Cartan MASAs and Exact Sequences of Inverse Semigroups

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Cartan MASAs

Let $\mathcal{M}$ be a von Neumann algebra. A maximal abelian subalgebra (MASA) $\mathcal{D}$ in $\mathcal{M}$ is a *Cartan MASA* if

1. the unitaries $U \in \mathcal{M}$ such that $U\mathcal{D}U^* = U^*\mathcal{D}U = \mathcal{D}$ span a weak-* dense subset in $\mathcal{M}$;
2. there is a normal, faithful conditional expectation $E\colon \mathcal{M} \to \mathcal{D}$.

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1. the partial isometries $V \in \mathcal{M}$ such that $V\mathcal{D}V^*$, $V^*\mathcal{D}V \subseteq \mathcal{D}$ span a weak-* dense subset in $\mathcal{M}$;
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We will call the pair $(\mathcal{M}, \mathcal{D})$ a *Cartan pair*. We call the normalizing partial isometries *groupoid normalizers*, written $\mathcal{G}_{\mathcal{M}}(\mathcal{D})$.

Examples of Cartan Pairs

Example

Let M_n be the $n \times n$ complex matrices, and let D_n be the diagonal $n \times n$ matrices. Then (M_n, D_n) is a Cartan pair:

1. the matrix units normalize D_n and generate M_n;
2. The map
$$E\colon [a_{ij}] \mapsto \mathrm{diag}[a_{11}, \dots, a_{nn}]$$
gives a faithful normal conditional expectation.

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Example

Let $\mathcal{D} = L^\infty(\mathbb{T})$ and let α be an action of $\mathbb{Z}$ on $\mathbb{T}$ by irrational rotation. Then $L^\infty(\mathbb{T})$ is a Cartan MASA in $L^\infty(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$.

Examples of Cartan Pairs

Example

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\},$$

and let

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$

Then H is a normal subgroup of G and $L(H)$ is Cartan MASA in $L(G)$.

Feldman & Moore approach

Feldman and Moore (1977) explored Cartan pairs $(\mathcal{M}, \mathcal{D})$ where $\mathcal{M}_*$ is separable and $\mathcal{D} = L^\infty(X, \mu)$. They showed:

1. there is a measurable equivalence relation R on X with countable equivalence classes and a 2-cocycle σ on R s.t.

$$\mathcal{M} \simeq \mathbf{M}(R, \sigma) \text{ and } \mathcal{D} \simeq \mathbf{A}(R, \sigma),$$

   where $\mathbf{M}(R, \sigma)$ are "functions on R" and $\mathbf{A}(R, \sigma)$ are the "functions" supported on diag. $\{(x, x) : x \in X\}$;
2. every sep. acting pair $(\mathcal{M}, \mathcal{D})$ arises this way.

A simple example

Consider the Cartan pair (M_3, D_3). Let $\mathcal{G} = \mathcal{G}_{M_3}(D_3)$. E.g., an element of $\mathcal{G}$ could look like

$$V = \begin{bmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & \gamma \end{bmatrix},$$

with $\lambda, \mu, \gamma \in \mathbb{T}$.
Let $\mathcal{P} = \mathcal{G} \cap D_n$. And let $\mathcal{S} = \mathcal{G}/\mathcal{P}$. So elements of $\mathcal{S}$ are of the form

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From (M_n, D_n) we have 3 semigroups: $\mathcal{P}$, $\mathcal{G}$ and $\mathcal{S}$.

A simple example: continued

Conversely, starting with S, we can construct P: P is all the continuous functions from the idempotents of S into $\mathbb{T}$. From S and P we can construct G, since every element of G is the product of an element in S and an element in P. From G we can construct (M_n, D_n) as the span of G.

Our Objective: Give an alternative approach using algebraic rather than measure theoretic tools which

- conceptually simpler;
- applies to the non-separably acting case.

Inverse Semigroups

A semigroup S is an *inverse semigroup* if for each $s \in S$ there is a unique "inverse" element $s^\dagger$ such that

$$ss^\dagger s = s \text{ and } s^\dagger s s^\dagger = s^\dagger.$$

We denote the idempotents in an inverse semigroup S by $\mathcal{E}(S)$. The idempotents form an abelian semigroup. For any element $s \in S$, $ss^\dagger \in \mathcal{E}(S)$.

