Positivity in Function Algebras

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What is a functional analyst doing at Intel?

Not functional analysis.

- I work on the Open-source 3-D graphics driver team
- Modern graphics cards are specialized processors that perform moderate calculations millions of times per second.
- My work has focused on the compiler for Intel GPUs
- My work so far has been:
  - 20% Graph Theory
  - 15% Algebraic Identities/Reductions
  - 65% Problem Solving and writing C Code

Overview

- Introduction
  - Problem Statement
  - Notation
- Positivity
  - Positivity in the Disc
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  - Positivity in more General Domains
- Connections with Representation Theory
- Future Work
- References

Problem Statement

Let $\mathcal{A}(\mathbb{D})$ be the disc algebra and give $\mathcal{A}(\mathbb{D})$ the involution

$$f \mapsto f^*; \quad f^*(z) = \overline{f(\bar{z})}$$

This yields a Banach $*$-algebra that is *not* a C^*-algebra.

Properties of $\mathcal{A}(\mathbb{D}, *)$

- $\mathcal{A}(\mathbb{D})$ (without the involution) is a norm-closed subalgebra of $\mathcal{C}(\mathbb{T})$ so it is an operator algebra
- $\mathcal{A}(\mathbb{D}, *)$ is a $*$-subalgebra of $\mathcal{C}[-1, 1]$
- For every $f \in \mathcal{A}(\mathbb{D})$, $\sigma(f) = f(\mathbb{D}^-)$

We wish to study the positive elements of $\mathcal{A}(\mathbb{D}, *)$.

Definition

Let $\mathcal{A}$ be a general $*$-algebra (no assumptions of norm). Then the set of *positive* elements of $\mathcal{A}$, denoted $\mathcal{A}_+$, is given by

$$\mathcal{A}_+ = \left\{ \sum_k a_k^* a_k : a_k \in \mathcal{A} \right\}.$$

Definition

Let $\mathcal{A}$ be a unital C^*-algebra. Then an element $a \in \mathcal{A}$ is said to be *positive* if $a^* = a$ and $\sigma(a) \subseteq \mathbb{R}_+$.

What is a good definition of positivity in $\mathcal{A}(\mathbb{D}, *)$?

Definition

Let $f \in \mathcal{A}(\mathbb{D}, *)$. Then f is said to be *positive* if

$$f([-1, 1]) \subseteq \mathbb{R}_+.$$

Is this the right definition?

Theorem (Ekstrand & Peters, 2013)

Let $f \in \mathcal{A}(\mathbb{D}, *)$. Then f is positive (as defined above) if and only if $f = g^*g$ for some $g \in \mathcal{A}(\mathbb{D})$.

Notation

For a domain $G \subseteq \mathbb{C}$, we have the following algebras:

- $\mathcal{H}(G)$ of holomorphic functions on G
- $H^\infty(G)$ of bounded holomorphic functions on G
- $\mathcal{A}(G)$ of bounded holomorphic functions on G which have continuous extension to G^-

If $f : G \to \mathbb{C}$ and $r\mathbb{T} \subseteq G$ and, we define the function

$$f_r : [-\pi, \pi] \to \mathbb{C}; \quad f_r(t) = f(re^{it}).$$

When it makes sense, we define the p^{th} Hardy space

$$H^p(G) = \{f \in \mathcal{H}(G) : \|f_r\|_p \text{ is bounded in } r\}$$

Positivity in the Disc

We begin with the case of non-vanishing functions.

Let $f \in \mathcal{A}(\mathbb{D})$ be non-vanishing. Since $\mathbb{D}$ is simply connected,

$$f(z) = e^{h(z)} \text{ for some } h \in \mathcal{H}(\mathbb{D}).$$

However, h need be neither bounded nor continuous on $\mathbb{D}^-$.

Lemma

Suppose $h : \mathbb{D} \to \mathbb{C}$ is continuous and that there is a continuous function $F : \mathbb{D}^- \to \mathbb{C}$ with $F = e^h$ on $\mathbb{D}$. If K is the set of zeros of F on $\mathbb{T}$ then h can be continuously extended to $\mathbb{D}^- \setminus K$.

