Math 104-004 — Answers to Test # 2b — 15 Oct 07

. 24 1. Find $f'(x)$ for the following functions. (You need not simplify your answers.)

(a) $f(x) = \dfrac{(4x+1)^4}{3x^2-1}$

(b) $f(x) = (x^2+4)^4 e^{5x^2+1}$

(c) $f(x) = e^{-x^3+x} + \ln(3e^{x^2} - 2x)$

Solution. For part (a), we use the quotient rule to get

$$f'(x) = \frac{(3x^2-1)4(4x+1)^3 \cdot 4 - (4x+1)^4 6x}{(3x^2-1)^2}.$$

For part (b), we use the product rule and then the chain rule to get

$$f(x) = 4(x^2+4)^3 \cdot 2x \cdot e^{5x^2+4} + (x^2+1)^4 \cdot e^{5x^2+1} \cdot 10x.$$

For part (c), we use the rules for $e^{h(x)}$ and $\ln h(x)$ to get

$$f'(x) = e^{-x^3+x}(-3x^2+1) + \frac{3e^{x^2}(2x) - 2}{3e^{x^2} - 2x}.$$

. 10 2. Find the equation of the tangent line to the graph of the curve $y = f(x) = (3x^2 - 10)^3$ at $x = 2$.

Solution. First, we find $f(2) = (3 \cdot 2^2 - 10)^3 = 2^3 = 8$ and then $f'(x) = 3(3x^2 - 9)^2(3x)$ so $f'(1) = 3(3 \cdot 2^2 - 10)^2(6) = 72$. Thus, the equation of the tangent line is

$$y - 8 = 72(x - 2).$$

. 10 3. Cesium-137 has a half-life of 30.07 years. How much of a 40 gram mass is left after 50 years?

Solution. Let $C(t)$ be the amount of Cesium, in grams, after t years. Then we know that $C(0) = 40$, $C(30.07) = 20$ and we have to find $C(50)$.

The general formula for $C(t)$ is Ae^{kt} and to find A we notice that $40 = C(0) = Ae^{k\cdot 0} = A$. To find k, we solve

$$\begin{aligned} 20 &= C(2.065) = 40e^{k\cdot 30.07} \\ \frac{1}{2} &= e^{k\cdot 30.07} \\ \ln\left(\frac{1}{2}\right) &= k \cdot 30.07 \\ k = \frac{\ln\left(\frac{1}{2}\right)}{30.07} &= -.0231 \end{aligned}$$

Thus, $C(50) = 40e^{-.0231\cdot 50} = 12.63$ grams.

. 12 4. Solve the following equations for x:

(a) $3e^{2x+10} = 18$

(b) $27^{x-1} = 3^{4x}$

Solution. For part (b), we have

$$\begin{aligned} e^{2x+10} &= \frac{18}{3} = 6 \\ \ln(e^{2x+10}) &= \ln 6 \\ 2x + 10 &= \ln 6 \\ 2x &= (\ln 6) - 10 \\ x &= \frac{(\ln 6) - 10}{2} \end{aligned}$$

For part (a), we have

$$\begin{aligned} (2^4)^{-x+1} &= (2^3)^{2x} \\ 2^{4(-x+1)} &= 2^{6x} \\ 4(-x+1) &= 6x \\ -4x + 4 &= 6x \\ 4 &= 10x, \end{aligned}$$

and so $x = 4/10$.

. 8 5. I need to have \$50,000 in 21 years to pay my daughter's college tuition. How much must I invest now in an account paying 8% compounded annually to have the required amound in 21 years?

Solution. Using the equation

$$P = A\left(1 + \frac{r}{m}\right)^{mt}$$

with $P = 50000$, $r = .08$, $m = 1$, and $t = 21$, we have $50,000 = A(1.09)^{21}$ and so $A = 50,000/(1.09)^{21}$, which gives \$9,932.79 as the required amount to invest now.

. 18 6. Let $f(x) = \frac{1}{4}x^4 - 3x^3 + 9x^2 + 2$, for all numbers x.

(a) Find all critical numbers of $f(x)$.

(b) Chart $f'(x)$ on a number line.

(c) List the open intervals on which $f(x)$ is an increasing function.

(d) List the open intervals on which $f(x)$ is an decreasing function.

Solution. Notice that

$$\begin{aligned} f'(x) &= x^3 - 9x^2 + 18x \\ &= x(x^2 - 9x + 18) \\ &= x(x-3)(x-6) \end{aligned}$$

Since $f'(x)$ is never undefined, the only critical numbers are where $f'(x) = 0$, that is, 0, 3, and 6.

Since $f(x)$ is never undefined, the only points where we have to split the number line are 0, 3, and 6. For the interval $(-\infty, 0)$, we pick $x = -1$ and get $f'(-1) = (-1)(-4)(-7) < 0$. For the interval $(0, 3)$, we pick $x = 1$ and get $f'(1) = 1 \cdot (-2)(-5) > 0$. For the interval $(3, 6)$, we pick $x = 4$ and get $f'(1) = 4 \cdot 1 \cdot (-2) < 0$. For the interval $(6, +\infty)$, we pick $x = 7$ and get $f'(7) = 7 \cdot 4 \cdot 1 > 0$.

Thus, $f(x)$ is increasing on $(0, 3)$ and $(6, +\infty)$ and it is decreasing on $(-\infty, 0)$ and on $(3, 6)$.

. 6 7. If $\log_b 2 = a$ and $\log_b 7 = c$, express $\log_b 98$ in terms of a and c?

Solution. Since $98 = 2 \cdot 7^2$, by the properties of logarithms,

$$\begin{aligned} \log_b 98 &= \log_b(2 \cdot 7^2) \\ &= \log_b 2 + \log_b 7^2 \\ &= \log_b 2 + 2\log_b 5 = a + 2c. \end{aligned}$$

. 12 8. How long does it take (in years) for \$30,000 to double in value, if it is invested in an account paying 8% compounded quarterly?

Solution. Using the equation

$$P = A\left(1 + \frac{r}{m}\right)^{mt}$$

with $A = 30000$, $P = 60000$, $r = .08$, and $m = 4$, we have

$$\begin{aligned} 60000 &= 30000(1.02)^{4t} \\ 2 &= (1.02)^{4t} \\ \ln 2 &= \ln(1.02)^{4t} \\ \ln 2 &= 4t \ln 1.02 \\ t &= \frac{\ln 2}{4 \ln 1.02} = 8.7506 \end{aligned}$$

Thus, it will take 8.7506 years or, to be precise, 9 years. (After 8 years and 3 quarters, the amount is \$59,996.69.)