MAC 2103

Module 10

Inner Product Spaces I

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Learning Objectives

Upon completing this module, you should be able to:

1. Define and find the inner product, norm and distance in a given inner product space.
2. Find the cosine of the angle between two vectors and determine if the vectors are orthogonal in an inner product space.
3. Find the orthogonal complement of a subspace of an inner product space.
4. Find a basis for the orthogonal complement of a subspace of $\Re^n$ spanned by a set of row vectors.
5. Identify the four fundamental matrix spaces of an m x n matrix A, and know the column rank of A, the row rank of A, and the rank of A.
6. Know the equivalent statements of an n x n matrix A.

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General Vector Spaces II

The major topics in this module:

Inner Product Spaces, Inner Products,
Norm, Distance, Fundamental Matrix Spaces,
Rank, and Orthogonality

Definition of an Inner Product and Orthogonal Vectors

A. Inner Product: Let $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$
be any function from $V \times V$ into $\mathbb{R}$ that satisfies the following conditions for any **u**, **v, z** in V:

$a)\ \langle\vec{u},\vec{v}\rangle = \langle\vec{v},\vec{u}\rangle,$
$b)\ \langle\vec{u}+\vec{v},\vec{z}\rangle = \langle\vec{u},\vec{z}\rangle + \langle\vec{v},\vec{z}\rangle,$
$c)\ \langle k\vec{u},\vec{v}\rangle = k\langle\vec{u},\vec{v}\rangle,$
$d)\ \langle\vec{v},\vec{v}\rangle \geq 0,$ and $\langle\vec{v},\vec{v}\rangle = 0$ iff $\vec{v} = 0.$

Then $\langle\vec{u},\vec{v}\rangle$ defines an inner product for all **u**, **v** in V.

u and **v** in V are Orthogonal Vectors iff $\langle\vec{u},\vec{v}\rangle = 0.$

Definition of a Norm of a Vector

B. Norm: Let $\|\cdot\|: V \to \mathbb{R}^+ = [0,\infty)$

be any function from V into $\mathbb{R}^+$ that satisfies the following conditions for any **u**, **v** in V:

$$a)\ \|\vec{u}\| \ge 0, \quad and\ \|\vec{u}\| = 0\ iff\ \vec{u} = \vec{0},$$

$$b)\ \|s\vec{u}\| = |s|\|\vec{u}\|, \quad and$$

$$c)\ \|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\| \qquad Triangle \quad inequality.$$

Then $\|\vec{u}\|$ defines a norm for all **u** in V.

The special norm that is induced by an inner product is

$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}.$$

Definition of the Distance Between Two Vectors

C. Distance: Let $d(\cdot,\cdot): V \times V \to \mathbb{R}^+ = [0,\infty)$

be any function from $V \times V$ into $\mathbb{R}^+$ that satisfies the following conditions for any **u**, **v, w** in V:

$$a)\ d(\vec{u},\vec{v}) \ge 0, \quad and\ d(\vec{u},\vec{v}) = 0\ iff\ \vec{u} = \vec{v},$$

$$b)\ d(\vec{u},\vec{v}) = d(\vec{v},\vec{u}), \quad and$$

$$c)\ d(\vec{u},\vec{v}) \le d(\vec{u},\vec{w}) + d(\vec{w},\vec{v}) \qquad Triangle \quad inequality.$$

Then $d(\vec{u},\vec{v})$ defines the distance for all **u**, **v** in V.

The special distance that is induced by a norm is

$$d(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\| = \sqrt{\langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle}.$$

How to Find an Inner Product, Norm, and Distance for Vectors in $\Re^n$?

For **u** and **v** in $\Re^n$, we define

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n,$$

$$\|\vec{u}\| = \langle \vec{u}, \vec{u} \rangle^{\frac{1}{2}} = \sqrt{(\vec{u} \cdot \vec{u})} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2},$$

and $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

$$= \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle^{\frac{1}{2}} = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}.$$

Note: These are our usual norm and distance in $\Re^n$. $\Re^n$ is an inner product space with the dot product as its inner product.

How to Find an Inner Product, Norm, and Distance for Vectors in $\Re^n$? (Cont.)

Example 1: Let **u** = (1,2,-1) and **v** = (2,0,1) in $\Re^3$. Find the inner product, norms, and distance.

