1. Find the total mass and the center of mass for the region bounded by the graphs of $f(x) = 4 - x^2$ and $g(x) = x + 2$, if the density if $\delta(x, y) = x^2$.

   Solution. Notice that the curves intersect at $(-2, 0)$ and $(1, 3)$. We set up the integrals in terms of vertical slices. The vertical slice has upper endpoint $4 - x^2$ and lower endpoint $x + 2$. Since the slice has uniform density, the center of mass of the slice is $(\tilde{x}, \tilde{y}) = (x, (6+x-x^2)/2)$. The slice has width dx, length $(4 - x^2) - (x + 2) = 2 - x - x^2$, area $(2 - x - x^2)\,dx$, and mass $dm = x^2(2 - x - x^2)\,dx$. Thus, the total mass is

   $$\int_{-2}^{1} dm = \int_{-2}^{1} x^2(2 - x - x^2)\,dx = \frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}\Big|_{-2}^{1} = \frac{63}{20}.$$

   The moment about each axis is

   $$M_x = \int_{-2}^{1} \tilde{y}\,dm = \int_{-2}^{1} \frac{6 + x - x^2}{2} x^2(2 - x - x^2)\,dx = \frac{351}{70}$$

   $$M_y = \int_{-2}^{1} \tilde{x}\,dm = \int_{-2}^{1} xx^2(2 - x - x^2)\,dx = \frac{-18}{5}$$

   Thus, the center of mass is $(-8/7, 78/49)$.

2. Determine if $a_n = \dfrac{\cos(n\pi)}{n^2}$ converges or diverges. If it converges, find the limit.

   Solution. As $\cos(n\pi) = (-1)^n$, a_n is $(-1)^n/n^2$. Since $\lim\limits_{n\to\infty} \dfrac{1}{n^2} = 0$ and $\lim\limits_{n\to\infty} \dfrac{-1}{n^2} = 0$, by the Squeeze Theorem, $\lim\limits_{n\to\infty} a_n = 0$.

3. Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ converges or diverges.

   Solution. Since $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1 \neq 0$, by the nth term test, the series diverges.

4. Determine if the following series converge or diverge and, if one converges, find its sum.

   $$\text{a)} \sum_{n=1}^{\infty} \frac{5(2^n)}{3^n}, \qquad \text{b)} \sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^3 - 2n^2 + 1}.$$

   Solution. For part a), we note that

   $$\sum_{n=1}^{\infty} \frac{5(2^n)}{3^n} = \sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n = \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \cdots$$

and so this is a geometric series with first term 10/3 and ratio 2/3. Since $2/3 < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{10/3}{1-2/3} = 10.$$

For part b), we use the limit comparision test with

$$a_n = \frac{n^2-1}{3n^3-2n^2+1}, \qquad b_n = \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3-n}{3n^3-2n^2+1} = \frac{1}{3} \neq 0$$

Since $\sum_{n=1}^{\infty} 1/n$ diverges (it is the Harmonic series), by the limit comparision test, the series

$$\sum_{n=1}^{\infty} \frac{n^2-1}{3n^3-2n^2+1}$$

diverges also.

5. Determine whether or not the following series converge or diverge:

$$\text{a)} \sum_{n=1}^{\infty} (-1)^n \ln(n), \qquad \text{b)} \sum_{k=2}^{\infty} \frac{3}{k(\ln k)^2}$$

Solution. For a), notice that the terms do not go to zero as n goes to infinity. In fact, $\lim\limits_{n\to\infty} (-1)^n \ln(n)$ does not exist. Thus, the series diverges by the nth term test.

For b), we use the integral test. Notice first that the function is continuous, nonnegative and decreasing, so that we can apply the Integral Test. Next, we compute

$$\begin{aligned}
\int_2^{\infty} \frac{3}{x(\ln x)^2}\,dx &= \lim_{R\to\infty} \int_2^R \frac{3}{x(\ln x)^2}\,dx \\
&= \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{3}{u^2}\,du \quad \begin{cases} u = \ln x \\ du = 1/x\,dx \end{cases} \\
&= \lim_{R\to\infty} \left. \frac{-3}{u} \right|_{\ln 2}^{\ln R} \\
&= \lim_{R\to\infty} \frac{3}{\ln 2} - \frac{3}{\ln R} = \frac{3}{\ln 2}.
\end{aligned}$$

Since the integral is finite, the series converges.

