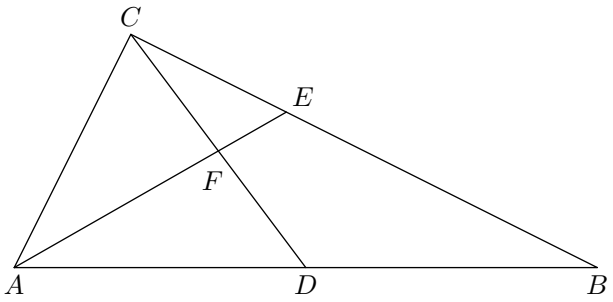
Explain your reasoning. A correct answer poorly explained will not get full marks.

. 5 1. In the triangle $\triangle ABC$ below, the D is the midpoint of $\overline{AB}$ and E is one third of the way from C to B. Use vectors to prove that F is the midpoint of $\overline{CD}$.



HINT: Suppose G is the midpoint of $\overline{CD}$ and show $\overrightarrow{AG}$ and $\overrightarrow{AE}$ have the same direction (are postive scalar multiples of one another).

Solution. We write the two vectors $\overrightarrow{AG}$ and $\overrightarrow{AE}$ in terms of $\overrightarrow{CA}$ and $\overrightarrow{CB}$. First, since D is the midpoint of AB, we have

$$\overrightarrow{CD} = \frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{CB}\right).$$

Since E is one third of thw way from C to B, $\overrightarrow{CE} = \frac{1}{3}\overrightarrow{CB}$.
Now, observe that

$$\begin{aligned}\overrightarrow{AE} &= -\overrightarrow{CA} + \overrightarrow{CE} \\ &= -\overrightarrow{CA} + \frac{1}{3}\overrightarrow{CB},\end{aligned}$$

while

$$\begin{aligned}\overrightarrow{AG} &= -\overrightarrow{CA} + \overrightarrow{CG} \\ &= -\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CD} \\ &= -\overrightarrow{CA} + \frac{1}{2}\left(\frac{1}{2}\left(\overrightarrow{CA} + \overrightarrow{CB}\right)\right) \\ &= -\frac{3}{4}\overrightarrow{CA} + \frac{1}{4}\overrightarrow{CB} \\ &= \frac{3}{4}\left(-\overrightarrow{CA} + \frac{1}{3}\overrightarrow{CB}\right) \\ &= \frac{3}{4}\overrightarrow{AE}.\end{aligned}$$

Since one vector is 3/4 of the other, they both point in the same direction. This means that G is on the straight line from A to E, i.e., the midpoint of $\overline{CD}$ is given by the intersection of $\overline{AE}$ and $\overline{CD}$, as required.