Time is 2 hours. Do the first two questions and **only three** of the remaining four.

. 20 1. State and prove the Bolzano-Weierstrass Theorem. Explain clearly your use of any lemmas.

. 20 2. For each of the following statements, determine if it is true or false and provide either a proof or a counterexample, as appropriate.

   (a) For $n \in \mathbb{N}$ and $A \subseteq \mathbb{R}$, define A^n to be $\{a^n : a \in A\}$. If A is a bounded-above nonempty set of nonnegative real numbers. then, for $n \in \mathbb{N}$, $\sup A^n = (\sup A)^n$.

   (b) If (x_n) has the property that every subsequence of (x_n) has a convergent subsubsequence, then (x_n) converges.

. 20 3. If $\lim\limits_{n\to\infty} a_n = a$ and there are infinitely many terms of (a_n) which are greater than a, then there is an decreasing subsequence of a_n which converges to a.

. 20 4. Suppose the sequence (a_n) is decreasing and $a_n - a_{n-1} > -1/n^2$ for all $n \in \mathbb{N}$. Prove that (a_n) converges.

. 20 5. Prove that every conditionally convergent series has a rearrangement that diverges to $+\infty$, i.e., the sequence of partial sums diverges to $+\infty$.

. 20 6. Suppose that (n_k) is a strictly increasing sequence of positive integers so that

$$\lim_{k\to\infty} \frac{n_k}{n_1 n_2 \cdots n_{k-1}} = +\infty.$$

   Prove that $\sum\limits_{i=1}^{\infty} \frac{1}{n_i}$ converges to an irrational number.