Math 107-250/350 Answers to Homework #2 19 Jan '07

. 5 1. On page T-4 at the back of the textbook, is the reduction formula

$$\int \sec^n ax\,dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\,dx, \quad n \neq 1$$

Using integration by parts and a suitable trig identity, derive this formula.

As usual, a is a constant.

Solution. Since there is a $\tan ax$ on the right hand side, we factor a $\sec^2 ax$ out of the integrand and use integration by parts with $dv = \sec^2 ax$:

$$\begin{aligned}
\int \sec^n ax\,dx &= \int \sec^{n-2} ax \sec^2 ax\,dx \\
&\quad \text{use} \quad \begin{cases} u = \sec^{n-2} ax & du = (n-2)\sec^{n-3} ax(\sec ax \tan ax)a\,dx \\ v = \frac{1}{a}\tan ax & dv = \sec^2 ax\,dx \end{cases} \\
&= \frac{1}{a}\sec^{n-2} ax \tan ax - (n-2)\int \frac{1}{a}\tan ax \sec^{n-2} ax \tan axa\,dx \\
&= \frac{1}{a}\sec^{n-2} ax \tan ax - (n-2)\int \tan^2 ax \sec^{n-2} ax\,dx
\end{aligned}$$

and now, using $\tan^2 ax = \sec^2 ax - 1$, we obtain

$$\begin{aligned}
&= \frac{1}{a}\sec^{n-2} ax \tan ax - (n-2)\int (\sec^2 - 1)ax \sec^{n-2} ax\,dx \\
&= \frac{1}{a}\sec^{n-2} ax \tan ax - (n-2)\int \sec^n ax\,dx + (n-2)\int \sec^{n-2} ax\,dx
\end{aligned}$$

We have an infinite loop, so we let $I = \int \sec^n ax\,dx$. Then

$$\begin{aligned}
I &= \frac{1}{a}\sec^{n-2} ax \tan ax - (n-2)I + (n-2)\int \sec^{n-2} ax\,dx \\
(n-1)I &= \frac{1}{a}\sec^{n-2} ax \tan ax + (n-2)\int \sec^{n-2} ax\,dx \\
I &= \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\,dx
\end{aligned}$$

This establishes the formula.