Math 104-004 **Answers to Test # 1b** **21 Sept 07**

. 18 1. Find $f'(x)$ for the following functions. (You need not simplify your answers.)

(a) $f(x) = 8x^5 - 2x^3 + \dfrac{3}{x}$

(b) $f(x) = 2 + 2\sqrt{x} - \dfrac{1}{\sqrt{x}}$

(c) $f(x) = \dfrac{x^2 - 3x + 1}{x}$

Solution. For part (a), we use the power rule with $n = 5$, $n = 3$, and $n = -1$.

$$f'(x) = 8 \cdot 5x^4 - 2 \cdot 3x^2 + 3(-1)x^{-2} = 40x^4 - 6x^2 - \frac{3}{x^2}.$$

For part (b), notice $f(x) = 2 + 2x^{1/2} - x^{-1/2}$ and so, using the power rule with $n = 0$, $n = 1/2$, and $n = -1/2$, we have

$$f'(x) = 0 + 2 \cdot \frac{1}{2}x^{-1/2} - \frac{-1}{2}x^{-3/2} = \frac{1}{\sqrt{x}} + \frac{1}{2x^{3/2}}.$$

For part (c), we rewrite $f(x)$ using

$$f(x) = \frac{x^2}{x} - \frac{3x}{x} + \frac{1}{x} = x - 3 + x^{-1}$$

and then use the power rule to get

$$f'(x) = 1 + 0 - x^{-2}.$$

. 18 2. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to+\infty} 5 + \frac{x^2 - 16}{5x^2 - 20}$.

(b) $\displaystyle\lim_{x\to 2^+} \frac{2x - 5}{x^2 - 3x + 2}$.

(c) $\displaystyle\lim_{x\to 9} \frac{\sqrt{x} - 3}{x - 9}$.

Solution. For part (a),

$$\begin{aligned}
\lim_{x\to+\infty} 5+\frac{x^2-16}{5x^2-20} &= \lim_{x\to+\infty} 5+\lim_{x\to+\infty}\frac{x^2-16}{5x^2-20}\\
&= 5+\lim_{x\to+\infty}\frac{\frac{x^2}{x^2}-\frac{16}{x^2}}{\frac{5x^2}{x^2}-\frac{20}{x^2}}\\
&= 5+\lim_{x\to+\infty}\frac{1-\frac{16}{x^2}}{5-\frac{20}{x^2}} = 5+\frac{1}{5},
\end{aligned}$$

and so the limit is $26/5 = 5.2$.

For part (b), substituting $x = 2$ into the numerator and denominator gives -1 and 0, respectively. Since we're dividing a nonzero number by zero, there is a vertical asymptote and the limit is either $+\infty$ or $-\infty$. Factoring the denominator, we have

$$\lim_{x\to2^+}\frac{2x-5}{x^2-3x+2} = \lim_{x\to2^+}\frac{2x-5}{(x-1)(x-2)}$$

As x approaches 2 from above, $2x-5$ approaches -1, $x-1$ approaches 1 and $x-2$ approaches 0 from above (is positive). Since we have two positive quantities and one negative, the limit is $-\infty$.

For part (c),

$$\lim_{x\to9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to9}\frac{\sqrt{x}-3}{(\sqrt{x}+3)(\sqrt{x}-3)}\lim_{x\to9}\frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

. 10 3. Find the horizontal and vertical asymptotes for

$$f(x) = \frac{2x^2+8}{x^2+3x-10}.$$

As always, explain your reasoning.

Solution. To find the vertical asymptotes, we look for values of x where the denominator is zero. If the numerator is not zero, then there is a vertical asymptote and if the numerator is zero, we need to take limits to figure out what is happening.

So, we solve $x^2+3x-10 = 0$, which factors as $(x+5)(x-2) = 0$, giving $x = -5$ and $x = 2$ as possible asymptotes. Since $2x^2+8$ is not zero for $x = -5$ or $x = 2$, these are both vertical asymptotes.

To find the horizontal asymptotes, we compute

$$\lim_{x\to\infty}\frac{2x^2+8}{x^2+3x-10}=\lim_{x\to\infty}\frac{\frac{2x^2}{x^2}+\frac{8}{x^2}}{\frac{x^2}{x^2}+\frac{3x}{x^2}-\frac{10}{x^2}}=\lim_{x\to\infty}\frac{2+\frac{8}{x^2}}{1+\frac{3}{x}-\frac{10}{x^2}}=2$$

So there is a horizontal asymptote, $y=2$.

. 10 4. Find the equation of the tangent line to the graph of $y=f(x)=4/x+5x$ when $x=2$.

