. 30   1. For each of the following, determine if the series converges or diverges. As always, give reasons for your answer.

$$\text{a) } \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}, \qquad \text{b) } \sum_{n=2}^{\infty} \frac{1}{n \ln n}, \qquad \text{c) } \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Solution. These questions are Section 8.5, # 17, Section 8.4, # 13 (changed to remove the square root) and Section 8.3, #13.

For a), use the ratio test. Notice that $a_{n+1} = \dfrac{(n+2)(n+3)}{(n+1)!}$ and so

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)(n+3)}{(n+1)!} \frac{n!}{(n+1)(n+2)} = \frac{n+3}{(n+1)^2} = \frac{n+3}{n^2 + 2n + 1}.$$

Thus, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$ and so, by the ratio test, the series converges. For b), use the integral test. Notice that $1/(x \ln x)$ is continuous and decreasing and

$$\int_2^{\infty} \frac{1}{x \ln x} = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x} \, dx \quad \begin{cases} u = \ln x \\ du = \frac{dx}{x} \end{cases}$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u} \, du$$

$$= \lim_{b\to\infty} \ln u \big|_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2) = \infty$$

Because this improper integral diverges, so does the series.

For c), use the alternating series test. Since the terms $\dfrac{1}{\sqrt{n}}$ are positive and decrease as n increases, we need only observe that $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$ to apply the alternating series test and conclude the series converges.

. 12   2. Find the center of mass of a thin plate of constant density δ covering the region bounded by $y = 1/x$ and $y = -1/x$ and by the lines $x = 2$ and $x = 5$.

Solution. This problem is similar to Section 6.7, #13. Because the region is symmetric about the x-axis and the density (being constant) is symmetric about the x-axis, then y-coordinate of the center of mass, $\bar{y}$, is 0.

After graphing the region, it is clear that using vertical slices means only one kind of slice we will arise. These slices have width dx, length $2/x$, and mass $dm = \delta(2/x)dx$. The center of mass of a slice is $(\tilde{x}, \tilde{y}) = (x, 0)$. The total mass is

$$M = \int_2^5 \delta \frac{2}{x} \, dx = 2\delta \ln x \big|_2^5 = 2\delta(\ln 5 - \ln 2)$$

and the moment about the y-axis is

$$M_y = \int_2^5 x\delta \frac{2}{x} \, dx = \delta \int_2^5 2 \, dx = 6\delta.$$

Thus, $\bar{x} = M_y/M = 3/(\ln 5 - \ln 2) \approx 3.27407$.

. 12

3. Determine if $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{3^n} \right)$ converges or diverges and, if it converges, find its sum.

Solution.   This is similar to Section 8.2, # 13. Recall that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{3^n} \right) = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}.$$

meaning that the original series converges if each of the two geometric series converge and, in this case, its sum is giving by adding the sums of the two geometric series. For the first series, the first term is $1/2$ and ratio is $1/2$, so it converges and its sum is $(1/2)/(1 - 1/2) = 1$. while for the second series, the first term is $-1/3$ and ratio is $-1/3$, so it converges and its sum is $(-1/3)/(1 - (-1/3)) = -1/4$.

Thus, the original series converges and its sum is $3/4$.

. 20

4. For each of the following, find the first four nonzero terms of the Taylor series (or give a formula for the whole series):

$$\text{a) } \cos(4x) \text{ at } x = 0, \qquad \text{b) } \frac{1}{x^3} \text{ at } x = 1.$$

Solution.   There problems are similar to Section 8.8, # 13 and # 25.
For a), recall that the first four nonzero terms of the Taylor series for $\cos y$ are

$$1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots.$$

Letting $y = 4x$, we obtain

$$1 - \frac{16x^2}{2} + \frac{256x^4}{4!} - \frac{4096x^6}{6!} + \cdots = 1 - 8x^2 + \frac{32x^4}{3} - \frac{256x^6}{45} + \cdots$$

For b), we use the formula for the Taylor series of $f(x)$ at $x = a$:

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f'''(a)}{6}(x - a)^3 + \cdots.$$

Notice $a = 1$ is this problem. Since $f(x) = x^{-3}$, we have $f'(x) = -3x^{-4}$, $f''(x) = 12x^{-4}$, and $f'''(x) = -60x^{-5}$. Thus, $f(1) = 1$, $f'(1) = -3$, $f''(1) = 12$, and $f'''(1) = -60$. Putting these into the formula with $a = 1$, we obtain

$$1 - 3(x - 1) + 6(x - 1)^2 - 10(x - 1)^3 + \cdots.$$

. 16

5. Find the radius and interval of convergence for $\sum_{n=1}^{\infty} \frac{n}{5^n}(x+3)^n$. Also, for which values of x does the series converge absolutely?

Solution.   This is similar to Section 8.7, # 17.
First, we find the radius of convergence using the ratio test. Notice that
$$|a_n| = \frac{n|x+3|^n}{5^n}, \qquad |a_{n+1}| = \frac{(n+1)|x+3|^{n+1}}{5^{n+1}}.$$
Thus,
$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)|x+3|^{n+1}}{5^{n+1}} \frac{5^n}{n|x+3|^n} = \frac{(n+1)|x+3|}{5n}$$
and so $\lim n \to \infty \frac{|a_{n+1}|}{|a_n|} = |x+3|/5 \lim_{n\to\infty} \frac{n+1}{n} = |x+3|/5$. For x to converge, we need $|x+3|/5 < 1$ or $|x+3| < 5$. So the radius of convergence is 5 and we know the series converges absolutely for x between -8 and 2, and it does not converge if $x < -8$ or $x > 2$.
It remains to check $x = -8$ and $x = 2$. For $x = 2$, we have
$$\sum_{n=1}^{\infty} \frac{n}{5^n}(2+3)^n = \sum_{n=1}^{\infty} n$$
which clearly diverges by the nth term test, since $\lim_{n\to\infty} n \neq 0$. For $x = -8$, we have
$$\sum_{n=1}^{\infty} \frac{n}{5^n}(-8+3)^n = \sum_{n=1}^{\infty} n\left(\frac{-5}{5}\right)^n = \sum_{n=1}^{\infty}(-1)^n n$$
which also diverges by the nth term test, since $\lim_{n\to\infty}(-1)^n n \neq 0$. Thus, the interval of convergence is $(-8, 2)$ and the series converges absolutely for all x in this interval.

. 10

6. Using suitable Taylor series, find $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$.

Solution.   This is similar to Section 8.9, # 31 and Question 10 a on the review sheet. We recall the first few terms of the Taylor series for $\sin x$ and $\cos x$:
$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots, \qquad x - \frac{x^3}{6} + \frac{x^5}{5!} - \cdots.$$
Thus, the first few terms of the Taylor series for $x\cos x - \sin x$ is
$$x\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right) - \left(x - \frac{x^3}{6} + \frac{x^5}{5!} - \cdots\right) = -\frac{x^3}{3} + \frac{x^5}{30} + \cdots.$$
Dividing by x^3, the series for $(x\cos x - \sin x)/x^3$ is
$$-\frac{1}{3} + \frac{x^2}{30} + \cdots.$$
Thus,
$$\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} = \lim_{x\to 0} -\frac{1}{3} + \frac{x^2}{30} + \cdots = -\frac{1}{3}.$$