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = (1)(2) + (2)(0) + (-1)(1) = 1,$$

$$\|\vec{u}\| = \langle \vec{u}, \vec{u} \rangle^{1/2} = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6},$$

$$\|\vec{v}\| = \langle \vec{v}, \vec{v} \rangle^{1/2} = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}, \quad and$$

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle^{1/2}$$

$$= \sqrt{(1-2)^2 + (2-0)^2 + (-1-1)^2} = \sqrt{9} = 3.$$

How to Find an Inner Product, Norm, and Distance for Vectors (Matrices) in M_{22}?

In M_{22}, the vectors are 2 x 2 matrices. Thus, given two matrices U and V in M_{22}, we define

$$\langle U,V\rangle = tr(U^T V) = tr(V^T U) = \langle V,U\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 .$$

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}, V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}.$$

Recall: tr(A) = sum of the entries on the main diagonal of A.

$$\|U\| = \langle U,U\rangle^{1/2} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}, \quad and$$

$$d(U,V) = \|U - V\| = \langle U - V, U - V\rangle^{1/2} = \sqrt{\langle U - V, U - V\rangle}$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 + (u_4 - v_4)^2} .$$

How to Find an Inner Product, Norm, and Distance for Vectors (Matrices) in M_{22}? (Cont.)

Example 2: Let

$$U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}, V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 4 & 6 \end{bmatrix}$$

be matrices in the vector space M_{22}. Find the inner product, norms and distance for U and V in M_{22}.

$$\langle U,V\rangle = tr(U^T V) = tr(V^T U) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

$$= (2)(0) + (3)(4) + (-1)(9) + (1)(6) = 9.$$

Recall: tr(A) = sum of the entries on the main diagonal of A.

How to Find an Inner Product, Norm, and Distance for Vectors (Matrices) in M_{22}? (Cont.)

$$\|U\| = \langle U,U\rangle^{1/2} = \sqrt{u_1^2+u_2^2+u_3^2+u_4^2} = \sqrt{2^2+(-1)^2+3^2+1^2} = \sqrt{15},$$
$$\|V\| = \langle V,V\rangle^{1/2} = \sqrt{u_1^2+u_2^2+u_3^2+u_4^2} = \sqrt{0^2+9^2+4^2+6^2} = \sqrt{133}, \quad and$$

$$d(U,V) = \|U-V\| = \langle U-V,U-V\rangle^{1/2} = \sqrt{\langle U-V,U-V\rangle}$$
$$= \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2+(u_4-v_4)^2}$$
$$= \sqrt{(2-0)^2+((-1)-9)^2+(3-4)^2+(1-6)^2}$$
$$= \sqrt{4+100+1+25} = \sqrt{130}.$$

How to Find an Inner Product, Norm, and Distance for Vectors (Polynomials) in P_2?

Let P_2 be the set of all polynomials of degree less than or equal to two. For two polynomials $\vec{p},\vec{q} \in P_2$,

$$p(x) = a_0x^0 + a_1x^1 + a_2x^2,$$
$$q(x) = b_0x^0 + b_1x^1 + b_2x^2$$

we define
$$\langle \vec{p},\vec{q}\rangle = a_0b_0 + a_1b_1 + a_2b_2,$$
$$\|\vec{p}\| = \langle \vec{p},\vec{p}\rangle^{1/2} = \sqrt{a_0^2+a_1^2+a_2^2}, \quad and$$
$$d(\vec{p},\vec{q}) = \|\vec{p}-\vec{q}\| = \langle \vec{p}-\vec{q},\vec{p}-\vec{q}\rangle^{1/2}$$
$$= \sqrt{(a_0-b_0)^2+(a_1-b_1)_1^2+(a_3-b_3)^2}.$$

How to Find an Inner Product, Norm, and Distance for Vectors (Polynomials) in P_2? (Cont.)