6. Estimate the error in approxmating $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}5}{n^3}$ by $S_{10} = \frac{5}{1} - \frac{5}{8} + \cdots + \frac{5}{10^3}$.

Solution. The error estimate for the Alternating Series Test tell us the error is at most $\left| \frac{(-1)^1 25}{11^3} \right| = 5/1,331 \approx .00376$.

7. Find the radius of convergence and the interval of convergence for $\displaystyle\sum_{k=1}^{\infty} \frac{k}{4^k} x^k$.

   Solution. Let $a_k = kx^k/4^k$. Then

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)|x|}{k4}.$$

   and so

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \frac{|x|}{4}.$$

   Thus, the radius of convergence is 4.

   Letting $x = 4$, the series becomes $\displaystyle\sum_{k=1}^{\infty} k$ which clearly diverges by the kth term test. Letting $x = -4$, the series becomes $\displaystyle\sum_{k=1}^{\infty} (-1)^k k$ which also diverges by the kth term test. Thus, the interval of convergence is $(-4, 4)$.

8. Find a Taylor series for $f(x) = x^2 \cos\sqrt{x}$ centered at $c = 0$. You can give either the whole series or the first four nonzero terms.

   Solution. The series for $\cos(y)$ is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} y^{2k}$. Letting $y = \sqrt{x}$ shows the Taylor series for $\cos(\sqrt{x})$ is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^k$. Finally, multiplying by x^2 gives the Taylor series for $x^2 \cos\sqrt{x}$, which is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{k+2}.$$

9. Find a Taylor series for $g(x) = \dfrac{2}{4+x}$ centered at $c = 0$ and determine its radius of convergence.

   Solution. Notice that

$$\frac{2}{4+x} = \frac{1}{2}\,\frac{1}{1+x/4}.$$

   This suggest obtained the Taylor series for g by making the substitution $y = -x/4$ in the Taylor series of

$$\frac{1}{1-y} = \sum_{k=0}^{\infty} y^k.$$

   Making this substitution gives

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2 \cdot 4^k} x^k.$$

To compute the radius of convergence, note that

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(-1)^{k+1}x^{k+1}}{2\cdot 4^{k+1}}}{\frac{(-1)^k}{2\cdot 4^k}}\right| = \frac{|x|}{4}.$$

Thus, the radius of convergence is 4.

10. Using an appropriate Taylor series, solve each of the following problems:

    (a) Evaluate $\lim_{x \to 0} \dfrac{e^{-2x} - 1}{x}$.

    (b) Find $e^{.4}$ to 10^{-5}.

    Solution. For a), we have the series $e^y = 1 + y + y^2/2 + y^3/6 + \cdots$. Substituting $y = -2x$ into the series gives $1 - 2x + 2x^2 - 4x^3/3 + \cdots$. Thus,

    $$\frac{e^{-2x} - 1}{x} = -2 + 2x - \frac{4}{3}x^2 + \cdots$$

    This shows that $\lim_{x \to 0} \dfrac{e^{-2x} - 1}{x} = -2$.

    For b), notice that the derivative of e^x is always e^x, so, for any n, the maximum of e^z for z between 0 and .4 is $e^{.4} < e < 3$. Thus, the error estimate for Taylor Polnomials shows that

    $$|e^{.4} - P_n(.4)| \le \frac{3(.4)^{n+1}}{(n+1)!}.$$

    Using a calculator shows that if $n = 5$, then the error is at most 1.71×10^{-5} and if $n = 6$, then the error is at most 9.75×10^{-7}. Thus, the required approximation to $e^{.4}$ is

    $$1 + .4 + \frac{.4^2}{2} + \frac{.4^3}{6} + \frac{.4^4}{24} + \frac{.4^5}{120} + \frac{.4^6}{720} = 1.491824356$$

    which compares to $e^{.4} = 1.491824698\ldots$.