Theorem

Let $f \in \mathcal{H}(\mathbb{D})$ be positive with no roots in $\mathbb{D}$. Then, for every integer $n > 0$ there is a unique positive function $g \in \mathcal{H}(\mathbb{D})$ such that $f = g^n$. If $f \in H^p(\mathbb{D})$ for some $1 \leq p \leq \infty$, then $g \in H^{np}(\mathbb{D})$. If $f \in \mathcal{A}(\mathbb{D})$, then $g \in \mathcal{A}(\mathbb{D})$.

Sketch of proof.

- $f = e^h$ for some $h \in \mathcal{H}(\mathbb{D})$; let $g = e^{h/n}$ on $\mathbb{D}$
- Define $x : \mathbb{T} \to \mathbb{C}$ as $x = e^{h/n}$ on $\mathbb{T} \setminus K$ and $x = 0$ on K
- Then x is continuous on $\mathbb{T}$ and x is a.e. the boundary values of g so $g \in \mathcal{A}(\mathbb{D})$. $\square$

BSF Factorization

For any function $f \in H^1(\mathbb{D})$, we can write $f = BSF$ where

$$F(z) = \lambda \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f(e^{i\theta})|\, d\theta\right],$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and

$$B(z) = z^{p_0} \prod_{n=1}^{\infty} \left[\frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}\right]^{p_n}$$

where $\{\alpha_n\}$ are the roots of f with multiplicities p_n and

$$S(z) = \exp\left[-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)\right]$$

for some singular positive measure μ on $[-\pi, \pi]$.

If we are going to use the BSF factorization, we need to handle the positivity and continuity of the different pieces.

Theorem

Let $f \in \mathcal{A}(\mathbb{D})$ and decompose f as $f = gB$ where $g \in H^\infty(\mathbb{D})$ and B is a Blaschke product. Then $g \in \mathcal{A}(\mathbb{D})$ and g has the same zeros on $\mathbb{T}$ as f.

Theorem

Let $f \in \mathcal{A}(\mathbb{D})$ and let B be a Blaschke product such that $f(z) = 0$ whenever z is a limit point of the roots of B. Then $fB \in \mathcal{A}(\mathbb{D})$.

Theorem

Let B be the Blaschke product. If B has the same roots as some positive $f \in \mathcal{H}(\mathbb{D})$, then there is another Blaschke product B_+ with $B = B_+^* B_+$.

Theorem

Let $f \in H^p(\mathbb{D})$ for some $1 \leq p \leq \infty$. Then f is positive if and only if there exists $g \in H^{2p}(\mathbb{D})$ so that $f = g^* g$. If $f \in \mathcal{A}(\mathbb{D})$ then g may also be chosen to be in $\mathcal{A}(\mathbb{D})$.

Positivity in the Annulus

Definition

Fix $0 < r_0 < 1$ and define the annulus

$$A = \{z \in \mathbb{C} : r_0 < |z| < 1\}.$$

We define the following algebras:

- $\mathcal{H}(A)$ of all holomorphic functions on A,
- $H^p(A)$ of all holomorphic functions on A with $\|f_r\|_p$ bounded for $r_0 < r < 1$,
- $\mathcal{A}(A)$ of all holomorphic functions on A with continuous extension to A^-.

Properties of $\mathcal{H}(A)$

Given a function $f \in \mathcal{H}(A)$, we have the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$$

so $f(z) = g(z) + h(r_0/z)$ where $g, h \in \mathcal{H}(\mathbb{D})$.

