Math 325-001 **Assignment # 5** **11 Feb 09**

Due: Feb 18th

1. For each of the following sequences, find the limit and prove your answer is correct using the definition:

$$\text{a)}\ \lim_{n\to\infty} \frac{1}{n^{1/3}}, \qquad \text{b)}\ \lim_{n\to\infty} \frac{n}{n^2+1}, \qquad \text{c)}\ \lim_{n\to\infty} \frac{2n+4}{5n+2}.$$

2. Suppose that the sequences (s_n) and (t_n) satisfy $\lim_{n\to\infty} s_n = 0$ and there is $M > 0$ so that for all $n \in \mathbf{N}$, $|t_n| \leq M$. Prove that $\lim_{n\to\infty} s_n t_n = 0$.

3. Let (s_n) be a sequence of real numbers so that $\lim_{n\to\infty} s_n = s \in \mathbb{R}$ and let $a \in \mathbb{R}$. Prove the following:
   (a) If, for all but finitely many $n \in \mathbf{N}$, $s_n \geq a$, then $s \geq a$.
   (b) If $s > a$, then for all but finitely many $n \in \mathbf{N}$, $s_n \geq a$.
   (c) Give an example of a sequence (s_n) and s as above, along with a number a, so that $s \geq a$ and there are infinitely many $n \in \mathbf{N}$ with $s_n < a$.

4. Suppose that there is N_0 so that for all $n \geq N_0$, $s_n \leq t_n$.
   (a) Prove that if $\lim_{n\to\infty} s_n = +\infty$ then $\lim_{n\to\infty} t_n = +\infty$.
   (b) Prove that if $\lim_{n\to\infty} t_n = -\infty$ then $\lim_{n\to\infty} s_n = -\infty$.
   (c) Prove the if $\lim_{n\to\infty} s_n = s \in \mathbb{R} \cup \{\pm\infty\}$ and $\lim_{n\to\infty} t_n = t \in \mathbb{R} \cup \{\pm\infty\}$, then $s \leq t$.

5. Show that if the sequence (s_n) satisfies $|s_n - s_{n+1}| < 2^{-n}$, then (s_n) is Cauchy and so converges.

6. Extra Credit: Fix two real numbers a and b. Define a sequence (x_n) by $x_1 = a$, $x_2 = b$ and $x_n = (x_{n-1} + x_{n-2})/2$ for $n \geq 2$. Find $\lim_{n\to\infty} x_n$.