Example 3: Find the inner product, norms, and distance for **p** and **q** in P_2, where

$$p(x) = 8 - 3x + 5x^2 \ \ and \quad q(x) = 1 - 2x + 7x^2.$$

$$\langle \vec{p},\vec{q}\rangle = a_0b_0 + a_1b_1 + a_2b_2 = (8)(1) + (-3)(-2) + (5)(7) = 8 + 6 + 35 = 49,$$

$$\|\vec{p}\| = \langle \vec{p},\vec{p}\rangle^{1/2} = \sqrt{a_0^2 + a_1^2 + a_2^2} = \sqrt{8^2 + (-3)^2 + 5^2} = \sqrt{98} = 7\sqrt{2},$$

$$\|\vec{q}\| = \langle \vec{q},\vec{q}\rangle^{1/2} = \sqrt{b_0^2 + b_1^2 + b_2^2} = \sqrt{1^2 + (-2)^2 + 7^2} = \sqrt{54} = 3\sqrt{6}, \ \ and$$

$$d(\vec{p},\vec{q}) = \|\vec{p} - \vec{q}\| = \langle \vec{p} - \vec{q}, \vec{p} - \vec{q}\rangle^{1/2} = \sqrt{(a_0 - b_0)^2 + (a_1 - b_1)_1^2 + (a_3 - b_3)^2}$$

$$= \sqrt{(8-1)^2 + ((-3) - (-2))^2 + (5-7)^2} = \sqrt{49 + 1 + 4} = \sqrt{54} = 3\sqrt{6}.$$

How to Find an Inner Product, Norm, and Distance for Vectors (Functions) in C[a,b]?

Let C[a,b] be the set of all continuous functions on [a,b]. For all **f** and **g** in C[a,b], we define

$$\langle \vec{f},\vec{g}\rangle = \int_a^b f(x)g(x)\,dx,$$

$$\|\vec{f}\| = \langle \vec{f},\vec{f}\rangle^{1/2} = \left(\int_a^b [f(x)]^2\,dx\right)^{1/2} = \sqrt{\int_a^b f^2(x)\,dx},$$

$$and \ \ d(\vec{f},\vec{g}) = \|\vec{f} - \vec{g}\| = \langle \vec{f} - \vec{g}, \vec{f} - \vec{g}\rangle^{1/2} = \left(\int_a^b [f(x) - g(x)]^2\,dx\right)^{1/2}.$$

How to Find an Inner Product, Norm, and Distance for Vectors (Functions) in C[a,b]? (Cont.)

Example 4: Let **f** = f(x) = 1 and **g** = g(x) = -x. Find the inner product, norms, and distance on C[2,4].

$$\langle \vec{f},\vec{g}\rangle = \int_2^4 f(x)g(x)\,dx = \int_2^4 (1)(-x)\,dx = -\frac{x^2}{2}\Big|_2^4 = (-8)-(-2) = -6,$$

$$\|\vec{f}\| = \langle \vec{f},\vec{f}\rangle^{1/2} = \left(\int_2^4 [f(x)]^2\,dx\right)^{1/2} = \sqrt{\int_2^4 f^2(x)\,dx} = \sqrt{\int_2^4 dx} = \sqrt{x\big|_2^4} = \sqrt{2},$$

$$\|\vec{g}\| = \langle \vec{g},\vec{g}\rangle^{1/2} = \left(\int_2^4 [g(x)]^2\,dx\right)^{1/2} = \sqrt{\int_2^4 g^2(x)\,dx} = \sqrt{\int_2^4 (-x)^2\,dx} = \sqrt{\frac{x^3}{3}\Big|_2^4} = 2\sqrt{\frac{14}{3}},$$

$$and \;\; d(\vec{f},\vec{g}) = \|\vec{f}-\vec{g}\| = \langle \vec{f}-\vec{g},\vec{f}-\vec{g}\rangle^{1/2} = \left(\int_2^4 [f(x)-g(x)]^2\,dx\right)^{1/2}$$

$$= \sqrt{\int_2^4 (1+x)^2\,dx} = \sqrt{\int_2^4 (1+2x+x^2)\,dx} = \sqrt{(x+x^2+\frac{x^3}{3})\Big|_2^4} = 7\sqrt{\frac{2}{3}}.$$

The Cauchy-Schwarz Inequality

Cauchy-Schwarz Inequality: $\left|\langle \vec{f},\vec{g}\rangle\right| \le \|\vec{f}\|\|\vec{g}\|$

for all **f**, **g** in an inner product space V, with the norm induced by the inner product. We evaluate the Cauchy-Schwarz inequality for the previous examples.

