NEST ALGEBRAS WITH LOCALLY CONSTANT COCYCLES

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Introduction

The first examples of triangular AF algebras to be studied were the refinement algebra and the standard algebra. Both are analytic algebras with the property that the cocycle can be taken to be constant on the matrix units of the algebra. The latter property, called local constancy, is quite special and is still ill-understood. In the present paper we examine the class of nest algebras $\mathcal{T}$ in AF C*-algebras which share the distinctive features of the refinement algebra:

(1) $\mathcal{T}$ is a nest algebra in which the nest generates the diagonal, and
(2) $\mathcal{T}$ admits a locally constant cocycle.

Such algebras will be referred to as lcc nest algebras (see Definition 1.6).

Let $\mathfrak{A}$ be the CAR algebra UHF(2^∞), and $(A_n)_{n=1}^\infty$ a sequence of subalgebras of $\mathfrak{A}$ with A_n (isomorphic to) M_{2^n} and the embedding $A_n \to A_{n+1}$ the refinement embedding, which is defined on matrix units by $e_{i\,j}^{(n)} \to e_{2i-1\,2j-1}^{(n+1)} + e_{2i\,2j}^{(n+1)}$, $1 \le i, j \le 2^n$. The canonical form for the cocycle d which is constant on the matrix units is $d(e_{i\,j}^{(n)}) = \frac{j-i}{2^n}$. If τ is the tracial state of $\mathfrak{A}$, then $\tau(e_{k\,k}^{(n)}) = \frac{1}{2^n}$, for $1 \le k \le 2^n$. Thus

$$d\left(e_{i\,j}^{(n)}\right) = \tau\left(\sum_{l=i+1}^{j} e_{l\,l}^{(n)}\right).$$

On the other hand, $\sum_{l=i+1}^{j} e_{l\,l}^{(n)}$ is the interval of the nest with 'left endpoint' $e_{i\,i}^{(n)}$ and 'right endpoint' $e_{j\,j}^{(n)}$. In other words, the value of the cocycle on a matrix unit is the trace of an interval in the nest, the interval being given as the difference of the two elements of the nest which dominate the initial and final projections.

This simple idea carries over to all lcc nest algebras, allowing us to give a canonical form for the cocycle. If $\mathfrak{N}$ is a nest which generates a canonical masa $\mathfrak{D}$ in an AF algebra

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$\mathfrak{A}$, and if $p \in \mathfrak{D}$ is any projection, we will let $\mathcal{P}_{\mathfrak{N}}(p)$ denote the unique minimal projection $P \in \mathfrak{N}$ which dominates p. (If the nest $\mathfrak{N}$ has been fixed, we just write $\mathcal{P}(p)$.) Note that, if $\mathfrak{N}$ does not generate a masa, there need not be a unique minimal nest projection which dominates a given diagonal projection (e.g. Example 1.8). However, if $\mathfrak{N}$ generates $\mathfrak{D}$, then by [PeW; Lemma 3.1] there is a unique decomposition of each projection $p \in \mathfrak{D}$ as a sum of disjoint nest intervals, say $p = P_1 - Q_1 + \cdots + P_r - Q_r$ with $P_1 > Q_1 > \cdots > P_r > Q_r$. Thus, $\mathcal{P}_{\mathfrak{N}}(p) = P_1$. The generic form of the cocycle can now be given: if $\mathfrak{A} = \overline{\cup A_n}$ is a presentation of the simple (unital) AF algebra $\mathfrak{A}$ in which $\mathfrak{N} \cap A_n$ is a maximal nest in A_n $(n \geq 1)$, and for which there is *some* locally constant cocycle, then

$$d_{\mathfrak{N}}(v) = \tau\big(\mathcal{P}_{\mathfrak{N}}(v^*v) - \mathcal{P}_{\mathfrak{N}}(vv^*)\big)$$

(where v is a matrix unit in some A_n, and τ is a tracial state for $\mathfrak{A}$) is also a locally constant cocycle.

It is not *a priori* obvious that the only infinite dimensional examples of lcc nest algebras in UHF C*-algebras are not the refinement algebras, one for each UHF algebra. It so happens, perhaps surprisingly, that there are many such algebras. Further, lcc nest algebras in simple AF C*-algebras with finite trace can be classified by means of source-ordered Bratteli diagrams. We use source order to distinguish this ordering on Bratteli diagrams from the (different) ordering introduced in [Po1] and used in [HPS,PW]. Thus each simple AF algebra with finite trace contains at least one lcc nest algebra. We know, for instance, that the Fibonacci algebra contains at least three distinct lcc nest algebras (Example 3.5); the CAR algebra admits countably many distinct lcc nest algebras (Example 3.3), although we must use countably many Bratteli diagrams. Finally, from our use of the term 'lcc nest algebra' the reader should not infer that the classification of this paper treats all nest algebras which admit locally constant cocycles; Example 1.8 shows that there is a nest algebra in the CAR algebra which admits a locally constant cocycle, yet the nest fails to generate a canonical masa. Such algebras lie outside the scope of this investigation.

Section I contains the theorem giving the canonical form for the cocycle. Examples are included to show that the simplicity assumption of the enveloping AF algebra and the generation assumption on the nest are essential to the theorem. Section II deals with source-ordered Bratteli diagrams. It is shown how a source-ordered diagram defines a unique nest, and conversely. The main result of this section, Theorem 2.8 (which is the main result of the paper) shows that two lcc nest algebras are isomorphic if and only if their corresponding source-ordered Bratteli diagrams are order equivalent. The principal content of Section III is the examples: the notion of a locally minimal path in a source-ordered Bratteli diagram is introduced. This idea is employed to show that there are various distinct lcc nest algebras in the Fibonacci algebra, each of which has the same (unordered) Bratteli diagram. Also in this section is a description of the automorphism group of a nest algebra where the nest generates a canonical masa; this includes lcc nest algebras.

Preliminaries. Suppose that $\mathfrak{A}$ is an AF C*-Algebra containing a canonical masa $\mathfrak{D}$ and $\mathcal{T} \subseteq \mathfrak{A}$ is a closed $\mathfrak{D}$-bimodule. Define the *normaliser* of $\mathfrak{D}$ in $\mathcal{T}$, $N_{\mathfrak{D}}(\mathcal{T})$, by

$$N_{\mathfrak{D}}(\mathcal{T}) = \big\{a \in \mathcal{T} \mid a \text{ is a partial isometry and } a^*da, ada^* \in \mathfrak{D} \text{ for all } d \in \mathfrak{D}\big\}.$$

Let X be the spectrum of $\mathfrak{D}$, the maximal ideal space of $\mathfrak{D}$; for p a projection in $\mathfrak{A}$, let $\hat{p}$ denote $\{x \in X \mid x(p) = 1\}$.

Each $a \in N_{\mathfrak{D}}(\mathfrak{A})$ induces a partial homeomorphism on X, h_a from $\widehat{aa^*}$ to $\widehat{a^*a}$, where $h_a(x)$ is $d \mapsto x(ada^*)$. If $\hat{a}$ denotes the graph of h_a, then we can define a topological binary relation $R(\mathcal{T}) \subset X \times X$ by

$$R(\mathcal{T}) = \bigcup \big\{\hat{a} : a \in N_{\mathfrak{D}}(\mathcal{T})\big\}. \tag{1}$$

Notice that $R(\mathfrak{A})$ is a topological equivalence relation, in fact a groupoid, and $R(\mathcal{T})$ is an open subset of $R(\mathfrak{A})$. In [MS, Theorem 3.10], Muhly and Solel show that there is a 1-1 correspondence between closed $\mathfrak{D}$-bimodules in $\mathfrak{A}$ and such open subsets of $R(\mathfrak{A})$. Further, $R(\mathcal{T})$ is a complete invariant for isometric isomorphism, in that if $R(\mathcal{T})$ and $R(\mathcal{S})$ are isomorphic as topological binary relations, then $\mathcal{T}$ and $\mathcal{S}$ are isometrically isomorphic, [Po2, Theorem 7.5].

Suppose $\mathfrak{A} = \varinjlim(A_n, \alpha_n)$ with each A_n isomorphic to a finite dimensional C*-algebra, i.e., a direct sum of matrix algebras, and $\alpha_n\colon A_{n-1} \to A_n$ is a $*$-homomorphism. We may assume that each α_n is *regular*, that is, maps matrix units to sums of matrix units. As each element of $N_{\mathcal{D}}(\mathcal{T})$ is a partial isometry in $\mathcal{D}$ times a sum of matrix units [Po2, Lemma 5.5], we obtain the same set in (1) if we restrict to those a which are the matrix units of the subalgebras A_n.

Just as $\mathfrak{A}$ is an inductive limit, $R(\mathfrak{A})$ is an inverse limit. For each A_n, let $R(A_n)$ be the set of matrix units of A_n. The set $R(A_n)$ is a groupoid where the product of matrix units e and f is ef if $ef \neq 0$ and is undefined if $ef = 0$. Since α_n is regular, there is a partially defined surjective map $\tilde{\alpha}_n\colon R(A_n) \to R(A_{n-1})$ given by sending each matrix unit in the sum $\alpha_n(e)$ to the matrix unit e. Then

$$R(\mathfrak{A}) = \varprojlim(R(A_n), \tilde{\alpha}_n). \tag{2}$$

We will need two orderings on projections, each different from the usual partial ordering on projections of a C*-algebra. First, consider a triangular algebra $\mathcal{T}$ contained in an AF C*-algebra $\mathfrak{A}$, that is, $\mathcal{T} \cap \mathcal{T}^*$ is a masa in $\mathfrak{A}$. Let $\mathfrak{D} = \mathcal{T} \cap \mathcal{T}^*$. The *diagonal order* on the projections of $\mathfrak{D}$, is given by $e \prec f$ iff there is $v \in N_{\mathfrak{D}}(\mathcal{T})$ with $v^*v = e$ and $vv^* = f$.

