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PREFACE

This book provides an introduction both to real analysis and to a range of important applications that require this material. More than half the book is a series of essentially independent chapters covering topics from Fourier series and polynomial approximation to discrete dynamical systems and convex optimization. Studying these applications can, we believe, both improve understanding of real analysis and prepare for more intensive work in each topic. There is enough material to allow a choice of applications and to support courses at a variety of levels.

The first part of the book covers the basic machinery of real analysis, focusing on that part needed to treat the applications. This material is organized to allow a streamlined approach that gets to the applications quickly, or a more wide-ranging introduction. To this end, certain sections have been marked as enrichment topics or as advanced topics to suggest that they might be omitted. It is our intent that the instructor will choose topics judiciously in order to leave sufficient time for material in the second part of the book.

A quick look at the table of contents should convince the reader that applications are more than a passing fancy in this book. Material has been chosen from both classical and modern topics of interest in applied mathematics and related fields. Our goal is to discuss the theoretical underpinnings of these applied areas concentrating on the role of fundamental principles of analysis. This is not a methods course, although some familiarity with the computational or methods-oriented aspects of these topics may help the student appreciate how the topics are developed. In each application, we have attempted to get to a number of substantial results and to show how these results depend on the fundamental ideas of real analysis. In particular, the notions of limit and approximation are two sides of the same coin, and this interplay is central to the whole book.

We emphasize the role of normed vector spaces in analysis, as they provide a natural framework for most of the applications. This begins early with a separate treatment of $\mathbb{R}^n$. Normed vector spaces are introduced to study completeness and limits of functions. There is a separate chapter on metric spaces that we use as an opportunity to put in a few more sophisticated ideas. This format allows its omission, if need be.

The basic ideas of calculus are covered carefully, as this level of rigour is not generally possible in a first calculus course. One could spend a whole semester doing this material, which forms the basis of many standard analysis courses today. When we have taught a course from these notes, however, we have often chosen to omit topics such as the basics of differentiation and integration on the grounds that these topics have been covered adequately for many students. The goal of getting further into the applications chapters may make it worth cutting here.

We have treated only tangentially some topics commonly covered in real analysis texts, such as multivariate calculus or a brief development of the Lebesgue integral. To cover this material in an accessible way would have left no time, even in a one-year course, for the real goal of the book. Nevertheless, we deal throughout with functions on domains in $\mathbb{R}^n$, and we do manage to deal with issues of higher dimensions without differentiability. For example, the chapter on convexity and optimization yields some deep results on “nonsmooth” analysis that contain the standard differentiable results such as Lagrange multipliers. This is possible because the subject is based on directional derivatives, an essentially one-variable idea. Ideas from multivariate calculus appear once or twice in the advanced sections, such as the use of Green’s Theorem in the section on the isoperimetric inequality.

Not covering measure theory was another conscious decision to keep the material accessible and to keep the size of the book under control. True, we do make use of the L^2 norm and do mention the L^p spaces because these are important ideas. We feel, however, that the basics of Fourier series, approximation theory, and even wavelets can be developed while keeping measure theory to a minimum. Of course, this does not mean we think that the subject is unimportant. Rather we wished to aim the book at an undergraduate audience. To deal partially with some of the issues that arise here, we have included a section on metric space completion. This allows a treatment of L^p spaces as complete spaces of bona fide functions, by means of the Daniell integral. This is certainly an enrichment topic, which can be used to motivate the need for measure theory and to satisfy curious students.

This book began in 1984 when the first author wrote a short set of course notes (120 pages) for a real analysis class at the University of Waterloo designed for students who came primarily from applied math and computer science. The idea was to get to the basic results of analysis quickly and then illustrate their role in a variety of applications. At that time, the applications were limited to polynomial approximation, Newton’s method, differential equations, and Fourier series.

A plan evolved to expand these notes into a textbook suitable for one semester or a year-long course. We expanded both the theoretical section and the choice of applications in order to make the text more flexible. As a consequence, the text is not uniformly difficult. The material is arranged by topic, and generally each chapter gets more difficult as one progresses through it. The instructor can choose to omit some more difficult topics in the chapters on abstract analysis if they will not be needed later. We provide a flow chart indicating the topics in abstract analysis required for each part of the applications chapters. For example, the chapter on limits of functions begins with the basic notion of uniform convergence and the fundamental result that the uniform limit of continuous functions is continuous. It ends with much more difficult material, such as the Arzela–Ascoli Theorem. Even if one plans to do the chapter on differential equations, it is possible to stop before the last section on Peano’s Theorem, where the Arzela–Ascoli Theorem is needed. So both topics can be conveniently omitted. Although one cannot proceed linearly through the text, we hope there is some compensation in demonstrating that, even at a high level, there is a continued interplay between theory and application.

The background assumed for using this text is decent courses in both calculus and linear algebra. What we expect is outlined in the background chapter. A student should have a reasonable working knowledge of differential and integral calculus.

Multivariable calculus is an asset because of the increased level of sophistication and the incorporation of linear algebra; it is not essential. We certainly expect that the student is used to working with exponentials, logarithms, and trigonometric functions. Linear algebra is needed because we treat $\mathbb{R}^n$, $C(X)$, and $L^2(-\pi, \pi)$ as vector spaces. We develop the notion of norms on vector spaces as an important tool for measuring convergence. As such, the reader should be comfortable with the notion of a basis in finite-dimensional spaces. Familiarity with linear transformations is also sometimes useful. A course that introduces the student to proofs would also be an asset. Although we have attempted to address this in the background chapter (Chapter 1), we have no illusions that this text would be easy for a student having no prior experience with writing proofs.

While this background is in principle enough for the whole book, sections marked with a $\bullet$ require additional mathematical maturity or are not central to the main development, and sections marked with a $\star$ are more difficult yet. By and large, the various applications are independent of each other. However, there are references to material in other chapters. For example, in the wavelets chapter (Chapter 15), it seems essential to make comparisons with the classical approximation results for Fourier series and for polynomials.

It is also possible to use an application chapter on its own for a student seminar or other topics course. We have included several modern topics of interest in addition to the classical subjects of applied mathematics. The chapter on discrete dynamical systems (Chapter 11) introduces the notions of chaos and fractals and develops a number of examples. The chapter on wavelets (Chapter 15) illustrates the ideas with the Haar wavelet. It continues with a construction of wavelets of compact support, and gives a complete treatment of a somewhat easier continuous wavelet. In the final chapter (Chapter 16), we study convex optimization and convex programming. Both of these latter chapters require more linear algebra than the others.

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We welcome comments on this book.

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