Purely infinite C^*-algebras associated to étale groupoids

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Purely infinite simple

Let A be a simple C^*-algebra.

- For $a \in M_n(A)$, $b \in M_m(A)$ be positive, we say a is *Cuntz below* b, $a \precsim b$, if there exist $x_k \in M_{m,n}(A)$ such that $x_k^* b x_k \to a$ in norm.
- $a \in A^+$ is *infinite* if there exists $b \in A^+$ $a \oplus b \precsim a$.
- A projection $p \in A$ is infinite if and only if it is Murray-von Neumann equivalent to a proper subprojection of itself. (KR00 Lemma 3.1)
- A is purely infinite if every $a \in A^+ - \{0\}$ is infinite. (KR00 Theorem 4.16)

Theorem (Kirchberg Phillips)

Let A and B be separable nuclear, purely infinite simple C^*-algebras satisfying the Universal Coefficient Theorem (UCT). Assume A and B are both unital or both nonunital. If there exists a graded isomorphism $\alpha : K_*(A) \to K_*(B)$, which (in the unital case) satisfies $\alpha([1_A]) = [1_B]$, then there exists an isomorphism $\phi : A \to B$

Graph C^*-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph.

- E is *row-finite* if $r^{-1}(v) < \infty \quad \forall v \in E^0$.
- E has *no sources* if $r^{-1}(v) \neq \emptyset \quad \forall v \in E^0$.
- $\alpha = \alpha_1\alpha_2\cdots$ is a path if $s(\alpha_i) = r(\alpha_{i+1})$: $r(\alpha) = r(\alpha_1)$.
- E^* is the set of finite paths, E^∞ is the set of infinite paths.
- For $\alpha \in E^*$
  - $|\alpha|$ is the length α;
  - $s(\alpha) = s(\alpha_{|\alpha|})$;
  - α is a *return path* if $r(\alpha) = s(\alpha)$.
    - A return path α has an *entrance* if there exists i and $e \in r^{-1}(r(\alpha_i)) - \{\alpha_i\}$.

Purely infinite graph algebras

KPRR 1997 construct a C^*-algebra $C^*(E)$ from E.

Theorem (KPR 1998)

$C^*(E)$ is purely infinite simple if and only if

1. *There exists a return path in E,*
2. *Every return path in E has an entrance, and*
3. $\forall v \in E^0,\ x = x_1 x_2 \cdots \in E^\infty\ \exists \alpha \in E^*,\ i \in \mathbb{N}$ *such that* $r(\alpha) = v,\ s(\alpha) = r(x_i)$. *(cofinal)*

- $C^*(E)$ simple iff items (2) and (3) above hold.
- That (1)-(3) are sufficient is a result about groupoids from Anantharaman-Delaroch 97.
- To show they are necessary, the authors show that if E satisfies (2) and (3) but not (1) then $C^*(E)$ is AF.
  - This dichotomy is particular for graphs.

If we try to generalize to k-graphs, A-D 97 still gives a sufficient condition, but no necessary condition is known.

Groupoids

A groupoid G can be defined as a small category in which every morphism is invertible.

- We identify the objects of the category with the identity morphisms and denote both by $G^{(0)}$; $G^{(0)}$ is called the unit space of G.
- Denote the range of a morphism γ by $r(\gamma)$ and its source by $s(\gamma)$; $r, s : G \to G^{(0)}$.
- We say a pair of morphisms (γ, η) is composable if and only if $s(\gamma) = r(\eta)$ and denote the composition by $\gamma\eta$.
- G acts on $G^{(0)}$ by $\gamma \cdot s(\gamma) = r(\gamma)$; for $C \subset G^{(0)}$ we denote $G \cdot C := \{r(\gamma) : s(\gamma) \in C\}$.

Topological groupoids

We say G is a topological groupoid if G has a topology in which composition and inversion of morphisms are continuous.

