Free Noncommutative Functions: An Introduction

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[1] A joint work with V. Vinnikov

Noncommutative (nc) multi-operator spectral theory: J. L. Taylor

Free probability: D.-V. Voiculescu...

NC free semi-algebraic geometry: J. W. Helton, M. Putinar, S. McCullough, I. Klep...

Representations of nc disk algebras, free holomorphic functions on nc domains: G. Popescu

Representations of tensor algebras over C^* correspondences: P. Muhly and B. Solel,

Noncommutative functions: J. Agler, J. E. McCarthy, N. Young ...

Let $\mathcal{R}$ be a commutative ring, and let $\mathcal{M}$ be a module over $\mathcal{R}$. We define the *nc space over* $\mathcal{M}$,

$$\mathcal{M}_{\mathrm{nc}} = \coprod_{n=1}^{\infty} \mathcal{M}^{n \times n}.$$

For $X \in \mathcal{M}^{n \times n}$ and $Y \in \mathcal{M}^{m \times m}$ their *direct sum* is

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{M}^{(n+m) \times (n+m)}.$$

A subset $\Omega \subseteq \mathcal{M}_{\mathrm{nc}}$ is called a *nc set* if it is closed under direct sums: denoting $\Omega_n = \Omega \cap \mathcal{M}^{n \times n}$, we have

$$X \in \Omega_n, Y \in \Omega_m \Longrightarrow X \oplus Y \in \Omega_{n+m}.$$

A nc set Ω is called *right admissible* if for every $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathcal{M}^{n \times m}$ there exists an invertible $r \in \mathcal{R}$ such that

$$\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Examples

Right admissible nc sets:

1. The set of upper triangular matrices over $\mathcal{M}$.
2. The set of matrices $X \in \mathbb{C}_{\rm nc}$ with $\sigma(X) \subseteq S$, for a given $S \subseteq \mathbb{C}$. In particular, the set of nilpotent matrices (for $S = \{0\}$); the set of invertible matrices (for $S = \mathbb{C} \setminus \{0\}$).
3. The nc unit ball $\mathbb{B}_{\rm nc}(0,1) = \{X \in \mathbb{R}_{\rm nc} \colon \|X\|_{2,2} < 1\}$.

Notice that matrices over $\mathcal{R}$ act from the right and from the left on matrices over $\mathcal{M}$ by the standard rules of matrix multiplication: if $X \in \mathcal{M}^{p\times q}$ and $T \in \mathcal{R}^{r\times p}$, $S \in \mathcal{R}^{q\times s}$, then

$$TX \in \mathcal{M}^{r\times q}, \quad XS \in \mathcal{M}^{p\times s}.$$

In the case of $\mathcal{M} = \mathcal{R}^d$, using the identification

$$\left(\mathcal{R}^d\right)^{p\times q} \cong \left(\mathcal{R}^{p\times q}\right)^d,$$

we have, for d-tuples $X = (X_1, \ldots, X_d) \in \left(\mathcal{R}^{n\times n}\right)^d$ and $Y = (Y_1, \ldots, Y_d) \in \left(\mathcal{R}^{m\times m}\right)^d$,

$$X \oplus Y = \left(\begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}, \ldots, \begin{bmatrix} X_d & 0 \\ 0 & Y_d \end{bmatrix} \right) \in \left(\mathcal{R}^{(n+m)\times(n+m)}\right)^d;$$

and for a d-tuple $X = (X_1, \ldots, X_d) \in \left(\mathcal{R}^{p\times q}\right)^d$ and matrices $T \in \mathcal{R}^{r\times p}$, $S \in \mathcal{R}^{q\times s}$,

$$TX = (TX_1, \ldots, TX_d) \in \left(\mathcal{R}^{r\times q}\right)^d, \quad XS = (X_1S, \ldots, X_dS) \in \left(\mathcal{R}^{p\times s}\right)^d.$$

When $\mathcal{R} = \mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and thus $\mathcal{M} = \mathcal{V}$ is a vector space, consider the following three topologies on the nc space $\mathcal{V}_{\rm nc}$:

1. **Finitely-open topology.** A set $\Omega \subseteq \mathcal{V}_{\rm nc}$ is *finitely-open* if for every $n \in \mathbb{N}$ and every finite-dimensional subspace $\mathcal{X} \subseteq \mathcal{V}^{n\times n}$ the set $\Omega_n \cap \mathcal{X}$ is open in the relative topology of $\mathcal{X}$.

2. **Norm topology.** Suppose each $\mathcal{V}^{n\times n}$ is a Banach space in a norm $\|\cdot\|_n$, and the system of norms $\|\cdot\|_n$, $n\in\mathbb{N}$, is *admissible*:

- For every $n, m \in \mathbb{N}$ there exist $C_1(n,m)$, $C_1'(n,m) > 0$ such that for all $X \in \mathcal{V}^{n\times n}$ and $Y \in \mathcal{V}^{m\times m}$,

$$\begin{aligned} C_1(n,m)^{-1}\max\{\|X\|_n, \|Y\|_m\} &\le \|X \oplus Y\|_{n+m} \\ &\le C_1'(n,m)\max\{\|X\|_n, \|Y\|_m\}. \end{aligned}$$

- For every $n \in \mathbb{N}$ there exists $C_2(n) > 0$ such that for all $X \in \mathcal{V}^{n\times n}$ and $S, T \in \mathbb{K}^{n\times n}$,

$$\|SXT\|_n \le C_2(n)\|S\|\,\|X\|_n\|T\|,$$

where $\|\cdot\|$ denotes the operator norm of $\mathbb{K}^{n\times n}$ with respect to the standard Euclidean norm of $\mathbb{K}^n$.

