Due: April 29th

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f has the intermediate value property.

   NOTE: of course, f is not continuous.

2. By directly verifying the $\delta - \epsilon$ condition, show that $f(x) = x^2$ is uniformly continuous on $[0, 5]$.

3. (a) Show that if $f : D \to \mathbb{R}$ is uniformly continuous on D and $S \subseteq D$ is a bounded set, then f is a bounded function on S, that is $\{f(x) : x \in S\}$ is a bounded set. HINT: assume not and use the Bolzano-Weierstrass Theorem and Theorem 19.4 (uniformly continuous functions preserve Cauchy sequences).

   (b) Use part (a) to give another proof that $f(x) = 1/x^2$ is not uniformly continuous on $(0, 1)$.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, that is, there is $d > 0$ so that $f(x + d) = f(x)$ for all $x \in \mathbb{R}$. Show that there are $x_{\min}, x_{\max} \in \mathbb{R}$ so that for all $x \in \mathbb{R}$,

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}).$$

   Further, show that f is uniformly continuous on $\mathbb{R}$.
   HINT: For the last part, show f is uniformly continuous on $[0, 2d]$ and use translation.

5. EXTRA CREDIT: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that f cannot take every real value exactly twice. Give an example to show that f can take every real value exactly three times.