DDPM Lower Bound

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1 Introduction

Derivation using Jensen's inequality [1]:

$$\begin{aligned}
\log p(X) &= \log \int_Z p(X,Z) \\
&= \log \int_Z p(X,Z) \frac{p(Z|X)}{p(Z|X)} \\
&= \log(\left(\mathbb{E}_q \left[\frac{p(X,Z)}{q(Z|X)} \right] \right)) \\
&\geq \mathbb{E}_q \left[\log \frac{p(X,Z)}{q(Z|X)} \right] \\
&= \mathbb{E}_q \left[\log p(X,Z) \right] + H[Z|X]
\end{aligned}$$

In the diffusion model, we replace by $X \sim x_0$ and $Z \sim x_{1:T}$, and since we have:

$$p(x_{0:T}) = p(x_T) \prod_{t=1}^{T} p(x_{t-1}|x_t) \tag{1}$$

and

$$q(x_{1:T|x_0}) = \prod_{t=1}^{T} q(x_t|x_{t-1}) \tag{2}$$

[1] https://xyang35.github.io/2017/04/14/variational-lower-bound/

we can derive the following:

$$
\begin{aligned}
\log p(x_0) &= \log \int p(x_0, x_{1:T}) \frac{q(x_{1:T}|x_0)}{q(x_{1:T}|x_0)} \\
&= \log \mathbb{E} \left(\frac{p(x_{0:T})}{q(x_{1:T}|x_0)} \right) \\
&\geq \mathbb{E}_q \left[\log \frac{p(x_{0:T})}{q(x_{1:T}|x_0)} \right] \\
&= \mathbb{E}_q \left[\log \frac{p(x_T) \prod_{t=1}^{T} p(x_{t-1}|x_t)}{\prod_{t=1}^{T} q(x_t|x_{t-1})} \right] \text{(using equation 1 and 2)} \\
&= \mathbb{E}_q \left[\log(p_T) \right] + \mathbb{E}_q \left[\frac{\prod_{t=1}^{T} p(x_{t-1}|x_t)}{\prod_{t=1}^{T} q(x_t|x_{t-1})} \right] \\
&= \mathbb{E}_q \left[\log(p_T) \right] + \mathbb{E}_q \left[\prod_{t=1}^{T} \frac{p(x_{t-1}|x_t)}{q(x_t|x_{t-1})} \right] \\
&= \mathbb{E}_q \left[\log(p_T) \right] + \sum_{t=1}^{T} \mathbb{E}_q \left[\frac{p(x_{t-1}|x_t)}{q(x_t|x_{t-1})} \right]
\end{aligned}
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