Second, if $\mathfrak{N}$ is a nest in $\mathfrak{A}$, then $\mathfrak{N}$ induces an order on the projections in the commutant of $\mathfrak{N}$ by $e \prec_{\mathfrak{N}} f$ iff $p \geq f$ implies $p \geq e$ for all $p \in \mathfrak{N}$. Notice that if $\mathfrak{N}$ is not maximal, then $\prec_{\mathfrak{N}}$ is not anti-symmetric. If $e \prec_{\mathfrak{N}} f$ and $f \prec_{\mathfrak{N}} e$, we call e and f *equivalent* in $\prec_{\mathfrak{N}}$.

I. Locally Constant Cocycles

Definition 1.1. By a *cocycle* for $\mathfrak{A}$ we mean a continuous function $c\colon R(\mathfrak{A}) \to \mathbb{R}$ so that $c(x,z) = c(x,y) + c(y,z)$ for all pairs $(x,y),(y,z) \in R(\mathfrak{A})$.

A subalgebra $\mathcal{T} \subset \mathfrak{A}$ is *analytic by a cocycle* c if $c^{-1}([0,+\infty)) = R(\mathcal{T})$.

We are most interested in cocycles that are *locally constant*, that is, for each point of the domain, there is a neighbourhood of the point on which the cocycle is constant. Algebras analytic by locally constant cocycles are inductive limits of finite-dimensional analytic algebras, in a sense we now make precise.

Definition 1.2. If $\mathfrak{A} = \varinjlim(A_n, \alpha_n)$ is an AF C*-algebra, we call a sequence of cocycles $\{c_n\}$ *compatible* with the presentation of $\mathfrak{A}$ if c_n is a cocycle for $R(A_n)$ and c_n and $c_{n-1} \circ \tilde{\alpha}_n$ agree on the domain of $\tilde{\alpha}_n$.

Although a cocycle c is a function on $R(\mathfrak{A})$, it is convenient to abuse notation by writing $c(\hat{v})$ if c is constant on $\hat{v}$.

Proposition 1.3. *Let $\mathfrak{A}$ be an AF* C*-*algebra.*

If $\mathfrak{A} = \varinjlim(A_n, \alpha_n)$ and $\{c_n\}$ is a compatible sequence of cocycles, then there is a locally constant cocycle c for $\mathfrak{A}$ with $c(\hat{e}) = c_n(e)$ for all $e \in R(A_n)$. Conversely, if c is a locally constant cocycle for $\mathfrak{A}$, then there is a presentation of $\mathfrak{A}$, $\varinjlim(A_n, \alpha_n)$, and a compatible sequence of cocycles $\{c_n\}$ with $c_n(e) = c(\hat{e})$ for all $e \in R(A_n)$.

Proof. Given a presentation, $\varinjlim(A_n, \alpha_n)$, and a compatible sequence of cocycles $\{c_n\}$, c can be constructed by (2) above and the universal property of inverse limits.

Conversely, suppose c is a locally constant cocycle for $\mathfrak{A}$. Choose a presentation $\mathfrak{A} = \varinjlim(A'_n, \alpha'_n)$. Let $A_n = \operatorname{span}\{v \in R(A'_n) \mid c \text{ is constant on } \hat{v}\}$. As $\alpha_n(A_{n-1}) \subset A_n$, we can form a new inductive limit, $\varinjlim(A_n, \alpha_n|_{A_{n-1}}) \subseteq \mathfrak{A}$.

Since c is locally constant on $R(\mathfrak{A})$, for every $a \in N_{\mathfrak{D}}(\mathfrak{A})$ the set $\hat{a}$ can be written as a disjoint union of sets $\hat{v}$ where c restricted to each $\hat{v}$ is constant. In particular, we may choose each v to be a matrix unit in some A_i and so $R(\mathfrak{A}) = \cup\{\hat{v} \mid v \in R(A_n)\}$. Since $R(\mathfrak{A})$ is an isomorphism invariant, $\mathfrak{A} = \varinjlim(A_n, \alpha_n|_{A_{n-1}})$. Define a cocycle, c_n, for each $R(A_n)$ by $c_n(v) = c(\hat{v})$. It follows easily that the sequence $\{c_n\}$ is compatible. $\square$

Lemma 1.4. *Suppose $\mathfrak{A}$ is an AF* C*-*algebra, c is a locally constant cocycle for $\mathfrak{A}$ and $\mathfrak{N} \subset \mathfrak{A}$ is a nest which generates a canonical masa and so that for all $p \in \mathfrak{N}$, c restricted to $\hat{p}$ is zero. Then there is a presentation of $\mathfrak{A} = \overline{\cup_n A_n}$, where each A_n is a finite-dimensional algebra and for all n,*

(1) *c is constant on $\hat{v}$ for all $v \in R(A_n)$, and*
(2) *$\mathfrak{N} \cap A_n$ is a maximal nest in A_n.*

Proof. By Proposition 1.3, we have a presentation of $\mathfrak{A}$, say $\mathfrak{A} = \overline{\cup_n B_n}$, satisfying the first condition.

Let $\mathfrak{D}$ be the canonical masa generated by $\mathfrak{N}$. While $\mathfrak{D} \cap B_n$ need not be generated by $\mathfrak{N} \cap B_n$, each minimal diagonal projection in B_n is a sum of differences of nest projections by Lemma 3.1 of [PeW]. Let E_n be the algebra generated by $\mathfrak{D} \cap B_n$ and all nest projections which appear in the decompositions of minimal projections of B_n.

Let A_n be the C*-algebra generated by E_n and B_n. Clearly, $A_n \cap \mathfrak{D} = E_n$ is a masa in A_n. It is easy to see that $\mathfrak{N} \cap A_n$ generates E_n and so is a maximal nest in A_n. The graphs of matrix units in A_n are restrictions of the graphs of matrix units in B_n (on which c was constant), so c is constant on the graphs of matrix units in each A_n. □

Theorem 1.5. *Let $\mathfrak{A}$ be a simple AF C*-algebra with finite trace τ and $\mathfrak{N}$ be a nest which generates a canonical masa. If $\mathcal{T} = \operatorname{Alg}\mathfrak{N}$ is analytic by a locally constant cocycle c, then there is a presentation of $\mathfrak{A}$, $\varinjlim(A_i, \alpha_i)$, so that $\mathcal{T}$ is also analytic by the locally constant cocycle $d_{\mathfrak{N}}$ satisfying $d_{\mathfrak{N}}(\hat{v}) = \tau\big(\mathcal{P}(v^*v) - \mathcal{P}(vv^*)\big)$ for all matrix units v in each A_i.*

Proof. Let $\varinjlim(A_i, \alpha_i)$ be the presentation given by Lemma 1.4. Let $\mathcal{N}_n = \mathfrak{N} \cap A_n$ and let $\prec_n$ denote the ordering on minimal projections of A_n induced by $\mathcal{N}_n$. Notice that $e \prec f$ implies $e \prec_n f$ but the converse holds if and only if e and f are in the same factor of A_n.

For $n \geq 1$, define $d_n \colon R(A_n) \to \mathbb{R}$ by $d_n(v) = \tau\big(\mathcal{P}(v^*v) - \mathcal{P}(vv^*)\big)$. Clearly, $d_n(v) \geq 0$ if and only if $v \in \operatorname{Alg}\mathfrak{N}$. By the definition of $d_{\mathfrak{N}}$, $d_n(v) = d_{\mathfrak{N}}(\hat{v})$. Thus

$$\mathcal{T} = \operatorname{Alg}\mathfrak{N} = \overline{\bigcup_n \operatorname{Alg}(\mathfrak{N} \cap A_n)} = \overline{\bigcup_n \operatorname{span} d_n^{-1}([0,+\infty))} = \overline{\operatorname{span} d_{\mathfrak{N}}^{-1}([0,+\infty))},$$

so $\mathcal{T}$ is analytic by $d_{\mathfrak{N}}$, provided $d_{\mathfrak{N}}$ is a locally constant cocycle. To show this, we prove that $d_{\mathfrak{N}}$ and the sequence $\{d_n\}$ satisfy the conditions of Proposition 1.3.

The only non-trivial condition is that d_n and $d_{n-1} \circ \tilde{\alpha}_n$ agree. In other words, if $v \in R(A_{n-1})$ and $\alpha_n(v) = \sum_{i=1}^{l} v_i$ with $v_i \in R(A_n)$, then we must show that $d_{n-1}(v) = d_n(v_i)$ for $1 \leq i \leq l$.

Since $\mathcal{N}_{n-1} \subset \mathcal{N}_n$, we have $v_i v_i^* \prec_n v_j^* v_j$ for all i and j. Index the v_i so that

$$v_1 v_1^* \prec_n v_2 v_2^* \prec_n v_3 v_3^* \prec_n \cdots \prec_n v_l v_l^*.$$

<u>Claim</u>: We also have $v_1^* v_1 \prec_n v_2^* v_2 \prec_n v_3^* v_3 \prec_n \cdots \prec_n v_l^* v_l$.

Suppose, to the contrary, that there are i, j with $i < j$ and $v_j^* v_j \prec_n v_i^* v_i$. While it does not follow that $v_j^* v_j \prec v_i^* v_i$, there is some m with $m \geq n$ so that v_i, v_j are partially embedded in the same factor of A_m, since $\mathfrak{A}$ is simple. Let u_i, u_j be the restrictions of v_i, v_j which lie in this factor of A_m. As $\mathcal{N}_n \subset \mathcal{N}_m$, we necessarily have $u_i u_i^* \prec_m u_j u_j^*$ and $u_j^* u_j \prec_m u_i^* u_i$. Since u_i and u_j are in the same factor of A_m, it does follow that

$$u_i u_i^* \prec u_j u_j^* \quad \text{and} \quad u_j^* u_j \prec u_i^* u_i.$$

Let $w, z \in \mathcal{T} \cap A_m$ be the elements implementing these relations, according to the definition of the diagonal order. A short calculation shows that $(w u_j z)(w u_j z)^* = u_i u_i^*$ and

$(wu_jz)^*(wu_jz) = u_i^*u_i$. Thus u_i and wu_jz are matrix units of A_m with the same initial and final projections, and so $u_i = wu_jz$. By our choice of the presentation, c is constant on $\hat{v}$ and hence on $\hat{u}_i$ and $\hat{u}_j$. Thus,

$$c(\hat{u}_i) = c(\widehat{wu_jz}) = c(\hat{w}\hat{u}_j\hat{z}) = c(\hat{w}) + c(\hat{u}_j) + c(\hat{z}).$$

Since $i \neq j$, $w, z \notin \mathfrak{D}$ and so $c(\hat{w}), c(\hat{z}) > 0$. Thus $c(\hat{u}_i) > c(\hat{u}_j)$. However, since c is constant on $\hat{v}$, $c(\hat{u}_i) = c(\hat{u}_j)$. This contradiction proves the claim.

