Math 106-350/550 **Answers to Homework #9** **10 Nov '06**

. 10 1. Consider the function $f(x) = 3x^2$ on the interval $[0, 2]$.

(a) Find the Riemann sum for this function using a partition of $[0, 2]$ into n equal subintervals and the righthand rule.

(b) Find the Riemann sum for this function using a partition of $[0, 2]$ into n equal subintervals and the lefthand rule.

(c) The the limits of these two Riemann sums as n goes to infintiy.

Hint: Look carefully at Example 5 on page 328.

Solution. For both (a) and (b), we divide the interval $[0, 2]$ into n intervals

$$\left[0, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{4}{n}\right] \left[\frac{4}{n}, \frac{6}{n}\right], \ldots, \left[\frac{2n-2}{n}, \frac{2n}{n}\right].$$

So, for each "rectangle" we have a base of $2/n$.

For part (a), we are evaluating the function at righthand endpoint of each interval, i.e., $2/n$ on $[0, 2/n]$, $4/n$ on $[2/n, 4/n]$, and so on. Thus, the formula for the kth term, which is for the interval $[2(k-1)/n, 2k/n]$, is

$$f\left(\frac{2k}{n}\right) \cdot \frac{2}{n} = 3\frac{4k^2}{n^2}\frac{2}{n} = 24\frac{k^2}{n^3}.$$

The Riemann sum then is

$$\begin{aligned}
\sum_{k=1}^{n} f\left(\frac{2k}{n}\right) \cdot \frac{2}{n} &= \sum_{k=1}^{n} 24\frac{k^2}{n^3} \\
&= \frac{24}{n^3}\sum_{k=1}^{n} k^2 \\
&= \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{4(2n^2+3n+1)}{n^2} = 8 + \frac{12}{n} + \frac{4}{n^2}
\end{aligned}$$

For part (b), we are evaluating the function at lefthand endpoint of each interval, i.e., 0 on $[0, 2/n]$, $2/n$ on $[2/n, 4/n]$, and so on Thus, the formula for the kth term, which is for the interval $[2(k-1)/n, 2k/n]$, is

$$f\left(\frac{2k-2}{n}\right) \cdot \frac{2}{n} = 3\frac{4(k-1)^2}{n^2}\frac{2}{n} = 24\frac{(k-1)^2}{n^3}.$$

The Riemann sum then is

$$
\begin{aligned}
\sum_{k=1}^{n} f\left(\frac{2k-2}{n}\right) \cdot \frac{2}{n} &= \sum_{k=1}^{n} 24\frac{(k-1)^2}{n^3} \\
&= \frac{24}{n^3}\sum_{k=1}^{n}(k^2-2k+1) \\
&= \frac{24}{n^3}\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right) \\
&= \frac{24}{n^3}\left(\frac{2n^3+3n^2+n-6(n^2+n)+6n}{6}\right) \\
&= \frac{4(2n^3-3n^2+n)}{n^3} \\
&= \frac{4(2n^2-3n+1)}{n^2} = 8 - \frac{12}{n} + \frac{4}{n^2}
\end{aligned}
$$

For part (c), we have for the righthand rule,

$$
\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{2k}{n}\right) \cdot \frac{2}{n} = \lim_{n\to\infty} 8 + \frac{12}{n} + \frac{4}{n^2} = 8.
$$

and for the lefthand rule,

$$
\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{2k-2}{n}\right) \cdot \frac{2}{n} = \lim_{n\to\infty} 8 - \frac{12}{n} + \frac{4}{n^2} = 8.
$$