Further analysis of the Cartan abelian core

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Plan

I. Introduction

II. Graph and k-graph algebras

III. Uniqueness theorems

IV. Cartan subalgebras

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- $\mathscr{M}$ is in certain cases a Cartan subalgebra.

Goal Analyze the pair $(C^*(\mathscr{G}), \mathscr{M})$ in the context of Renault's theory of Cartan inclusions.

Graph Algebras

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CK1	$T_e^* T_e = T_v,$	where $v = s(e)$.
CK2	$\displaystyle\sum_{r(e)=w} T_e T_e^* = T_w$	(assuming $0 < \lvert r^{-1}(w) \rvert < \infty$).

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Q: When is the natural map $\pi : C^*(\Lambda) \to C^*(T_\lambda)$ injective?

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$$\begin{tikzcd} \bullet \arrow[r, "e"] & \bullet \arrow[loop right, "f"] \end{tikzcd}$$

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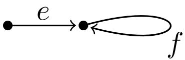

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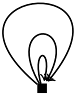

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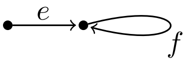

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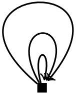

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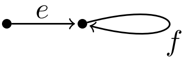

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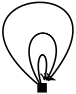

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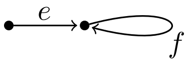

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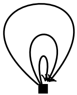

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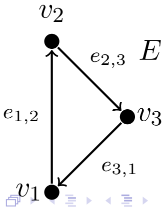
Example where uniqueness fails:
$T_{v_i} = \varepsilon_{ii}, \;\; T_{e_{i,j}} = \varepsilon_{ji}$ (matrix units)
$C^*(T_\lambda) = M_3(\mathbb{C}) \not\cong C(\mathbb{T}, M_3(\mathbb{C})) = C^*(E)$.
The graph has a *cycle without entry*.



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Uniqueness theorems for other combinatorial algebras:
Inverse semigroups (LaLonde, Milan), Steinberg algebras (Clark-Exel-Pardo), ultragraph algebras (Gonçalves, Li, Royer).

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1986 Kumjian: notion of C^*-diagonal subalgebra pair arising from a twisted equivalence relation.

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$$\mathcal{G}_E = \{(\overset{r}{\alpha} y, m, \overset{s}{\beta} y) \mid y \in E^\infty,\ \alpha, \beta \in E^*,\ m = d(\alpha) - d(\beta)\}$$

$$(x, m, y)(y, m', z) = (x, m + m', z) \quad (x, m, y)^{-1} = (y, -m, x)$$

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Basis for topology:

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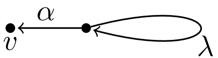

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Brown, Li, Yang: Concrete necessary and sufficient conditions on a k-graph for $\mathrm{Iso}(\mathcal{G})^\circ$ to be closed. It's not always!

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\bullet & \bullet \arrow[l, "\alpha"]
\end{tikzcd}
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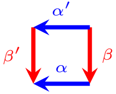

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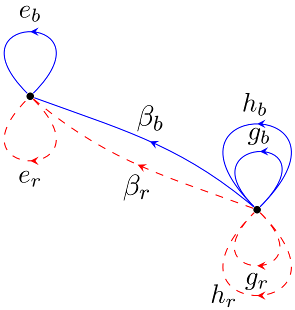

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$$e_b\beta_r = e_r\beta_b$$

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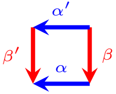

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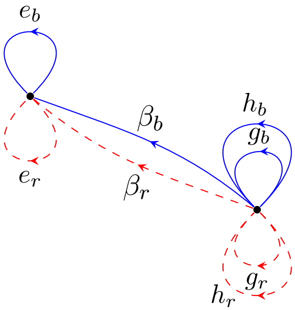

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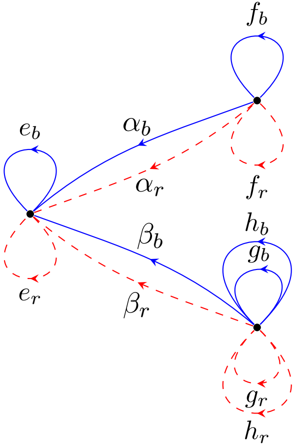

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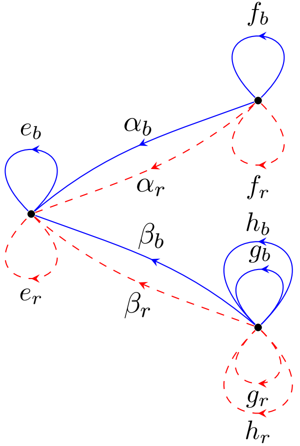

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Here $(e_r(e_be_r)^\infty, (1,-1), e_b(e_be_r)^\infty) \in \overline{\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ} \setminus \mathrm{Iso}(\mathcal{G})^\circ$.

Thank you!

A groupoid twist is a groupoid extension

$$\mathbb{T} \times \mathcal{G}^{(0)} \leftarrowtail \Sigma \twoheadrightarrow \mathcal{G}$$

defined via a 2-cocycle $\omega : \mathcal{G}^{(2)} \to \mathbb{T}$ on the set of composable pairs in $\mathcal{G}$, with product topology and $r(z, \gamma) = (1, r(\gamma))$, $s(z, \gamma) = (1, s(\gamma))$, and

$$(s, \eta)(t, \gamma) = (st\omega(\eta, \gamma), \eta\gamma) \quad (z, \eta)^{-1} = (z^{-1}\omega(\eta, \eta^{-1}), \gamma^{-1})$$

For $f, g \in C_c(\mathcal{G}, \Sigma)$, define

$$f * g(\gamma) = \int_{\mathcal{G}} f(\eta)g(\eta^{-1}\gamma)\omega(\eta, \eta^{-1}\gamma)d\lambda^{r(\gamma)}(\eta)$$

and

$$f^*(\gamma) = \overline{f(\gamma^{-1})\omega(\gamma, \gamma^{-1})}$$

Again, $C^*(\mathcal{G}, \Sigma)$ is the completion of $C_c(\mathcal{G}, \Sigma)$ in the usual norm.

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$$t_e = \begin{cases} \chi_{Z_{\mathcal{G}}(e,s(e))} & \text{if } e \notin E_\circ^1 \\ \sum_{z \in \mathbb{T}} z \chi_{Z_{\mathcal{G}_E}(e,s(e)) \times \{z\}} & \text{if } e \in E_\circ^1 \end{cases}$$

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(i) each $\psi \in \mathcal{S}$ extends uniquely to a state $\tilde{\psi}$ on A, and
(ii) the direct sum $\oplus_{\psi \in S} \pi_{\tilde{\psi}}$ of the GNS representations associated to the extensions to A of elements in S is faithful on A.

Then a $*$-homomorphism $\Phi : A \to B$ is injective iff $\Phi|_M$ is injective.

Our proof of the main theorem applies the AUT to the set S of pure states of $C_r^*(\mathrm{Iso}(\mathcal{G})^\circ)$ that factor through some $C_r^*(\mathcal{G}_u^u)$ with $\mathcal{G}_u^u = \mathrm{Iso}(\mathcal{G})_u^\circ$ (where $\mathcal{G}_u^u = \mathrm{Iso}(\mathcal{G}) \cap r^{-1}(u)$).

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