Math 107-250/350 — Answers to Homework #8 — 23 Mar '07

Explain your reasoning. A correct answer poorly explained will not get full marks.

. 5 1. We approximate $\int_0^{\pi/2} \cos(\sqrt{x})\,dx$ using a power series.

(a) Find a Taylor polynomial, $p(x)$, that approximates $f(x) = \cos(\sqrt{x})$ to within $2 \cdot 10^{-4}$ i.e., so that $|p(x) - f(x)| \le 2 \cdot 10^{-4}$, for all x in $[0, \pi/2]$.

(b) If $P(t) = \int_0^t p(x)\,dx$, then the polynomial $P(t)$ approximates $F(t) = \int_0^t \cos(\sqrt{x})\,dx$. Find $P(t)$ and give a reasonable bound for error $|P(t) - F(t)|$ for t in $[0, \pi/2]$. Hint: You can write $P(t) - F(t)$ as an integral of $p(x) - f(x)$.

(c) Find $P(\pi/2)$ and an error bound for how well it approximates $\int_0^{\pi/2} \cos(\sqrt{x})\,dx$.

Solution. For a), we start with the Taylor series for $\cos(y)$, which is

$$1 - \frac{y^2}{2} + \frac{y^4}{24} - \frac{y^6}{720} + \frac{y^8}{40,320} - \cdots$$

Substituting $y = \sqrt{x}$, we have

$$1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40,320} - \cdots$$

Notice that this is an alternating series, so the error is at most the first term omitted, by the error estimate for alternating series. Since x is in the interval between 0 and $\pi/2$, the largest error will occur when $x = \pi/2$. If we use $1 - x/2 + x^2/24$, then the error is at most

$$\frac{(\pi/2)^3}{720} \approx 0.00538303.$$

This is not less than $2 \cdot 10^{-4}$, so we try using $p(x) = 1 - x/2 + x^2/24 + x^3/720$, when the error is at most

$$\frac{(\pi/2)^4}{40,320} \approx 0.0001509938 < 2 \cdot 10^{-4}.$$

For b), we have

$$\begin{aligned} P(t) = \int_0^t p(x)\,dx &= \int_0^t 1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720}\,dx \\ &= t - \frac{t^2}{4} + \frac{t^3}{72} - \frac{t^4}{2880} \end{aligned}$$

To bound the error, we have

$$\begin{aligned} |P(t) - F(t)| &= \left| \int_0^t p(x) - f(x)\,dx \right| \\ &\le \int_0^t |p(x) - f(x)|\,dx \\ &\le \int_0^t 2 \cdot 10^{-4}\,dx = t \cdot 2 \cdot 10^{-4} \end{aligned}$$

Since t is at most $\pi/2$, the error is at most $\pi \cdot 10^{-4} \approx .00031415$.

For c), we have

$$P\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{(\pi/2)^2}{4} + \frac{(\pi/2)^3}{72} - \frac{(\pi/2)^4}{2880} \approx 1.005662480$$

The error bound is still .00031415, as in part b).

For comparison, the integral (to 10 digits) is 1.005709234 so the actual error is 0.000046754, much less than our bound of .00031415.