Observation

- *$f \in H^p(A)$ if and only if $g, h \in H^p(\mathbb{D})$*
- *$f \in \mathcal{A}(A)$ if and only if $g, h \in \mathcal{A}(\mathbb{D})$*
- *$f \in H^p(A)$ can be recovered from its boundary values*

Positivity in $H^P(A)$

Definition

Let $f \in \mathcal{H}(A)$. Then f is said to be *positive* if

$$f(x) \geq 0 \text{ for all } x \in A \cap \mathbb{R}.$$

- How do we study positive functions on A?
- For $f \in \mathcal{H}(A)$, $f(z) = g(z) + h(r_0/z)$ where $g, h \in \mathcal{H}(D)$. However, f positive does not imply that g or h is positive.
- $f \in \mathcal{H}(A)$ non-vanishing does not imply $f = e^g$.
- How do we replace our use of the BSF factorization?

Non-vanishing functions in $\mathcal{H}(A)$

The problem here is that A is not simply connected.

Theorem

Let G be a domain and f be holomorphic on G. Suppose f is non-vanishing and

$$\oint_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for every simple closed curve γ. Then there exists a holomorphic function g on G so that $f = e^g$.

Definition

For $f \in \mathcal{H}(A)$ non-vanishing, define the *winding number* of f by

$$\mathrm{wn}(f) = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f'(z)}{f(z)}\, dz$$

where $\gamma_r(t) = re^{it}$ for $t \in [-\pi, \pi]$ and $r_0 < r < 1$.

Theorem (Ekstrand, 2014)

Let $f \in \mathcal{H}(A)$ be positive and non-vanishing. Then $\mathrm{wn}(f)$ is an even number.

For any positive $f \in \mathcal{H}(A)$, the function $g(z) = f(z)z^{-\mathrm{wn}(f)}$ is positive with $\mathrm{wn}(g) = 0$.

Theorem (Ekstrand, 2014)

Let $f \in \mathcal{H}(A)$ be positive and non-vanishing. Then there exists a function $g \in \mathcal{H}(A)$ so that $f = g^*g$. Furthermore, if $f \in H^p(A)$, then $g \in H^{2p}(A)$ for $1 \leq p \leq \infty$ and, if $f \in \mathcal{A}(A)$, then $g \in \mathcal{A}(A)$.

Sketch of proof.

- Let $f_0(z) = f(z)z^{-\mathrm{wn}(f)}$; $\mathrm{wn}(f_0) = 0$.
- $f_0 = e^h$ for some $h \in \mathcal{H}(A)$.
- Define g by $g(z) = e^{h(z)/2}z^{\mathrm{wn}(f)/2}$.
- Continuity is similar to the disc case. $\square$

H^p spaces of an annulus (Sarason, 1965)

In his 1965 work, Sarason studies holomorphic functions on A and tries to recover a BSF factorization for the annulus.

- Sarason's work focuses on the universal covering surface

$$\hat{A} = \{(r,t) \in \mathbb{R}^2 : r_0 < r < 1\}$$

with the covering map

$$\varphi : \hat{A} \to A; \quad \varphi(r,t) = re^{it}.$$

- Sarason develops a BSF factorization for *modulus automorphic* functions $\hat{A}$
- Unfortunately, these result don't translate easily to $H^p(A)$

Blaschke Products on A

Sarason's construction is enough to get us the following:

Theorem (Sarason, 1965; Ekstrand, 2014)

Let $f \in H^\infty(A)$ that is not identically zero and let $\{a_n\}_{n=1}^\infty$ be the set of zeros of f repeated according to multiplicity. Then

$$\sum_{n=1}^{\infty} \min\left(1 - |a_n|, 1 - \frac{r_0}{|a_n|}\right) < \infty.$$

Theorem (Ekstrand, 2014)

Let $f \in H^\infty(A)$ that is not identically zero and let $\{a_n\}$ be the roots of f repeated according to multiplicity. Then the Blaschke products

$$B_1(z) = \prod_{|a_n| \geq \sqrt{r_0}} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z} \quad \text{and} \quad B_2(z) = \prod_{|a_n| < \sqrt{r_0}} \frac{a_n}{|a_n|} \frac{r_0/a_n - z}{1 - (r_0/\bar{a}_n) z}$$

converge and we may decompose f as $f(z) = g(z) B_1(z) B_2(r_0/z)$ where g is bounded, holomorphic, and non-vanishing on A. If f has a continuous extension to A^- then so does g.

