C^*-Extreme Points of the Generalized State Space of a Commutative C^*-Algebra

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$\mathcal{H}$ - Hilbert Space, $\mathcal{B}(\mathcal{H})$ - bounded linear operators on $\mathcal{H}$

X - compact, Hausdorff

$C(X) = \{f : X \to \mathbb{C} \,|\, f$ is continuous$\}$

Definition 1. The **state space** of $C(X)$ is
$S_{\mathbb{C}}(C(X)) = \{\phi : C(X) \to \mathbb{C} \,|\, \phi(1) = 1,\ \phi \text{ a positive linear map}\}$

Definition 2. The **generalized state space** of $C(X)$ is
$S_{\mathcal{H}}(C(X)) = \{\phi : C(X) \to \mathcal{B}(\mathcal{H}) \,|\, \phi(1) = I, \phi \text{ a positive linear map}\}$

Definition 3. If $s, y_1, \ldots, y_n \in S$ and $t_1, \ldots, t_n \in (0, 1)$ with $t_1 + \cdots + t_n = 1$ then

$$s = t_1 y_1 + \cdots + t_n y_n$$

expresses x as a **convex combination** of $y_1, \ldots y_n$

Definition 4. If $\phi, \psi_1, \ldots, \psi_n \in S_{\mathcal{H}}(C(X))$ and $t_1, \ldots, t_n$ are invertible operators in $\mathcal{B}(\mathcal{H})$ with $t_1^* t_1 + \cdots + t_n^* t_n = I$ then

$$\phi(\cdot) = t_1^* \psi_1(\cdot) t_1 + \cdots + t_n^* \psi_n(\cdot) t_n$$

expresses ϕ as a C^*-**convex combination** of $\psi_1, \ldots, \psi_n$.

Definition 5. $s \in S$ is **extreme** if whenever

$$s = t_1 y_1 + \cdots + t_n y_n$$

where $t_j \in (0,1)$ and $y_j \in S$, then

$$s = y_j \quad \forall j$$

Definition 6. $\phi \in S_{\mathcal{H}}(C(X))$ is C^*-**extreme** if whenever

$$\phi = t_1^* \psi_1 t_1 + ... + t_n^* \psi_n t_n$$

where $\psi_j \in S_{\mathcal{H}}(C(X))$ and $t_j \in \mathcal{B}(\mathcal{H})$ are invertible with $t_1^* t_1 + \cdots + t_n^* t_n = I$, then

$$\psi_j \sim \phi \quad \forall j$$

Other non-commutative convexity

- matrix convexity
  (Wittstock, Effros-Winkler, Winkler-Webster)

- CP-convexity (Fujimoto)

CP-states:

$Q_{\mathcal{H}}(\mathcal{A}) = \{\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \mid \phi$ is completely positive and $\|\phi\|_{cb} \leq 1\}$

CP-convex combination

$$\phi = \sum t_i^* \psi_i t_i,$$

$t_i \in \mathcal{B}(\mathcal{H})$ (need not be invertible), $\sum t_i^* t_i \leq I$
sum converges in BS-topology

CP-extreme states of $Q_{\mathcal{H}}(\mathcal{A}) \subsetneq C^*$-extreme states of $Q_{\mathcal{H}}(\mathcal{A})$

Definition 7. *matrix convex set* (Wittstock, 1983)
$K = \{K_n\}_{n\in\mathbb{N}}$, $K_n \subseteq M_n(V)$ convex satisfying:

1. $\alpha \in M_{r,n}$ with $\alpha^*\alpha = 1 \Rightarrow \alpha^* K_r \alpha \subseteq K_n$

2. for $m, n \in \mathbb{N}$, $K_m \oplus K_n \subseteq K_{m+n}$.