Equivalently, injections,

$$\iota_{ij} \colon \mathcal{V} \to \mathcal{V}^{n\times n}, \quad \iota_{ij} \colon v \mapsto E_{ij} v,$$

and projections,

$$\pi_{ij} \colon \mathcal{V}^{n\times n} \to \mathcal{V}, \quad \pi_{ij} \colon X \mapsto X_{ij},$$

are bounded for all n, i, j.

(When C_1, C_1', C_2 are independent of n, m, all ι_{ij} and π_{ij} are uniformly completely bounded.)

A set $\Omega \subseteq \mathcal{V}_{\mathrm{nc}}$ is *open* if for every $n \in \mathbb{N}$ the set $\Omega_n \subseteq \mathcal{V}^{n\times n}$ is open in norm $\|\cdot\|_n$.

3. **Uniform topology.** A vector space $\mathcal{V}$ is called an *operator space* if $\mathcal{V}_{\rm nc}$ is equipped with an admissible system of norms $\|\cdot\|_n$ on $\mathcal{V}^{n\times n}$, $n\in\mathbb{N}$, and $C_1(n,m)=C_1'(n,m)=C_2(n)=1$ for all $n,m\in\mathbb{N}$. In particular, all injections ι_{ij} are complete isometries and all projections π_{ij} are complete co-isometries.

The *nc ball centered at $Y\in\mathcal{V}^{s\times s}$ of radius ϵ* is

$$B_{\rm nc}(Y,\epsilon):=\coprod_{m=1}^{\infty}\left\{X\in\mathcal{V}^{sm\times sm}\colon\left\|X-\bigoplus_{\alpha=1}^{m}Y\right\|_{sm}<\epsilon\right\}.$$

NC balls form a base for the uniform topology on $\mathcal{V}_{\rm nc}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be modules over a unital commutative ring $\mathcal{R}$, and let $\Omega \subseteq \mathcal{M}_{\rm nc}$ be a nc set. A mapping $f\colon \Omega \to \mathcal{N}_{\rm nc}$ with $f(\Omega_n) \subseteq \mathcal{N}^{n\times n}$ is called a *nc function* if

- *f respects direct sums*:

$$f(X \oplus Y) = f(X) \oplus f(Y), \qquad X, Y \in \Omega. \tag{1}$$

- *f respects similarities*: if $X \in \Omega_n$ and $S \in \mathcal{R}^{n\times n}$ is invertible with $SXS^{-1} \in \Omega_n$, then

$$f(SXS^{-1}) = Sf(X)S^{-1}. \tag{2}$$

Proposition

A mapping $f\colon \Omega \to \mathcal{N}_{\rm nc}$ with $f(\Omega_n) \subseteq \mathcal{N}^{n\times n}$ respects direct sums and similarities, i.e., (1) *and* (2) *hold* **iff** *f respects intertwinings: for any $X \in \Omega_n$, $Y \in \Omega_m$, and $T \in \mathcal{R}^{n\times m}$ such that $XT = TY$,*

$$f(X)T = Tf(Y). \tag{3}$$

Examples

NC functions.

1. NC polynomials viewed as functions on matrices, e.g.,

$$p(X) = X_1X_2 - X_2X_1, \quad X = (X_1, X_2) \in (\mathcal{R}^2)_{\mathrm{nc}},$$

so that $p\colon (\mathcal{R}^2)_{\mathrm{nc}} \to \mathcal{R}_{\mathrm{nc}}$. In general,

$$p(X) = \sum_{w \in \mathcal{G}_d\colon |w| \le N} p_w X^w, \quad X = (X_1, \ldots, X_d) \in (\mathcal{R}^d)_{\mathrm{nc}},$$

where $\mathcal{G}_d$ is the *free monoid on d generators* $g_1, \ldots, g_d$, with the unit element $\emptyset$ (the *empty word*); $X^w = X_{i_1} \cdots X_{i_k}$ for a word $w = g_{i_1} \cdots g_{i_k}$ (and $X^\emptyset = I$); the *length of the word* $w = g_{i_1} \cdots g_{i_k}$ is $|w| = k$ (and $|\emptyset| = 0$); $p_w \in \mathcal{R}$ for every $w \in \mathcal{G}_d$. So, $p\colon (\mathcal{R}^d)_{\mathrm{nc}} \to \mathcal{R}_{\mathrm{nc}}$.

2. NC power series evaluated on matrices:

$$f(X) = \sum_{w \in \mathcal{G}_d} f_w X^w, \quad X = (X_1, \ldots, X_d) \in \Omega,$$

where $\Omega \subseteq (\mathbb{K}^d)_{\mathrm{nc}}$ is a nc set where the series converges (in some sense), $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, thus $f : \Omega \to \mathbb{K}_{\mathrm{nc}}$.

3. NC rational expressions, e.g., realization formulas

$$f(X) = D(X) + C(X)(I - A(X))^{-1} B(X),$$

where $A(X)$, $B(X)$, $C(X)$, $D(X)$ are linear functions, or perhaps even higher degree nc polynomials. See [J. A. Ball, G. Groenewald, and T. Malakorn].

