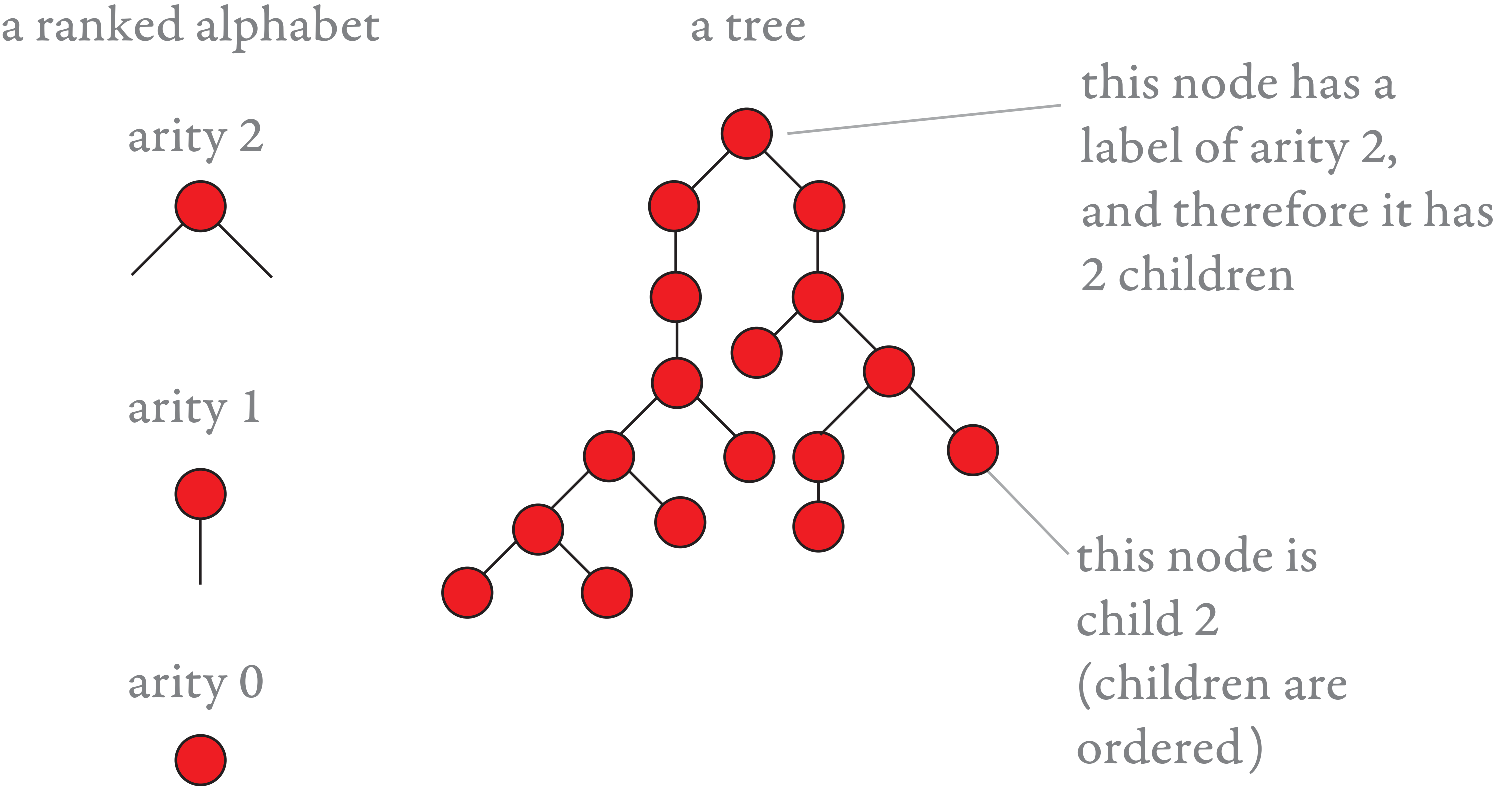
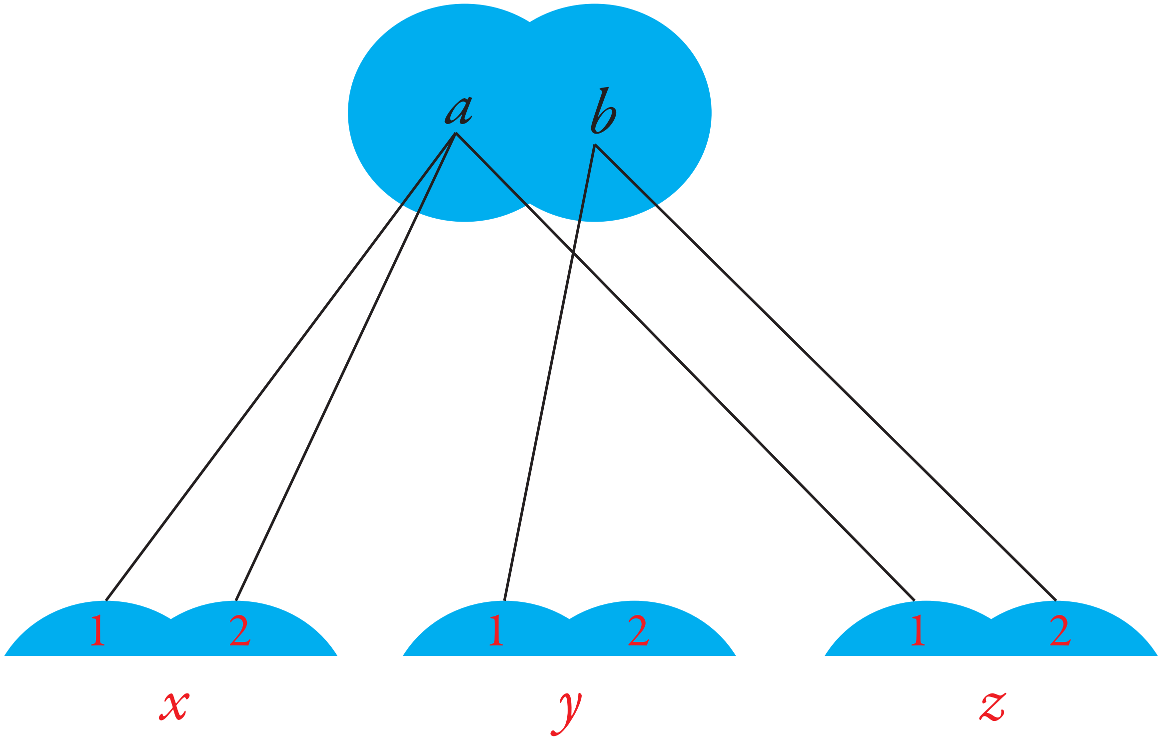