Let $\{e_i\}_{1 \leq i \leq [n]}$ be the diagonal matrix units of A_n, indexed according to the total ordering $\prec_n$. Suppose $v_lv_l^* = e_{i_0}$ and $v_l^*v_l = e_{i_1}$. Then

$$\alpha_n\big(\mathcal{P}(v^*v) - \mathcal{P}(vv^*)\big) = \sum_{i_0 < i \leq i_1} e_i = \mathcal{P}(v_l^*v_l) - \mathcal{P}(v_lv_l^*)$$

so $d_{n-1}(v) = d_n(v_l)$.

Suppose $e \in A_{n-1}$ is a diagonal matrix unit. As we ordered the e_i with the ordering induced by $\mathcal{N}_n$ and $\mathcal{N}_{n-1} \subset \mathcal{N}_n$, it follows that $\alpha_n(e)$ is a consecutive sequence of the e_i, i.e., $\alpha_n(e) = \sum_{i=a}^{b} e_i$. By the claim, $v_{l-1}^*v_{l-1} \prec_n v_l^*v_l$ and $v_{l-1}v_{l-1}^* \prec_n v_lv_l^*$, so $v_{l-1}v_{l-1}^* = e_{i_0-1}$ and $v_{l-1}^*v_{l-1} = e_{i_1-1}$. Thus,

$$\mathcal{P}(v_{l-1}^*v_{l-1}) - \mathcal{P}(v_{l-1}v_{l-1}^*) = \sum_{i_0 \leq i < i_1} e_i = \mathcal{P}(v_l^*v_l) - \mathcal{P}(v_lv_l^*) + v_{l-1}v_{l-1}^* - v_{l-1}^*v_{l-1}.$$

Since $\tau(v_{l-1}v_{l-1}^* - v_{l-1}^*v_{l-1}) = 0$, $d_n(v_{l-1}) = d_{n-1}(v)$. Similarly, $d_n(v_i) = d_{n-1}(v)$ for all i. $\square$

Definition 1.6. Let $\mathfrak{A}$ be a simple (unital) AF algebra with finite trace. A nest algebra $\operatorname{Alg}\mathfrak{N}$ in $\mathfrak{A}$ which satisfies the assumptions of Theorem 1.5 will be called an *lcc nest algebra.*

The next two examples show that the assumptions that $\mathfrak{A}$ is simple and that $\mathfrak{N}$ generates a canonical masa are necessary.

Example 1.7. This shows that the assumption of simplicity is needed in Theorem 1.5. Let $A_n = \sum_{i=1}^{n} \oplus A_{n,i}$ where $A_{n,i} = M_2$ for all n and i. The embedding $\alpha_n \colon A_n \to A_{n+1}$ is given by $a_1 \oplus \cdots \oplus a_n \mapsto a_1 \oplus \cdots \oplus a_n \oplus a_n$. Let $\mathcal{N}_n$ be the maximal nest in the diagonal of A_n given by ordering the minimal diagonal projections as follows:

$$\begin{array}{ll}
\mathcal{N}_1 & \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \\[1em]
\mathcal{N}_2 & \begin{bmatrix} 1 & \\ & 4 \end{bmatrix} \oplus \begin{bmatrix} 2 & \\ & 3 \end{bmatrix} \\[1em]
\mathcal{N}_3 & \begin{bmatrix} 1 & \\ & 6 \end{bmatrix} \oplus \begin{bmatrix} 2 & \\ & 5 \end{bmatrix} \oplus \begin{bmatrix} 3 & \\ & 4 \end{bmatrix}
\end{array}$$

and so on. Note that $\alpha_n(\mathcal{N}_n) \subset \mathcal{N}_{n+1}$ for all n. Also, $\mathfrak{A} = \varinjlim(A_n, \alpha_n)$ is a finite trace AF C*-algebra. Indeed, the (normalised) trace of the central projection of $A_{n,i}$ is $1/2^i$ for $1 \le i < n$ and that of $A_{n,n}$ is $1/2^{n-1}$.

Letting $\mathfrak{N} = \cup_n \mathcal{N}_n$, it is clear that $\mathrm{C}^*(\mathfrak{N})$ is the diagonal and that $\operatorname{Alg}\mathfrak{N}$ is the inductive limit of $\operatorname{Alg}\mathcal{N}_n$. We claim that $\operatorname{Alg}\mathfrak{N}$ is analytic by a locally constant cocycle, d. Let $e_{j,k}^{(n,i)}$ be the (j,k) matrix unit of $A_{n,i}$. Set $d(e_{j,k}^{(n,i)}) = k - j$.

To show, finally, that the formula $d_{\mathfrak{N}} = \tau(\mathcal{P}(v^*v) - \mathcal{P}(vv^*))$ does not define a locally constant cocycle, take $v = e_{1,2}^{(1,1)}$. Then $e_{1,2}^{(1,1)} = e_{1,2}^{(2,1)} + e_{1,2}^{(2,2)}$ while $d_{\mathfrak{N}}(e_{1,2}^{(2,1)}) = 3/4$ and $d_{\mathfrak{N}}(e_{1,2}^{(2,2)}) = 1/4$. Similarly, $e_{1,2}^{(2,2)} = e_{1,2}^{(3,2)} + e_{1,2}^{(3,3)}$, $d_{\mathfrak{N}}(e_{1,2}^{(3,2)}) = 3/8$ and $d_{\mathfrak{N}}(e_{1,2}^{(3,3)}) = 1/8$. Continuing, we see that

$$d_{\mathfrak{N}}\left(e_{1,2}^{(n,n-1)}\right) > \frac{1}{2^n}, \qquad d_{\mathfrak{N}}\left(e_{1,2}^{(n,n)}\right) = \frac{1}{2^n}.$$

Since $e_{1,2}^{(n,n-1)}$ and $e_{1,2}^{(n,n)}$ are submatrix units of $e_{1,2}^{(1,1)}$, we never reach stability and so the formula does not give a locally constant cocycle.

Can there be some other presentation, say $\mathfrak{A} = \overline{\cup_n B_n}$, for which the cocycle formula works? One can see that B_n can be taken as $\mathrm{C}^*(A_n, D_{m_n})$ for some $m_n > n$, where D_m is the diagonal of A_m. (This is a general fact, not particular to this example.) But observe that in this example $\mathrm{C}^*(A_n, D_{n+1}) = A_{n+1}$. Hence if the formula $d_{\mathfrak{N}}$ gives a locally constant cocycle for $\operatorname{Alg}\mathfrak{N}$ in any presentation, then it gives a locally constant cocycle in the presentation $\varinjlim(A_n, \alpha_n)$.

Example 1.8. This example shows that there is a maximal nest $\mathcal{N}$ such that $\operatorname{Alg}\mathcal{N}$ is analytic TAF, but $\mathcal{N}$ does not generate the diagonal. Moreover, $\operatorname{Alg}\mathcal{N}$ is analytic by a locally constant cocycle. (Examples of strongly maximal, non-analytic nest algebras whose nests don't generate the diagonal are already known– e.g., the $\mathcal{T}_\alpha$ of [PeW].)

Take $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \phi_n)$, where $\mathfrak{A}_n = M_{2^n}$ and $\phi_n = \operatorname{Ad}P_n \circ \rho_n$, $\rho_n : M_{2^n} \mapsto M_{2^{n+1}}$ is the refinement embedding, and P_n is the $2^{n+1} \times 2^{n+1}$ permutation matrix such that $\operatorname{Ad}P_n$ interchanges the $2^n, 2^n+1$ rows and the $2^n, 2^n+1$ columns. Let $\{e_{ij}^{(n)}\}_{1 \le i,j \le 2^n}$ be matrix units for $\mathfrak{A}_n$; write the diagonal matrix units as $e_i^{(n)}$. Set $p_0^{(n)} = 0, p_i^{(n)} = \sum_{j=1}^i e_j^{(n)}, 1 \le i \le 2^n$. Let $\mathcal{N}_n = \{p_i^{(n)} : 0 \le i < 2^{n-1}, \text{ or } 2^{n-1} < i \le 2^n\}$. Set $\mathcal{N} = \cup \mathcal{N}_n$. (Note: $\phi_n(\mathcal{N}_n) \subset \mathcal{N}_{n+1}$.) Let T_n be the upper triangular subalgebra of $\mathfrak{A}_n$. Observe that $\phi_n(T_n) \subset T_{n+1}$. Indeed, $\phi_n(e_{ij}^{(n)}) = \rho_n(e_{ij}^{(n)})$ if $i,j \notin \{2^{n-1}, 2^{n-1}+1\}$. Now $\phi_n(e_{2^{n-1}\,2^{n-1}+1}^{(n)}) = e_{2^n-1\,2^n}^{(n+1)} + e_{2^n+1\,2^n+2}^{(n+1)} \in T_{n+1}$. Similarly, elements of the form $e_{i\,2^{n-1}}^{(n)}, e_{i\,2^{n-1}+1}^{(n)}, e_{2^{n-1}\,i}^{(n)}, e_{2^{n-1}+1\,i}^{(n)}$ in T_n map under ϕ_n to T_{n+1} .

<u>Claim</u>: $\operatorname{Alg}\mathcal{N}$ is TAF; that is, $\operatorname{Alg}\mathcal{N} \cap \mathfrak{A}_n = T_n$. Now $\operatorname{Alg}\mathcal{N} \cap \mathfrak{A}_n \subset \operatorname{Alg}\mathcal{N}_n \cap \mathfrak{A}_n$, and the only lower triangular matrix unit of the algebra on the right is $e_{2^{n-1}+1\,2^{n-1}}^{(n)}$. However this is mapped by ϕ_n to a sum of two matrix units in $\mathfrak{A}_{n+1}$, one of which is $e_{2^n+2\,2^n+1}^{(n+1)}$, which is not in $\operatorname{Alg}\mathcal{N}_{n+1} \cap \mathfrak{A}_{n+1}$. This completes the claim.

