Math 107-250/350 Answers to Homework #3 26 Jan '07

. 5 1. Evaluate $\displaystyle\int \frac{7x^3 - 12x^2 - 6x + 19}{(x-2)(x+3)(x^2-4x+5)}\,dx.$

To evaluate one of the integrals in the partial fractions decomposition, you might want to complete the square and substitute for $x-2$.

Solution. Notice that $x^2 - 4x + 5 = x^2 - 4x + 4 + 1 = (x-2)^2 + 1$; this is an irreducible quadratic factor since it has no real roots.
The form of the partial fractions decomposition is

$$\frac{7x^3 - 12x^2 - 6x + 19}{(x-2)(x+3)(x^2-4x+5)} = \frac{A}{x-2} + \frac{B}{x+3} + \frac{Cx+D}{(x-2)^2+1}.$$

Multiplying by $(x-2)(x+3)(x^2-4x+5)$, we get

$$\begin{aligned}7x^3 - 12x^2 - 6x + 19 &= A(x+3)((x-2)^2+1) + B(x-2)((x-2)^2+1)\\ &\quad + (Cx+D)(x-2)(x+3).\end{aligned}$$

Letting $x = 2$, we get

$$\begin{aligned}7\cdot 8 - 12\cdot 4 - 6\cdot 2 + 19 &= A\cdot 5\cdot 1\\ 15 &= 5A\end{aligned}$$

and so $A = 3$.
Letting $x = -3$, we get

$$\begin{aligned}7\cdot(-27) - 12\cdot 9 - 6\cdot(-3) + 19 &= B\cdot(-5)\cdot 26\\ -260 &= -130B\end{aligned}$$

and so $B = 2$.
Next, we let $x = 0$, to get $19 = 15A - 10B - 6D$. Since $A = 3$ and $B = 2$, we have $6D = 15A - 10B - 19 = 45 - 20 - 19 = 6$ and so $D = 1$. Finally, let $x = 1$, to get

$$\begin{aligned}7 - 12 - 6 + 19 &= A\cdot 8 + B\cdot(-2) + (C+D)\cdot(-4)\\ 8 &= 8A - 2B - 4C - 4D\end{aligned}$$

Dividing by 4 and solving for C, $C = 2A - B/2 - D - 2 = 6 - 1 - 1 - 2 = 2$. Thus,

$$\frac{7x^3 - 12x^2 - 6x + 19}{(x-2)(x+3)(x^2-4x+5)} = \frac{3}{x-2} + \frac{2}{x+3} + \frac{2x+1}{(x-2)^2+1}.$$

Notice

$$\frac{2x+1}{(x-2)^2+1} = \frac{2(x-2)+5}{(x-2)^2+1}.$$

and so using the substitution $u = x - 2$, we have

$$\begin{aligned}
\int \frac{2x+1}{(x-2)^2+1}\,dx &= \int \frac{2u+5}{u^2+1}\,du \\
&= \int \frac{2u}{u^2+1}\,du + \int \frac{5}{u^2+1}\,du \\
&= \ln|u^2+1| + 5\arctan(u) + C \\
&= \ln|(x-2)^2+1| + 5\arctan(x-2) + C
\end{aligned}$$

Thus,

$$\begin{aligned}
\int \frac{7x^3-12x^2-6x+19}{(x-2)(x+3)(x^2-4x+5)}\,dx = 3\ln|x-2| + 2\ln|x+3| + \ln|(x-2)^2+1| \\
+ 5\arctan(x-2) + C
\end{aligned}$$