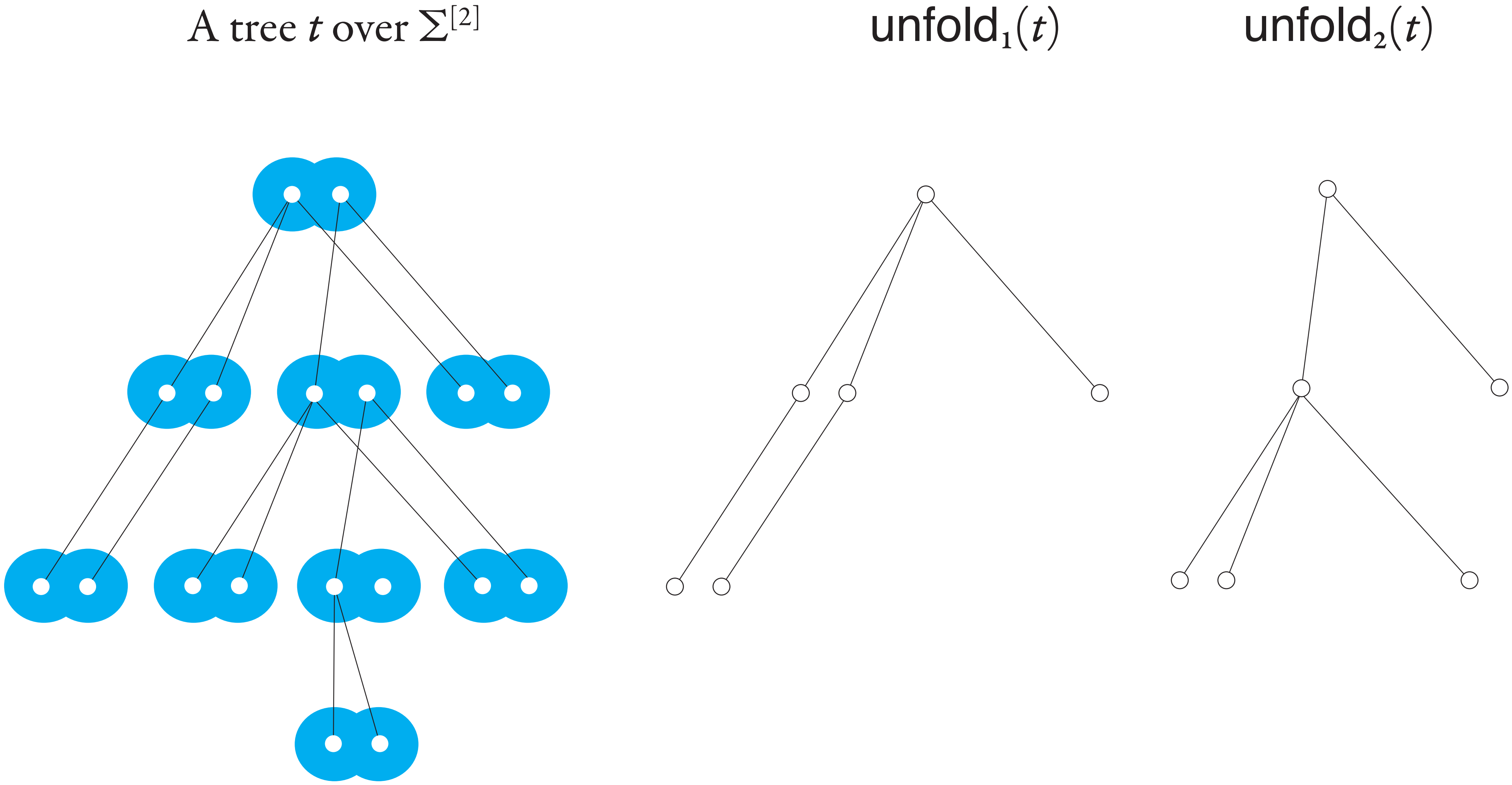
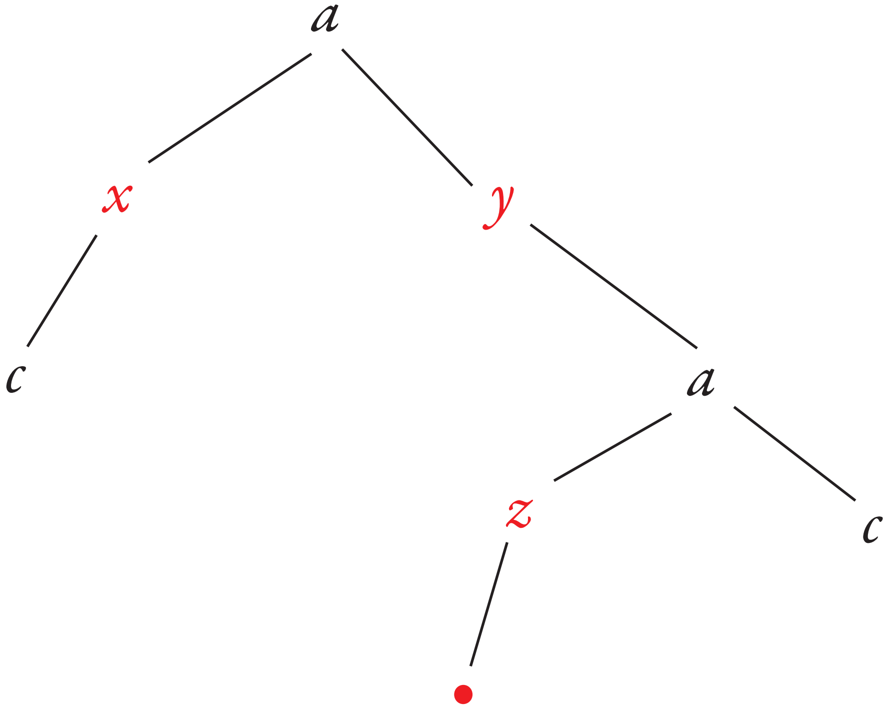