- This implies r and s are continuous.
- We assume this topology is second countable locally compact and Hausdorff.
- G is étale if it is a topological groupoid such that r and s are local homeomorphisms.
  - G étale implies $G^{(0)}$ is open and closed in G.
  - we call open sets B such that r, s are homeomorphisms on B *bisections*.
- G is *topologically principal* if $\{u \in G^{(0)} : r^{-1}(u) \cap s^{-1}(u) = \{u\}\}$ is dense in $G^{(0)}$.
- G is *minimal* if $G \cdot U \subset U$ and U open implies $U \in \{\emptyset, G^{(0)}\}$.
  - Minimal implies: for $x \in G^{(0)}$, U open there exists an open bisection B such that $x \in B \cdot U$.

The groupoid of a directed graph

Let be a row-finite directed graph with no sources. Define:

$$G_E := \{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \exists N \text{ with } x_{i+k} = y_i \text{ for } i \geq N\}.$$

- If $\alpha = x_1 x_2 \cdots x_{N+k}$, $\beta = y_1 y_2 \cdots y_N$ and $z = y_{N+1} y_{N+2} \cdots$, then $(x, k, y) = (\alpha z, |\alpha| - |\beta|, \beta z)$.
- G_E is a groupoid with unit space E^∞:
  - (x, k, y) is a morphism from y to x,
  - composition is given by $(x, k, y)(y, \ell, z) = (x, k + \ell, z)$,
  - $(x, k, y)^{-1} = (y, -k, x)$.
- The sets

  $$Z(\alpha, \beta) := \{(\alpha x, |\alpha| - |\beta|, \beta x) : r(x) = s(\alpha) = s(\beta)\} \quad \text{for } \alpha, \beta \in E^*$$

  form a basis for a locally compact Hausdorff topology on G_E:
  - $Z(\alpha, \beta)$ is compact.
- G_E topologically principal iff every return path in E has an entrance
- G_E minimal iff E cofinal.

Groupoid C^*-algebras

For an étale groupoid G we define a convolution algebra structure on $C_c(G)$ by

$$f * g(\gamma) = \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma) \quad f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

We define the regular representation of $C_c(G)$ at $u \in G^{(0)}$ on $\ell^2(Gu)$ by

$$\pi_u(f)\delta_\gamma = \sum_{s(\eta)=r(\gamma)} f(\eta)\delta_{\gamma\eta}$$

- $C_r^*(G)$ is the completion of $C_c(G)$ in the norm

$$\|f\|_r = \sup_{u \in G^{(0)}} \|\pi_u(f)\|.$$

$C_r^*(G)$ is simple if G is topologically principal and minimal (Renault 1980).

Conditional expectation

Let G be an étale groupoid.

- $G^{(0)}$ is open and closed in G.
- $f \in C_c(G^{(0)})$ the $f \in C_c(G)$ (extend by 0)
  - This extends to an embedding of $C_0(G^{(0)})$ into $C_r^*(G)$.
- $f \in C_c(G) \implies f|_{G^{(0)}} \in C_0(G^{(0)})$.
  - $f \mapsto f|_{G^{(0)}}$ extends to a faithful conditional expectation

$$E : C_r^*(G) \to C_0(G^{(0)}).$$

  - That is:
    - $E(ba) = bE(a) \ \forall \ b \in C_0(G^{(0)}),\ a \in C_r^*(G)$
    - $E(a) \in C_0(G^{(0)})^+ \ \forall \ a \in C_r^*(G)^+$
    - $E(a^*a) = 0 \implies a = 0$.

Purely infinite simple étale groupoids

Given $a \in C_r^*(G)^+$, $E(a) \in C_0(G^{(0)})^+$.

- Take
$$h \text{``}=\text{''} \max\{E(a/\|a\|) - 1/2, 0\}$$
- Then $h \precsim a$ by Lemma 2.2 of Kirchberg Rørdam 2000.
- So if h is infinite then so is a.
- Thus if every element in $C_0(G^{(0)})^+$ is infinite then every element $C_r^*(G)^+$ is.

Theorem (B., Clark, Sierakowski)

If G is a locally compact étale groupoid that is topologically principal and minimal then $C_r^*(G)$ is purely infinite if and only if every element of $C_0(G^{(0)})^+$ is infinite (in $C_r^*(G)$).