Theorem (Ekstrand, 2014)

An element $f \in H^p(A)$ is positive if and only if $f = g^* g$ for some $g \in H^{2p}(A)$. Furthermore, if f is continuous on A^-, then g may be chosen continuous on A^-.

Generalizations to other domains

Definition

Let G be a domain. We say that G is *symmetric* if

$$G = G^* = \{\bar{z} : z \in G\}.$$

Theorem (Ekstrand, 2014)

Let G be a symmetric domain where ∂G is the union of finitely many disjoint Jordan curves and let $f \in H^\infty(G)$. Then f is positive if and only if there is some $g \in H^\infty(G)$ so that $f = g^* g$. Furthermore, if $f \in \mathcal{A}(G)$ then g may be chosen in $\mathcal{A}(G)$.

Connections with Representation Theory

Definition

Let $\mathcal{A}$ be a $*$-algebra. Then a *$*$-representation* of $\mathcal{A}$ is a pair $(\mathcal{H}, \varphi)$ where $\mathcal{H}$ is a Hilbert space and $\varphi : \mathcal{A} \to B(\mathcal{H})$ is a $*$-homomorphism.

What about $\mathcal{A}(G, *)$?

- If $(\mathcal{H}, \varphi)$ is a $*$-representation of $\mathcal{A}$ then, for all $g \in \mathcal{A}$, $\varphi(g^* g) = \varphi(g)^* \varphi(g)$ is positive in $\mathcal{H}$.
- The one-dimensional $*$-representations of $\mathcal{A}(G, *)$ are exactly the point-evaluations on $G \cap \mathbb{R}$.
- $f \in \mathcal{A}(G, *)$ is positive if and only if $\varphi(f) \geq 0$ for every one-diimensional $*$-representation φ of $\mathcal{A}(G, *)$.

Theorem (Ekstrand, 2014)

Let G be a region so that ∂G is the union of finitely many disjoint Jordan curves in $\mathbb{C}_\infty$. For each $f \in \mathcal{A}(G)$, TFAE:

1. *f is positive, i.e., $f(G \cap \mathbb{R}) \geq 0$,*
2. *$f = g^* g$ for some $g \in \mathcal{A}$,*
3. *$f = \sum_{i=1}^{n} g_i^* g_i$ for some $g_1, \ldots, g_n \in \mathcal{A}$,*
4. *$f = \lim_{n\to\infty} f_n$ where each f_n is of the form given in 3.*
5. *$\varphi(f) \geq 0$ for every one-dimensional $*$-rep. $(\mathbb{C}, \varphi)$ of $\mathcal{A}(G)$*

5. is equivalent to $\sigma(a) \geq 0$ in abelian C^*-algebras

Future Work

1. Extend the results to even more general domains
   - While the restriction that ∂G is the union of finitely many disjoint Jordan curves is sufficient, I have no proof that it is necessary.
   - Unfortunately, such an extension would probably need a new technique.
2. Try and extend these results to a non-abelian case
   - These definitions extend fairly easily to $\mathcal{M}_{n\times n}(\mathcal{A}(G))$
3. Consider domains not in $\mathbb{C}$ such as Riemann surfaces
   - There is a 1965 paper by Voichick and Zalcman that gives a BSF factorization for a certain class of Riemann surfaces

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Thank You!

Idea Behind the Proof

Theorem (Riemann Mapping Theorem)

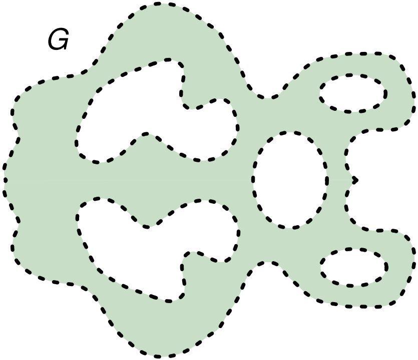
Let $G \subseteq \mathbb{C}$ be a simply connected region that is not the whole plane and let $a \in G$. Then there is a unique holomorphic bijection $\phi : G \to \mathbb{D}$ so that $\phi(a) = 0$ and $\phi'(a) > 0$.