Theorem

Let $f : \Omega \to \mathcal{N}_{\mathrm{nc}}$ be a nc function on a right admissible nc set Ω. Let $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathcal{M}^{n\times m}$ be such that $\left[\begin{smallmatrix} X & Z \\ 0 & Y \end{smallmatrix}\right] \in \Omega_{n+m}$. Then

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & \Delta_R f(X,Y)(Z) \\ 0 & f(Y) \end{bmatrix},$$

where the off-diagonal block entry $\Delta_R f(X,Y)(Z)$ as a function of Z has a unique extension to a linear function on $\mathcal{M}^{n\times m}$.

Theorem

$$\begin{aligned} f(X) - f(Y) &= \Delta_R f(Y,X)(X-Y) \\ &= \Delta_R f(X,Y)(X-Y), \quad n \in \mathbb{N},\ X,Y \in \Omega_n. \end{aligned}$$

Δ_R is called the *right nc difference-differential operator*.The linear mapping $\Delta_R f(Y,Y)(\cdot)$ plays the role of a nc differential.

Under appropriate continuity conditions, $\Delta_R f(Y,Y)(Z)$ is the directional derivative of f at Y in the direction Z.
In the case where $\mathcal{M} = \mathcal{R}^d$, the finite difference formula turns into

$$f(X) - f(Y) = \sum_{i=1}^{N} \Delta_{R,i} f(Y,X)(X_i - Y_i), \quad X, Y \in \Omega_n,$$

with the *right partial difference-differential operators* $\Delta_{R,i}$:

$$\Delta_{R,i} f(Y,X)(C) := \Delta_R f(Y,X)(0,\ldots,0, \underbrace{C}_{i^{\mathrm{th}}\ \mathrm{place}}, 0,\ldots,0).$$

The left nc full and partial difference-differential operators Δ_L, $\Delta_{L,i}$, $i = 1,\ldots,d$, are defined analogously.

Theorem

For any $X \in \Omega_n$, $Y \in \Omega_m$, $T \in \mathcal{R}^{n\times m}$, $n, m \in \mathbb{N}$:

$$f(X)T - Tf(Y) = \Delta_R f(Y, X)(XT - TY).$$

Proof. Let $r \in \mathcal{R}$ be invertible and such that

$$\begin{bmatrix} X & r(TY - XT) \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Then

$$\begin{bmatrix} X & r(TY - XT) \\ 0 & Y \end{bmatrix} \begin{bmatrix} rT \\ I_m \end{bmatrix} = \begin{bmatrix} rT \\ I_m \end{bmatrix} Y.$$

Then

$$f\left(\begin{bmatrix} X & r(TY - XT) \\ 0 & Y \end{bmatrix}\right) \begin{bmatrix} rT \\ I_m \end{bmatrix} = \begin{bmatrix} rT \\ I_m \end{bmatrix} f(Y),$$

i.e.,

$$\begin{bmatrix} f(X) & \Delta_R f(X,Y)(r(TY-XT)) \\ 0 & f(Y) \end{bmatrix} \begin{bmatrix} rT \\ I_m \end{bmatrix} = \begin{bmatrix} rT \\ I_m \end{bmatrix} f(Y).$$

Comparing the 1st block entries in the matrix products on the right-hand side and on the left-hand side, we obtain

$$rf(X)T + r\Delta_R f(X,Y)(TY-XT)) = rTf(Y).$$

Cancelling r and rearranging, we obtain

$$f(X)T - Tf(Y) = \Delta_R f(X,Y)(XT-TY).$$

$\square$

As a function of X and Y, $\Delta_R f(X,Y)(\cdot)$ respects direct sums and similarities, or equivalently, *respects intertwinings*: if $X \in \Omega_n$, $Y \in \Omega_m$, $\widetilde{X} \in \Omega_{\tilde{n}}$, $\widetilde{Y} \in \Omega_{\tilde{m}}$, and $T \in \mathcal{R}^{n\times\tilde{n}}$, $S \in \mathcal{R}^{m\times\tilde{m}}$ are such that

$$XT = T\widetilde{X}, \quad YS = S\widetilde{Y},$$

and $Z \in \mathcal{M}^{\widetilde{n}\times m}$, then

$$\Delta_R f(X,Y)(TZ)S = T\Delta_R(\widetilde{X},\widetilde{Y})(ZS).$$

Example

Let $f\colon \mathcal{R}_{\mathrm{nc}} \to \mathcal{R}_{\mathrm{nc}}$, $f(X) = X^2$. Since

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}^2 = \begin{bmatrix} X^2 & XZ + ZY \\ 0 & Y^2 \end{bmatrix},$$

$\Delta_R f(X, Y)(Z) = XZ + ZY$. If $XT = T\widetilde{X}$, $YS = S\widetilde{Y}$, then

$$\begin{aligned}\Delta_R f(X, Y)(TZ)S &= (XTZ + TZY)S = XTZS + TZYS \\ &= T\widetilde{X}ZS + TZS\widetilde{Y} = T(\widetilde{X}ZS + ZS\widetilde{Y}) = T\Delta_R f(\widetilde{X}, \widetilde{Y})(ZS).\end{aligned}$$

We denote the class of functions h on $\Omega \times \Omega$ whose values on $\Omega_n \times \Omega_m$ are linear mappings $\mathcal{M}^{n\times m} \to \mathcal{N}^{n\times m}$ satisfying the property above (with $\Delta_R f$ replaced by h) as

$$\mathcal{T}^1 = \mathcal{T}^1(\Omega; \mathcal{N}_{\rm nc}, \mathcal{M}_{\rm nc}).$$

Thus for a nc function f, $\Delta_R f \in \mathcal{T}^1$.

