. 5 1. (a) How many subintervals must you use in Simpson's rule to find $\int_0^\pi \sin(x^2)\,dx$ to 10 digits accuracy, that is, with an error less than $\frac{1}{2}\cdot 10^{-10}$.
You may use, without justification, the fact that, if $f(x)=\sin(x^2)$, then $|f^{(4)}(x)| \le 1200$ for $x \in [0,\pi]$.

(b) Evaluate $\int_0^\infty \frac{2}{x^2+4x+3}\,dx$.
Hint: use properties of logarithms to simplify the definite integral and remember that ln is a continuous function.

Solution. For the first question, we take the formula from the textbook and class, that the error in using Simpson's Rule for a function $f(x)$ on an interval $[a,b]$ with n equal subintervals is

$$\frac{M(b-a)^5}{180n^4},$$

where M is the maximum of $|f^{(4)}(x)|$ for x in $[a,b]$. In our problem, $M=1200$ and $b-a=\pi$, so we have

$$\begin{aligned} \frac{1200\pi^5}{180n^4} &\le \frac{10^{-10}}{2}, \\ \frac{20\pi^5}{3\left(\frac{10^{-10}}{2}\right)} &\le n^4, \\ 4.0603\cdot 10^{13} &\le n^4. \end{aligned}$$

Taking fourth roots of both sides, we have $n \ge 2527.39$, so we need to have n at least 2528.

For the second question, observe that $x^2+4x+3=(x+1)(x+3)$, so there are no vertical asymptotes on $[0,+\infty)$. Next, we find the partial fractions decomposition of $2/(x^2+4x+3)$. The form is

$$\frac{2}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3}.$$

Multiplying by $(x+1)(x+3)$, we get $2=A(x+3)+B(x+1)$. Letting $x=-1$, $2=2A$, so $A=1$ and letting $x=-3$, $2=-2B$, so $B=-1$. Thus,

$$\begin{aligned} \int \frac{2}{x^2+4x+3}\,dx &= \int \frac{1}{x+1}\,dx - \int \frac{1}{x+3}\,dx \\ &= \ln|x+1| - \ln|x+3| + C \end{aligned}$$

and now we use the property $\ln a - \ln b = \ln(a/b)$ to simplify this to

$$= \ln\left|\frac{x+1}{x+3}\right| + C$$

Thus,

$$
\begin{aligned}
\int_0^\infty \frac{2}{x^2+4x+3}\,dx &= \lim_{b\to\infty} \int_0^b \frac{2}{x^2+4x+3}\,dx \\
&= \lim_{b\to\infty} \ln\left|\frac{x+1}{x+3}\right|\Bigg|_0^b \\
&= \lim_{b\to\infty} \ln\left|\frac{b+1}{b+3}\right| - \ln\left(\frac{1}{3}\right)
\end{aligned}
$$

and now, using the continuity of the function $\ln|x|$, we have

$$
\begin{aligned}
&= \ln\left|\lim_{b\to\infty} \frac{b+1}{b+3}\right| - \ln\left(\frac{1}{3}\right) \\
&= \ln 1 - \ln 1/3 = \ln 3.
\end{aligned}
$$