Purely infinite ample groupoids

A groupoid is ample if it has a basis of compact open bisections.

- Ample groupoids are étale and locally compact.
  - So the previous theorem applies to them.
- Ample groupoids have lots of projections:
  - If U compact open bisection then χ_U is continuous.
  - For U, V compact open bisections $\chi_U * \chi_V = \chi_{UV}$ so if $U \subset G^{(0)}$ we have

$$\chi_U \in C_0(G^{(0)}) \quad \text{and} \quad \chi_U^2 = \chi_U.$$

- If $h \in C_0(G^{(0)})^+$, then there exists U and $s \in \mathbb{R}_{>0}$ such that $h|_U \geq s$.
- Therefore $h \geq s\chi_U$ and so $\chi_U \precsim h$.
- Thus if χ_U is infinite then h is infinite.

Theorem (B., Clark, Sierakowski)

If G is a locally compact ample groupoid that is topological principal and minimal and $\mathcal{B}$ is a basis of compact open sets for $G^{(0)}$ then $C_r^*(G)$ is purely infinite if and only if χ_U is infinite for all $U \in \mathcal{B}$.

More fun with ample groupoids

Theorem (B., Clark, Sierakowski)

If G is a locally compact ample groupoid that is topological principal and minimal and $\mathcal{B}$ is a basis of compact open sets for $G^{(0)}$ then $C_r^*(G)$ is purely infinite if and only if χ_U is infinite for all $U \in \mathcal{B}$.

Now G is minimal.

- So if $x \in G^{(0)}$ and $U \in \mathcal{B}$ there exists a compact open bisection such that $x \in r(B)$ and $s(B) \subset U$.
- Since $\chi_B^* \chi_B \leq \chi_U$ and $\chi_B \chi_B^* = \chi_{r(B)}$, we have χ_U infinite if $\chi_{r(B)}$ infinite.

Thus

Corollary (B., Clark, Sierakowski)

$C^*(G)$ is purely infinite if and only if χ_V is infinite for all $V \in \mathcal{N}$ where $\mathcal{N}$ is a neighborhood basis at some point $x \in G^{(0)}$ of compact open sets.

k-graphs

A k-graph Λ is a generalization of a graph where paths have "shape" given by elements of $\mathbb{N}^k$.

- $C^*(\Lambda)$ is the universal C^*-algebra generated by a Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$;
  - in particular $s_\lambda^* s_\lambda = s_{s(\lambda)}$.

Define

$$G_\Lambda := \{(x, n, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : \sigma^l(x) = \sigma^m(y), n = l - m\}$$

where σ is the shift map and x and y are infinite paths.

- The unit space of G_Λ is Λ^∞ the set of infinite paths.

For $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$, the sets

$$Z(\lambda, \mu) := \{(\lambda z, d(\lambda) - d(\mu), \mu z) : z \in \Lambda^\infty(s(\lambda))\}.$$

give a basis of a second countable locally compact Hausdorff topology of compact open sets.

- $\phi : C^*(\Lambda) \to C^*(G_\Lambda) \quad s_\lambda \mapsto \chi_{Z(\lambda, s(\lambda))}$ is an isomorphism.

Purely infinite k-graphs

- For $x \in \Lambda^\infty$ the sets $Z(\lambda, \lambda)$ where λ ranges over initial segments of X is a neighborhood basis at x.
- $s_\lambda s_\lambda^*$ is infinite iff $\phi(s_\lambda s_\lambda^*) = \chi_{Z(\lambda,\lambda)}$ is infinite.
- $s_{s(\lambda)} = s_\lambda s_\lambda^*$ is equivalent to $s_\lambda s_\lambda^*$.
- So $s_\lambda s_\lambda^*$ is infinite if and only if $s_{s(\lambda)}$ is.

Theorem (B., Clark, Sierakowski)

Let Λ be a row-finite k-graph with no sources then $C^*(\Lambda)$ is purely infinite if and only if there exists $x \in \Lambda^\infty$ such that s_v is infinite for all vertices v in x.

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