More generally, we define the class of nc functions of order k,

$$\mathcal{T}^k = \mathcal{T}^k(\Omega; \mathcal{N}_{0,\mathrm{nc}}, \mathcal{N}_{1,\mathrm{nc}}, \ldots, \mathcal{N}_{k,\mathrm{nc}})$$

as a class of functions on Ω^{k+1}, where $\Omega \subseteq \mathcal{M}_{\mathrm{nc}}$ is a nc set, whose values on $\Omega_{n_0} \times \cdots \times \Omega_{n_k}$ are k-linear forms

$${\mathcal{N}_1}^{n_0 \times n_1} \times \cdots \times {\mathcal{N}_k}^{n_{k-1} \times n_k} \to {\mathcal{N}_0}^{n_0 \times n_k},$$

and which respect direct sums and similarities, or equivalently, respect intertwinings: if $X^j \in \Omega_{n_j}$, $\widetilde{X}^j \in \Omega_{\widetilde{n}_j}$, $T_j \in \mathcal{R}^{n_j \times \widetilde{n}_j}$ satisfy

$$X^j T_j = T_j \widetilde{X}^j, \quad j = 0, 1, \ldots, k,$$

and $Z^j \in \mathcal{M}^{\widetilde{n}_j \times n_{j+1}}$, $j = 1, \ldots, k$, then

$$f(X^0, \ldots, X^k)(T_0 Z^1, \ldots, T_{k-1} Z^k) T_k \\ = T_0 f(\widetilde{X}^0, \ldots, \widetilde{X}^k)(Z^1 T_1, \ldots, Z^k T_k).$$

$\mathcal{T}^0 = \mathcal{T}^0(\Omega; \mathcal{N}_{\mathrm{nc}})$ is the class of nc functions $f \colon \Omega \to \mathcal{N}_{\mathrm{nc}}$.

Example

Let $f_0, \dots, f_k \in \mathcal{T}^0(\mathcal{R}_{\rm nc}; \mathcal{R}_{\rm nc})$. We define $f \in \mathcal{T}^k(\mathcal{R}_{\rm nc}; \mathcal{R}_{\rm nc}, \dots, \mathcal{R}_{\rm nc})$ by

$$f(X^0, \dots, X^k)(Z^1, \dots, Z^k) = f_0(X^0)Z^1 f_1(X^1)Z^2 \cdots Z^k f_k(X^k).$$

Suppose that

$$X^j T_j = T_j \widetilde{X}^j, \quad j = 0, 1, \dots, k.$$

Then

$$\begin{aligned} & f(X^0, \dots, X^k)(T_0 Z^1, \dots, T_{k-1} Z^k) T_k \\ & \qquad = f(X^0) T_0 Z^1 f_1(X^1) T_1 Z^2 \cdots T_{k-1} Z^k f_k(X^k) T_k \\ & \qquad = T_0 f_0(\widetilde{X}^0) Z^1 T_1 f(\widetilde{X}^1) Z^2 \cdots Z^k T_k f(\widetilde{X}^k) \\ & \qquad = T_0 f(\widetilde{X}^0, \dots, \widetilde{X}^k)(Z^1 T_1, \dots, Z^k T_k). \end{aligned}$$

We define $\Delta_R \colon \mathcal{T}^k \to \mathcal{T}^{k+1}$ as follows:

$$\begin{aligned} & f\left(X^0, \ldots, X^{k-1}, \begin{bmatrix} X^{k\prime} & Z \\ 0 & X^{k\prime\prime} \end{bmatrix}\right)\left(Z^1, \ldots, Z^{k-1}, \operatorname{row}\left[Z^{k\prime}, Z^{k\prime\prime}\right]\right) \\ & \quad = \operatorname{row}\Big[f\left(X^0, \ldots, X^{k-1}, X^{k\prime}\right)\left(Z^1, \ldots, Z^{k-1}, Z^{k\prime}\right), \\ & \quad \Delta_R f\left(X^0, \ldots, X^{k-1}, X^{k\prime}, X^{k\prime\prime}\right)\left(Z^1, \ldots, Z^{k-1}, Z^{k\prime}, Z\right) \\ & \qquad + f\left(X^0, \ldots, X^{k-1}, X^{k\prime\prime}\right)\left(Z^1, \ldots, Z^{k-1}, Z^{k\prime\prime}\right)\Big]. \end{aligned}$$

Notice, that $\Delta_R = {}_k\Delta_R$. We define ${}_j\Delta_R$ similarly for $j = 0, \ldots, k-1$.