Solution. To find the equation of the tangent line, we need to find $f(2)$, as the line goes through the point $(2,f(2))$, and we need to find $f'(2)$, since this is the slope of the tangent line at $x=2$.
First, $f(2)=4/2+5\cdot 2=2+10=12$, and second, rewrite $f(x)$ as $f(x)=4x^{-1}+5x$, so

$$f'(x)=-4x^{-2}+5=\frac{-4}{x^2}+5.$$

Thus, $f'(2)=-4/2^2+5=-1+5=4$. So the equation of the line is

$$y-12=4(x-2).$$

. 12 5. Let $f(x)=\begin{cases}\dfrac{2x^2+x-20}{x-4} & \text{if } x<4,\\ 5x-4 & \text{if } x\geq 4.\end{cases}$.

(a) Find $\lim\limits_{x\to 4^+} f(x)$ and $\lim\limits_{x\to 4^-} f(x)$.

(b) Is $f(x)$ continuous at 4? Why or why not?

Solution. For part a), we have

$$\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+} 5x-4=5(4)-4=16$$

and

$$\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}\frac{2x^2+x-20}{x-4}$$

Substituting $x=4$ in the fraction makes the denominator zero and the numerator $2\cdot 4^2+4-20=16$. Since the numerator is not zero, this limit is either $+\infty$ or $-\infty$. In fact, since the numerator is positive and denominator is negative (as x is a bit less than 4), the limit is $-\infty$.
For part b), since

$$\lim_{x\to 4^-} f(x)\neq\lim_{x\to 4^+} f(x),$$

we have $\lim\limits_{x\to 4} f(x)$ does not exist and so the function cannot be continuous at 4.

. 20 6. Suppose the cost of producing x calculus textbooks is given by $C(x) = 50 + x - \frac{x^2}{100}$, in hundreds of dollars.

(a) What is the **average cost** of a book if 150 books are made?

(b) What is the **marginal cost** function?

(c) What is the **marginal cost** when $x = 20$?

(d) What is the exact cost of the 21st book?

(e) If the book sells for \$95 each, what are the **revenue** function $R(x)$ and the **profit** function $P(x)$, in hundreds of dollars?

Solution. For part a), there are two acceptable answers. Using the usual meaning of average cost, it is

$$\frac{C(150)}{100} = \frac{50 + 150 - \frac{150^2}{100}}{100} = \frac{-25}{100} = -.25$$

which means the cost per book is -\$25 (since are units are hundreds of dollars). Using the average rate of change from $x = 0$ to $x = 150$, we have

$$\frac{C(150) - C(0)}{100 - 0} = \frac{-25 - 100}{100} = -1.25$$

which means the average rate of change of the cost in making the first hundred and fifty books is -\$125 per book.

For part b), the marginal cost function is

$$C'(x) = 1 - \frac{2x}{100} = 1 - \frac{x}{50}.$$

For part c), we have

$$C'(20) = 1 - \frac{20}{50} = .6$$

which, with units of hundreds of dollars, is \$60.

For part d), the exact cost is the cost of making 21 books minus the cost of making the first 20, i.e.,

$$C(21) - C(20) = \left(50 + 21 - \frac{21^2}{100}\right) - \left(50 + 20 - \frac{20^2}{100}\right) = 1 - \frac{41}{100} = .59$$

which, with units of hundreds of dollars, is \$59.

For part e), the revenue is $R(x) = .95x$ and the profit is

$$P(x) = R(x) - C(x) = .95x - \left(50 + x - \frac{x^2}{100}\right) = -50 - .05x + \frac{x^2}{100}.$$

. 12 7. **Using the definition**, find the derivative of the function $f(x) = 2x^2 - 7x - 5$ when $x = 3$.

Solution. By the definition,

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

Now,

$$\begin{aligned} f(2+h) &= 2(3+h)^2 - 7(3+h) - 5 \\ &= 2(9 + 6h + h^2) - 21 - 7h - 5 \\ &= 18 + 12h + 2h^2 - 21 - 7h - 5 = -8 + 5h + 2h^2 \end{aligned}$$

while $f(3) = 2(3)^2 - 7 \cdot 3 - 5 = 18 - 21 + 5 = -8$. Thus,

$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{-8 + 5h + 2h^2 - (-8)}{h} \\ &= \lim_{h \to 0} \frac{5h + 2h^2}{h} \\ &= \lim_{h \to 0} 5 + 2h = 5 \end{aligned}$$

So $f'(3) = 5$.

As a check, we compute $f'(x) = 4x - 7$ according to the rules, and so $f'(3) = 4 \cdot 3 - 7 = 5$, so the two ways of computing $f'(3)$ agree.