Claim: $\mathcal{N}$ is a maximal nest. Let p be a projection in $\mathfrak{A}$ such that $\{p\} \cup \mathcal{N}$ is a nest. Since p commutes with every projection in $\mathcal{N}$, $p \in (\operatorname{Alg}\mathcal{N}) \cap (\operatorname{Alg}\mathcal{N})^*$. Thus, $p \in \mathfrak{D}$ and so $p \in D_n$, the diagonal of T_n, for some n. Since $\mathcal{N}_n \cup \{p\}$ must be a nest, it follows that either p is $p^{(n)}_{2^{n-1}}$ or $p^{(n)}_{2^{n-1}-1} + e^{(n)}_{2^{n-1}+1}$. In the second case, $\phi_n(p) = p^{(n+1)}_{2^n-2} + e^{(n+1)}_{2^n} + e^{(n+1)}_{2^n+2}$ so that $\mathcal{N}_{n+1} \cup \{\phi_n(p)\}$ is not a nest. In the first case, $\phi_n(p) = p^{(n+1)}_{2^n-1} + e^{(n+1)}_{2^n+1}$ and again $\mathcal{N}_{n+1} \cup \{\phi_n(p)\}$ is not a nest. Thus, in neither case can $\{p\} \cup \mathcal{N}$ be a nest.

To show that $\mathrm{C}^*(\mathcal{N}) \neq \mathfrak{D}$, observe that $e^{(1)}_1 \notin \mathrm{C}^*(\mathcal{N})$. Indeed, if $e^{(1)}_1$ were in $\mathrm{C}^*(\mathcal{N})$, then by Lemma 3.1 of [PeW] $e^{(1)}_1$ could be written as a sum of disjoint nest intervals (uniquely). Thus, we would have $e^{(1)}_1$ in $\mathrm{C}^*(\mathcal{N}_n)$ for some n in $\mathbb{Z}^+$. As $e^{(1)}_1$ is a sum of $e^{(n)}_i$ which either includes $e^{(n)}_{2^{n-1}-1}, e^{(n)}_{2^{n-1}+1}$ but not $e^{(n)}_{2^{n-1}}$ (n even) or else includes $e^{(n)}_{2^{n-1}}, e^{(n)}_{2^{n-1}+2}$ but not $e^{(n+1)}_{2^{n-1}+1}$ (n odd), we have a contradiction, since any interval in $\mathcal{N}_n$ which contains one of $e^{(n)}_{2^{n-1}}, e^{(n)}_{2^{n-1}+1}$ must contain the other.

It is now straightforward to apply Prop. 3.20 and Thm. 3.16 of [PePW2] to conclude that $\operatorname{Alg}\mathcal{N}$ is analytic. It remains only to show that $\operatorname{Alg}\mathcal{N}$ is analytic by a locally constant cocycle. By Proposition 1.3, it suffices to give a sequence $\{c_n\}$ so that c_n is a cocycle for $\mathfrak{A}_n$ with $\operatorname{span} c_n^{-1}([0, +\infty)) = T_n$ for each n and the sequence is compatible with $\varinjlim(\mathfrak{A}_n, \phi_n)$.

To specify a cocycle $c_n \colon R(\mathfrak{A}_n) \to \mathbb{R}$ with $\operatorname{span} c_n^{-1}([0, +\infty)) = T_n$, it suffices to give the $(2^n - 1)$-tuple of positive values $(c_n(e^{(n)}_{1,2}), \dots, c_n(e^{(n)}_{2^n-1,2^n}))$. For clairity, let $x(n,i) = c_n(e^{(n)}_{i,i+1})$ for $i = 1, \dots, 2^n - 1$. Then compatibility between c_{n-1} and c_n requires that

$$\begin{aligned} x(n, 2^{n-1}+1) = x(n, 2^{n-1}-1) &= x(n-1, 2^{n-2}) \\ x(n, 2^{n-1}) + x(n, 2^{n-1}+1) + x(n, 2^{n-1}+2) &= x(n-1, 2^{n-2}+1) \\ x(n, 2^{n-1}-2) + x(n, 2^{n-1}-1) + x(n, 2^{n-1}) &= x(n-1, 2^{n-2}-1) \\ x(n, 2i-1) + x(n, 2i) &= x(n-1, i) \end{aligned}$$

for $i = 1, \dots, 2^{n-2}-1, 2^{n-2}+2, \dots, 2^{n-1}-1$, and

$$x(n, 2i) + x(n, 2i+1) = x(n-1, i)$$

for $i = 1, \dots, 2^{n-2}-2, 2^{n-2}+1, \dots, 2^{n-1}-1$.

Let $y_n = x(n, 2^{n-1})$ for all n. Solving the above equations in terms of y_n, y_{n-1}, y_{n-2} and $x(n-1, j)$ for $j = 1, \dots, 2^{n-1}-1$, gives

$$x(n, 2^{n-1}) = y_n, \qquad x(n, 2^{n-1} \pm 1) = y_{n-1}, \qquad x(n, 2^{n-1} \pm 2) = y_{n-2} - (y_{n-1} + y_n)$$

and $x(n,i)$ for all other values of i is either $y_{n-1} + y_n$ (for i odd) or $x(n-1, i/2) - (y_{n-1} + y_n)$ (for i even). By induction on n, all $x(n,i)$ are determined by the sequence $\{y_n\}$.

Define $\{y_n\}$ by $y_1 = 1$ and $y_{n+1} = y_n/2^n$ for $n \geq 1$. We will show by induction on n that for all $n \geq 4$ and all i in $\{1, \dots, 2^{n-1}-2, 2^{n-1}+2, \dots, 2^n - 1\}$, $x(n,i) \geq y_{n-1} + y_n$. Notice that this holds for $n = 4$ since $x(4,i)$, for $i = 1, \dots, 15$ are respectively

$$\frac{9}{64}, \frac{31}{64}, \frac{9}{64}, \frac{15}{64}, \frac{9}{64}, \frac{23}{64}, \frac{1}{8}, \frac{1}{64}, \frac{1}{8}, \frac{23}{64}, \frac{9}{64}, \frac{15}{64}, \frac{9}{64}, \frac{31}{64}, \frac{9}{64}.$$

while $y_3 + y_4 = 9/64$. Suppose the conclusion holds for $x(n-1,i)$; we prove it for $x(n,i)$ for all required values of i. Since $y_{n-2} \geq 2(y_{n-1} + y_n)$ for $n \geq 4$, the formula for $x(n, 2^{n-1} \pm 2)$ above proves the conclusion for $i = 2^{n-1} \pm 2$. For all odd i with $i \neq 2^{n-1} - 1$, $2^{n-1} + 1$, it holds trivially, since $x(n,i) = y_n + y_{n-1}$ in this case. For i even with $i \neq 2^{n-1} - 2$, 2^{n-1}, $2^{n-1} + 2$, we have

$$x(n,i) = x(n-1, i/2) - (y_{n-1} + y_n) \geq (y_{n-2} + y_{n-1}) - (y_{n-1} + y_n)$$

by our inductive hypothesis. Since $y_{n-2} + y_{n-1} \geq 2(y_{n-1} + y_n)$, the inequality follows, finishing the induction.

In particular, it follows that all $x(n,i)$ are positive. Hence we have constructed a compatible sequence of cocycles for $\varinjlim(\mathfrak{A}_n, \phi_n)$ and so $\operatorname{Alg}\mathcal{N}$ is analytic by a locally constant cocycle.

Remarks. 1. Notice that the formula for the canonical cocycle $d_{\mathfrak{N}}$ is ill-defined for this nest. Indeed, $\mathcal{P}_{\mathfrak{N}}(e^{(1)}_{1\,1})$ is not defined, as there is no unique minimal nest projection majorizing $e^{(1)}_{1\,1}$.

2. This example shows that the existence of a locally constant cocycle for a nest algebra does not imply that the nest generates the diagonal. An lcc nest algebra is, by assumption, one in which both conditions are satisfied.

3. Example 1.8 suggests a duality between the condition, for a nest TAF algebra analytic by a locally constant cocycle, that the nest generates the diagonal and the condition, for a $\mathbb{Z}$-analytic TAF algebra, that the counting cocycle is continuous. Indeed, consider the algebra given by changing the maps $\phi_n = \operatorname{Ad} P_n \circ \rho_n$ above to $\operatorname{Ad} P_n \circ \sigma_n$ where $\sigma_n \colon M_{2^n} \to M_{2^{n+1}}$ is the standard embedding (given by $a \mapsto a \oplus a$). This is the example in [PePW2, p. 396] due to Hopenwasser and Donsig and shown in [PePW2] to be a $\mathbb{Z}$-analytic TAF algebra with discontinuous counting cocycle.

II. Source-Ordered Bratteli Diagrams

In this section, we construct a correspondence between lcc nest algebras and Bratteli diagrams with a partial order. We need some preliminary results about finite-dimensional algebras and diagrams.

Definition 2.1. Given non-empty finite sets V and W, a *diagram* from V to W is a finite set E and maps $r \colon E \to W$ and $s \colon E \to V$. A *source ordering* on a diagram (E, r, s) is a partial order on E such that $e, e' \in E$ are comparable if and only if $s(e) = s(e')$.

Call two diagrams (E, r, s) and (E', r', s') *isomorphic* if there is a bijection $\Phi: E \to E'$ so that $r'(\Phi(e)) = r(e)$ and $s'(\Phi(e)) = s(e)$. Call them (*source*) *order isomorphic* (denoted $\cong^{ord}$) if they have source orderings and Φ preserves this ordering.

As usual, we think of $e \in E$ as an edge from $s(e)$ to $r(e)$. It is well-known that diagrams induce $*$-homomorphisms.

Examples. The simplest interesting diagram is $V = \{1\}$, $W = \{1\}$ and E is some finite set; r and s are then determined. Each source ordering on (E, r, s) is a total ordering of E and any pair of source orderings are order isomorphic.