Theorem

Let $f \in \mathcal{T}^0(\Omega; \mathcal{N}_{\mathrm{nc}})$. *Then*

$$\Delta_R^\ell f(X^0, \ldots, X^\ell)(Z^1, \ldots, Z^\ell) = f\left(\begin{bmatrix} X^0 & Z^1 & 0 & \cdots & 0 \\ 0 & X^1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & X^{\ell-1} & Z^\ell \\ 0 & \cdots & \cdots & 0 & X^\ell \end{bmatrix}\right)_{1,\ell+1}.$$

Generalized finite difference formulas:

$$
\begin{aligned}
f&\left(X^0,\ldots,X^{k-1},X\right)\left(Z^1,\ldots,Z^k\right)T\\
&\qquad - f\left(X^0,\ldots,X^{k-1},Y\right)\left(Z^1,\ldots,Z^kT\right)\\
&= \Delta_R f\left(X^0,\ldots,X^{k-1},X,Y\right)\left(Z^1,\ldots,Z^k,XT-TY\right),
\end{aligned}
$$

$$
\begin{aligned}
f(X,X^1,\ldots,X^k)&(TZ^1,\ldots,Z^k)-Tf(Y,X^1,\ldots,X^k)(Z^1,Z^2,\ldots,Z^k)\\
&= {}_0\Delta_R f(X,Y,X^1,\ldots,X^k)(XT-TY,Z^1,\ldots,Z^k),
\end{aligned}
$$

$$
\begin{aligned}
&f(X^0,\ldots,X^{j-1},X,X^{j+1},\ldots,X^k)(Z^1,\ldots,Z^j,TZ^{j+1},\ldots,Z^k)\\
&-f(X^0,\ldots,X^{j-1},Y,X^{j+1},\ldots,X^k)(Z^1,\ldots,Z^{j-1},Z^jT,Z^{j+1},\ldots,Z^k)\\
&\qquad = {}_j\Delta_R f(X^0,\ldots,X^{j-1},X,Y,X^{j+1},\ldots,X^k)\\
&\qquad\qquad (Z^1,\ldots,Z^j,XT-TY,Z^{j+1},\ldots,Z^k), \quad 0<j<k.
\end{aligned}
$$

We use the calculus of higher order nc difference-differential operators to derive a nc analogue of Taylor's formula, which we call the *Taylor–Taylor (TT) formula*.

Theorem

Let $f \in \mathcal{T}^0(\Omega; \mathcal{N}_{\mathrm{nc}})$ with $\Omega \subseteq \mathcal{M}_{\mathrm{nc}}$ a right admissible nc set, $s \in \mathbb{N}$, and $Y \in \Omega_s$. Then for every $m \in \mathbb{N}$, $N \in \mathbb{Z}_+$, and $X \in \Omega_{sm}$,

$$f(X) = \sum_{\ell=0}^{N} \Delta_R^{\ell} f \underbrace{\Big(\bigoplus_{\alpha=1}^{m} Y, \ldots, \bigoplus_{\alpha=1}^{m} Y\Big)}_{\ell+1 \text{ times}} \underbrace{\Big(X - \bigoplus_{\alpha=1}^{m} Y, \ldots, X - \bigoplus_{\alpha=1}^{m} Y\Big)}_{\ell \text{ times}}$$

$$+ \Delta_R^{N+1} f \underbrace{\Big(\bigoplus_{\alpha=1}^{m} Y, \ldots, \bigoplus_{\alpha=1}^{m} Y, X\Big)}_{N+1 \text{ times}} \underbrace{\Big(X - \bigoplus_{\alpha=1}^{m} Y, \ldots, X - \bigoplus_{\alpha=1}^{m} Y\Big)}_{N+1 \text{ times}}.$$

In the case where $\mathcal{V} = \mathcal{R}^d$, we obtain

$$f(X) = \sum_{\ell=0}^{N} \sum_{w=g_{i_1}\cdots g_{i_\ell}} \Delta_R^{w^\top} f\underbrace{\Big(\bigoplus_{\alpha=1}^{m} Y, \dots, \bigoplus_{\alpha=1}^{m} Y\Big)}_{\ell+1 \text{ times}}$$

$$\Big(X_{i_1} - \bigoplus_{\alpha=1}^{m} Y_{i_1}, \dots, X_{i_\ell} - \bigoplus_{\alpha=1}^{m} Y_{i_\ell}\Big)$$

$$+ \sum_{w=g_{i_1}\cdots g_{i_{N+1}}} \Delta_R^{w^\top} f\underbrace{\Big(\bigoplus_{\alpha=1}^{m} Y, \dots, \bigoplus_{\alpha=1}^{m} Y}_{N+1 \text{ times}}, X\Big)$$

$$\Big(X_{i_1} - \bigoplus_{\alpha=1}^{m} Y_{i_1}, \dots, X_{i_{N+1}} - \bigoplus_{\alpha=1}^{m} Y_{i_{N+1}}\Big),$$

where for a word $w = g_{i_1} \cdots g_{i_\ell}$, $\Delta_R^{w^\top} := \Delta_{R,i_\ell} \cdots \Delta_{R,i_1}$.

If $Y = (\mu_1 I_s, \ldots, \mu_d I_s)$, this reduces to a genuine nc power expansion

$$f(X) = \sum_{\ell=0}^{N} \sum_{|w|=\ell} (X - \mu I_{sm})^w \, \Delta_R^{w^\top} f(\underbrace{\mu, \ldots, \mu}_{\ell+1 \text{ times}}) + \sum_{|w|=N+1} (X - \mu I_{sm})^w \, \Delta_R^{w^\top} f(\underbrace{\mu, \ldots, \mu}_{N+1 \text{ times}}, X).$$