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An inverse semigroup S has a natural partial order defined by

$$s \leq t \text{ if and only if } s = te$$

for some idempotent $e \in \mathcal{E}(S)$.

Matrix example

Example

Consider the Cartan pair (M_n, D_n) again. Again, let

$$\begin{aligned} G &= \mathcal{G}_{M_n}(D_n) \\ &= \{\text{partial isometries } V \in M_n \colon VD_nV^* \subseteq D_n,\ V^*D_nV \subseteq D_n\}. \end{aligned}$$

Then G is an inverse semigroup:

- if $V, W \in G$ then

  $$(VW)D_n(VW)^* = V(WD_nW^*)V^* \subseteq D_n,$$

  so $VW \in G$;

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  so $VW \in G$;
- the "inverse" of V is V^*;
- the idempotents are the projections in D_n;
- $V \leq W$ if $V = WP$ for some projection $P \in D_n$.

Bigger Matrix example

More generally...

Example

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair. Then the groupoid normalizers $\mathcal{G}_M(\mathcal{D})$ form an inverse semigroup.

- if $V, W \in \mathcal{G}_M(\mathcal{D})$ then

$$(VW)\mathcal{D}(VW)^* = V(W\mathcal{D}W^*)V^* \subseteq \mathcal{D},$$

  so $VW \in \mathcal{G}_M(\mathcal{D})$;
- the "inverse" of V is V^*;
- the idempotents are the projections in $\mathcal{D}$;
- $V \leq W$ if $V = WP$ for some projection $P \in D$.

Extensions of Inverse Semigroups

Let S and P be inverse semigroups. And let

$$\pi\colon P \to S,$$

be a surjective homomorphism such that $\pi|_{\mathcal{E}(P)}$ is an isomorphism from $\mathcal{E}(P)$ to $\mathcal{E}(S)$.
An *idempotent separating extension of* S *by* P is an inverse semigroup G with

$$P \xhookrightarrow{\iota} G \xrightarrow{q} S$$

and

- ι is an injective homomorphism;
- q is a surjective homomorphism;
- $q(g) \in \mathcal{E}(S)$ if and only if $g = \iota(p)$ for some $p \in P$;
- $q \circ \iota = \pi$.

Note that $\mathcal{E}(P) \cong \mathcal{E}(G) \cong \mathcal{E}(S)$.

The Munn Congruence

Let G be an inverse semigroup. Define an equivalence relation (*the Munn congruence*) $\sim$ on G by

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Thus $S = G/\sim$ is an inverse semigroup.
Let $P = \{v \in G \colon v \sim e \text{ for some } e \in \mathcal{E}(G)\}$. Then P is an inverse semigroup.
And G is an extension of S by P:

$$P \hookrightarrow G \to S.$$

From Cartan Pairs to Extensions of Inverse Semigroups

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair. Let

$$\begin{aligned} G &= \mathcal{G}_{\mathcal{M}}(\mathcal{D}) \\ &= \{v \in \mathcal{M} \text{ a partial isometry} \colon v\mathcal{D}v^* \subseteq \mathcal{D} \text{ and } v^*\mathcal{D}v \subseteq \mathcal{D}\}. \end{aligned}$$

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Let $S = G/\sim$, where $\sim$ is the Munn congruence on G and let

$$P = \{V \in G \colon V \sim P,\ P \in \mathrm{Proj}(\mathcal{D})\}.$$

Definition

We call the extension

$$P \hookrightarrow G \to S,$$

the *extension associated to the Cartan pair* $(\mathcal{M}, \mathcal{D})$.

Properties of associated extensions

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair, and let

$$P \hookrightarrow G \to S,$$

be the associated extension.
Then $P = \mathcal{G}_{\mathcal{M}}(\mathcal{D}) \cap \mathcal{D}$, i.e. P is simply the partial isometries in $\mathcal{D}$.

