Math 106, Sect. 350/550 Answers to Test #1 14 Sept '06

. 18 1. A local math professor climbs to the top of the Memorial Stadium and throws a small rock vertically upward at 48 ft/sec. If the height of the rock at time t seconds after throwing the rock is $s(t) = 160 + 48t - 16t^2$ feet.

(a) When does the rock hit the ground?

(b) What is the rock's maximum height?

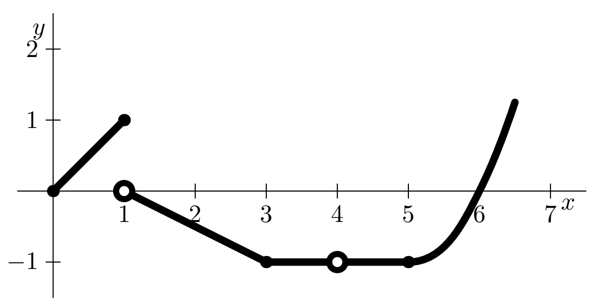
Include units in your answers.

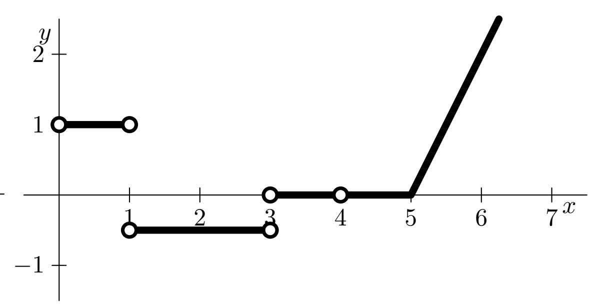
Solution. For (a), the rock hits the ground at the time t when $s(t) = 0$. So we have to solve $-16t^2 + 48t + 160 = 0$, which simplifies to $t^2 - 3t - 10 = 0$. This factors as $(t - 5)(t + 2) = 0$. We are only interested in positive time in this problem, so $t = 5$ is the solution and the rock hits the ground 5 sec after being dropped.

For (b), the maximum height occurs when the velocity is zero. Observe that $v(t) = s'(t) = 48 - 32t$. To solve $48 - 32t = 0$, we divide by 16 and solve to get $t = 3/2$ sec. The maximum height is $s(3/2) = 160 + 48 \cdot (3/2) - 16 \cdot (3/2)^2 = 196$ feet.

. 10 2. For the graph of $y = f(x)$ given below, list the discontinuities of f and their type, and then sketch the graph of $y = f'(x)$ on the second set of axes.

Solution.





The discontinuities of f are at $x = 1$ and at $x = 4$. The first is a jump discontinuity and the second is removable.

. 8 3. Find the derivative of $f(x) = \dfrac{\cos x}{1 + e^x}$.

Solution. Using the quotient rule, we have

$$f'(x) = \frac{(1 + e^x) \cdot (-\sin x) - (\cos x)(e^x)}{(1 + e^x)^2} = -\frac{\sin x + e^x(\sin x + \cos x)}{(1 + e^x)^2}$$

. 16

4. Find the horizontal and vertical asymptotes of $y = \dfrac{x^2+1}{2x^2-x-6}$ and sketch its graph. (Be sure to compute appropriate limits to justify your asymptotes).

Solution. To find the horizontal asymptotes, we compute

$$\lim_{x\to\infty} \frac{x^2+1}{2x^2-x-6} = \lim_{x\to\infty} \frac{1+1/x^2}{2-1/x-6/x^2} = \frac{1}{2}.$$

and the calculation for $\displaystyle\lim_{x\to-\infty} \frac{x^2+1}{2x^2-x-6} = \frac{1}{2}$ is exactly the same. So the line $y = 1/2$ is a horizontal asymptote.

To find the vertical asymptotes, we first factor the denominator, giving $2x^2 - x - 6 = (2x+3)(x-2)$ So $x = -3/2$ and $x = 2$ are *possible* vertical asymptotes. Since the numerator is never zero, it is not zero at these points and so they are asymptotes, although we still need to compute the one-sided limits at $x = 2$ and $x = -3/2$.

Notice that for all values of x, the numerator is always positive, so the sign of the denominator determines the limit.

For $x \to 2^+$, the denominator is positive $(7 \cdot \text{positive})$, so we have $\displaystyle\lim_{x\to 2^+} \frac{x^2+1}{2x^2-x-6} = +\infty$. For $x \to 2^-$, the denominator is negative $(7 \cdot \text{negative})$, so we have $\displaystyle\lim_{x\to 2^+} \frac{x^2+1}{2x^2-x-6} = -\infty$.

For $x \to -3/2^+$, the denominator is negative $(\text{positive}\cdot(-7/2))$, so we have $\displaystyle\lim_{x\to -3/2^+} \frac{x^2+1}{2x^2-x-6} = -\infty$. For $x \to -3/2^-$, the denominator is positive $(\text{negative}\cdot(-7/2))$, so we have $\displaystyle\lim_{x\to -3/2^+} \frac{x^2+1}{2x^2-x-6} = +\infty$.