Definition 8. (Webster-Winkler, 1999) $v \in K_n$ *matrix extreme point* if whenever

$$v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i$$

$v_i \in K_{n_i}$, $\gamma_i \in M_{n_i,n}$ right invertible, and $\sum \gamma_i^* \gamma_i = I$, then each $n_i = n$ and $v_i \sim v$

matrix extreme points of $S_{\mathbb{C}^n}(\mathcal{A}) \subsetneq C^*$-extreme

Example 9. (Webster, Winkler, 1999)

- $\{S_{\mathbb{C}^n}(\mathcal{A})\}_{n\in\mathbb{N}}$ is a matrix convex set
- (Example 2.3) matrix extreme points of $S_{\mathcal{C}^n}(\mathcal{A})$ = pure maps in $S_{\mathbb{C}^n}(\mathcal{A})$

$S_{\mathbb{C}^n}(C(X))$ contains no matrix extreme points for $n > 1$

(Farenick, Morenz, 1997)
$S_{\mathbb{C}^n}(\mathcal{A})$ is the closed C^*-convex hull of its C^*-extreme points
(closure w.r.t. the bounded weak topology)

- In $S_{\mathbb{C}}(C(X))$ extreme points are multiplicative

- (1969) Arveson characterized extreme points of $S_{\mathcal{H}}(\mathcal{A})$

- structure theorem for extreme points of $S_{\mathbb{C}^n}(C(X))$

- there are non-multiplicative extreme points in $S_{\mathbb{C}^n}(C(X))$

Some known results when $\mathcal{H} = \mathbb{C}^n$ finite dimensional (D. Farenick, P. Morenz, 1997):

- $\phi \in S_{\mathbb{C}^n}(\mathcal{A})$ C^*-extreme
  $\Leftrightarrow$ $\phi \sim \phi_1 \oplus \cdots \oplus \phi_n$, ϕ_j pure maps

- $\phi \in S_{\mathbb{C}^n}(\mathcal{A})$ C^*-extreme $\Rightarrow$ ϕ extreme

- $\phi \in S_{\mathbb{C}^n}(C(X))$
  C^*-extreme $\Leftrightarrow$ ϕ multiplicative

$S_{\mathbb{C}}(C(X))$	extreme	$=$	C^*-extreme	$=$	pure	$=$	mult.
$S_{\mathbb{C}}(\mathcal{A})$	extreme	$=$	C^*-extreme	$=$	pure	$\supsetneqq$	mult.
$S_{\mathbb{C}^n}(C(X))$	extreme	$\supsetneqq$	C^*-extreme	$=$	mult.		
$S_{\mathbb{C}^n}(\mathcal{A})$	extreme	$\supsetneqq$	C^*-extreme	$\supsetneqq$	pure		
			C^*-extreme	$\supsetneqq$	mult.		
$\phi : C(X) \to \mathcal{K}^+$	extreme	$\supsetneqq$	C^*-extreme	$=$	mult.		
$S_{\mathcal{H}}(C(X))$	extreme	?	C^*-extreme	$\supsetneqq$	mult.		
$S_{\mathcal{H}}(\mathcal{A})$	extreme	?	C^*-extreme	$\supsetneqq$	pure		
			C^*-extreme	$\supsetneqq$	mult.		

Recall:
$\phi \in S_{\mathbb{C}}(C(X))$ positive, linear
$\Rightarrow \exists$ a unique positive Borel measure μ s.t.

$$\phi(f) = \int_X f d\mu$$

$\forall f \in C(X)$

Compare: (Paulsen)
$\phi \in S_{\mathcal{H}}(C(X))$ a positive linear map
$\Rightarrow \exists$ a positive operator-valued measure

$$\mu : \text{Borel sets of } X \to \mathcal{B}(\mathcal{H})$$

s.t

$$\int_X f d\mu_\phi = \phi(f)$$

Fix $\phi \in S_{\mathcal{H}}(C(X))$

for each pair of vectors $x, y \in \mathcal{H}$; the map

$$\begin{aligned} C(X) &\to (C) \\ f \quad &\mapsto \langle \phi(f)x, y \rangle \end{aligned}$$

corresponds to $\mu_{x,y}$ on X

$$\int_X f d\mu_{x,y} := \langle \phi(f)x, y \rangle \text{ for any } f \in C(X)$$

B a Borel set of X

$$(x, y) \mapsto \mu_{x,y}(B)$$

is a sesquilinear form

let x, y range over $\mathcal{H}$ determines an operator $\mu(B)$

define *operator-valued measure*

$$\mu : \text{ Borel sets } \longrightarrow \mathcal{B}(\mathcal{H})$$

μ_ϕ is

1. *weakly countably additive*, i.e. $\{B_i\}_{i=1}^{\infty}$ prwise disjt Borel sets,
$$\left\langle \mu \left(\bigcup_{i=1}^{\infty} B_i \right) x, y \right\rangle = \sum_{i=1}^{\infty} \langle \mu(B_i) x, y \rangle$$
for every $x, y \in \mathcal{H}$.