Another diagram which we will use in Section III, the Fibonacci diagram, has $V = \{1, 2\}$ and $W = \{1, 2\}$. There is one edge from 1 to 1, one from 1 to 2, and one from 2 to 1. There are two (non-isomorphic) source orderings on this diagram, depending on how we order to two edges from 1.

To save repetition, we fix the following notation.

Context. Let (E, r, s) be a diagram from $V = \{1, 2, \ldots, a\}$ and $W = \{1, 2, \ldots, b\}$. Let

$$A = \bigoplus_{i=1}^{a} M_{m_i}, \qquad B = \bigoplus_{j=1}^{b} M_{n_j} \qquad \text{where } n_j = \sum_{r(e)=j} m_{s(e)}.$$

Let $\mathcal{M}$ be a maximal nest in the diagonal of A.

Let $\phi: A \to B$ be a unital regular $*$-homomorphism induced by the diagram (E, r, s) and let τ_A and τ_B be traces on A and B so that $\tau_B \circ \phi = \tau_A$.

Lemma 2.2. *In the Context above, there is a surjective map* $\Omega: \{f \mid f = f^* \in R(B)\} \to E$ *so that*

(1) $s(\Omega(f)) = i$ *if and only if* $f \leq \phi(1_{M_{m_i}})$,
(2) $r(\Omega(f)) = j$ *if and only if* $f \in M_{n_j}$, *and*
(3) $\Omega(f) = \Omega(g)$ *if and only if* $\phi(v) f \phi(v)^* = g$ *for some* $v \in R(A)$.

Moreover, Ω *is unique up to conjugation by unitary in the commutant in* B *of* $\phi(A)$.

Proof. Let $\{e_{k,l}^{(i)}, 1 \leq k, l \leq m_i, 1 \leq i \leq a\}$ be a system of matrix units for A. Fix a summand of B, say M_{n_j}, and consider the image in this summand of $\phi(e_{1,1}^{(i)})$, where $1 \leq i \leq a$. To satisfy condition (2), every diagonal matrix unit f in this summand of B must be assigned an edge e with $r(e) = j$. To satisfy condition (1), each matrix unit f subordinate to $\phi(e_{1,1}^{(i)})$ must be assigned an edge e with $s(e) = i$. Since ϕ is induced by the diagram (E, r, s), the number of edges e with $r(e) = j$ and $s(e) = i$ equals the number of diagonal matrix units in M_{n_j} subordinate to $\phi(e_{1,1}^{(i)})$. Define Ω on these matrix units by choosing any bijection onto $\{e \in E \mid r(e) = j, s(e) = i\}$.

To define Ω on the matrix units of M_{n_j} subordinate to $\phi(e_{k,k}^{(i)})$, conjugate these matrix units by $\phi(e_{k,1}^{(i)})$. This maps them to matrix units of M_{n_j} subordinate to $\phi(e_{1,1}^{(i)})$. By condition (3), we must define Ω on the original matrix units to agree with their conjugates subordinate to $\phi(e_{1,1}^{(i)})$. Letting k, i and j vary we have defined Ω, as required.

The only choice in constructing Ω lies in the choice of bijection between the matrix units of M_{m_j} subordinate to $\phi(e_{1,1}^{(i)})$ and the set $\{e \in E \mid r(e) = j, s(e) = i\}$. It is a straightforward calculation to show there is a permutation unitary in the commutant of $\phi(A)$ that carries any one of these bijections onto any other. □

Definition 2.3. The *nest induced by a source order* on (E, r, s) is the nest $\mathcal{N} \subset B$ with $\phi(\mathcal{M}) \subset \mathcal{N}$ so that if $f, g \in B$ are diagonal matrix units equivalent in $\prec_{\phi(\mathcal{M})}$, then $f \prec_{\mathcal{N}} g$ if and only if $\Omega(f)$ is less than $\Omega(g)$ in the source order.

Formally, $\mathcal{N}$ depends on the choice of Ω but changing Ω gives a nest equivalent by a unitary in the commutant of $\phi(A)$.

To make this concrete, consider the diagram from $V = \{1\}$ to $W = \{1\}$ with n edges, mentioned above. If $\phi\colon M_m \to M_{mn}$ is the $*$-homomorphism induced by the diagram, conjugated so that $\operatorname{Alg}\mathcal{M} = T_m$ and $\operatorname{Alg}\mathcal{N} = T_{mn}$, then ϕ is the refinement embedding of multiplicity n.

Given $A = \oplus_{i=1}^{a} M_{m_i}$ with maximal nest $\mathcal{M}$ and trace τ_A, there is a cocycle induced by $\mathcal{M}$, $d_{\mathcal{M}}$, given by

$$d_{\mathcal{M}}(v) = \tau_A\big(P(v^*v) - P(vv^*)\big)$$

where $P(p) = \inf\{m \in \mathcal{M} \mid m \geq p\}$.

Theorem 2.4. *In the Context above, each source order on (E, r, s) induces a maximal nest $\mathcal{N} \subset B$ so that $\phi(\operatorname{Alg}\mathcal{M}) \subset \operatorname{Alg}\mathcal{N}$ and $d_{\mathcal{N}}$ and $d_{\mathcal{M}} \circ \tilde{\phi}$ agree on the domain of $\tilde{\phi}$.*

Further, this is a bijection between source orders on (E, r, s) (up to order isomorphism) and maximal nests $\mathcal{N}$ so that $\phi(\mathcal{M}) \subset \mathcal{N}$ and $d_{\mathcal{N}}$ and $d_{\mathcal{M}} \circ \tilde{\phi}$ agree on the domain of $\tilde{\phi}$ (up to conjugation by unitaries in the commutant of $\phi(A)$).

Proof. First, we show that $\mathcal{N}$ is maximal. Suppose f, g are diagonal matrix units of B. Since $\mathcal{M}$ is a maximal nest and ϕ is unital, either f and g are ordered by $\phi(\mathcal{M})$ or they are both subordinate to the image of one matrix unit in A. By condition (1) of Lemma 2.2, $\Omega(f)$ and $\Omega(g)$ are comparable in the source order. If $\Omega(f) = \Omega(g)$, then condition (3) implies that either f and g are subordinate to the images of different matrix units or they are equal. Thus, $\mathcal{N}$ totally orders the matrix units of B and so is maximal.

That $\phi(\operatorname{Alg}\mathcal{M}) \subset \operatorname{Alg}\mathcal{N}$ is a routine consequence of $\phi(\mathcal{M}) \subset \mathcal{N}$ and the maximality of $\mathcal{M}$ in A.

Let f be a matrix unit of B in the domain of $\tilde{\phi}$ with $g = \tilde{\phi}(f)$ and $F = \phi(g)$. Since

$$d_{\mathcal{M}}(g) = \tau_A(\mathcal{P}_{\mathcal{M}}(g^*g) - \mathcal{P}_{\mathcal{M}}(gg^*)) = \tau_B \circ \phi(\mathcal{P}_{\mathcal{M}}(g^*g) - \mathcal{P}_{\mathcal{M}}(gg^*))$$

$$= \tau_B\big(\mathcal{P}_{\phi(\mathcal{M})}(\phi(g)^*\phi(g)) - \mathcal{P}_{\phi(\mathcal{M})}(\phi(g)\phi(g)^*)\big) = \tau_B\big(\mathcal{P}(F^*F) - \mathcal{P}(FF^*)\big),$$

to show $d_{\mathcal{M}}(g) = d_{\mathcal{N}}(f)$ it suffices to show that

$$\tau_B\big(\mathcal{P}(F^*F) - \mathcal{P}(FF^*)\big) = \tau_B\big(\mathcal{P}(f^*f) - \mathcal{P}(ff^*)\big).$$

This is equivalent to

$$\tau_B\big(\mathcal{P}(F^*F) - \mathcal{P}(f^*f)\big) = \tau_B\big(\mathcal{P}(FF^*) - \mathcal{P}(ff^*)\big). \tag{3}$$

Observe that $\mathcal{P}(F^*F) - \mathcal{P}(f^*f)$ is the sum of all diagonal matrix units k with k a restriction of F^*F and $\Omega(k) > \Omega(f^*f)$. Similarly, $\mathcal{P}(FF^*) - \mathcal{P}(ff^*)$ is the sum of all diagonal matrix units h with h a restriction of FF^* and $\Omega(h) > \Omega(ff^*)$. Conditions (1) and (3) of Lemma 2.2 imply that conjugation by $F = \phi(g)$ carries the set of such k to the set of such h. Thus, equation (3) holds and so $d_{\mathcal{N}}$ and $d_{\mathcal{M}} \circ \tilde{\phi}$ agree.

Suppose $\mathcal{N} \subset B$ is a maximal nest so that $\phi(\mathcal{M}) \subset \mathcal{N}$ and $d_{\mathcal{N}}$ and $d_{\mathcal{M}} \circ \tilde{\phi}$ agree on the domain of $\tilde{\phi}$. Let $\Omega: \{f \mid f = f^* \in R(B)\} \to E$ be as in Lemma 2.2. Suppose f_1, f_2 and g_1, g_2 are two pairs of diagonal matrix units in B with each pair equivalent in $\prec_{\phi(\mathcal{M})}$. Also, suppose that $e_i = \Omega(f_i) = \Omega(g_i)$ for $i = 1, 2$.

Claim: If $s(e_1) = s(e_2)$ and $f_1 \prec_{\mathcal{N}} f_2$ then $g_1 \prec_{\mathcal{N}} g_2$. Since f_1 and f_2 are equivalent in $\prec_{\phi(\mathcal{M})}$ they are subordinates of $\phi(a)$ for some diagonal matrix unit $a \in A$. Similarly, g_1 and g_2 are subordinates of $\phi(b)$ for some matrix units b. Since $s(e_1) = s(e_2)$, a and b appear in the same summand of A and so there is a matrix unit $v \in A$ with initial projection one of a or b and final projection the other. Suppose $v^*v = a$ and $vv^* = b$; the other case is similar. Let v_i be the matrix unit of B with $v_i^*v_i = f_i$ and $v_iv_i^* = g_i$, for $i = 1, 2$; notice that each v_i is a restriction of $\phi(v)$.