1. $|\langle \vec{u},\vec{v}\rangle| = 1 \le \sqrt{6}\sqrt{5} = \|\vec{u}\|\|\vec{v}\| \quad from \;\; Example \;\; 1.$

2 $|\langle U,V\rangle| = 9 \le \sqrt{15}\sqrt{133} = \|U\|\|V\| \quad from \;\; Example \;\; 2.$

3. $|\langle \vec{p},\vec{q}\rangle| = 49 \le (7\sqrt{2})(3\sqrt{6}) = \|\vec{p}\|\|\vec{q}\| \quad from \;\; Example \;\; 3.$

4. $|\langle \vec{f},\vec{g}\rangle| = 6 \le (\sqrt{2})(2\sqrt{\frac{14}{3}}) = \|\vec{f}\|\|\vec{g}\| \quad from \;\; Example \;\; 4.$

Orthogonality for Vectors (Functions) in C[a,b]

Example 5: Let **f** = f(x) = 1 and **g** = g(x) = e^x. Find the inner product on C[0,2].

$$\langle \vec{f}, \vec{g} \rangle = \int_0^2 f(x)g(x)\,dx = \int_0^2 (1)(e^x)\,dx = e^2 - e^0 = e^2 - 1 \neq 0$$

Since the inner product is not zero, **f** and **g** are not orthogonal functions in C[0,2].

Example 6: Let **f** = f(x) = 5 and **g** = g(x) = cos(x). Find the inner product on C[0,π].

$$\langle \vec{f}, \vec{g} \rangle = \int_0^\pi f(x)g(x)\,dx = \int_0^\pi 5\cos(x)\,dx == 5\int_0^\pi \cos(x)\,dx$$

$$= 5[\sin(\pi) - \sin(0)] = 5[0-0] = 0$$

Since the inner product is zero, **f** and **g** are orthogonal functions in C[0,π].

Orthogonality for Vectors (Functions) in C[a,b] (Cont.)

Example 7: Let **f** = f(x) = 4x and **g** = g(x) = x^2. Find the inner product on C[-1,1].

$$\langle \vec{f}, \vec{g} \rangle = \int_{-1}^1 f(x)g(x)\,dx = \int_{-1}^1 (4x)(x^2)\,dx == 4\int_{-1}^1 x^3\,dx = \left(x^4\right)\Big|_{-1}^1 = 1 - 1 = 0$$

Since the inner product is zero, **f** and **g** are orthogonal functions in C[-1,1].

Inner Product and Orthogonality for Vectors (Functions) in C[a,b] (Cont.)

Example 8: Let **f** = f(x) = 1 and **g** = g(x) = sin(2x). Find the inner product on C[-π,π].

$$\langle \vec{f},\vec{g}\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx = \int_{-\pi}^{\pi} (1)(\sin(2x))dx = \int_{-\pi}^{\pi} \sin(2x)dx = -\frac{1}{2}\cos(2x)\Big|_{-\pi}^{\pi}$$

$$= \left[-\frac{1}{2}\cos(2\pi)\right] - \left[-\frac{1}{2}\cos(-2\pi)\right] = -\frac{1}{2}\cos(2\pi) + \frac{1}{2}\cos(2\pi) = 0,$$

$$\cos(2x) \quad is \quad even.$$

Since the inner product is zero, **f** and **g** are orthogonal functions in C[-π,π].

The Generalized Theorem of Pythogoras

Generalized Theorem of Pythogoras: If **f** and **g** are orthogonal vectors in an inner product space V, then $\|\vec{f}+\vec{g}\|^2 = \|\vec{f}\|^2 + \|\vec{g}\|^2$.

Proof: Let f and g be orthogonal in V, then $\langle \vec{f},\vec{g}\rangle = 0.$

$$Thus,\ \|\vec{f}+\vec{g}\|^2 = \langle \vec{f}+\vec{g},\vec{f}+\vec{g}\rangle = \langle \vec{f},\vec{f}+\vec{g}\rangle + \langle \vec{g},\vec{f}+\vec{g}\rangle$$

$$= \langle \vec{f},\vec{f}\rangle + \langle \vec{f},\vec{g}\rangle + \langle \vec{g},\vec{f}\rangle + \langle \vec{g},\vec{g}\rangle$$

$$= \|\vec{f}\|^2 + 2\langle \vec{f},\vec{g}\rangle + \|\vec{g}\|^2$$

$$= \|\vec{f}\|^2 + \|\vec{g}\|^2.$$

The Generalized Theorem of Pythogoras (Cont.)

