3(2)^2 - 5 \cdot 2 + 1 = 12 - 10 + 1 = 3$. Thus,

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{3 + 7h + 3h^2 - 3}{h} \\ &= \lim_{h \to 0} \frac{7h + 3h^2}{h} \\ &= \lim_{h \to 0} 7 + 3h = 7 \end{aligned}$$

So $f'(2) = 7$.

As a check, we compute $f'(x) = 6x - 5$ according to the rules, and so $f'(2) = 6 \cdot 2 - 5 = 7$, so the two ways of computing $f'(2)$ agree.