1. One can interpret the ℓ-linear mappings

$$\Delta_R^\ell f(Y, \ldots, Y)\colon \left(\mathcal{M}^{s\times s}\right)^\ell \to \mathcal{N}^{s\times s}$$

as linear mappings

$$\Delta_R^\ell f(Y, \ldots, Y)\colon \left(\mathcal{M}^{s\times s}\right)^{\otimes\ell} \to \mathcal{N}^{s\times s},$$

and then extend them to matrices over the tensor algebra $\mathbf{T}(\mathcal{M}^{s\times s})$ as acting entrywise. So,

$$\Delta_R^\ell f\Big(\bigoplus_{\alpha=1}^m Y, \ldots, \bigoplus_{\alpha=1}^m Y\Big)\Big(X - \bigoplus_{\alpha=1}^m Y, \ldots, X - \bigoplus_{\alpha=1}^m Y\Big) = \Big(X - \bigoplus_{\alpha=1}^m Y\Big)^{\odot_s \ell} \Delta_R^\ell f(Y, \ldots, Y).$$

Here the "faux" product $\odot_s$ is understood as a product in $\mathbf{T}(\mathcal{M}^{s\times s})$, i.e., for $Z', Z'' \in \mathcal{M}^{sm\times sm} \cong (\mathcal{M}^{s\times s})^{m\times m}$,

$$(Z' \odot_s Z'')_{ij} = \sum_{k=1}^{m} Z'_{ik} \otimes Z''_{kj}.$$

For $Y \in \mathcal{M}^{s\times s}$, let $\mathrm{Nilp}(\mathcal{M}^{s\times s}, Y)$ denote the nc set of matrices $X \in \coprod_{m=1}^{\infty} \mathcal{M}^{sm\times sm}$ such that $\left(X - \bigoplus_{\alpha=1}^{m} Y\right)^{\odot_s \ell} = 0$ for some $\ell \in \mathbb{N}$.Then every nc function $f \colon \mathrm{Nilp}(\mathcal{M}^{s\times s}, Y) \to \left(\mathcal{N}^{s\times s}\right)_{\mathrm{nc}}$ has the Taylor–Taylor series expansion

$$f(X) = \sum_{\ell=0}^{\infty} \left(X - \bigoplus_{\alpha=1}^{m} Y\right)^{\odot_s \ell} \Delta_R^{\ell} f(Y, \ldots, Y).$$

Theorem

Given a nc function $f\colon \mathcal{M}_{\mathrm{nc}} \to \mathcal{N}_{\mathrm{nc}}$, for every $n, m \in \mathbb{N}$, $T \in \mathcal{R}^{sn\times sm}$, and $j = 0, \ldots, \ell - 1$, the ℓ-linear mappings $f_\ell = \Delta_R^\ell f(Y, \ldots, Y)\colon\ (\mathcal{M}^{s\times s})^\ell \to \mathcal{N}^{s\times s}$, $\ell = 0, 1, \ldots,$ satisfy

$$f_\ell\Big(Z^1, \ldots, Z^k\Big)T - f_\ell\Big(Z^1, \ldots, Z^k T\Big)$$

$$= f_{\ell+1}\Big(Z^1, \ldots, Z^k, \Big(\bigoplus_{\alpha=1}^{n} Y\Big)T - T\Big(\bigoplus_{\beta=1}^{m} Y\Big)\Big),$$

$$f_\ell(TZ^1, \ldots, Z^k) - Tf_\ell(Z^1, \ldots, Z^k)$$

$$= f_{\ell+1}\Big(\Big(\bigoplus_{\alpha=1}^{n} Y\Big)T - T\Big(\bigoplus_{\beta=1}^{m} Y\Big), Z^1, \ldots, Z^k\Big),$$

$$f_\ell(Z^1, \ldots, Z^j, TZ^{j+1}, \ldots, Z^k) - f_\ell(Z^1, \ldots, Z^{j-1}, Z^j T, Z^{j+1}, \ldots, Z^k)$$

$$= f_{\ell+1}\Big(Z^1, \ldots, Z^j, \Big(\bigoplus_{\alpha=1}^{n} Y\Big)T - T\Big(\bigoplus_{\beta=1}^{m} Y\Big), Z^{j+1}, \ldots, Z^k\Big),\ 0 < j < k.$$

Conversely, given a sequence of ℓ-linear mappings $f_\ell\colon \left(\mathcal{M}^{s\times s}\right)^{\ell} \to \mathcal{N}^{s\times s}$, $\ell = 0, 1, \ldots$, satisfying the conditions above, the mapping $f : \mathrm{Nilp}(\mathcal{M}^{s\times s}, Y) \to \left(\mathcal{N}^{s\times s}\right)_{\mathrm{nc}}$ defined by

$$f(X) = \sum_{\ell=0}^{\infty} \left(X - \bigoplus_{\alpha=1}^{m} Y\right)^{\odot_s \ell} f_\ell$$

is a nc function and $f_\ell = \Delta_R^{\ell} f(Y, \ldots, Y)$ are the Taylor–Taylor coefficients of f.

Not every nc power series centered at Y defines a nc function on $\mathrm{Nilp}(\mathcal{M}^{s\times s}, Y)$!

Example

Let $Y \in \mathbb{C}^{2\times 2}$. In general,

$$p(X) = \Big(X - \bigoplus_{\alpha=1}^{m} Y\Big)^2 = X^2 - X\Big(\bigoplus_{\alpha=1}^{m} Y\Big) - \Big(\bigoplus_{\alpha=1}^{m} Y\Big)X + \bigoplus_{\alpha=1}^{m} Y^2$$

is not a nc function: it does not respect similarities.

2.