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The inverse semigroup S has the following properties

1. S is *fundamental*: $\mathcal{E}(S)$ is maximal abelian in S;
2. $\mathcal{E}(S)$ is a hyperstonean boolean algebra, i.e. the idempotents are the projection lattice of an abelian W^*-algebra;
3. S is a meet semilattice under the natural partial order on S;
4. for every pairwise orthogonal family $\mathcal{F} \subseteq S$, $\bigvee \mathcal{F}$ exists in S.
5. S contains 1 and 0.

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5. S contains 1 and 0.

Definition

An inverse semigroup S, satisfying the conditions above is called a *Cartan inverse monoid*.

Matrix example

Example

In the matrix example (M_n, D_n), the semigroups $P,\ G$ and S are the semigroups discussed earlier:

1. G is the partial isometries V such that $VD_nV^*,\ V^*D_nV \subseteq D_n$;
2. P is the partial isometries in D_n;
3. S is the matrices in G with only 0 and 1 entries.

Equivalent Extensions of Cartan Inverse monoid

Let $\alpha\colon S_1 \to S_2$ be an isomorphism of Cartan inverse monoids. Then $\mathcal{E}(S_i)$ is the lattice of projections for a W^*-algebra, $\mathcal{D}_i = C(\widehat{\mathcal{E}(S_i)})$. The isomorphism α induces an isomorphism $\widetilde{\alpha}$ from $\mathcal{D}_1$ to $\mathcal{D}_2$.

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Definition

Let S_1 and S_2 be isomorphic Cartan inverse monoids. Let P_i be the partial isometries in $\mathcal{D}_i$. Extensions G_i of S_i by P_i are *equivalent* if there is an isomorphism $\underline{\alpha}\colon G_1 \to G_2$ such that

$$\begin{array}{ccccc}
P_1 & \xrightarrow{\iota_1} & G_1 & \xrightarrow{q_1} & S_1 \\
\widetilde{\alpha}\big\downarrow & & \underline{\alpha}\big\downarrow & & \alpha\big\downarrow \\
P_2 & \xrightarrow{\iota_2} & G_2 & \xrightarrow{q_2} & S_2.
\end{array}$$

commutes.

More on Extensions of Inverse Monoids

It was shown by Laush (1975) that there is one-to-one correspondence between extensions of S by P and the second cohomology group $H^2(S, P)$.
It is also shown that every extension of S by P is determined by cocycle function $\sigma\colon S \times S \to P$.

Uniqueness of Extension

Theorem

Let $(\mathcal{M}_1, \mathcal{D}_1)$ *and* $(M_2, \mathcal{D}_2)$ *be two Cartan pairs with associated extensions*

$$P_i \hookrightarrow G_i \to S_i$$

for $i = 1, 2$.
There is a normal isomorphism $\theta \colon \mathcal{M}_1 \to \mathcal{M}_2$ *such* $\theta(\mathcal{D}_1) = \mathcal{D}_2$ *if and only if the two associated extensions are equivalent.*

Going in the other direction

Let S be a Cartan inverse monoid. Let $\mathcal{D} = C(\widehat{\mathcal{E}(S)})$, and let P be the partial isometries in $\mathcal{D}$. Given an extension

$$P \hookrightarrow G \to S$$

we want to construct a Cartan pair $(\mathcal{M}, \mathcal{D})$ with associated extension (equivalent to) $P \hookrightarrow G \to S$.

A $\mathcal{D}$-valued Reproducing kernel space

Let j be an order-preserving map, $j\colon S \to G$ such that $j \circ q = \mathrm{id}$. That is $j(s) \leq j(t)$ when $s \leq t$ and $j\colon \mathcal{E}(S) \to \mathcal{E}(G)$ is an isomorphism.

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Define a map

$$K\colon S \times S \to \mathcal{D}$$

by $K(s,t) = j(s^\dagger t \wedge 1)$.

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The idempotent $s^\dagger t \wedge 1$ is the minimal idempotent e such that

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Thus $K(s,t)$ is the idempotent in G defining $j(s) \wedge j(t)$.