Since $d_{\mathcal{N}}$ agrees with $d_{\mathcal{M}} \circ \tilde{\phi}$, $d_{\mathcal{N}}(v_1) = d_{\mathcal{N}}(v_2)$. In other words, the number of minimal projections in $\mathcal{P}(g_1) - \mathcal{P}(f_1)$ equals the number in $\mathcal{P}(g_2) - \mathcal{P}(f_2)$. Since $f_1 \prec_{\mathcal{N}} f_2$, $\mathcal{P}(f_2) \geq \mathcal{P}(f_1)$ and hence $\mathcal{P}(g_2) \geq \mathcal{P}(g_1)$. This is exactly $g_1 \prec_{\mathcal{N}} g_2$, proving the claim.

Define a source ordering on (E, r, s) by ordering edges $e_1, e_2 \in E$ with $s(e_1) = s(e_2)$ as f_1 and f_2 are ordered by $\prec_{\mathcal{N}}$ where f_1, f_2 are equivalent in $\prec_{\phi(\mathcal{M})}$ and $\Omega(f_i) = e_i$ for $i = 1, 2$. By the claim, this is well-defined. Since $\mathcal{N}$ is maximal, this gives a source ordering. $\square$

To fix notation, here is a formal definition of a Bratteli diagram, following [PW].

Definition 2.5. A unital *Bratteli diagram* is a pair $(\mathcal{V}, \mathcal{E})$ where

$$\mathcal{V} = V_0 \cup V_1 \cup \cdots,$$

a disjoint union of finite sets with V_0 a singleton, and

$$\mathcal{E} = \{(E_n, r_n, s_n) \mid n \geq 1\}$$

with each (E_n, r_n, s_n) a diagram from V_{n-1} to V_n. We assume that (E_0, r_0, s_0) consists of one edge to each element of V_1. A *source ordering* on a Bratteli diagram is a set of source orderings, one for each diagram.

Let $AF(\mathcal{V}, \mathcal{E})$ denote $\varinjlim(A_n, \alpha_n)$ where $A_0 = \mathbb{C}$ and $\alpha_n: A_{n-1} \to A_n$ is the unital $*$-homomorphism induced by (E_n, r_n, s_n).

Notice that A_n is determined by A_{n-1}, the diagram (E_n, r_n, s_n) and the fact that α_n is unital.

Theorem 2.6. *Let $(\mathcal{V},\mathcal{E})$ be a Bratteli diagram so that $AF(\mathcal{V},\mathcal{E})$ is simple and has a tracial state. Then each source order on $(\mathcal{V},\mathcal{E})$ induces a nest $\mathfrak{N}$ so that $\mathfrak{N}$ generates a canonical masa and* $\operatorname{Alg}\mathfrak{N}$ *is analytic by a locally constant cocycle.*

Conversely, if $\mathfrak{N}$ is such a nest in a simple AF C*-*algebra* $\mathfrak{A}$ *where $\mathfrak{A}$ has a trace, then there is a Bratteli diagram $(\mathcal{V},\mathcal{E})$ with a source order so that $AF(\mathcal{V},\mathcal{E}) = \mathfrak{A}$ and $\mathfrak{N}$ is induced by the source order.*

Proof. For $(\mathcal{V},\mathcal{E})$, let $\varinjlim(A_n,\alpha_n)$ be the presentation given by $(\mathcal{V},\mathcal{E})$. By Theorem 2.4, each source order on (E_n,r_n,s_n) gives a maximal nest N_n in A_n so that $N_{n-1} \subset N_n$ for all n and the sequence of cocycles induced by N_n, say d_n, are compatible. If $\mathfrak{N} = \cup_n N_n$, then by Proposition 1.3, $\operatorname{Alg}\mathfrak{N}$ is analytic by $d_{\mathfrak{N}}$. Since each N_n is maximal in A_n, $\mathfrak{N}$ generates a canonical masa.

Conversely, if $\mathfrak{N}$ is such a nest in $\mathfrak{A}$, then by Theorem 1.5, there is presentation of $\mathfrak{A}$, say $\varinjlim(A_n,\alpha_n)$, so that the cocycles in A_{n-1} and A_n induced by $\mathfrak{N}\cap A_{n-1}$ and $\mathfrak{N}\cap A_n$ are compatible. If $(\mathcal{V},\mathcal{E})$ is the Bratteli diagram that for this presentation, then by Theorem 2.4, there is a source order on each (E_n,r_n,s_n) so that this source order and $\mathfrak{N}\cap A_{n-1}$ induces $\mathfrak{N}\cap A_n$. Thus, we have a Bratteli diagram $(\mathcal{V},\mathcal{E})$ with $AF(\mathcal{V},\mathcal{E}) = \mathfrak{A}$ and a source order on the Bratteli diagram that induces $\mathfrak{N}$. □

Definitions 2.7. Given two diagrams with source orderings, (E_1,r_1,s_1) from V_1 to V_2 and (E_2,r_2,s_2) from V_2 to V_3, the *contraction* is a diagram (E,r,s) from V_1 to V_3 with

$$E = \{(e_1,e_2) \in E_1 \times E_2 \,|\, r_1(e_1) = s_2(e_2)\},$$

$s(e_1,e_2) = s_1(e_1)$ and $r(e_1,e_2) = r_2(e_2)$. The source ordering is given by comparing edges in E_1 and if they are equal, then edges in E_2.

Denote E by $E_2 \circ E_1$.

Two Bratteli diagrams with source orderings, $(\mathcal{V},\mathcal{E})$ and $(\mathcal{W},\mathcal{F})$, are *order equivalent* if there are strictly increasing functions $f,g\colon \mathbb{N}\to\mathbb{N}$ with $f(0)=g(0)=0$ and diagrams with source orderings E'_n from V_n to $W_{f(n)}$ and F'_n from W_n to $V_{g(n)}$ so that

$$F'_{f(n)} \circ E'_n \cong^{ord} E_{g(f(n))} \circ \cdots \circ E_{n+1}$$

and

$$E'_{g(n)} \circ F'_n \cong^{ord} F_{f(g(n))} \circ \cdots \circ F_{n+1}$$

for every n (recall $\cong^{ord}$ is from Definition 2.1).

Theorem 2.8. *Two lcc nest algebras are isomorphic by a trace-preserving isometric isomorphism if and only if the associated Bratteli diagrams with their source orderings are order equivalent.*

Proof. If two Bratteli diagrams with source orderings are order equivalent, it is straightforward to show that there is a trace-preserving isometric isomorphism between the associated algebras. Applying Theorem 2.4 to each E'_n and F'_n in the definition of order equivalence gives corresponding maps between the finite-dimensional algebras of the two AF C*-algebras. The universal property of inductive limits now gives the required isomorphism. By Theorem 2.4, the intertwining maps are compatible with the cocycles and so the isomorphism carries one lcc nest algebra to the other.

Conversely, suppose there is an trace preserving isometric isomorphism, $\Phi: \mathfrak{A} \to \mathfrak{B}$, and two nests generating canonical masas, $\mathfrak{M} \subset \mathfrak{A}$ and $\mathfrak{N} \subset \mathfrak{B}$, so that $\operatorname{Alg} \mathfrak{M}$ is analytic by a locally constant cocycle c and $\operatorname{Alg} \mathfrak{N}$ is analytic by a locally constant cocycle d. By Theorem 1.5, we can choose presentations $\mathfrak{A} = \varinjlim(A_i, \alpha_i)$ and $\mathfrak{B} = \varinjlim(B_i, \beta_i)$ so that c and d are constant on the matrix units of each A_i and B_i respectively, and agree with the cocycles induced by the maximal nests $\mathfrak{N} \cap A_i$ and $\mathfrak{M} \cap B_i$. By Theorem 2.6, there are Bratteli diagrams with source orderings associated to these two presentations.

A short argument using any of [PeW, Corollary 1.14], [Po2, Theorem 5.3] or [DH, Theorem 19] shows that there are increasing functions $f, g: \mathbb{N} \to \mathbb{N}$ and families of embeddings $\phi_n: A_n \to B_{f(n)}$ and $\psi_n: B_n \to A_{g(n)}$ so that

$$\psi_{f(n)} \circ \phi_n = \alpha_{g(f(n))} \circ \cdots \circ \alpha_{n+1} \tag{4}$$

and

$$\phi_{g(n)} \circ \psi_n = \beta_{f(g(n))} \circ \cdots \circ \beta_{n+1} \tag{5}$$

for each n. Moreover, we can identify each A_n with a subalgebra of $\mathfrak{A}$ and each B_n with a subalgebra of $\mathfrak{B}$ so that $\phi_n = \Phi|_{A_n}$ and $\psi_n = \Phi^{-1}|_{B_n}$.

Since $\Phi(\operatorname{Alg} \mathfrak{M}) = \operatorname{Alg} \mathfrak{N}$, Φ is an isomorphism, and maximal nests are reflexive [PeW, Corollary 2.5], we have that $\Phi(\mathfrak{M}) = \mathfrak{N}$. Hence for a matrix unit $v \in A_i$, we have $c(v) = d(\Phi(v))$ and similarly for a matrix unit $v \in B_i$. Thus each ϕ_n and ψ_n is compatible with the cocycles induced by the appropriate nests and so by Theorem 2.4, is induced by a source ordering on the appropriate diagram. By replacing each map in equations (4) and (5) with the corresponding diagram with its source ordering, we have that the diagrams for $\mathfrak{A}$ and $\mathfrak{B}$ are order equivalent. $\square$

Example 2.9. Let $(\mathcal{V}, \mathcal{E})$ be a Bratteli diagram where each vertex set is a singleton. It is easy to see that all source orderings on $(\mathcal{V}, \mathcal{E})$ are equivalent since there is only one way, up to order isomorphism, to order the edges of each diagram in $(\mathcal{V}, \mathcal{E})$. Given a source ordering, let $\mathfrak{N}$ be the nest in $AF(\mathcal{V}, \mathcal{E}) = \varinjlim(A_n, \alpha_n)$ produced by Theorem 2.6. If for each n we arrange $A_n \cap \operatorname{Alg} \mathfrak{N}$ to be the upper-triangular matrices in A_n, then each α_n is a refinement embedding, as noted before. Thus $\operatorname{Alg} \mathfrak{N}$ is the (unique) refinement embedding algebra in the UHF C*-algebra $AF(\mathcal{V}, \mathcal{E})$.

