The Choquet boundary of an operator system

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Operator systems and completely positive maps

Definition

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For a non-self-adjoint subalgebra (or subspace) $\mathcal{A}$ contained in a unital C^*-algebra, can consider corresponding operator system $\mathcal{S} = \mathcal{A} + \mathcal{A}^* + \mathbb{C}1$.

Definition

For operator systems $\mathcal{S}_1, \mathcal{S}_2$, a map $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ induces maps $\phi_n : \mathcal{M}_n(\mathcal{S}_1) \to \mathcal{M}_n(\mathcal{S}_2)$ by

$$\phi_n([s_{ij}]) = [\phi(s_{ij})].$$

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The collection of operator systems forms a category, **the category of operator systems**. The morphisms between operator systems are the completely positive maps. The isomorphisms are the unital completely positive maps with unital completely positive inverse.

Stinespring (1955) introduces the notion of a completely positive map and proves his dilation theorem.

> *W.F. Stinespring, Positive functions on C*-algebras, Proceedings of the AMS 6 (1955), No 6, 211–216.*

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Arveson (1969/1972) uses completely positive maps as the basis of his work on non-commutative dilation theory and non-self-adjoint operator algebras.

W.B. Arveson, Subalgebras of C-algebras, Acta Math. 123 (1969), 141–224.*

W.B. Arveson, Subalgebras of C-algebras II, Acta Math. 128 (1972), 271–308.*

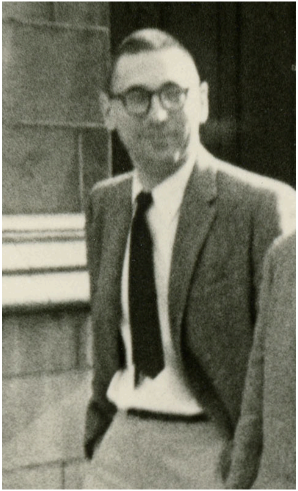

Figure: Stinespring and Arveson

A **dilation** of a UCP (unital completely positive) map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a UCP map $\psi : \mathcal{S} \to \mathcal{B}(K)$, where $K = H \oplus K'$ and

$$\psi(S) = \begin{pmatrix} \phi(S) & * \\ * & * \end{pmatrix}, \quad \forall S \in \mathcal{S}.$$

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$$\psi(S) = \begin{pmatrix} \phi(S) & * \\ * & * \end{pmatrix}, \quad \forall S \in \mathcal{S}.$$

Theorem (Stinespring's dilation theorem)

*Let $\mathcal{A}$ be a C^*-algebra. Every UCP map $\phi : \mathcal{A} \to \mathcal{B}(H)$ dilates to a *-representation of $\mathcal{A}$.*

Arveson proved an operator system analogue of the Hahn-Banach theorem.

Theorem (Arveson's Extension Theorem)

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is CP (completely positive) and $\mathcal{S} \subseteq \mathcal{T}$, then there is a CP map $\psi : \mathcal{T} \to \mathcal{B}(H)$ extending ϕ, i.e.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\phi"] \arrow[d, hook] & \mathcal{B}(H) \\
\mathcal{T} \arrow[ur, "\exists\psi"'] &
\end{tikzcd}$$

We can combine Stinesprings' dilation theorem and Arveson's extension theorem.

Theorem (Arveson-Stinespring Dilation Theorem)

*Let $\mathcal{S}$ be an operator system. Every UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ dilates to a *-representation of $\mathrm{C}^*(\mathcal{S})$.*

Boundary representations and the C*-envelope

Arveson's Philosophy

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2. Decouple the structure of the operator system from any particular representation as operators.

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2. Decouple the structure of the operator system from any particular representation as operators.

Somewhat analogous to the theory of concrete vs abstract C*-algebras, and concrete von Neumann algebras vs W*-algebras.