The Generalized Theorem of Pythogoras $\|\vec{f}+\vec{g}\|^2 = \|\vec{f}\|^2 + \|\vec{g}\|^2$

We evaluate the theorem for the previous examples. From example 6, we have

$$\|\vec{f}+\vec{g}\|^2 = \int_0^\pi [5+\cos(x)]^2\,dx = \int_0^\pi (25+(2)(5)\cos(x)+\cos^2(x))\,dx$$

$$= \int_0^\pi 25\,dx + 2\int_0^\pi (5)\cos(x)\,dx + \int_0^\pi \cos^2(x)\,dx$$

$$= \|\vec{f}\|^2 + 2\langle 5,\cos(x)\rangle + \|\vec{g}\|^2 = \|\vec{f}\|^2 + 2(0) + \|\vec{g}\|^2 = \|\vec{f}\|^2 + \|\vec{g}\|^2 .$$

The Generalized Theorem of Pythogoras (Cont.)

From Example 7, we have

$$\|\vec{f}+\vec{g}\|^2 = \int_{-1}^{1} [4x+x^2]^2\,dx = \int_{-1}^{1} (16x^2+(2)(4x)x^2+x^4)\,dx$$

$$= \int_{-1}^{1} 16x^2\,dx + 2\int_{-1}^{1} 4x^3\,dx + \int_{-1}^{1} x^4\,dx$$

$$= \|\vec{f}\|^2 + 2\langle 4x,x^2\rangle + \|\vec{g}\|^2 = \|\vec{f}\|^2 + 2(0) + \|\vec{g}\|^2 = \|\vec{f}\|^2 + \|\vec{g}\|^2 .$$

Notice that we have used orthogonality to obtain our results without directly computing the integrals.

Next, we will compute the integrals directly to obtain the result.

The Generalized Theorem of Pythogoras (Cont.)

$$\|\vec{f}\|^2 = \langle \vec{f}, \vec{f} \rangle = \int_{-1}^{1} f^2(x)dx = \int_{-1}^{1} [f(x)]^2\, dx = \int_{-1}^{1} (4x)(4x)dx == 16\int_{-1}^{1} x^2\, dx = (16)\frac{x^3}{3}\bigg|_{-1}^{1} = \frac{32}{3}$$

$$\|\vec{g}\|^2 = \langle \vec{g}, \vec{g} \rangle = \int_{-1}^{1} g^2(x)dx = \int_{-1}^{1} [g(x)]^2\, dx = \int_{-1}^{1} (x^2)(x^2)dx == \int_{-1}^{1} x^4\, dx = \frac{x^5}{5}\bigg|_{-1}^{1} = \frac{2}{5}$$

$$\|\vec{f}+\vec{g}\|^2 = \langle \vec{f}+\vec{g}, \vec{f}+\vec{g} \rangle = \int_{-1}^{1} [f(x)+g(x)]^2\, dx = \int_{-1}^{1} (4x+x^2)(4x+x^2)dx$$

$$= \int_{-1}^{1} (16x^2+8x^3+x^4)dx = \left(\frac{16x^3}{3}+\frac{8x^4}{4}+\frac{x^5}{5}\right)\bigg|_{-1}^{1} = \frac{166}{15} = \frac{32}{3}+\frac{2}{5} = \|\vec{f}\|^2 + \|\vec{g}\|^2.$$

Thus,

$$\|\vec{f}+\vec{g}\|^2 = \|\vec{f}\|^2 + \|\vec{g}\|^2.$$

How to Find the Cosine of the Angle Between Two Vectors in an Inner Product Space?

Example 9: Let $\Re^5$ have the Euclidean inner product.
Find the cosine of the angle between
u = (2,1,0,-1,5) and **v** = (1,-1,-2,3,1).

$$\cos(\theta) = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|\|\vec{v}\|} = \frac{u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 + +u_5v_5}{\sqrt{u_1^2+u_2^2+u_3^2+u_4^2+u_5^2}\sqrt{v_1^2+v_2^2+v_3^2+v_4^2+v_5^2}}$$

$$= \frac{(2)(1)+(1)(-1)+(0)(-2)+(-1)(3)+(5)(1)}{\sqrt{2^2+1^2+0+(-1)^2+5^2}\sqrt{1^2+(-1)^2+(-2)^2+(3)^2+1^2}}$$

$$= \frac{3}{4\sqrt{31}}.$$

How to Find the Cosine of the Angle Between Two Vectors in an Inner Product Space? (Cont.)