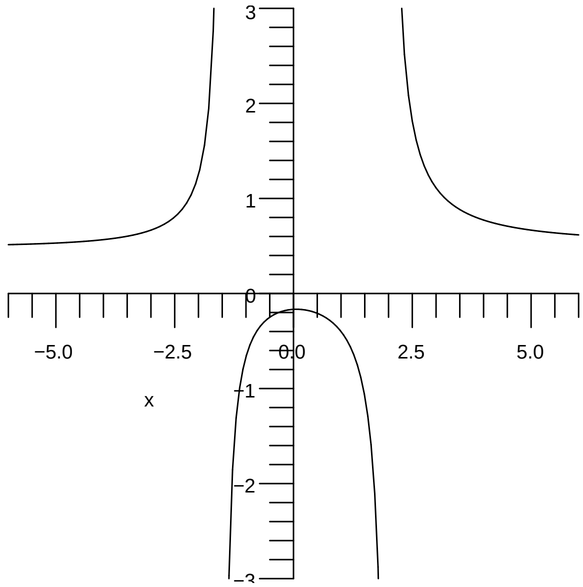


Figure 1: Graph of $f(x) = \dfrac{x^2+1}{2x^2-x-6}$.

. 15 5. Suppose that u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 3, \quad u'(0) = -2, \quad v(0) = -1, \quad v'(0) = 5.$$

Find the values of the following derivatives at $x = 0$:

$$\text{(a)}\ \frac{d}{dx}\left(\frac{u}{v}\right), \qquad \text{(b)}\ \frac{d}{dx}(uv - 3v).$$

Solution. For (a), we use the quotient rule to obtain

$$\left(\frac{u}{v}\right)'(0) = \frac{v(0)u'(0) - u(0)v'(0)}{v(0)^2} = \frac{(-1)\cdot(-2) - 3\cdot 5}{(-1)^2} = -13,$$

and for (b), we use the product rule (along with the sum and constant multiple rules, but you don't have mention them), to get

$$(uv - 3v)'(0) = \big(u(0)v'(0) + u'(0)v(0)\big) - 3v'(0) = 3\cdot 5 + (-2)\cdot(-1) - 3\cdot 5 = 2.$$

. 15 6. Using the definition, find the derivative of $f(x) = \dfrac{x}{2x+1}$. Give the equation of the tangent line at $x = 1/2$.

Check your derivative using the "rules".

Solution. By definition

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{x+h}{2x+2h+1} - \dfrac{x}{2x+1}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(2x+1) - x(2x+2h+1)}{(2x+2h+1)(2x+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{2x^2+2xh+x+h-2x^2-2xh-x}{(2x+2h+1)(2x+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{h}{(2x+2h+1)(2x+1)}\right) \\
&= \lim_{h\to 0} \frac{1}{(2x+2h+1)(2x+1)} = \frac{1}{(2x+1)^2}
\end{aligned}$$

To check this answer using the quotient rule, compute

$$f'(x) = \frac{(2x+1)\cdot 1 - x(2)}{(2x+1)^2} = \frac{1}{(2x+1)^2},$$

so our answer is correct.

At $x = 1/2$, $f(x) = (1/2)/(2(1/2)+1) = 1/4$ and $f'(1/2) = (2(1/2)+1)^{-2} = 1/4$, so this is the line through $(1/2, 1/4)$ with slope $1/4$. Using the point slope form, it is $y - 1/4 = (1/4)(x - 1/2)$ which simplifies to $y = \frac{x}{4} + \frac{1}{8}$.

. 18 7. Evaluate the following limits. (You can use the simplest limit laws, like those for sums and products, without comment but you should clearly indicate when you are using more complicated properties.)

(a) $\displaystyle\lim_{x\to -1} \frac{x+1}{x^4-1}$

(b) $\displaystyle\lim_{x\to 2} \sin\left(\frac{x-2}{x^2-4}\right)$

(c) $\displaystyle\lim_{x\to 0} \frac{x^2 \cot 4x}{\tan 3x}$

Solution. For (a),

$$\lim_{x\to -1}\frac{x+1}{x^4-1}=\lim_{x\to -1}\frac{x+1}{(x^2+1)(x-1)(x+1)}=\lim_{x\to -1}\frac{1}{(x^2+1)(x-1)}=\frac{1}{-4}=-\frac{1}{4},$$

where the evaluation of the limit is possible by the quotient limit law, since the denominator is not zero.
For (b), we use the continuity of the sin function to write

$$\begin{aligned}\lim_{x\to 2}\sin\left(\frac{x-2}{x^2-4}\right)&=\sin\left(\lim_{x\to 2}\frac{x-2}{x^2-4}\right)\\&=\sin\left(\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}\right)\\&=\sin\left(\lim_{x\to 2}\frac{1}{x+2}\right)=\sin(1/4)\end{aligned}$$

For (c), first observe that

$$\frac{x^2\cot 4x}{\tan 3x}=\frac{x^2\dfrac{\cos 4x}{\sin 4x}}{\dfrac{\sin 3x}{\cos 3x}}=\frac{x^2\cos 3x\cos 4x}{\sin 3x\sin 4x},$$

and so

$$\lim_{x\to 0}\frac{x^2\cot 4x}{\tan 3x}=\lim_{x\to 0}\frac{\cos 3x\cos 4x}{12}\frac{4x}{\sin 4x}\frac{3x}{\sin 3x}=\frac{1}{12}.$$

where, in evaluating the limit, we've used $\lim_{x\to 0}\cos nx=1$ for any number n and $\lim_{\theta\to 0}(\sin\theta)/\theta=1$.