2. *bounded*, i.e. $\|\mu\| := \sup\{\|\mu(B)\| : B \in \mathcal{S}\} < \infty$

3. *regular*, i.e. $\forall x, y \in \mathcal{H}$, $\mu_{x,y}$ is regular, where
$$\mu_{x,y}(B) = \langle \mu_\phi(B) x, y \rangle$$

Proposition 10. (Paulsen, *Completely Bounded Maps*) Given an operator valued measure μ and its associated linear map ϕ,

1. ϕ is self-adjoint if and only if μ is self-adjoint,
2. ϕ is positive if and only if μ is positive,
3. ϕ is a homomorphism if and only if $\mu(B_1 \cap B_2) = \mu(B_1)\mu(B_2)$ for all Borel sets B_1, B_2,
4. ϕ is a $*$-homomorphism if and only if μ is spectral (i.e., projection-valued).

Moreover,

- $\mu_1 \sim \mu_2 \Leftrightarrow \phi_1 \sim \phi_2$

- μ is C^*-extreme $\Leftrightarrow \phi$ is C^*-extreme

- range $\mu_\phi \subseteq$ WOT-cl range ϕ

- $\mu_\phi(F)$ is a projection $\Rightarrow \mu_\phi(F) \in \phi(C(X))'$

Theorem 11. $\phi : C(X) \longrightarrow \mathcal{B}(\mathcal{H})$ a unital, positive map. If ϕ is C^*-extreme, then for every Borel set $F \subset X$, either

(1) $\sigma(\mu_\phi(F)) \subseteq \{0, 1\}$
(i.e. $\mu_\phi(F)$ is a projection), or

(2) $\sigma(\mu_\phi(F)) = [0, 1]$.

Assume $\exists\, F \subseteq X$ with

$$\sigma(\mu_\phi(F)) \subsetneq [0,1]$$

and $\mu_\phi(F)$ not a projection.

Choose an interval (a,b) with

$$(a,b) \cap \sigma(\mu_\phi(F)) = \varnothing$$

Let $Q_k = \frac{1}{2}\mu_\phi(F) + s_k \mu_\phi(F^C)$,

where $\max\left\{\frac{1}{4}, \frac{1}{2}\left(\frac{a-ab}{b-ab}\right)\right\} < s_1 < \frac{1}{2}$ and $s_2 = 1 - s_1$

Construct μ_1, μ_2 from μ_ϕ by:

$$\mu_k(B) = Q_k^{-\frac{1}{2}} \left(\frac{1}{2} \mu_\phi(B \cap F) + s_k \mu_\phi(B \cap F^C) \right) Q_k^{-\frac{1}{2}}$$

μ_k are unital and positive, and

$$\mu_\phi = Q_1^{\frac{1}{2}} \mu_1 Q_1^{\frac{1}{2}} + Q_2^{\frac{1}{2}} \mu_2 Q_2^{\frac{1}{2}}$$

show μ_k and μ_ϕ not unitarily equivalent:

Compute

$$\begin{aligned}\mu_k(F) &= Q_k^{-1/2}\left(\frac{1}{2}\mu_\phi(F)\right)Q_k^{-1/2} \\ &= \frac{1}{2}\mu_\phi(F)\left(s_k I + \left(\frac{1}{2} - s_k\right)\mu_\phi(F)\right)^{-1} \\ &= f_k(\mu_\phi(F)),\end{aligned}$$

f_1 continuous, increasing, concave down on $(0,1)$

$\sigma(\mu_\phi(F))$:

$\sigma(\mu_1(F))$:

Theorem 11 also shows:

$\lambda \in (0,1)$ an eigenvalue of $\mu_\phi(F) \Rightarrow (0,1) \subseteq \sigma_{pt}(\mu_\phi(F))$

Note: $\mathcal{H}$ separable
ϕ C^*-extreme $\Rightarrow \mu_\phi(F)$ has no eigenvalues in $(0,1)$

Corollary 12. (Farenick, Morenz)
$\phi \in S_{\mathcal{H}}(C(X))$ is C^*-extreme $\Leftrightarrow$ it is a $*$-homomorphism.