Example 10: Let P_2 have the previous inner product. Find the cosine of the angle between **p** and **q**.

$$\vec{p} = p(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 = 2 + 6x + 3x^2,$$

$$\vec{q} = q(x) = b_0 x^0 + b_1 x^1 + b_2 x^2 = -2 - x + 4x^2$$

$$\cos(\theta) = \frac{\langle \vec{p}, \vec{q} \rangle}{\|\vec{p}\|\|\vec{q}\|} = \frac{a_0 b_0 + a_1 b_1 + a_2 b_2}{\sqrt{a_0^2 + a_1^2 + a_2^2}\sqrt{b_0^2 + b_1^2 + b_2^2}}$$

$$= \frac{(2)(-2) + (6)(-1) + (3)(4)}{\sqrt{2^2 + 6^2 + 3^2}\sqrt{(-2)^2 + (-1)^2 + (4)^2}}$$

$$= \frac{2}{7\sqrt{21}}.$$

Quick Review on a Basis for a Vector Space and the Dimension of a Vector Space

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be nonzero vectors in V. Then for $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, we have that W = span(S) = {all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$} is a subspace of V.

If S contains some linearly dependent vectors, then the dim(W) < n.

If S is a linearly independent set of vectors, then S is a basis for the vector space W. W is a proper subspace of V, if dim(W) < dim(V). If dim(W) = dim(V) = n, then S is a basis for V.

A non-trivial vector space V always have two subspaces: the trivial vector space, {**0**} and V.

Quick Review on a Basis for a Vector Space and the Dimension of a Vector Space

So, we need two conditions for the set S to be a basis of a vector space, it must be a linearly independent set and it must span the vector space. The number of basis vectors in S is the dimension of the vector space.

For example: $\dim(\Re^2)=2$, $\dim(\Re^3)=3$, $\dim(\Re^5)=5$, $\dim(P_2)=3$, $\dim(M_{22})=4$, and $\dim(C[a,b])$ is infinite.

Rank, Row Rank, and Column Rank of a Matrix

The dimensions of the row space of A, column space of A, and nullspace of A are also called the row rank of A, column rank of A, and nullity of A, respectively.

The rank of A = the row rank of A = the column rank of A, and is the number of linearly independent columns in the matrix A.

The nullity of A = dim(null(A)) and is the number of free variables in the solution space.

The Orthogonal Complement of a Subspace W

The orthogonal complement of W, $W^{\perp}$ is the set of all vectors in a finite dimensional inner product space V that are orthogonal to every vector in W.

Both W and $W^{\perp}$ are subspaces of V where dim(W) + dim($W^{\perp}$) = dim(V).

The only vector common to W and $W^{\perp}$ is **0**, and $W = (W^{\perp})^{\perp}$.

The Four Fundamental Matrix Spaces

Let A be an m x n matrix, then:

1.The null(A) and the row(A) are orthogonal complements in $\Re^n$ with respect to the Euclidean inner product.

2.The null(A^T) and the col(A) are orthogonal complements in $\Re^m$ with respect to the Euclidean inner product.

Note that row(A) = col(A^T) and col(A) = row(A^T).

The four fundamental matrix spaces are:

1. row(A) = col(A^T),
2. col(A) = row(A^T),
3. null(A), and
4. null(A^T).

How to Find a Basis for the Orthogonal Complement of the Subspace of $\Re^n$ Spanned by the Vectors?

Example 11: Find a basis for the orthogonal complement of the subspace of $\Re^n$ spanned by the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$.

Let V = span({$\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$}). Then, V is a subspace of of $\Re^n$ if $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3} \in \Re^n$. Let

$$\vec{v}_1 = (1,-1,3), \vec{v}_2 = (5,-4,-4), \vec{v}_3 = (7,-6,2).$$

From the last module, we know that the null(A) is the orthogonal complement to the row(A) = col(A^T), and that the null(A) is the solution space of the homogeneous system, A**x** = **0**. Let $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ be the row vectors of the matrix A. Then, row(A) is the orthogonal complement to null(A) which is the solution space to A**x** = **0**. We shall use Gauss Elimination to reduce A to a row echelon form to find the basis for null(A).