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Thus $K(s,t)$ is the idempotent in G defining $j(s) \wedge j(t)$.
The map K is positive: that is for $c_1, \ldots, c_k \in \mathbb{C}$ and $s_1, \ldots, s_k \in S$

$$\sum_{i,j} \overline{c_i} c_j K(s_i, s_j) \geq 0.$$

A $\mathcal{D}$-valued Reproducing kernel space

For each $s \in S$ define a "kernel-map" $k_s \colon S \to \mathcal{D}$ by

$$k_s(t) = K(t, s).$$

Let $\mathfrak{A}_0 = \mathrm{span}\{k_s \colon s \in S\}$. The positivity of K shows that the

$$\langle \sum c_i k_{s_i}, \sum d_j k_{t_j} \rangle = \sum_{i,j} \overline{c_i} d_j K(s_i, t_j)$$

defines a $\mathcal{D}$-valued inner product on $\mathfrak{A}_0$. Let $\mathfrak{A}$ be completion of $\mathfrak{A}_0$.
Thus $\mathfrak{A}$ is a reproducing kernel Hilbert $\mathcal{D}$-module of functions from S into $\mathcal{D}$.

A left representation of G

For $g \in G$ define an adjointable operator $\lambda(g)$ on $\mathfrak{A}$ by

$$\lambda(g)k_s = k_{q(g)s}\sigma(g, s),$$

where $\sigma\colon G \times S \to P$ is a "cocycle-like" function (related to the cocycles of Lausch). This is determined by the equation

$$gj(s) = j(q(g)s)\sigma(g, s),$$

i.e. elements of the form $gj(s)$ can be factored into the product of an element in $j(S)$ by an element in P.

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i.e. elements of the form $gj(s)$ can be factored into the product of an element in $j(S)$ by an element in P. The mapping

$$\lambda\colon G \to L(\mathfrak{A})$$

is a representation of G by partial isometries.

A left representation of G on a Hilbert space

Let π be a faithful representation of $\mathcal{D}$ on a Hilbert space $\mathcal{H}$. We can form a Hilbert space $\mathfrak{A} \otimes_\pi \mathcal{H}$ by completing $\mathfrak{A} \otimes \mathcal{H}$ with respect to the inner product

$$\langle a \otimes h, b \otimes k \rangle := \langle h, \pi(\langle a, b \rangle) k \rangle.$$

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Then π determines a faithful representation $\hat{\pi}$ of $L(\mathfrak{A})$ on the Hilbert space $\mathfrak{A} \otimes_\pi \mathcal{H}$ by

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Thus, we have a faithful representation of G on the hilbert space $\mathfrak{A} \otimes_\pi \mathcal{H}$ by

$$\lambda_\pi \colon g \mapsto \hat{\pi}(\lambda(g)).$$

Creating Cartan pairs

Let $\mathcal{M}_q = \lambda(G)''$, and $\mathcal{D}_q = \lambda(\mathcal{E}(S))''$. Then $(\mathcal{M}_q, \mathcal{D}_q)$ is a Cartan pair such that

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3. The conditional expectation $E\colon \mathcal{M}_q \to \mathcal{D}_q$ is induced from the map

$$\begin{aligned} S &\to \mathcal{E}(S) \\ s &\mapsto s \wedge 1. \end{aligned}$$

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$$\begin{aligned} S &\to \mathcal{E}(S) \\ s &\mapsto s \wedge 1. \end{aligned}$$
4. The extension associated to $(\mathcal{M}_q, \mathcal{D}_q)$ is equivalent to
$$P \hookrightarrow G \xrightarrow{q} S$$
(the extension we started with).

Main Theorem

Theorem (Feldman-Moore; Donsig-F-Pitts)

- *If S is a Cartan inverse monoid and $P \hookrightarrow G \xrightarrow{q} S$ is an extension of S by $P := p.i.(C^*(\mathcal{E}(S))$, then the extension determines a Cartan pair $(\mathcal{M}, \mathcal{D})$ which is unique up to isomorphism. Equivalent extensions determine isomorphic Cartan pairs.*
- *Every Cartan pair $(\mathcal{M}, \mathcal{D})$ determines uniquely an extension of a Cartan inverse semigroup S by P, $P \hookrightarrow G \xrightarrow{q} S$.*