Fix $n, k \in \mathbb{N}$. A multindex of length k means a k-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ with each α_i an integer between 1 and n. We denote the length of multindex α by $|\alpha|$.

For a vector $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ and a multindex $\alpha = (\alpha_1, \ldots, \alpha_k)$, we define $\mathbf{v}_\alpha$ to be the product

$$v_{\alpha_1} v_{\alpha_2} \cdots v_{\alpha_k}.$$

Notice that, for $t \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n$, and $|\alpha| = k$, $(t\mathbf{v})_\alpha = t^k \mathbf{v}_\alpha$.

For a function f defined on a subset of $\mathbb{R}^n$ and a multindex $\alpha = (\alpha_1, \ldots, \alpha_k)$, we define $D_\alpha f(\mathbf{a})$ to be partial derivative

$$\frac{\partial^k f}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}}(\mathbf{a}).$$

Let $S \subset \mathbb{R}^n$, $\mathbf{a} \in S$, and $f : S \to \mathbb{R}^m$. The kth order differential of f at $\mathbf{a}$ is a map $d^{(k)} f_{\mathbf{a}}$ from $\mathbb{R}^n \to \mathbb{R}^m$ given by

$$d^{(k)} f_{\mathbf{a}}(\mathbf{v}) = \sum_{|\alpha|=k} D_\alpha f(\mathbf{a}) \mathbf{v}_\alpha.$$

From the equation after the defintion of $\mathbf{v}_\alpha$, it follows that $d^{(k)} f_{\mathbf{a}}$ is a k-ary form, i.e., for $t \in \mathbb{R}$,

$$d^{(k)} f_{\mathbf{a}}(t\mathbf{v}) = t^k d^{(k)} f_{\mathbf{a}}(\mathbf{v}).$$

By the equality of mixed partials, $D_\alpha f(\mathbf{a})$ does not depend on the order of the entries in α. We can use this to obtain an expresion for $d^{(k)} f_{\mathbf{a}}$ with significantly fewer terms.

Call a multindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ ordered if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$. Define $\big(\alpha\big)$, the binomial coefficient associated to a multindex α, by first letting j_i be the number of entries of α which are equal to i, for $i = 1, \ldots, n$ and defining

$$\big(\alpha\big) = \frac{k!}{j_1! j_2! \cdots j_n!}.$$

Thus, if all entries of α are the same, then $\big(\alpha\big) = 1$ while if all entries are distinct, then $\big(\alpha\big) = k!$.

With these definitions, it is routine to show that

$$d^{(k)} f_{\mathbf{a}}(\mathbf{v}) = \sum_{\substack{|\alpha|=k \\ \alpha \text{ ordered}}} \big(\alpha\big) D_\alpha f(\mathbf{a}) v_\alpha.$$

Finally, we define $D^{(k)}_{\mathbf{v}} f(\mathbf{a})$ to the kth directional derivative of f in the direction v, that is,

$$D^{(k)}_{\mathbf{v}} f(\mathbf{a}) = \underbrace{D_{\mathbf{v}} D_{\mathbf{v}} \cdots D_{\mathbf{v}}}_{k \text{ times}} f(\mathbf{a}).$$

The 'kth differentials and kth directional derivatives' Theorem states that, for a function $f : S \to \mathbb{R}^m$ which is C^k on an open set $S \subset \mathbb{R}^n$, then

$$d^{(k)} f_{\mathbf{a}}(\mathbf{v}) = D^{(k)}_{\mathbf{v}} f(\mathbf{a}),$$

for all $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{a} \in S$.