Math 107-250/350 - Analytic Geometry & Calculus I **2nd Semester, '06-'07**

Partial Fractions Example

Evaluate $\int \frac{3x^3+4x^2-3x+4}{(x-1)(x+1)(x^2+1)^2}\,dx$.

The partial fractions decomposition has the form

$$\frac{3x^3+4x^2-3x+4}{(x-1)(x+1)(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

and then multiplying by the denominator of the lefthand side gives

$$3x^3+4x^2-3x+4 = A(x+1)(x^2+1)^2+B(x-1)(x^2+1)^2+(Cx+D)(x-1)(x+1)(x^2+1)+(Ex+F)(x-1)(x+1)$$

Letting $x=1$, we get the equation $8=8A$, so $A=1$.

Letting $x=-1$, we get the equation $8=-8B$, so $B=-1$.

To find C, D, E, and F, we need to multiply out the polynomials and equate coefficients.

$$3x^3+4x^2-3x+4 = (x+1)(x^2+1)^2-(x-1)(x^2+1)^2+(Cx+D)(x^2-1)(x^2+1)+(Ex+F)(x^2-1)$$

Bringing the first two terms of the righthand side to the left and expanding, we have

$$\begin{aligned}3x^3+4x^2-3x+4-(x+1)(x^4+2x^2+1)+(x-1)(x^4+2x^2+1)\\ = (Cx+D)(x^2-1)(x^2+1)+(Ex+F)(x^2-1),\end{aligned}$$

and so

$$\begin{aligned}3x^3+4x^2-3x+4-[x^5+2x^3+x+x^4+2x^2+1]+[x^5+2x^3+x-x^4-2x^2-1]\\ = (Cx+D)(x^4-1)+(Ex+F)(x^2-1)\end{aligned}$$

and finally

$$-2x^4+3x^3-3x+2 = Cx^5+Dx^4-Cx-D+Ex^3+Fx^2-Ex-F.$$

Equating coefficients gives $C=0$, $D=-2$, $E=3$, $F=0$, $-C-E=-3$, and $-D-F=2$. Clearly the last two equations are redundant and the partial fractions decomposition is

$$\frac{3x^3+4x^2-3x+4}{(x-1)(x+1)(x^2+1)^2} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} + \frac{3x}{(x^2+1)^2}$$

and so

$$\int \frac{3x^3+4x^2-3x+4}{(x-1)(x+1)(x^2+1)^2}\,dx = \int \frac{dx}{x-1} - \int \frac{dx}{x+1} - \int \frac{2\,dx}{x^2+1} + \int \frac{3x\,dx}{(x^2+1)^2}$$

and using the substitution $u=x^2+1$, $du=2dx$ in the last integral gives

$$\int \frac{3x\,dx}{(x^2+1)^2} = \frac{3}{2}\int \frac{1}{u^2}du = -\frac{3}{2}u^{-1}+C = -\frac{3}{2(x^2+1)}+C.$$

Thus, the final answer is

$$\int \frac{3x^3+4x^2-3x+4}{(x-1)(x+1)(x^2+1)^2}\,dx = \ln|x-1| - \ln|x+1| - 2\arctan(x) - \frac{3}{2(x^2+1)} + C$$