Math 104-004 **Answers to Test # 3b** **9 Nov 07**

. 8 1. Find $f''(-2)$ if $f(x) = \dfrac{4x}{2x+3}$.

Solution. Using the quotient rule, we have

$$f'(x) = \frac{(2x+3)4 - (4x)(2)}{(2x+3)^2} = \frac{8x + 12 - 8x}{(2x+3)^2} = \frac{12}{(2x+3)^2} = 12(2x+3)^{-2}.$$

Thus, using the generalized power rule, we have

$$f''(x) = 12(2x+3)^{-3} \cdot 2 = \frac{24}{(3x+1)^3}.$$

In particular,

$$f''(-2) = \frac{24}{(2 \cdot (-2) + 3)^3} = \frac{2}{4}(-1)^3 = -24.$$

. 10 2. Use calculus methods to find the absolute maximum value M and the absolute minimum value m of the function $f(x) = x^3 + 3x^2 - 9x + 1$ on the interval $[-6, 2]$.

Solution. Taking the derivative, we have

$$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x + -3) = 3(x+3)(x-1).$$

so $x = -3$ and $x = 1$ are critical numbers. So we have to evaluate $f(x)$ at $x = -6$, $x = -3$, $x = 1$, and $x = 2$. Notice that $f(-6) = -53$, $f(-3) = 28$, $f(1) = -4$, and $f(2) = 3$.
So the absolute maximum is $M = 28$ and the absolute minimum is -53.

. 14 3. Let $y = f(x)$ be a function such that $f'(x) = x(x+3)^2(x+5)(x+3)(x-1)^2$ for all $x \in (-\infty, \infty)$.

(a) Find the critical numbers,

(b) Chart $f'(x)$.

(c) Find the open intervals on which f is increasing, and

(d) Find all x coordinates so that $(x, f(x))$ is a relative maximum of $y = f(x)$.

(e) Find all x coordinates so that $(x, f(x))$ is a relative minimum of $y = f(x)$.

Solution. For part (a), the critical numbers are zeros of $f'(x)$, which are -5, -3, 0, 1, 3.

For part (b), the chart of $f'(x)$ is

For part (c), $f'(x)$ is increasing on $(0, 1)$, $(1, 3)$, and $(3, +\infty)$.

For part (d), to have a relative max, $f'(x)$ must go from positive to negative, so the relative maxima are at $x = -5$ and $x = 3$.

For part (e), $f'(x)$ must go from negative to positive, so the only relative min is at $x = 0$.

. 10 4. Given the cost function $C(x) = 3x^2 + 5x + 300$ dollars, use calculus methods to determine the number of units x that should be produced in order to minimize the **average cost per unit**.

Solution. The average cost function is

$$\overline{C}(x) = \frac{3x^2 + 5x + 300}{x} = 3x + 5 + \frac{300}{x},$$

and so

$$\overline{C}'(x) = 3 - \frac{300}{x^2} = \frac{3x^2 - 300}{x^2}.$$

Thus, $\overline{C}'(x) = 0$ when $3x^2 = 300$, or $x^2 = 100$, or $x = \pm 10$. Since x is positive, the only meaningful critical number is $x = 10$.

So $x = 10$ units should be produced to minimize the average cost per unit.

. 10 5. Let $y = f(x) = \frac{1}{4}x^4 - 2x^3 - 18x^2 + \frac{7}{2}x + 3$ for all $x \in (-\infty, +\infty)$.

(a) Find $f''(x)$.

(b) Chart $f''(x)$.

(c) Find the open intervals where the graph of f is concave up, and

(d) Find the points $(x, f(x))$ on the graph of f which are inflection points.

Solution. First, $f'(x) = x^3 - 6x^2 - 36x + \frac{7}{2}$ and so

$$f''(x) = 3x^2 - 12x - 36 = 3(x^2 - 4x - 12) = 3(x - 6)(x + 2).$$

To make the chart we notice that $f''(x)$ is zero when $x = -2$ and $x = 6$. Using $f''(-3) = 27$, $f''(0) = -36$, and $f''(7) = 9$, we chart $f''(x)$.

For part (c), f is concave up on $(-\infty, -2)$ and on $(6, +\infty)$.

For part (d), since there are sign changes at both $x = -2$ and $x = 6$, there are inflection points for these x-values. Notice that $f(-2) = 4+16-72-7+3 = -56$ and $f(6) = 324-432+648+21+3 = -732$. Thus, the inflection points are $(-2, -56)$ and $(6, -732)$.

. 8 6. For the function $f(x) = 4x^5 - 2x + 5$, find dy if $x = -1$ and $dx = \Delta x = .03$

Solution. Notice that $f'(x) = 20x^4 - 2$, so $f'(-1) = 18$. Thus, at $x = -1$,

$$dy = 18dx = 18(.03) = .56.$$

. 15 7. Sketch the graph of function $y = f(x)$ with the following properties

(a) $f'(x) > 0$ for x in $(-3, -1)$, and $(7, +\infty)$,
(b) $f'(x) < 0$ for x in $(-\infty, -3)$, and $(-1, 7)$,
(c) $f''(x) > 0$ for x in $(-2, 4)$,
(d) $f''(x) < 0$ for x in $(-\infty, -2)$, and $(4, +\infty)$.

Solution.

. 15 8. Using calculus-based methods, find positive numbers x and y with $x + y = 20$ so that $x^4 y$ is as large as possible.

Solution. Notice that $y = 20 - x$, so we are maximizing the function $f(x) = x^4(20 - x) = 20x^4 - x^5$. Now

$$f'(x) = 80x^3 - 5x^4 = 5x^3(16 - x)$$

so $f'(x) = 0$ when $x = 0$ or $x = 16$. Next, observe that since x and y must both be positive, x must be in the interval $[0, 20]$. Thus, we must check $f(x)$ for $x = 0$, $x = 16$ and $x = 20$. We have $f(0) = 0$, $f(15) = 262,144$ and $f(20) = 0$.
Thus when $x = 16$ and $y = 4$, then $x^4 y$ is as large as possible.

. 10 9. Use differentials to approximate $\sqrt{103}$. Using your calculator, find the error in your approximation.

Solution. Notice that $103 = 100 + 3$, as $\sqrt{100} = 10$, find the derivative of $f(x) = \sqrt{x}$ at $x = 100$ and then find dy when $dx = 3$.
As $f(x) = x^{1/2}$, $f'(x) = 1/2x^{-1/2}$ and so $f'(100) = 1/(2 \cdot 10) = 1/20$. Thus,

$$dy = \frac{1}{20}dx = \frac{3}{20}.$$

So $\sqrt{103}$ is about $10 + \frac{3}{20} = 10.15$.
For comparision, $\sqrt{103}$ is 10.14889, so the error is about .00111.