. 5

1. Does $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ converge or diverge?

Solution. We use the integral test. The associated improper integral is

$$\begin{aligned}
\int_1^{\infty} \frac{e^x}{1+e^{2x}}\,dx &= \lim_{b\to\infty} \int_1^b \frac{e^x}{1+e^{2x}}\,dx \\
&= \lim_{b\to\infty} \int_e^{e^b} \frac{1}{1+u^2}\,du \begin{cases} u = e^x \\ du = e^x\,dx \end{cases} \\
&= \lim_{b\to\infty} \arctan u \Big|_e^{e^b} \\
&= \lim_{b\to\infty} \arctan(e^b) - \arctan(e) = \frac{\pi}{2} - \arctan(e) < \infty
\end{aligned}$$

Since the improper integral converges, so does the series.

. 5

2. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(\ln n)}$ converge or diverge?

Solution. We know that $\ln n < n$, since $y = \ln x$ has a smaller derivative than $y = x$ and $\ln 1 < 1$. Applying ln to both sides, we get $\ln(\ln n) < \ln n$, and then inverting everything gives

$$\frac{1}{\ln(\ln n)} > \frac{1}{\ln n} > \frac{1}{n}.$$

Since the Harmonic Series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by the comparison test, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(\ln n)}$.