Theorem

Let $\mathcal{R} = \mathbb{K}$ be an infinite field, and let $f\colon (\mathbb{K}^d)_{\mathrm{nc}} \to \mathbb{K}_{\mathrm{nc}}$ be a nc function such that for every $n \in \mathbb{N}$ each matrix entry of $f(X_1, \ldots, X_d)$ is a polynomial in $(X_i)_{j,k}$, $i = 1, \ldots, d$, $j, k = 1, \ldots, n$, of a uniformly (in n) bounded degree. Then f is a nc polynomial, i.e.,

$$f(X) = \sum_{w \in \mathcal{G}_d, |w| \le N} f_w X^w.$$

Proof Let $f(X_1, \ldots, X_d)$ be a polynomial in $(X_i)_{j,k}$, $i = 1, \ldots, d$, $j, k = 1, \ldots, n$, of total degree N. For $Y = (0, \ldots, 0) \in \mathbb{K}^d$, write

$$f(X) = \sum_{\ell=0}^{N} \sum_{w \in \mathcal{G}_d \colon |w| \leq N} X^w \Delta_R^{w^\top} f(0, \ldots, 0)$$

$$+ \sum_{|w|=N+1} X^w \Delta_R^{w^\top} f(0, \ldots, 0, X).$$

Since for $w = g_{i_1} \cdots g_{i_{N+1}}$,

$$X^w \Delta_R^{w^\top} f(0, \ldots, 0, X) = f\left(\begin{bmatrix} 0 & e_{i_1} \otimes X_{i_1} & & \\ & \ddots & \ddots & \\ & & 0 & e_{i_{N+1}} \otimes X_{i_{N+1}} \\ & & & X \end{bmatrix}\right)_{1,N+2}$$

is a polynomial in $(X_i)_{jk}$ of total degree at least $N+1$, we must have

$$\sum_{|w|=N+1} X^w \Delta_R^{w^\top} f(0,\ldots,0,X) = 0,$$

so that f is a nc polynomial. $\square$

The assumption of boundedness of degrees cannot be omitted!

Example

Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of homogeneous nc polynomials vanishing on $(\mathbb{K}^{n\times n})^2$. E.g.,

$$p_n = \sum_{\pi\in S_{n+1}} \mathrm{sign}(\pi) x_1^{\pi(1)-1} x_2 \cdots x_1^{\pi(n+1)-1} x_2.$$

$\alpha_n := \deg p_n = \frac{(n+1)(n+2)}{2}$. Then

$$f(X_1, X_2) = \sum_{n=0}^{\infty} p_n(X_1, X_2)$$

is a nc function which is a polynomial in matrix entries for each n. However, f is not a nc polynomial.

Theorem

Let f be a nc function on $(\mathbb{K}^d)_{\rm nc}$, where $\mathbb{K}$ is an infinite field, with values in $\mathcal{N}_{\rm nc}$. Assume that for each n, $f(X_1, \ldots, X_d)$ is a polynomial function of degree L_n in dn^2 commuting variables $(X_i)_{jk}$, $i = 1, \ldots, d$; $j, k = 1, \ldots, n$, with values in $\mathcal{N}^{n\times n}$. Then there exists a unique sequence of homogeneous nc polynomials $f_j \in \mathcal{N}\langle x_1, \ldots, x_d\rangle$ of degree j, $j = 0, 1, \ldots,$ such that for all $n \in \mathbb{N}$,

- *f_j vanishes on $(\mathbb{K}^{n\times n})^d$ for all $j > L_n$,*
- *f_{L_n} does not vanish identically on $(\mathbb{K}^{n\times n})^d$,*

and for $X \in (\mathbb{K}^{n\times n})^d$,

$$f(X) = \sum_{j=0}^{\infty} f_j(X) = \sum_{j=0}^{L_n} f_j(X).$$

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$ (resp., a Banach space equipped with an admissible system of matrix norms over $\mathcal{V}$, operator space), let $\mathcal{W}$ be a Banach space equipped with an admissible system of matrix norms over $\mathcal{W}$, and let $\Omega \subseteq \mathcal{V}_{\rm nc}$ be a finitely-open (resp., norm-open, uniformly open) nc set.

Facts. 1. If a nc function $f \colon \Omega \to W_{\rm nc}$ is *locally bounded on slices*, i.e., for every $Y \in \Omega_n$ and $Z \in \mathcal{V}^{n\times n}$, $f(Y + \lambda Z)$ is bounded for $|\lambda| < \epsilon$, then f is *analytic on slices*, i.e., for every $Y \in \Omega_n$ and $Z \in \mathcal{V}^{n\times n}$, $f(Y + \lambda Z)$ is analytic as a function of λ.

2. If a nc function $f \colon \Omega \to W_{\rm nc}$ is locally norm-bounded, then f is *analytic* (= locally norm-bounded + Gâteaux differentiable).

3. If a nc function $f \colon \Omega \to W_{\rm nc}$ is *locally uniformly bounded*, i.e., for every $Y \in \Omega$, f is bounded in some nc ball $B_{\rm nc}(Y, \epsilon)$, then f is *uniformly analytic* (= locally uniformly bounded + Gâteaux differentiable).

