Math 106-350/550 Answers to Test #2 25 Oct '06

. 10 1. Using calculus, find the extreme values, if any, of $y = \arcsin(x^2)$ and where they occur.

Solution. This is almost exactly the same as problem 53 in section 4.1 and is same basic format as assigned problems 41 and 47.

Recall that the domain of arcsin is $[-1, 1]$. So for $\arcsin(x^2)$ to be defined, x must be in the interval $[-1, 1]$. Next,

$$y' = \frac{1}{\sqrt{1-(x^2)^2}} 2x = \frac{2x}{\sqrt{1-x^4}}.$$

Observe that y' is zero when the numerator is zero, i.e., $x = 0$ and is undefined when the denominator is zero, i.e., $x = \pm 1$. So our critical points are $-1, 0, 1$.

x	y
-1	$\arcsin(1) = \pi/2$
0	$\arcsin(0) = 0$
1	$\arcsin(1) = \pi/2$

Based on the table, the absolute minimum is 0 at $x = 0$ and the absolute maximum is $\pi/2$, which occurs at $x = 1$ and at $x = -1$.

Instead of the table, you can look at the sign of y' to show which values are the maximum and minimum.

. 12 2. For the parametric equations $x = \sqrt{t}$, $y = \sqrt{t+1}$, $t \geq 0$, identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation and indicate the direction of motion.

Solution. This is essentially the same as problem 90 in section 3.5 and is same basic format as assigned problems 81, 85, and 89.

Solving for t, $x^2 = t$ and substituting this into the equation for y gives

$$\begin{aligned} y &= \sqrt{x^2+1} \\ y^2 &= x^2+1 \\ y^2 - x^2 &= 1 \end{aligned}$$

Notice that if $t \geq 0$, then x and y are both positive.

To the graph is the part of the hyperbola $y^2 - x^2 = 1$ with $x \geq 0$ and $y \geq 0$.

13 3. The U.S. mint decides that all \$1 coins they make must be within 1/250 of their ideal weight. How much variation, dr, in the radius of a \$1 coin can be tolerated? Assume that the thickness does not vary and that weight is proportional to volume.

Solution. This is essentially the same as assigned problem 57 in section 3.10.

Because weight and volume are proportional, we can work with the volume. The first sentence says that

$$|\Delta V| \leq \frac{1}{250} V. \tag{1}$$

The formula for volume is $V = \pi r^2 h$ and the last sentence says that h is constant.

We assume that ΔV is the same as the differential dV. Now,

$$dV = \pi(2r\,dr)h = 2\pi h\,r\,dr.$$

Substituting the formulas for dV and V into the equation (1), we have

$$\begin{aligned} |2\pi h\,r\,dr| &\leq \frac{1}{250}\pi r^2 h \\ 2\pi h r|dr| &\leq \frac{1}{250}\pi r^2 h \\ |dr| &\leq \frac{1}{500} r \end{aligned}$$

Thus, the radius has to be within 1/500 of its ideal value.

12 4. Find the equation of the line **normal** to the curve $x^2 - \sqrt{3}xy + 2y^2 = 5$ through $(\sqrt{3}, 2)$. Hint: implicit differentiation.

Solution. This is problem 34(b) in Section 3.6 and is the same format as assigned problems 31 and 35.

Recall that the normal line is perpendicular to the tangent line. So we start by finding the slope of the tangent line through $(\sqrt{3}, 2)$. First, we apply implicit differentiation to the equation and then solve for $\frac{dy}{dx}$.

$$\begin{aligned} 2x - \sqrt{3}x\frac{dy}{dx} - \sqrt{3}1 \cdot y + 4y\frac{dy}{dx} &= 0 \\ (-\sqrt{3}x + 4y)\frac{dy}{dx} &= -2x + \sqrt{3}y \\ \frac{dy}{dx} &= \frac{-2x + \sqrt{3}y}{-\sqrt{3}x + 4y} \end{aligned}$$

Substituting $x = \sqrt{3}$ and $y = 2$ into this equation gives

$$\left.\frac{dy}{dx}\right|_{(\sqrt{3},2)} = \frac{0}{-1} = 0$$

This shows that tangent line is a horizontal line through $(\sqrt{3}, 2)$ and so the normal line is a vertical line through $(\sqrt{3}, 2)$. That is, the normal line is $x = \sqrt{3}$.

