Math 106-350/550 Answers to Homework #2 1 Sept '06

. 5 1. Evaluate $\lim_{s\to 0} \dfrac{s \sin 2s \cot 4s}{\tan 3s}$, being sure to say which limit laws or other results you are using.

Solution. Using the definitions of tangent and cotangent in terms of sine and cosine, we have

$$\begin{aligned}
\frac{s \sin 2s \cot 4s}{\tan 3s} &= \frac{s \sin 2s \frac{\cos 4s}{\sin 4s}}{\frac{\sin 3s}{\cos 3s}} \\
&= \frac{s \sin 2s \cos 4s \cos 3s}{\sin 4s \sin 3s}
\end{aligned}$$

To write this expression using in terms of $\sin\theta/\theta$, we introduce $2s$, $3s$, and $4s$, giving

$$= s \cos 4s \cos 3s \frac{\sin 2s}{2s} \frac{4s}{\sin 4s} \frac{3s}{\sin 3s} \frac{2s}{(4s)(3s)}$$

and then cancelling out the common factors gives

$$= \frac{\cos 4s \cos 3s}{6} \frac{\sin 2s}{2s} \frac{4s}{\sin 4s} \frac{3s}{\sin 3s}$$

Using this equation and the limit laws, we have

$$\begin{aligned}
\lim_{s\to 0} \frac{\sin 2s \cot 4s}{\tan 3s} &= \lim_{s\to 0} \frac{\cos 4s \cos 3s}{6} \frac{\sin 2s}{2s} \frac{4s}{\sin 4s} \frac{3s}{\sin 3s} \\
&= \lim_{s\to 0} \frac{\cos 4s \cos 3s}{6} \lim_{s\to 0} \frac{\sin 2s}{2s} \lim_{s\to 0} \frac{4s}{\sin 4s} \lim_{s\to 0} \frac{3s}{\sin 3s} \\
&= \frac{1 \cdot 1}{6} \cdot 1 \cdot 1 \cdot 1 = \frac{1}{6}.
\end{aligned}$$

. 5 2. Find the equations of all horizontal and vertical asymptotes of $y = \dfrac{2x^2+3}{x^2-4}$. You should justify your answer using algebra and limits.

Draw a clear sketch of the graph of this function and mark the asymptotes.

Solution. To find the horizontal asymptotes, we compute

$$\lim_{x\to+\infty} \frac{2x^2+3}{x^2-4} = \lim_{x\to+\infty} \frac{2+\frac{3}{x^2}}{1-\frac{4}{x^2}} = 2.$$

A very similar calculation also shows that $\lim_{x\to-\infty} \dfrac{2x^2+3}{x^2-4} = 2$.

To find the vertical asymptotes, we look at the points where the denominator is zero. As $0 = x^2 - 4 = (x+2)(x-2)$, the two possible asymptotes are $x = -2$ and $x = 2$. To see if these are asymptotes we look at the one-sided limits at these points.

If we let $x = 2$ in $(2x^2+3)/(x^2-4)$ we get $11/0$ so there is an asymptote at this point. For x a bit bigger than 2, the numerator and and denominator are both positive. For x a bit less than 2, the numerator is still positive but the denominator is negative.

$$\lim_{x\to 2^+} \frac{2x^2+3}{x^2-4} = +\infty, \qquad \lim_{x\to 2^-} \frac{2x^2+3}{x^2-4} = -\infty.$$

If we let $x = -2$ in $(2x^2+3)/(x^2-4)$ we get $11/0$ so there is an asymptote at $x = -2$. For x a bit bigger than -2, the numerator is positive and the denominator is negative. For x a bit less than -2, the numerator and and denominator are both positive.

$$\lim_{x\to -2^+} \frac{2x^2+3}{x^2-4} = -\infty, \qquad \lim_{x\to -2^+} \frac{2x^2+3}{x^2-4} = +\infty.$$

To complete the sketch, we look for intercepts. If $x = 0$, $y = -3/4$ and if $y = 0$ then $2x^2+3 = 0$, which has no solutions and so there are no x-axis intercepts. The sketch of the graph is given in Figure 2.