Theorem (Carathéodory)

Let $G \subseteq \mathbb{C}$ be a simply connected region whose boundary is a Jordan curve. Then the Riemann map $\phi : G \to \mathbb{D}$ extends to a homeomorphism $\Phi : G^{-} \to \mathbb{D}^{-}$.

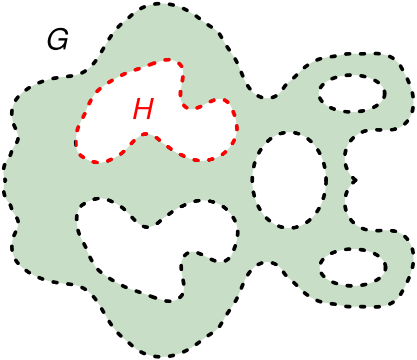
Sketch of Proof

Start with some symmetric region G and $f \in H^\infty(G)$



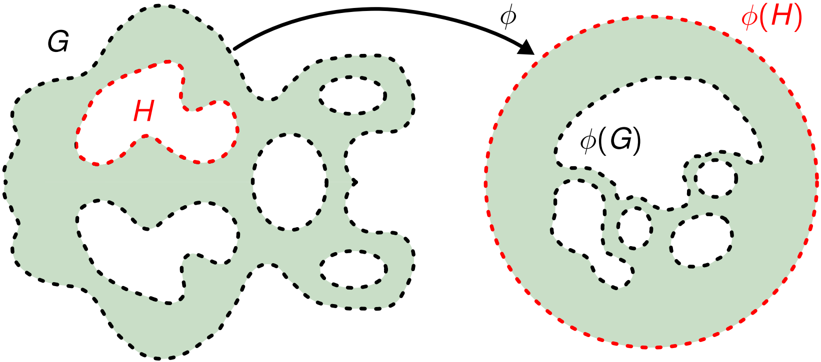
Sketch of Proof

Pick a single hole H in G



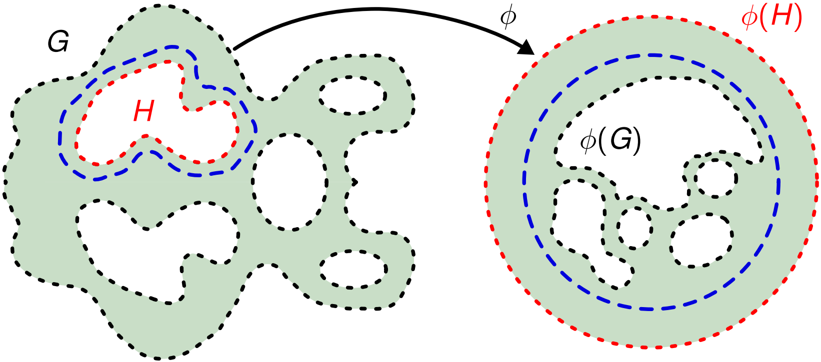
Sketch of Proof

Define a Carathéodory map $\phi : \mathbb{C} \setminus H \to \mathbb{D}^-$



Sketch of Proof

Pick r_0 so that $\{z \in \mathbb{C} : r_0 \leq |z| < 1\} \subseteq \phi(G)$



Sketch of Proof

- This gives us an annulus $A = \{z \in \mathbb{C} : r_0 \leq |z| < 1\}$.
- We can factor $f \circ \phi^{-1}$ as $f \circ \phi^{-1} = gB$ where $g \in H^\infty(A)$ and B is a Blaschke product.
- Translating back to G, $f = (g \circ \phi)(B \circ \phi)$.
- A similar trick can be used to ensure $\mathrm{wn}(f \circ \varphi^{-1}) = 0$.
- Decompose f, square root the non-vanishing part and put it back together as we did before.
- Thanks to the Carathéodory theorem, ϕ is a homeomorphism of $\mathbb{C} \setminus H$ and $\mathbb{D}^-$ so continuity follows from results in the annulus. $\square$