. 17 5. A plane is located 20 miles (horizontally) away from an airport and is flying at a constant altitude of 4 miles. The radar system at the airport indicates that the straight-line distance between plane and airport is decreasing at a rate of 240 miles per hour (mph). What is the (horizontal) velocity of the plane?

Be sure to draw a relevant picture and define your variables (in a sentence or two).

Solution. This is a related rates problem (Section 3.9).

Let $x(t)$ be the horizontal distance from the plane to the airport and $s(t)$ be the straight-line distance. We want to find $x'(t)$.

I will include a diagram eventually, but I don't have time to include one now. Think of a right triangle with a horizontal side for the ground, of length x, a vertical side with the airplane at the top, of height 4, and a hypotenuse connected the airplane to the airport, of length s.

We are told that $\frac{ds}{dt} = -240$ mph and $x = 20$. We are asked to find $\frac{dx}{dt}$.

By right triangles, $16 + x^2 = s^2$. Differentiating this relation with respect to time gives

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}.$$

Solving, we have

$$\frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt}.$$

When $x = 20$, $s = \sqrt{416} = 20.4$ and so $\frac{dx}{dt} = 20.4/20 \cdot (-240) = -244.75$ mph. So the airplane's horizontal velocity is 244.75 mph towards the airport.

. 20 6. Using the procedure developed in class, sketch the graph of $y = xe^{-x}$. Include the coordinates of local extreme points and inflection points.

Solution. This is problem 38 from Section 4.4 and is the same format as assigned problems 15, 25, 29, 35, and 41.

First we note that the domain is $(-\infty, \infty)$ and there are no obvious symmetries. Next,

$$y' = e^{-x} + xe^{-x}(-1) = e^{-x}(1-x).$$
$$y'' = e^{-x}(-1) + e^{-x}(-1)(1-x) = e^{-x}(x-2).$$

To find critical points, notice that y' is never undefined and $e^{-x}(1-x) = 0$ is only possible if $1 - x = 0$ since e^{-x} is never zero. Thus, the only critical point is $x = 0$.

To work out the sign of y', we have two intervals, $(-\infty, 1)$ and $(1, +\infty)$. For the first, substituting $x = 0$ into y' gives $1 > 0$, so y' is positive (and hence y is increasing) on $(-\infty, 1)$. For the second interval, subsituting $x = 2$ into y' gives $e^{-2}(-1) < 0$, so y' is negative (and hence y is decreasing) on $(1, +\infty)$. Since y' goes from positive to negative, $x = 0$ gives local max.

To find inflection points, notice that y'' is never undefined and $e^{-x}(2-x) = 0$ is only possible if $2 - x = 0$ since e^{-x} is never zero. Thus the only inflection point occurs at $x = 2$.

To work out the sign of y'', we have two intervals, $(-\infty, 2)$ and $(2, +\infty)$. For the first, substituting $x = 0$ into y'' gives $-2 < 0$, so y'' is negative (and hence y is concave down) on $(-\infty, 2)$. For the second interval, subsituting $x = 3$ into y' gives $e^{-3} > 0$, so y'' is positive (and hence y is concave up) on $(1, +\infty)$.

I will include a graph eventually, but I don't have time to include one now. In the meantime, you can graph the function on your calculator.

Notice that you should mark the points $(1, 1/e)$ (the absolute max) and $(2, 2/e^3)$ (the inflection point) on the graph.

. 6 7. The function $f(x) = x^3 - ax^2 - 9x + 5$ has an inflection point at $x = 4$. What is a and why?

Solution. This is a new problem which is not similar to assigned problems but can be solved by using the ideas we have covered. (Similar problems in the text include Section 4.5, problems 28 and 30.)

We know that if f has an inflection point at $x = 4$, then $f''(4) = 0$. Since $f'(x) = 3x^2 - 2ax - 9$ and $f''(x) = 6x - 2a$, we have

$$\begin{aligned} f''(4) &= 0 \\ 6 \cdot 4 - 2a &= 0 \\ 2a &= 24 \\ a &= 12 \end{aligned}$$

So a must be 12 because only this value will make $f''(x)$ equal to zero at $x = 4$.