Math 325-001 **Assignment # 9** **1 Apr 09**

Due: April 8th

1. For both of the following, decide if the series converges or diverges, proving your answer.

$$\text{a)} \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}, \qquad \text{b)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}.$$

2. Find two series, one divergent and one convergent, so that the root test does not show whether or not the series converges or diverges, i.e., $\sum_{n=1}^{\infty} a_n$ so that $\limsup |a_n|^{1/n} = 1$.

3. Suppose that (a_n) is a sequence of nonnegative numbers. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

4. From class, we know that $\sum_{n=1}^{\infty} (-1)^n/n = \ln 2$. The goal of this exercise is to show that the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots$$

converges to $(\ln 2)/2$. This series can be written in the form $\sum_{n=1}^{\infty} a_n$ with $a_{3n-2} = 1/(2n-1)$, $a_{3n-1} = -1/(4n-2)$, and $a_{3n} = -1/4n$. Let $s_n = \sum_{k=1}^{n} a_k$.

   (a) Show that $a_{3n-2} + a_{3n-1} + a_{3n} = 1/(4n(2n-1))$. Find an expression for s_{3n}.

   (b) Using $\frac{1}{4k(2k-1)} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k}\right)$, show that $\lim_{n\to\infty} s_{3n} = (\ln 2)/2$.

   (c) Show that $|s_{3n} - s_{3n\pm1}| \le 1/(2n)$. Deduce that $\lim_{n\to\infty} s_{3n\pm1} = \lim_{n\to\infty} s_{3n}$.

   (d) Show that $\sum a_n$ converges to $(\ln 2)/2$, i.e., $\lim_{n\to\infty} s_n = (\ln 2)/2$.