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Definition

The **C*-envelope** $\mathrm{C}^*_e(\mathcal{S})$ is the C^*-algebra generated by an isomorphic copy $\iota(\mathcal{S})$ of $\mathcal{S}$ with the following universal property:

For every isomorphic copy $\phi(\mathcal{S})$ of $\mathcal{S}$, there is a surjective *-homomorphism

$$\pi : \mathrm{C}^*(\phi(\mathcal{S})) \to \mathrm{C}^*_e(\mathcal{S})$$

such that $\pi \circ j = \iota$, i.e.

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\iota"] \arrow[dr, "\phi"'] & \mathrm{C}^*_e(\mathcal{S}) \\
& \mathrm{C}^*(\phi(\mathcal{S})) \arrow[u, "\pi"']
\end{tikzcd}$$

Example

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. The disk algebra is

$$\begin{aligned} A(\mathbb{D}) &= \{\text{functions analytic on } \mathbb{D} \text{ and continuous on } \partial\mathbb{D} = \mathbb{T}\} \\ &= \overline{\mathbb{C}[z]}^{\|\cdot\|_\infty}. \end{aligned}$$

Clearly $A(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$. But by max. modulus principle, norm on $A(\mathbb{D})$ is completely determined on $\partial\mathbb{D}$, so restriction map $A(\mathbb{D}) \to C(\partial\mathbb{D})$ is completely isometric. But no smaller space norms $A(\mathbb{D})$. Hence $\mathrm{C}^*_e(A(\mathbb{D})) = C(\partial\mathbb{D})$.

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An irreducible representation $\sigma : \mathrm{C}^*(\mathcal{S}) \to \mathcal{B}(H)$ is a **boundary representation** for $\mathcal{S}$ if the restriction $\sigma|_{\mathcal{S}}$ of σ to $\mathcal{S}$ has a *unique* UCP extension to $\mathrm{C}^*(\mathcal{S})$.

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Boundary representations give irreducible representations of $\mathrm{C}_e^*(\mathcal{S})$. So if there are enough boundary representations, then we can use them to construct $\mathrm{C}_e^*(\mathcal{S})$ from $\mathcal{S}$.

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Theorem (Arveson)

If there are sufficiently many boundary representations $\{\sigma_\lambda\}$ to completely norm $\mathcal{S}$, then letting $\sigma = \oplus\sigma_\lambda$,

$$\mathrm{C}_e^*(\mathcal{S}) = \mathrm{C}^*(\sigma(\mathcal{S})).$$

Example

Let $\mathcal{A} \subseteq C(X)$ be a function system. The irreducible representations of $C(X)$ are the point evaluations δ_x for $x \in X$, which are given by representing measures μ on $\mathcal{A}$,

$$f(x) = \int_X f \mathrm{d}\mu, \quad \forall f \in \mathcal{A}.$$

Thus δ_x is a boundary representation for $\mathcal{A}$ if and only if x has a unique representing measure on $\mathcal{A}$. The set of such points is precisely the classical **Choquet boundary** of X with respect to $\mathcal{A}$.

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Arveson calls the set of boundary representations of an operator system $\mathcal{S}$ the **(non-commutative) Choquet boundary.**

Two big problems

In his 1969 paper, Arveson was unable to construct boundary representations, and hence the C*-envelope, in general. The following questions were left unanswered.

Questions

1. Does every operator system have sufficiently many boundary representations?
2. Does every operator system have a C*-envelope?

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Very difficult to "get your hands on" this construction. Does not give boundary representations.

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Theorem (Dritschel-McCullough 2005)

There are maximal representations $\{\sigma_\lambda\}$ such that letting $\sigma = \oplus \sigma_\lambda$,

$$\mathrm{C}^*_e(\mathcal{S}) = \mathrm{C}^*(\sigma(\mathcal{S})).$$

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Theorem (Arveson)

Every separable operator system has sufficiently many boundary representations.

Our results

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Proof is dilation-theoretic and works in complete generality. Very much in the style of Arveson's original work.

A completely positive map ϕ is **pure** if whenever $0 \leq \psi \leq \phi$ implies $\psi = \lambda\phi$.

Lemma (Arveson 1969)

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is pure and maximal, then it extends to a boundary representation.

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Lemma (Arveson 1969)

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is pure and maximal, then it extends to a boundary representation.