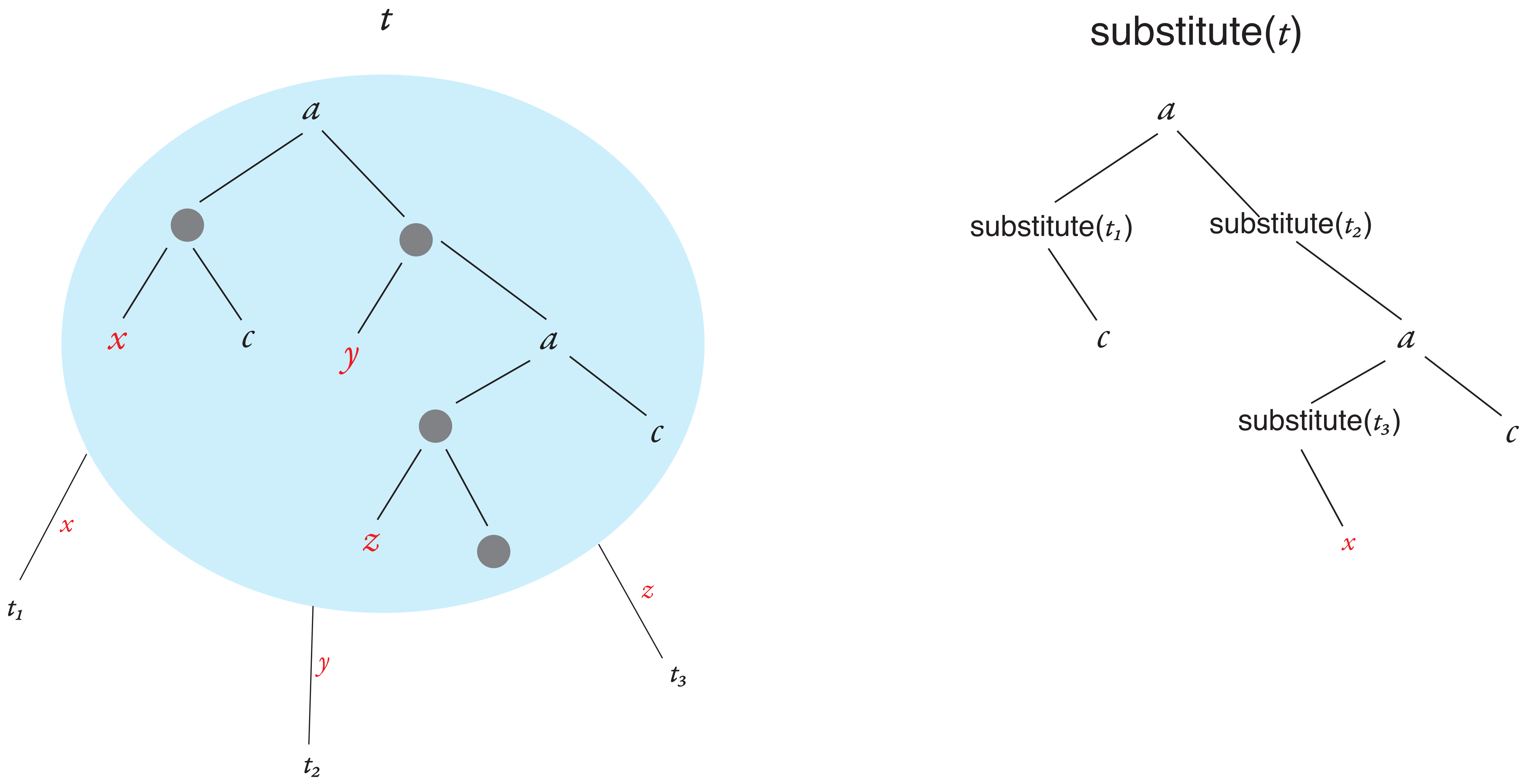

a ranked alphabet
a tree
arity 2
arity 1
arity 0
this node has a label