Math 325-001 **Assignment # 10** **15 Apr 09**

Due: April 22nd

1. Using the $\delta - \epsilon$ condition, show that $f(x) = x^3$ is continuous at an arbitrary $x_0 \in \mathbb{R}$. HINT: $x^3 - x_0^3 = (x - x_0)(x^2 + xx_0 + x_0^2)$.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x\chi_{\mathbb{Q}}(x)$. That is, $f(x)$ is 0 for x irrational and is x for $x \in \mathbb{Q}$. Show that f is continuous at 0 and that this is the only point where f is continuous.

3. (a) Suppose $f : (a, b) \to \mathbb{R}$ is continuous. Show that if $f(r) = 0$ for each rational number $r \in (a, b)$, then $f(x) = 0$ for all $x \in (a, b)$.

   (b) Suppose $f, g : (a, b) \to \mathbb{R}$ are continuous. Show that if $f(r) = g(r)$ for each rational number $r \in (a, b)$, then $f = g$.

4. Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous and $f(0) = f(2)$. Prove that there $x, y \in [0, 2]$ so that $|x-y| = 1$ and $f(x) = f(y)$. HINT: consider $g(x) = f(x+1)-f(x)$ for $x \in [0, 1]$.

5. Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [b, c] \to \mathbb{R}$ are both continuous and $f(b) = g(b)$. Define $h : [a, c] \to \mathbb{R}$ by
$$h(x) = \begin{cases} f(x) & \text{if } x \in [a, b], \\ g(x) & \text{if } x \in (b, c]. \end{cases}$$
   Show that h is continuous on $[a, c]$.

6. EXTRA CREDIT: Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(u + v) = f(u) + f(v)$ for all $u, v \in \mathbb{R}$. Show that if f is continuous, then there is some $m \in \mathbb{R}$ so that for all $x \in \mathbb{R}$, $f(x) = mx$.
   HINT: start with u and v integers.