Math 104-004 **Answers to Test # 3a** **9 Nov 07**

. 8 1. Find $f''(-1)$ if $f(x) = \dfrac{2x}{3x+1}$.

Solution. Using the quotient rule, we have

$$f'(x) = \frac{(3x+1)2 - (2x)(3)}{(3x+1)^2} = \frac{6x+2-6x}{(3x+1)^2} = \frac{2}{(3x+1)^2} = 2(3x+1)^{-2}.$$

Thus, using the generalized power rule, we have

$$f''(x) = 2(3x+1)^{-3} \cdot 3 = \frac{6}{(3x+1)^3}.$$

In particular,

$$f''(-1) = \frac{6}{(3 \cdot (-1)+1)^3} = \frac{6}{(-2)^3} = -\frac{3}{4}.$$

. 10 2. Use calculus methods to find the absolute maximum value M and the absolute minimum value m of the function $f(x) = x^3 - 12x^2 + 36x + 5$ on the interval $[-1, 7]$.

Solution. Taking the derivative, we have

$$f'(x) = 3x^2 - 24x + 36 = 3(x^2 - 8x + 12) = 3(x-2)(x-6).$$

and $x = 6$ are critical numbers. So we have to evaluate $f(x)$ at $x = -1$, $x = 2$, $x = 6$, and $x = 7$. Notice that $f(-1) = -44$, $f(2) = 37$, $f(6) = 5$, and $f(7) = 12$.

So the absolute maximum is $M = 37$ and the absolute minimum is -44.

. 14 3. Let $y = f(x)$ be a function such that $f'(x) = x^2(x+3)(x-4)^2(x+4)(x-2)$ for all $x \in (-\infty, \infty)$.

(a) Find the critical numbers,

(b) Chart $f'(x)$.

(c) Find the open intervals on which f is increasing, and

(d) Find all x coordinates so that $(x, f(x))$ is a relative maximum of $y = f(x)$.

(e) Find all x coordinates so that $(x, f(x))$ is a relative minimum of $y = f(x)$.

Solution. For part (a), the critical numbers are zeros of $f'(x)$, which are -4, -3, 0, 2, 4.

For part (b), the chart of $f'(x)$ is

For part (c), $f'(x)$ is increasing on $(-4,-3)$, $(2,4)$, and $(4,+\infty)$.

For part (d), to have a relative max, $f'(x)$ must go from positive to negative, so the only relative max is at $x=-3$.

For part (e), $f'(x)$ must go from negative to positive, so the relative mins are at $x=-4$ and $x=2$.

. 10 4. Given the cost function $C(x)=2x^2+7x+450$ dollars, use calculus methods to determine the number of units x that should be produced in order to minimize the **average cost per unit**.

Solution. The average cost function is

$$\overline{C}(x)=\frac{2x^2+7x+450}{x}=2x+7+\frac{450}{x},$$

and so

$$\overline{C}'(x)=2-\frac{450}{x^2}=\frac{2x^2-450}{x^2}.$$

Thus, $\overline{C}'(x)=0$ when $2x^2=450$, or $x^2=225$, or $x=\pm15$. Since x is positive, the only meaningful critical number is $x=15$.

So $x=15$ units should be produced to minimize the average cost per unit.

. 10 5. Let $y=f(x)=\frac{1}{4}x^4-3x^3+\frac{15}{2}x^2+4x+2$ for all $x\in(-\infty,+\infty)$.

(a) Find $f''(x)$.

(b) Chart $f''(x)$.

(c) Find the open intervals where the graph of f is concave up, and

(d) Find the points $(x,f(x))$ on the graph of f which are inflection points.

Solution. First, $f'(x)=x^3-9x^2+15x+4$ and so

$$f''(x)=3x^2-18x+15=3(x^2-6x+5)=3(x-1)(x-5).$$

To make the chart we notice that $f''(x)$ is zero when $x=1$ and $x=5$. Using $f''(0)=15$, $f''(2)=-9$ and $f''(6)=15$.

For part (c), f is concave up on $(-\infty,1)$ and on $(5,+\infty)$.

For part (d), since there are sign changes at both $x = 1$ and $x = 5$, there are inflection points for these x-values. Notice that $f(1) = 1/4 - 3 + 15/2 + 4 + 2 = 43/4 = 10.75$ and $f(5) = 625/4 - 375 + \frac{15}{2}25 + 2 = -37/4 = 9.75$. Thus, the inflection points are $(1, 43/4)$ and $(5, -37/4)$.

. 8 6. For the function $f(x) = 3x^3 - x^2 + 5x$, find dy if $x = 3$ and $dx = \Delta x = -.01$

Solution. Notice that $f'(x) = 9x^2 - 2x + 5$, so $f'(3) = 81 - 6 + 5 = 80$. Thus, at $x = 3$,

$$dy = 80dx = 80(-.01) = -.8.$$

. 15 7. Sketch the graph of function $y = f(x)$ with the following properties

(a) $f'(x) > 0$ for x in $(-\infty, -2)$, and $(1, 5)$,
(b) $f'(x) < 0$ for x in $(-2, 5)$, and $(5, +\infty)$,
(c) $f''(x) > 0$ for x in $(-1, 4)$,
(d) $f''(x) < 0$ for x in $(-\infty, -1)$, and $(4, +\infty)$.

Solution.

. 15 8. Using calculus-based methods, find positive numbers x and y with $x + y = 20$ so that x^3y is as large as possible.

Solution. Notice that $y = 20 - x$, so we are maximizing the function $f(x) = x^3(20 - x) = 20x^3 - x^4$. Now

$$f'(x) = 60x^2 - 4x^3 = 4x^2(15 - x)$$

so $f'(x) = 0$ when $x = 0$ or $x = 15$. Next, observe that since x and y must both be positive, x must be in the interval $[0, 20]$. Thus, we must check $f(x)$ for $x = 0$, $x = 15$ and $x = 20$. We have $f(0) = 0$, $f(15) = 16,875$ and $f(20) = 0$.
Thus when $x = 15$ and $y = 5$, then x^3y is as large as possible.

. 10 9. Use differentials to approximate $\sqrt{146}$. Using your calculator, find the error in your approximation.

Solution. Notice that $146 = 144 + 2$, as $\sqrt{144} = 12$, find the derivative of $f(x) = \sqrt{x}$ at $x = 144$ and then find dy when $dx = 2$.
As $f(x) = x^{1/2}$, $f'(x) = 1/2x^{-1/2}$ and so $f'(144) = 1/(2 \cdot 12) = 1/24$. Thus,

$$dy = \frac{1}{24}dx = \frac{2}{24} = \frac{1}{12}$$

So $\sqrt{146}$ is about $12 + \frac{1}{12} = 12.08333$
For comparision, $\sqrt{146}$ is 12.08304, so the error is about .00029.