Math 106, Sect. 350/550 Answers to Test #3 29 Nov '06

. 20 1. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. If the cross-sections perpendicular to the x-axis are semi-circles with diameters running from the x-axis to the curve $y = 2\sqrt{\sin x}$, then find the volume of this solid.

Draw a suitable diagram.

Solution. This is a variation on problem 5 in Section 6.1 (semi-circular cross-sections instead of triangles or squares), an assigned homework problem.

For a slice at a fixed value of x, the endpoints of the base in the x, y-plane are $(x, 0)$ and $(x, 2\sqrt{\sin x})$. The area of a semi-circle $\pi r^2/2$, so we need to find the radius, which is half the distance from $(x, 0)$ to $(x, 2\sqrt{\sin x})$, i.e., $\sqrt{\sin x}$. Thus, the area of each slice is $(\pi \sin x)/2$.

The limits of integration are $x = 0$ and $x = \pi$.

The volume of the solid is

$$V = \int_0^\pi \frac{\pi \sin x}{2}\, dx = \frac{\pi}{2} \int_0^\pi \sin x = \frac{\pi}{2} - \cos x\Big|_0^\pi = \frac{\pi}{2} (-\cos \pi + \cos 0) = \pi.$$

. 22 2. Using suitable substitutions, evaluate the following integrals:

$$\text{(a)} \int \frac{1}{x \ln x}\, dx \qquad \text{(b)} \int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^x \cos e^x\, dx$$

Solution. Part (a) is Problem 43 from Section 5.5, an assigned homework problem; part (b) is Problem 23 on page 380.

For (a), we use the substitution $u = \ln x$, with $du = \frac{1}{x}\, dx$, to get the indefinite integral

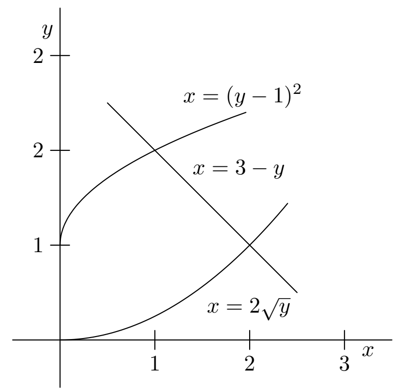
$$\int \frac{1}{u}\, du = \ln |u| + C = \ln |\ln x| + C$$

For (b), we use the substitution $u = e^x$, with $du = e^x\, dx$. Notice that if $x = \ln(\pi/6)$, then $u = \pi/6$ and if $x = \ln(\pi/2)$ then $u = \pi/2$. Thus

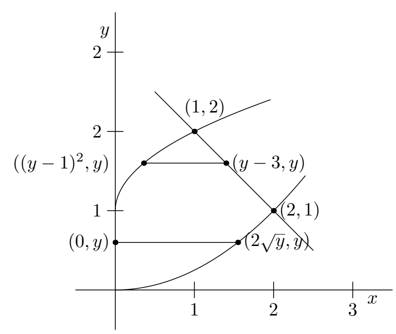
$$\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^x \cos e^x\, dx = \int_{\pi/6}^{\pi/2} 2 \cos x\, dx = 2 \sin x\Big|_{\pi/6}^{\pi/2} = 2\sin(\pi/2) - 2\sin(\pi/6) = 1.$$

. 15 3. Set up the integral(s) for the following area **BUT DO NOT evaluate** the integral(s).

The area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$ and above right by the line $x = 3 - y$.



Solution. This is the problem from Homework 10, except you don't have to evaluate the integral.



Notice that the intersection points of $x = 2\sqrt{y}$ and $x = 3 - y$ is $(2, 1)$ as can be checked by substituting the point into both curves. Similarly the intersection point of $x = (y-1)^2$ and $x = 3 - y$ is $(1, 2)$. For y from 0 to 1, the upper endpoint is $(2\sqrt{y}, y)$ and the lower endpoint is $(0, y)$, so the length is $2\sqrt{y}$. For y from 1 to 2, the upper endpoint is $(3 - y, y)$ and the lower endpoint is $((y-1)^2, y)$, so the length is $3 - y - (y-1)^2 = 2 + y - y^2$. Thus, the area is $\int_0^1 2\sqrt{y}\,dy + \int_1^2 2 + y - y^2\,dy$.