E.g., in the uniform topology setting we have

Theorem

Let a nc function $f\colon \Omega \to \mathcal{W}_{\mathrm{nc}}$ be uniformly locally bounded. For $s \in \mathbb{N}$, $Y \in \Omega_s$, let $\delta := \sup\{r > 0 \colon f \text{ is bdd on } B_{\mathrm{nc}}(Y, r)\}$. Then

$$f(X) = \sum_{\ell=0}^{\infty} \Big(X - \bigoplus_{\alpha=1}^{m} Y \Big)^{\odot_s \ell} \Delta_R^{\ell} f(\underbrace{Y, \ldots, Y}_{\ell+1 \text{ times}})$$

holds, with the TT series convergent absolutely and uniformly on every open nc ball $B_{\mathrm{nc}}(Y, r)$ with $r < \delta$.

Proof.

Let $r < \delta$ and $\|f(Z)\|_{sm_Z} \leq M$ for all $Z \in B_{\mathrm{nc}}\Big(Y, \frac{r+\delta}{2}\Big)$. For some $X \in B_{\mathrm{nc}}\Big(Y, r\Big)$, set

$$Z := \begin{bmatrix} \bigoplus_{\alpha=1}^{m_X} Y & \frac{r+\delta}{2r}\Big(X - \bigoplus_{\alpha=1}^{m_X} Y\Big) & & \\ & \ddots & \ddots & \\ & & \ddots & \frac{r+\delta}{2r}\Big(X - \bigoplus_{\alpha=1}^{m_X} Y\Big) \\ & & & \bigoplus_{\alpha=1}^{m_X} Y \end{bmatrix}.$$

Since $\|Z - \bigoplus_{\beta=1}^{(\ell+1)m_X} Y\|_{sm_X(\ell+1)} \leq \frac{r+\delta}{2}$, we have

$$\Big\|\Big(X - \bigoplus_{\alpha=1}^{m} Y\Big)^{\odot_s \ell} \Delta_R^{\ell} f(\underbrace{Y, \ldots, Y}_{\ell+1 \text{ times}})\Big\|_{sm_X} = \|f(Z)_{1,\ell+1}\|_{sm_X}\Big(\frac{2r}{r+\delta}\Big)^{\ell}$$

$$\leq M\Big(\frac{2r}{r+\delta}\Big)^{\ell}.$$

$\square$

Theorem

Let $\Omega \subseteq (\mathbb{C}^d)_{\mathrm{nc}}$ be uniformly-open, $\mathcal{W}$ an operator space, and a nc function $f\colon \Omega \to \mathcal{W}_{\mathrm{nc}}$ uniformly locally bounded. For every $s \in \mathbb{N}$, $Y \in \Omega_s$, let $\delta := \sup\{r > 0\colon f \text{ is bdd on } B_{\mathrm{nc}}(Y, r)\}$. Then

$$f(X) = \sum_{w \in \mathcal{G}_d} \Big(X - \bigoplus_{\alpha=1}^{m} Y\Big)^{\odot_s w} \Delta_R^{w^\top} f(\underbrace{Y, \dots, Y}_{\ell+1 \text{ times}})$$

with the TT series convergent absolutely and uniformly on every open nc diamond about Y of radius $r < \delta$,

$$\diamondsuit_{\mathrm{nc}}(Y, r) := \coprod_{m=1}^{\infty} \Big\{X \in \Omega_{sm}\colon \sum_{j=1}^{d} \Big\| X_j - \bigoplus_{\alpha=1}^{m} Y_j \Big\|_{sm} < r\Big\}.$$

Let $\mathcal{V}$ and $\mathcal{W}$ be operator spaces. Let $f_\ell\colon \mathcal{V}^\ell \to \mathcal{W}$ be a sequence of *completely bounded ℓ-linear functions* (in the sense of Christensen and Sinclair), that is,

$$\|f_\ell\|_{\mathrm{cb}} := \sup \|f_\ell(Z^1, \ldots, Z^\ell)\| < \infty,$$

where the sup is taken over all $n_0, \ldots, n_\ell \in \mathbb{N}$ and all matrices $Z^1 \in \mathcal{V}^{n_0 \times n_1}, \ldots, Z^\ell \in \mathcal{V}^{n_{\ell-1} \times n_\ell}$ of norm 1. Equivalently, the linear mappings $f_\ell\colon \mathcal{V}^{\otimes \ell} \to \mathcal{W}$ are completely bounded.

Let $Y \in \mathcal{V}^{s \times s}$ for some $s \in \mathbb{N}$. For the series

$$\sum_\ell \Big(X - \bigoplus_{\alpha=1}^{m} Y\Big)^{\odot_s \ell} f_\ell \tag{4}$$

we define the *Cauchy–Hadamard radius*

$$\rho_{\mathrm{cb}} = \Big(\lim_{\ell \to \infty} \sqrt[\ell]{\|f_\ell\|_{\mathrm{cb}}}\Big)^{-1}.$$

Theorem

The series (4) converges uniformly and absolutely on every nc ball $B_{\mathrm{nc}}(Y, r)$ with $r < \rho_{\mathrm{cb}}$. Moreover, the convergence is normal, *i.e.,*

$$\sum_{\ell=0}^{\infty} \sup_{W \in B_{\mathrm{nc}}(Y,r)} \left\| \left(W - \bigoplus_{\alpha=1}^{m_W} Y \right)^{\odot_s \ell} f_\ell \right\|_{sm_W} < \infty.$$

The series (4) fails to converge uniformly on every nc ball $B_{\mathrm{nc}}(Y, r)$ with $r > \rho_{\mathrm{cb}}$.

For more detail, see arXiv:1212.6345

THANK YOU!