Math 496-004 **Assignment # 6** **1 Oct 08**

Due: Oct 8th

1. Let $\mathbf{x} : (a,b) \to \mathbb{R}$ be a regular, twice differentiable curve. Recall that $\mathbf{T} = \mathbf{x}'/\|\mathbf{x}'\|$. Define the curvature vector $\mathbf{K}$ to be the derivative of $\mathbf{T}$ with respect to arc length, s.

   (a) Show that $\mathbf{K} = \mathbf{T}'/\|\mathbf{x}'\|$ and $\kappa = \|\mathbf{K}\|$. Hint: $\frac{ds}{dt} = \|\mathbf{x}'\|$.

   (b) For the curve $\mathbf{x}(t) = t\mathbf{i} + (t^3/3)\mathbf{j}$, verify the following
   $$\mathbf{T} = \frac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{1+t^4}}, \qquad \mathbf{K} = \frac{-2t(t^2\mathbf{i} - \mathbf{j})}{(1+t^4)^2}, \qquad \kappa = \frac{2|t|}{(1+t^4)^{3/2}}.$$

2. Let $\mathbf{x} : (a,b) \to \mathbb{R}^3$ be a regular, twice differentiable curve. Show that
   $$\kappa = \frac{\|x' \times x''\|}{\|x'\|^3}.$$
   Hint: Write x' and x'' in terms of $\frac{dx}{ds}$ and $\frac{d^2x}{ds^2}$ and use the properties of the latter derivatives, such as $\kappa = \|\frac{d^2x}{ds^2}\|$.

3. Do Problem 3.14 (page 89) in Edwards. That is, the following example illustrates the hazards of denoting functions by real variables. Let $w = f(x,y,z)$ and $z = g(x,y)$. Then
   $$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x},$$
   since $\partial x/\partial x = 1$ and $\partial y/\partial x = 0$. Hence $\partial w/\partial x\, \partial z/\partial x = 0$. But if $w = x+y+z$ and $z = x+y$, then $\partial w/\partial z = \partial z/\partial x = 1$, so we have $1 = 0$. Where is the mistake?

4. Do Problems 6.80 and 6.81 in Schaum's. We reword these problems slightly, to fit the discussion on page 80-81 of Edwards, which you might want to read.

   Suppose $u, v : \mathbb{R}^2 \to \mathbb{R}$ are functions with continuous second derivatives. Let $U = u \circ T$ and $V = v \circ T$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\rho,\phi) = (\rho\cos(\phi), \rho\sin(\phi))$. That is, T is the transformation between polar and rectangular coordinates, given by $x = \rho\cos\phi$ and $y = \rho\sin\phi$.

   (a) Show that, under this transformation, the equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ become $\dfrac{\partial U}{\partial \rho} = \dfrac{1}{\rho}\dfrac{\partial V}{\partial \phi}$, $\dfrac{\partial V}{\partial \rho} = -\dfrac{1}{\rho}\dfrac{\partial U}{\partial \phi}$.

   (b) Show that Laplace's equation, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ becomes $\dfrac{\partial^2 U}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial U}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 U}{\partial \phi^2} = 0$.