. 13

4. Use l'Hôpital's Rule to evaluate $\lim_{\theta \to \pi/2} \dfrac{1 - \sin\theta}{1 + \cos 2\theta}$.

   Be sure to justify the use of l'Hôpital's Rule.

   Solution. This is Problem 19 from Section 4.6, an assigned homework problem.
   Notice that $1 - \sin(\pi/2) = 0$ and $1 + \cos(\pi) = 0$, so this limit is a $0/0$ indeterminate form and we can apply l'Hôpital's Rule.

$$\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{1+\cos 2\theta} = \lim_{\theta\to\pi/2}\frac{-\cos(\theta)}{-2\sin 2\theta} \qquad \frac{-\cos(\pi/2)}{-2\sin(\pi)} = \frac{0}{0}$$

$$= \lim_{\theta\to\pi/2}\frac{\sin(\theta)}{-4\cos 2\theta} \qquad \text{by l'Hôpital's Rule again}$$

$$= \frac{\sin(\pi/2)}{-4\cos(\pi)} = \frac{1}{4}.$$

. 10 5. Consider the function $f(x) = 3 - x^2$ on the interval $[0, 1]$. Set up in Σ-notation **BUT DO NOT evaluate** the Riemann sum for this function using a partition of $[0, 1]$ into n equal subintervals and the righthand rule.

Solution. This is part of Homework 9 with the function $3x^2$ replaced by $3 - x^2$ and the interval $[0, 2]$ by $[0, 1]$.
We divide the interval $[0, 1]$ into n intervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right] \left[\frac{2}{n}, \frac{3}{n}\right], \ldots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

So, for each "rectangle" we have a base of $1/n$.
We are evaluating the function at righthand endpoint of each interval, i.e., $1/n$ on $[0, 1/n]$, $2/n$ on $[1/n, 2/n]$, and so on. Thus, the formula for the kth term, which is for the interval $[(k-1)/n, k/n]$, is

$$f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \left(3 - \frac{k^2}{n^2}\right)\frac{1}{n} = \frac{3}{n} - \frac{k^2}{n^3}$$

The Riemann sum then is $\displaystyle\sum_{k=1}^{n} \frac{3}{n} - \frac{k^2}{n^3}$.

. 20 6. You are designing a rectangular poster to contain 100 in^2 of printing with an 8 inch margin at top and bottom and 2 in margin at each side. What overall dimensions for the piece of paper will minimize the amount of paper used?

Be sure to draw a relevant diagram and name your variables.

Solution. This is a variation on Problem 11 from Section 4.5, an assigned homework problem.
Let the height and width of the poster be x and y respectively. We want to minimize the area, which is $A = xy$.
Then the area available for printing is $(x-16)(y-4)$, and this has to be 100 sq. in., so $(x-16)(y-4) = 100$. Solving this equation for y, we have $y = 4 + 100/(x-16)$. Substituting this into the the formula, we have

$$A(x) = x\left(4 + \frac{100}{x-16}\right) = 4x + \frac{100x}{x-16}.$$

Notice that x must be greater than 16 and can be as large as we like, so the domain is $(16, +\infty)$. Differentiating, we have

$$A' = 4 + \frac{(x-16)100 - 100x}{(x-16)^2} = 4 - \frac{1600}{(x-16)^2} = \frac{4(x-16)^2 - 1600}{(x-16)^2}.$$

In order for the derivative to be zero, we must have

$$\begin{aligned} 4(x-16)^2 - 1600 &= 0 \\ (x-16)^2 &= 400 \\ (x-16) &= 20 \\ x &= 36 \end{aligned}$$

and

$$y = 4 + \frac{100}{36-16} = 9$$

To see that this answer is indeed a minimum, observe that

$$A''(x) = \frac{d}{dx}\left(4 - \frac{1600}{(x-16)^2}\right) = \frac{3200}{(x-16)^3}$$

and so $A''(36) > 0$, so by the second derivative test, $A(x)$ has a minimum at $x = 36$.