It follows by Theorem 2.8 that a Bratteli diagram with a source order is order equivalent to a single vertex diagram if and only if the associated nest algebra is a refinement embedding algebra. Later, we will give a Bratteli diagram $(\mathcal{V}, \mathcal{E})$ so that $AF(\mathcal{V}, \mathcal{E})$ is a UHF C*-algebra and yet there is a source order on $(\mathcal{V}, \mathcal{E})$ that does not give a refinement embedding algebra.

III. Automorphisms, Inadequacy of ordered K_0, & Examples

We begin this section by describing the automorphism group of an lcc nest algebra in terms of the multiplicative group of $\mathbb{T}$-valued cocycles. This description is, in fact, valid for any nest algebra in which the nest generates a canonical masa. Then we turn to the question of whether the algebraic order on K_0-group of an lcc nest algebra is a complete invariant for isomorphism (it is not). The final example illustrates some of the constructions applied to an lcc nest algebra in the Fibonacci AF C*-algebra.

Theorem 3.1. *Let $\mathfrak{A}$ be a simple AF* C***-algebra with finite trace. Let $\mathfrak{N}$ be a nest in $\mathfrak{A}$ which generates (as a* C***-algebra) a canonical masa. Then*

$$\mathrm{Aut}\,(\mathrm{Alg}\,\mathfrak{N}) \cong Z^1(R(\mathfrak{A}), \mathbb{T}),$$

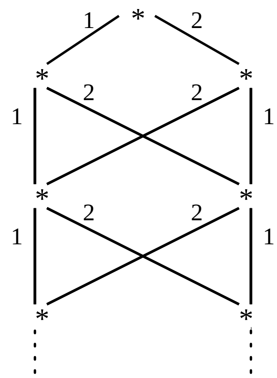
the multiplicative group of $\mathbb{T}$-valued cocycles on the groupoid $R(\mathfrak{A})$.

Proof. By [PePW2, Theorem 4.6], $\mathrm{Alg}\,\mathfrak{N}$ is analytic, hence strongly maximal. Let $\Phi \in \mathrm{Aut}\,(\mathrm{Alg}\,\mathfrak{N})$; by [Po2, Theorem 8.3], Φ extends to an automorphism of the enveloping C*-algebra $\mathfrak{A}$. If τ is a normalised trace on $\mathfrak{A}$, then so is $\tau \circ \Phi$, and by the uniqueness of the normalised trace in a simple C*-algebra we have that $\tau = \tau \circ \Phi$. By [PeW, Proposition 2.8], $\Phi(\mathfrak{N}) = \mathfrak{N}$. (Note that the results of [PeW] are stated in the context of UHF C*-algebras, but many of the results, and in particular this Proposition, are valid in the context of simple AF C*-algebras.) The map $\tau\colon \mathfrak{N} \to \mathbb{R}\colon P \mapsto \tau(P)$ is one to one and as $\tau \circ \phi(P) = \tau(P)$, we have $\Phi(P) = P$ for all $P \in \mathfrak{N}$. As the nest $\mathfrak{N}$ generates a canonical masa, Φ leaves this masa pointwise fixed. The conclusion now follows from [V, Theorem 3.5]. $\square$

Corollary 3.2. *Let $\mathfrak{A}, \mathfrak{B}$ be simple AF* C***-algebras with tracial states and let $\mathfrak{M} \subset \mathfrak{A}$, $\mathfrak{N} \subset \mathfrak{B}$ be nests which generate canonical masas. If $\mathrm{Alg}\,\mathfrak{M} \cong \mathrm{Alg}\,\mathfrak{N}$, then the isomorphism is unique, up to a $\mathbb{T}$-valued cocycle.*

Due to the rigid structure of lcc nest algebras, it is natural to ask if an invariant such as the algebraic order on the scale of the K_0-group [Po1] is able to distinguish all isomorphism classes. Before turning to this question, we mention a related matter. We say a TAF algebra $\mathfrak{S}$ is *strongly maximal in factors* if there is a sequence of finite-dimensional factors $\{A_n\}_{n=1}^{\infty}$ and unital $*$-homomorphisms $\phi_n\colon A_n \to A_{n+1}$ such that $S_n \equiv \mathfrak{S} \cap A_n$ is maximal triangular in A_n, $\phi_n(S_n) \subseteq S_{n+1}$ and $\mathfrak{S} = \varinjlim(S_n, \phi_n)$. Recall that there is only one source ordering on a single vertex Bratteli diagram (Example 2.9) and that such Bratteli diagrams give UHF C*-algebras. Do these facts imply that, in a given UHF C*-algebra, there is a unique lcc nest algebra which is strongly maximal in factors? The following example provides a negative answer to both questions.

Example 3.3. Consider the source-ordered Bratteli diagram



where we use the integer near each edge to indicate its ordering in the set $s^{-1}(v)$ for $v \in \mathcal{V}$.

It gives finite-dimensional algebras $A_0 = \mathbb{C}$, $A_{n+1} = M_{2^n} \oplus M_{2^n}$ for $n \geq 0$ and inductive limit $\mathfrak{A} = UHF(2^\infty)$. Let $\mathfrak{N}$ be the nest defined by the diagram and $\operatorname{Alg}\mathfrak{N}$ the lcc nest algebra in $\mathfrak{A}$. Denote by $\mathfrak{D}$ the canonical masa $\mathrm{C}^*(\mathfrak{N})$ and by $\mathcal{D}_n$ the masa $\mathfrak{D} \cap A_n$ in A_n. The embeddings defined by the Bratteli diagram can be factored

$$\cdots A_n \longrightarrow M_{2^n} \longrightarrow \mathrm{C}^*(M_{2^n}, \mathcal{D}_{n+1}) = A_{n+1} \longrightarrow M_{2^{n+1}} \longrightarrow A_{n+2} \cdots$$

Ordering the diagonal matrix units in A_n so that $\operatorname{Alg}\mathfrak{N} \cap A_n$ is upper-triangular, then the induced map $\phi_n \colon M_{2^n} \to M_{2^{n+1}}$ takes the upper-triangular subalgebra T_{2^n} to $T_{2^{n+1}}$. Hence $\operatorname{Alg}\mathfrak{N} = \varinjlim(T_{2^n}, \phi_n)$.

The maps ϕ_n can be described in several ways. For one, let I and V be the 2×2 matrices $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and define a sequence of permutation matrices $P_n \in M_{2^{n+1}}$ as follows

$$P_1 = \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}, \qquad P_{n+1} = \begin{bmatrix} P_n & 0 \\ 0 & \tilde{P}_n \end{bmatrix} \quad (n \geq 0),$$

where $\tilde{P}_n$ is the matrix P_n with all V and I interchanged. Then $\phi_n = \operatorname{Ad} P_n \circ \rho_n$ where $\rho_n \colon M_{2^n} \to M_{2^{n+1}}$ is the refinement embedding.

Alternatively, ϕ_n can be defined using the function $\chi \colon \mathbb{Z}^+ \to \{0, 1\}$ given by $\chi(k) = \sum a_i \pmod 2$ if $k = \sum_i a_i 2^i$ with all $a_i \in \{0, 1\}$. Then

$$\phi_n(e_{i,j}^{(n)}) = \begin{cases} e_{2i-1,2j-1}^{(n+1)} + e_{2i,2j}^{(n+1)} & \text{if } \chi(i-1) = \chi(j-1) \\ e_{2i-1,2j}^{(n+1)} + e_{2i,2j-1}^{(n+1)} & \text{if } \chi(i-1) \neq \chi(j-1) \end{cases}$$

Here $\{e_{i,j}^{(n)} \mid 1 \leq i,j \leq 2^n\}$ is a system of matrix units for M_{2^n}. If $\{f_{i,j}^{(n)} \mid 1 \leq i,j \leq 2^n\}$ is another system of matrix units for M_{2^n} then the map $e_{i,i}^{(n)} \mapsto f_{i,i}^{(n)}$ extends to an ismorphism between the diagonals of $\operatorname{Alg}\mathfrak{N}$ and the refinement algebra $\mathfrak{T} = \varinjlim(T_{2^n}, \rho_n)$. This isomorphism is a diagonal order isomorphism in the sense of [PePW1, Section 3] Equivalently, it gives an isomorphism of the scales of the K_0-groups that preserves the algebraic order.

We conclude this example by showing that $\operatorname{Alg}\mathfrak{N}$ is not isometrically isomorphic to the refinement algebra. It suffices to show the Bratteli diagram given above is not order equivalent to a single vertex Bratteli diagram. First, we make

Definition 3.4. Let $(\mathcal{V}, \mathcal{E})$ be a Bratteli diagram. A *path* consists of a sequence $\{e_n\}_{n=1}^{\infty}$ with $e_n \in E_n$ and $r(e_n) = s(e_{n+1})$ for $n \geq 1$. If $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ are paths, we say $\{e_n\}_{n=1}^{\infty}$ is *less than* $\{f_n\}_{n=1}^{\infty}$ if e_j is less than f_j in the source ordering on E_j, where $j = \min\{i : e_i \neq f_i\}$. The assumption that the vertex set V_0 is a singleton implies that any two paths are comparable and that there is a unique minimal and a unique maximal path.

Call a path $\{e_n\}_{n=1}^{\infty}$ *locally minimal* if there is some $N \in \mathbb{Z}^+$ so that if $\{f_n\}_{n=1}^{\infty}$ is a path with $f_n = e_n$ for all $n < N$, then $\{f_n\}_{n=1}^{\infty}$ is less than $\{e_n\}_{n=1}^{\infty}$. Observe that a locally minimal path is, under order equivalence of Bratteli diagrams, sent to a locally minimal path. Also, locally maximal paths can be defined similarly.

Two paths, $\{e_n\}_{n=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$, are *equivalent* if there is some $N \in \mathbb{Z}^+$ with $e_n = f_n$ for all $n \geq N$. It is not hard to see that a path equivalent to locally minimal path must itself be locally minimal. We call an equivalence class of paths *locally minimal* if one path in the class is locally minimal (and hence all paths in the class are locally minimal).