Corollary 13. $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ a von Neumann algebra, $\phi \in S_{\mathcal{H}}(C(X))$, range of ϕ in $\mathcal{M}$

If ϕ fails to meet the spectral condition described in Theorem 11, then

$$\phi = t_1^* \psi_1 t_1 + t_2^* \psi_2 t_2,$$

where each $t_k \in \mathcal{M}$, each $\psi_k : C(X) \longrightarrow \mathcal{M}$, and, for at least one choice of k, ψ_k is not unitarily equivalent to ϕ in $\mathcal{B}(\mathcal{H})$.

$\mathcal{K}$ - the ideal of compact operators in $\mathcal{B}(\mathcal{H})$

$\mathcal{K}^+$ - the C^*-algebra generated by $\mathcal{K}, I$

Theorem 11 implies:

Theorem 14. $\phi : C(X) \to \mathcal{K}^+$ unital, positive
ϕ is C^*-extreme $\Leftrightarrow$ ϕ is a $*$-homomorphism.

$q : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}$ the usual quotient map

Lemma 15. $\phi : C(X) \to \mathcal{K}^+$ unital, positive, C^*-extreme. Then $\tau = q \circ \phi$ is multiplicative.

$$
\begin{tikzcd}
C(X) \arrow[r, "\phi"] \arrow[dr, dashed, "\tau"'] & \mathcal{K}^+ \arrow[d, "q"] \\
& \mathbb{C}
\end{tikzcd}
$$

ϕ multiplicative $\Rightarrow$ ϕ C^*-extreme (Farenick-Morenz, 1993)

ϕ C^*-extreme $\Rightarrow$ $\tau(f) = f(x_0)$

Choose $x_1 \neq x_0$, $g \in C(X)$ as shown:

Then $\phi(g)$ is compact, and so is $\phi(\chi_{NC})$.

Theorem 11 $\Rightarrow \phi(\chi_{_{NC}})$ is a f.r. projection.

$B \not\ni x_0$ any Borel set, use the regularity of the measures $\mu_{x,x}$ to show $\mu_\phi(B)$ a projection

$\Rightarrow \mu_\phi$ projection valued

Theorem 14 yields:

If $\phi : C(X) \to \mathcal{K}^+$ is C^*-extreme $\Rightarrow$

- supp $\mu_\phi =$ discrete set $+$ one accumulation point x_0

- ϕ has the form

$$\phi(f) = \sum_{x \in \mathrm{supp}(\mu_\phi)} f(x) P_x$$

where $P_x = \mu_\phi(\{x\})$ is a f.r. projection for all $x \neq x_0$

Non-multiplicative C^*-extreme maps exist:

Example 16. (Arveson, 1969, Farenick-Morenz, 1993)

Consider the representation

$$\begin{aligned} \pi : C(\mathbb{T}) &\to \mathcal{B}(L^2(\mathbb{T}, m)) \\ f &\mapsto M_f \end{aligned}$$

Define

$$\begin{aligned} \phi : C(\mathbb{T}) &\to \mathcal{B}(H^2) \\ f &\mapsto PM_fP = T_f \end{aligned}$$

P is the projection of $L^2(\mathbb{T}, m)$ onto the Hardy space H^2.

$\mu_\pi(B) = M_{\chi_B}$, so $\mu_\phi(B) = PM_{\chi_B}P = T_{\chi_B}$

B a nontrivial Borel subset of X
$\sigma(\mu_\phi(B)) = [0, 1]$ (Hartman, Wintner, 1954)

So ϕ satisfies the conditions of Theorem 11.

Example 17. Define

$$\begin{aligned}\psi : C([0,2\pi]) &\to \mathcal{B}(H^2)\\ g \quad &\mapsto \phi(f)\end{aligned}$$

where $g(t) = f(e^{it})$

The End