Our strategy is to extend a pure UCP map in small steps, taking care to preserve purity, until we attain maximality.

Say a UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is **maximal** at $(s, x) \in \mathcal{S} \times H$ if, whenever $\psi : \mathcal{S} \to \mathcal{B}(K)$ dilates ϕ, $\|\psi(s)x\| = \|\phi(s)x\|$.

Key Lemma

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a pure UCP map and $(s, x) \in \mathcal{S} \times H$, then there is a *pure* UCP map $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ dilating ϕ that is maximal at (s, x).

Theorem

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Easy transfinite induction argument on the key lemma obtains dilation that is maximal at each pair $(s, x) \in \mathcal{S} \times H$.

If $\mathcal{S}$ is separable and $\dim H < \infty$, then can work entirely with finite rank maps.

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First proof uses C*-convexity of matrix states, a Krein-Milman type theorem of Webster-Winkler (1999) for C*-convex sets and a result of Farenick (2000) characterizing matrix extreme points as pure matrix states.

Shorter second proof suggested by Craig Kleski (thanks!) using the fact (following from our results) that the boundary representations of $\mathcal{M}_n(\mathcal{S})$ norm $\mathcal{M}_n(\mathcal{S})$, plus a result of Hopenwasser.

An application of these ideas

Let H be a Hilbert space completion of $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$. Let $M_z = (M_{z_1}, \ldots, M_{z_d})$ denote the d-tuple of multiplication operators on H,

$$M_{z_i} z^\alpha = z_i z^\alpha, \quad \forall \alpha \in \mathbb{N}^d.$$

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Let $I \lhd \mathbb{C}[z]$ be an ideal. Then $I^\perp$ is a coinvariant subspace for M_z, so we can write

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Theorem (Arveson, Müller-Vasilescu)

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Theorem (Arveson, Müller-Vasilescu)

Every contractive d-tuple of commuting operators $A = (A_1, \ldots, A_d)$ *arises in this way for suitable* H.

May need to consider vector-valued polynomials. But many interesting problems reduce to the scalar case.

Conjecture (Arveson-Douglas)

Let H be a "nice" completion of $\mathbb{C}[z]$. Then the d-tuple $A = (A_1, \ldots, A_d)$ is essentially normal, i.e.

$$A_i^* A_j - A_j A_i^* \in \mathcal{K}, \quad 1 \leq i, j \leq d.$$

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Motivation: We should expect connections between the structure of $A = (A_1, \ldots, A_d)$ and the geometric structure of the variety

$$V(I) = \{\lambda \in \mathbb{C}^d \mid p(\lambda) = 0 \ \forall p \in I\}.$$

Example: A positive solution to the Arveson-Douglas conjecture would imply the sequence

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow \mathrm{C}^*(A_1, \ldots, A_d) + \mathcal{K}\,(H) \longrightarrow \mathrm{C}(V(I) \cap \partial\mathbb{B}_d) \longrightarrow 0$$

is exact. The C*-algebra $\mathrm{C}^*(A_1, \ldots, A_d)$ gives rise to an invariant of $V(I)$, conjectured to be the fundamental class of $V(I) \cap \partial\mathbb{B}_d$.

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Theorem (K-Shalit 2013)

Let H be a Besov-Sobolev space (for example, Hardy space, the Bergman space or the Drury-Arveson space), and let $I \lhd \mathbb{C}[z]$ be a homogeneous ideal. Then $A = (A_1, \ldots, A_d)$ is essentially normal if and only if it is hyperrigid.

The future

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In recent years, a great deal of evidence has been compiled showing that noncommutative techniques are needed even in the classical commutative setting.

For example, the disk algebra on the Drury-Arveson space A_d has been much more tractable than the disk algebra on the Hardy space $A(\mathbb{B}_d)$. One explanation is that the C*-envelope of A_d is noncommutative, while the C*-envelope of $A(\mathbb{B}_d)$ is commutative. Classical notions of measure and boundary may not suffice for $d \geq 2$ variables.

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All restrictions have now been removed on the use of Arveson's ideas from 1969. Perhaps we can now realize his vision.

Thanks!