How to Find a Basis for the Orthogonal Complement of the Subspace of $\Re^n$ Spanned by the Vectors? (Cont.)

$$\begin{array}{c} r1 \\ r2 \\ r3 \end{array} \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} = A$$

$$\begin{array}{c} r1 \\ -5r1+r2 \\ -7r1+r3 \end{array} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$$

$$\begin{array}{c} r1 \\ r2 \\ -r2+r3 \end{array} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the null(A) is the solution space of the homogeneous system, A**x** = **0**, the general solution of the system is:

$$x_3 = s,$$
$$x_2 - 19x_3 = 0, x_2 - 19s = 0, x_2 = 19s.$$
$$x_1 - x_2 + 3x_3 = 0, x_1 - 19s + 3s = 0, x_1 = 16s.$$

Note that column 1 and column 2, with the red leading 1's, are linearly independent. The corresponding columns of A form a basis for the col(A). Then, dim(col(A)) = 2.

How to Find a Basis for the Orthogonal Complement of the Subspace of $\Re^n$ Spanned by the Vectors? (Cont.)

The nonzero rows with the leading 1's are linearly independent. These rows form a basis for row(A). Then, dim(row(A)) = 2.

Note that the row space and column space are both two dimensional, so rank(A) = 2. The rank(A) is the common dimension of the row space of A and the column space of A.

So, rank(A) = rowrank(A) = colrank(A) = dim(col(A)) =dim(row(A)) is always true for any m x n matrix A.

How to Find a Basis for the Orthogonal Complement of the Subspace of $\Re^n$ Spanned by the Vectors? (Cont.)

Thus, the solution vector is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16s \\ 19s \\ s \end{bmatrix} = s\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

and the vector $\vec{w}_1 = \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$

{**w**$_1$} forms a basis for the null(A). In other words, {**w**$_1$} = {(16,19,1)} forms a basis for the orthogonal complement of row(A).

How to Find a Basis for the Orthogonal Complement of the Subspace of $\Re^n$ Spanned by the Vectors? (Cont.)

Since B = {$\mathbf{w_1}$} is a basis for null(A), span(B) = null(A), and dim(null(A)) = 1 is the nullity of A. In this example, rank(A) is 2 and the nullity of A is 1, and the number of columns of A is 3.

For any m x n matrix A, rank(A) + nullity(A) is n, which is the number of columns of A.

Let's Review Some Equivalent Statements

If A is an n x n matrix, and if T_A: $\Re^n \rightarrow \Re^n$ is multiplication by A, than the following are equivalent. (See Theorem 6.2.7)

- A is invertible.
- A$\mathbf{x}$ = $\mathbf{0}$ has only the trivial solution
- The reduced row-echelon form of A is I_n.
- A is expressible as a product of elementary matrices.
- A$\mathbf{x}$ = $\mathbf{b}$ is consistent for every n x 1 matrix $\mathbf{b}$.
- A$\mathbf{x}$ = $\mathbf{b}$ has exactly one solution for every n x 1 matrix $\mathbf{b}$.
- det(A) $\neq$ 0.
- The range of T_A is $\Re^n$.
- T_A is one-to-one.

Let's Review Some Equivalent Statements (Cont.)

- The column vectors of A are linearly independent.
- The row vectors of A are linearly independent.
- The column vectors of A span $\Re^n$.
- The row vectors of A span $\Re^n$.
- The column vectors of A form a basis for $\Re^n$.
- The row vectors of A form a basis for $\Re^n$.
- A has rank n = row rank n = column rank n = dim(col(A)).
- A has nullity 0 = dim(null(A)).
- The orthogonal complement of the nullspace of A, null(A), is $\Re^n$.
- The orthogonal complement of the row space of A, row(A), is {**0**}.

What have we learned?

We have learned to:

1. Define and find the inner product, norm and distance in a given inner product space.
2. Find the cosine of the angle between two vectors and determine if the vectors are orthogonal in an inner product space.
3. Find the orthogonal complement of a subspace of an inner product space.
4. Find a basis for the orthogonal complement of a subspace of $\Re^n$ spanned by a set of row vectors.
5. Identify the four fundamental matrix spaces of an m x n matrix A, and know the column rank of A, the row rank of A, and the rank of A.
6. Know the equivalent statements of an n x n matrix A.

Credit

Some of these slides have been adapted/modified in part/whole from the following textbook:

• Anton, Howard: Elementary Linear Algebra with Applications, 9th Edition