Due: Wednesday, February 13th

1. Given a function $f : [a, b] \to \mathbb{R}$, of bounded variation, define, for $x \in [a, b]$, and a sequence $X = (x_0, \ldots, x_{2n})$ with $a = x_0 \le x_1 \le \cdots \le x_{2n} = x$,

$$f_+(x, X) = \sum_{j=1}^{n} f(x_{2j}) - f(x_{2j-1}), \qquad f_-(x, X) = \sum_{j=1}^{n} f(x_{2j-2}) - f(x_{2j-1}),$$

and

$$f_+(x) = \sup\{f_+(x, X) : X \text{ as above}\}, \qquad f_-(x) = \sup\{f_-(x, X) : X \text{ as above}\}.$$

First, verify that these suprema are finite and so the functions are well-defined.

   (a) Show that f_+ and f_- are increasing functions.

   (b) Show that $V_a^x f = f_+(x) + f_-(x)$ and $f = f(a) + f_+ - f_-$.
   HINT: For the second equation, first show $f(x) = f(a) + f_+(x, X) - f_-(x, X)$.

   (c) Show that if g and h are increasing functions from $[a, b]$ into $\mathbb{R}$ with $g(a) = h(a) = 0$ and $f = f(a) + g - h$, then $g - f_+$ and $h - f_-$ are increasing functions.

   It is possible to show that, if f is continuous, so are f_+ and f_-, but we leave this as an optional, extra credit question. (The key point is that the sets of points with jump discontinuities for f_+ and f_-, respectively, are disjoint.)

2. Do Exercise 4.4.L in the text.

3. Suppose that $S \subseteq [0, +\infty)$ and there is $K > 0$ so that for all finite sets $T \subseteq S$,

$$\sum_{s \in T} s < K.$$

   Prove that S is countable.

4. Do Exercise 7.1.H in the text.

5. Let F be the set of all sequences $\mathbf{x} = (x_n)$ of real numbers so that all but finitely many of the x_n are zero. Then F is a vector space with the operations

$$(x_n) + (y_n) := (x_n + y_n), \qquad k \cdot (x_n) := (kx_n).$$

   Show that each of the following defines a norm, that is, satisfies the homogenity condition and the triangle inequality, on this vector space:

$$\|\mathbf{x}\|_\infty = \max_{n \ge 1} |x_n|, \quad \|\mathbf{x}\|_w = \max_{n \ge 1} |nx_n|, \quad \|\mathbf{x}\|_1 = \sum_{n \ge 1} |x_n|, \quad \|\mathbf{x}\|_u = \sum_{n \ge 1} |nx_n|.$$

   Is there a constant $K > 0$ so that $\|\mathbf{x}\|_\infty \le K\|\mathbf{x}\|_w$? Is there a constant $L > 0$ so that $\|\mathbf{x}\|_w \le L\|\mathbf{x}\|_\infty$?