Returning to Example 3.3, we see that there are two equivalence classes of paths which are locally minimal. Clearly, a single vertex diagram can have only one locally minimal equivalence class. It is also clear that order equivalence of Bratteli diagrams preserves locally minimal equivalence classes. Thus, the source-ordered Bratteli diagram of the example is not order isomorphic to a single vertex Bratteli diagram and hence $\operatorname{Alg}\mathfrak{N}$ is not isometrically isomorphic to the refinement algebra.

Remark. The fact that the two algebras are not isomorphic is also a consequence of [PePW1, Theorem 4.3 (ii)].

For a single vertex Bratteli diagram, we have shown that there is only one source ordering and hence, up to isomorphism, only one lcc nest algebra associated with such a diagram. For more complex Bratteli diagrams, it is to be expected that the diagram might be ordered in different ways so as not to be order equivalent. That would mean that there are non-isomorphic lcc nest algebras with identical (unordered) Bratteli diagrams.

For example, the Bratteli diagram of Example 3.3 can be given a different source order, so that the lcc nest algebra it gives is the refinement algebra. So there are two non-isomorphic lcc nest algebras with this Bratteli diagram. Fix an integer n and consider the stationary Bratteli diagram with 2^n vertices at each level, and one edge between each pair of vertices on adjacent levels. Such a Bratteli diagram can have up to 2^n locally minimal equivalence classes and so gives at least 2^n distinct lcc nest algebras. The source-ordered Bratteli diagrams with more than 2^{n-1} locally minimal equivalence classes can not be obtained from the Bratteli diagrams given by smaller choices of n. Hence there are at least countably many non-isomorphic lcc nest algebras in the CAR algebra, although we have used countably many Bratteli diagrams.

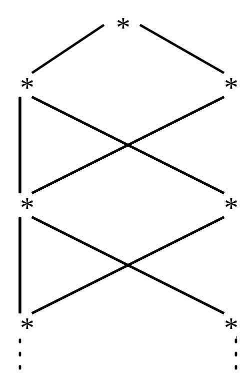
Some of the algebras in the next example have already appeared in the literature, as nest subalgebras associated to one-sided Markov chains [Po2, Chapter 9]. In particular, the nest algebras A_U^+ and A_V^+ of [Po2, Examples 9.9 (2)] arise as two of the nest subalgebras considered below.

As all nest subalgebras associated to one-sided Markov chains are lcc nest algebras, we describe the connection between Power's construction and source-ordered Bratteli diagrams. The construction of [Po2, Section 9.8] depends on a square matrix with entries either zero or one; call this matrix U and suppose it is r by r. Consider a unital Bratteli diagram with 1 vertex on the first level and r vertices on all other levels. For the first level, there is one edge from the first level vertex to each vertex on the second level. For other levels, there is an edge between the ith vertex of one level and the jth vertex of the next if and only if the (i, j) entry of U is one. In other words, U is the adjacency matrix for each level after the first. The source ordering on the diagram is given by setting (i, j) less than (i, k) if $j < k$. It is not hard to show that this source-ordered Bratteli diagram gives the same nest algebra as Power's construction.

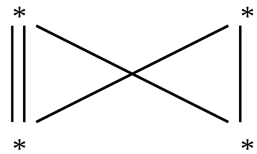
We should point out that there are lcc nest algebras which cannot be obtained by in this way—for instance, Example 3.3. If it could be so obtained, then the source-ordered Bratteli diagram given by the argument above would have a single locally minimal equivalence class (this uses the fact that the generated C*-algebra is simple). On the other hand, the Bratteli diagram given in Example 3.3 has two locally minimal equivalence classes, contradicting the order equivalence of the two Bratteli diagrams.

Example 3.5. *The Fibonacci Algebra.*

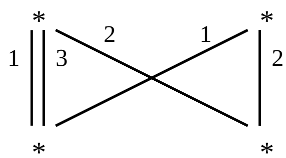
This is the AF C*-algebra given by the diagram below. We call this diagram the 'standard' diagram, for just as in the UHF case, there may be other diagrams which are not contractions of this diagram and yield the same AF C*-algebra. Denote the finite dimensional subalgebras associated with (the vertex sets of) this diagram by $A_n, n \geq 0$. They have the form $A_0 = \mathbb{C}$, $A_n = M_{p_n} \oplus M_{p_{n-1}}, (n \geq 1)$, where $p_0 = 1, p_1 = 1, p_n = p_{n-1} + p_{n-2}, (n \geq 2)$ is the Fibonacci sequence.

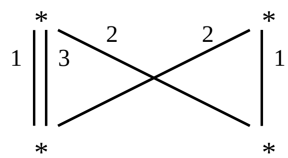


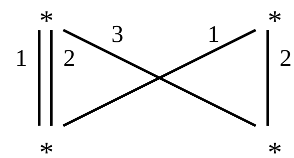
If we contract the Bratteli diagram to the odd vertex sets, we obtain a diagram which, after its first step, is stationary and is based on



If we make the source ordering stationary as well, there are six possible orderings. Notice that the three diagrams

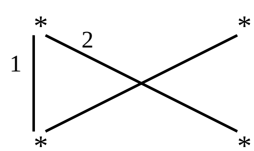


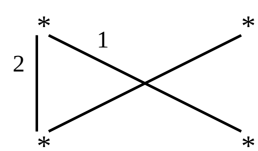




give Bratteli diagrams with, respectively, two locally maximal equivalence classes and one locally minimal, one locally maximal and two locally minimal, and lastly one locally maximal and one locally minimal. (The other three possible orderings yield no new combinations.) Thus, the Fibonacci algebra contains at least three non-isomorphic lcc nest algebras associated with the standard diagram.

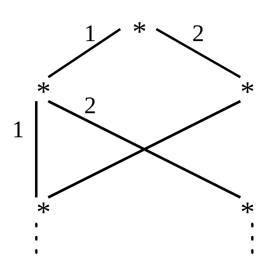
We have not been able to determine if there are only finitely many lcc nest algebras associated with the standard Fibonacci diagram. (Of course, there may be other lcc nest algebras associated with other diagrams, analogous to Example 3.3.) There is the relation $A \circ B \circ B = B \circ A \circ A$ where A and B are, respectively, the diagrams





However, for contractions of up to 13 copies of A or B, Maple calculations show that all other relations are generated by this relation. (The three stationary source orderings above correspond to $A \circ A$, $B \circ B$, and $B \circ A$, respectively.)

We conclude this example by illustrating how a source-ordered Bratteli diagram determines the embeddings and the canonical cocycle. Consider the stationary Bratteli diagram based on A (which is order equivalent to the one based on $A \circ A$ above). It is



Let $\mathfrak{A} = \overline{\cup_n A_n}$ and $\mathfrak{N}$ be the nest in $\mathfrak{A}$ derived from the diagram. If $\mathcal{N}_n = \mathfrak{N} \cap A_n$, then

$$\begin{aligned}
\mathcal{N}_0 &= \{0, 1\} \\
\mathcal{N}_1 &= \{0, [1] \oplus [0], 1\} \\
\mathcal{N}_2 &= \left\{0, \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \oplus [0], \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \oplus [1], 1\right\} \\
\mathcal{N}_3 &= \left\{0, \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & \\ & 0 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}, 1\right\}
\end{aligned}$$

The embeddings $\phi_n : A_n \to A_{n+1}$ are completely determined by the conditions:

(1) ϕ_n is a unital C*-embedding which maps matrix units to sums of matrix units, and
(2) $\phi_n(\mathrm{Alg}_n \mathcal{N}_n) \subseteq \mathrm{Alg}_{n+1} \mathcal{N}_{n+1}$ (where $\mathrm{Alg}_n \mathcal{N}_n = A_n \cap \mathrm{Alg}\, \mathcal{N}_n$).

For example, $\phi_2 : A_2 \to A_3$ is given by

$$\phi_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus [e]\right) = \begin{bmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{bmatrix} \oplus \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Now $\mathfrak{A}$ is a simple AF C*-algebra with finite trace τ whose values on minimal projections in $A_n = M_{p_n} \oplus M_{p_{n-1}}$ are given in the following way. Let x_n be the trace of a minimal projection in the first factor, M_{p_n}, of A_n. It follows from the diagram that x_{n+1} is also the trace of a minimal projection in second factor, $M_{p_{n-1}}$, of A_n, If $\xi = (\sqrt{5} - 1)/2$, then

$$x_1 = \xi, \qquad x_2 = 1 - \xi, \qquad x_{n+2} = x_n - x_{n+1} \quad (n > 0),$$

Let $\{e_{j,k}^{(n,i)}\}$ be a system of matrix units for $A_n = M_{p_n} \oplus M_{p_{n+1}}$. Then we can compute, for example,

$$d_{\mathfrak{N}}(e_{2,3}^{(3,1)}) = \tau\left(\mathcal{P}_{\mathfrak{N}}\left(e_{3,3}^{(3,1)}\right) - \mathcal{P}_{\mathfrak{N}}\left(e_{2,2}^{(3,1)}\right)\right).$$

where $\mathcal{P}_{\mathfrak{N}}(p)$ is the smallest projection in $\mathfrak{N}$ dominating p. Hence

$$\mathcal{P}_{\mathfrak{N}}(e_{3,3}^{(3,1)}) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \qquad \mathcal{P}_{\mathfrak{N}}(e_{2,2}^{(3,1)}) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$$

and so

$$d_{\mathfrak{N}}(e_{2,3}^{(3,1)}) = \tau\left(\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & \\ & 0 \end{bmatrix}\right) = x_3 = 2\xi - 1 = \sqrt{5} - 2.$$

On the other hand,

$$d_{\mathfrak{N}}(e_{1,2}^{(3,1)}) = \tau\left(\begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}\right) = x_3 + x_4 = 1 - \xi = \frac{3 - \sqrt{5}}{2}.$$

Thus, the cocycle $d_{\mathfrak{N}}$ need not take the same value on the superdiagonal matrix units of a factor.

Acknowledement. The nest algebra of Example 3.3 first arose in discussions with Alan Hopenwasser, as the inductive limit $\varinjlim(T_{2^n}, \phi_n)$. We also thank him for his comments on an earlier version of this paper.

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