of arity 2, and therefore it has 2 children
this node is child 2 (children are ordered)

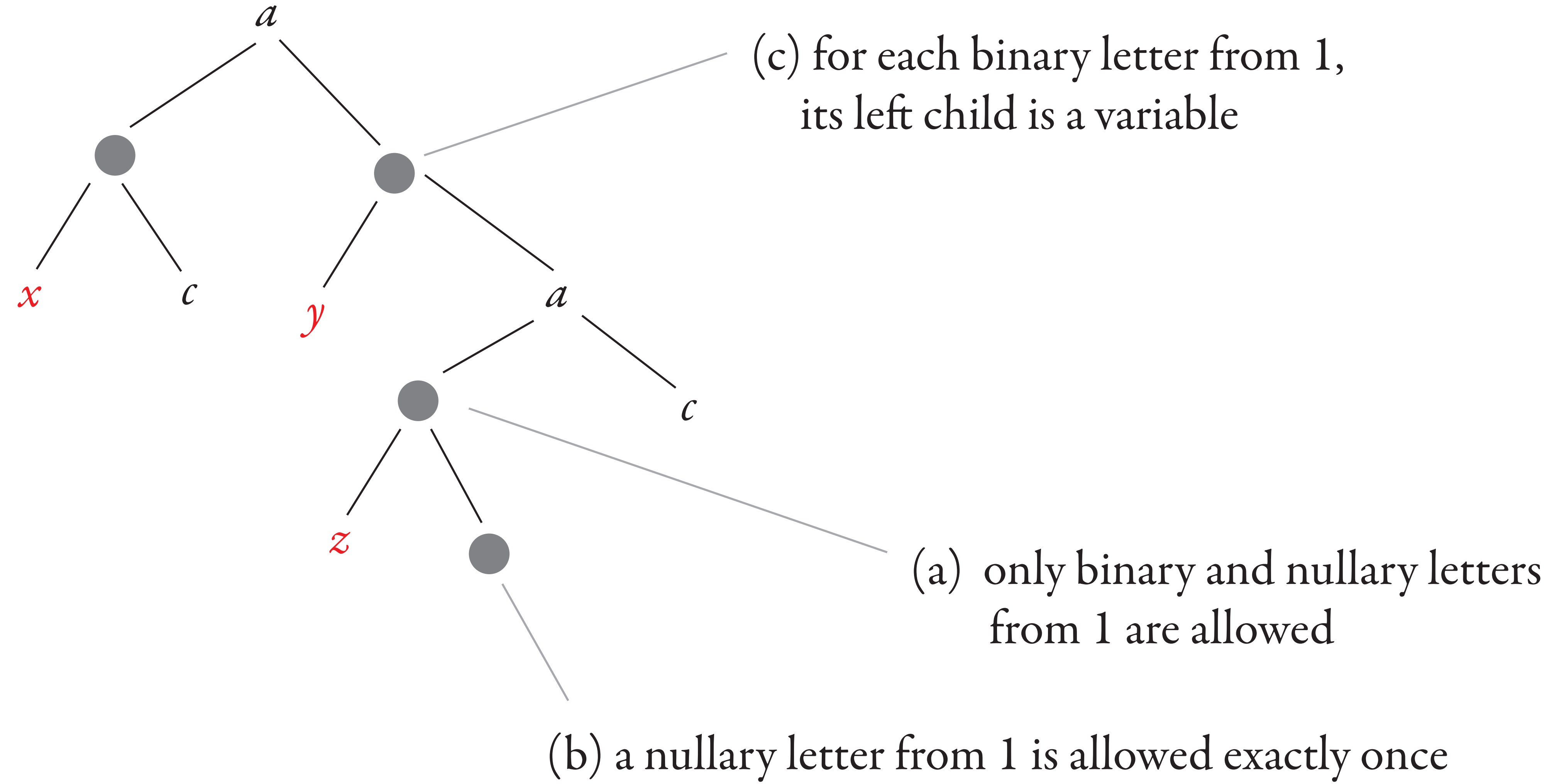

a
b
1
2
1
2
1
2
x
y
z

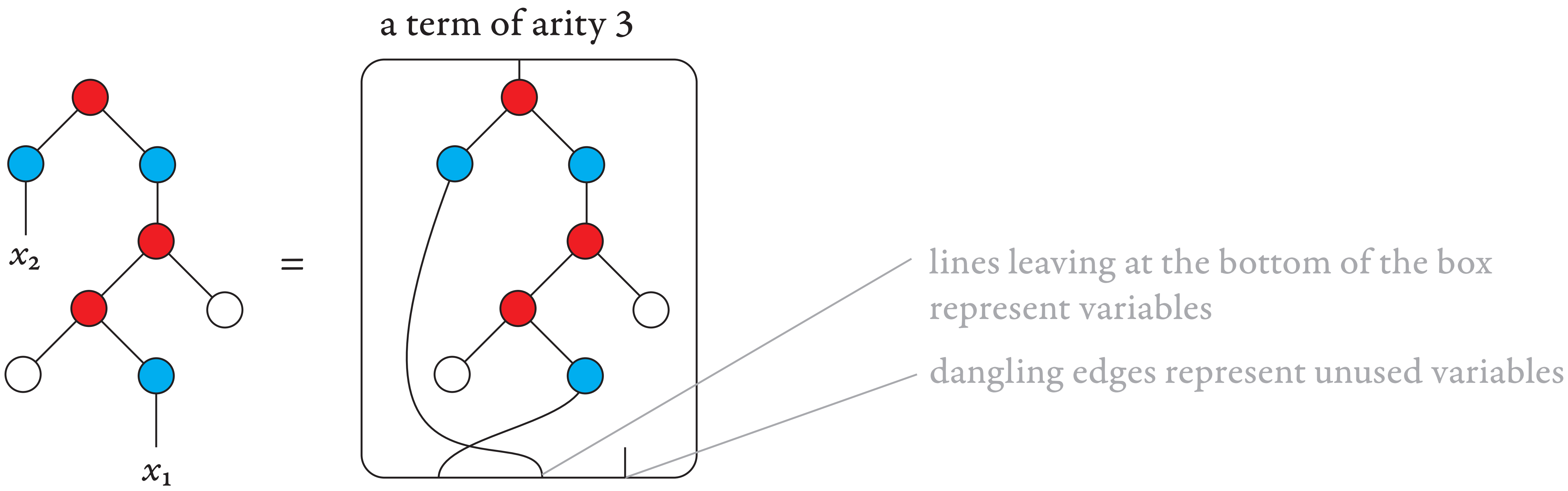

A tree t over $\Sigma^{[2]}$
$\mathsf{unfold}_1(t)$
$\mathsf{unfold}_2(t)$

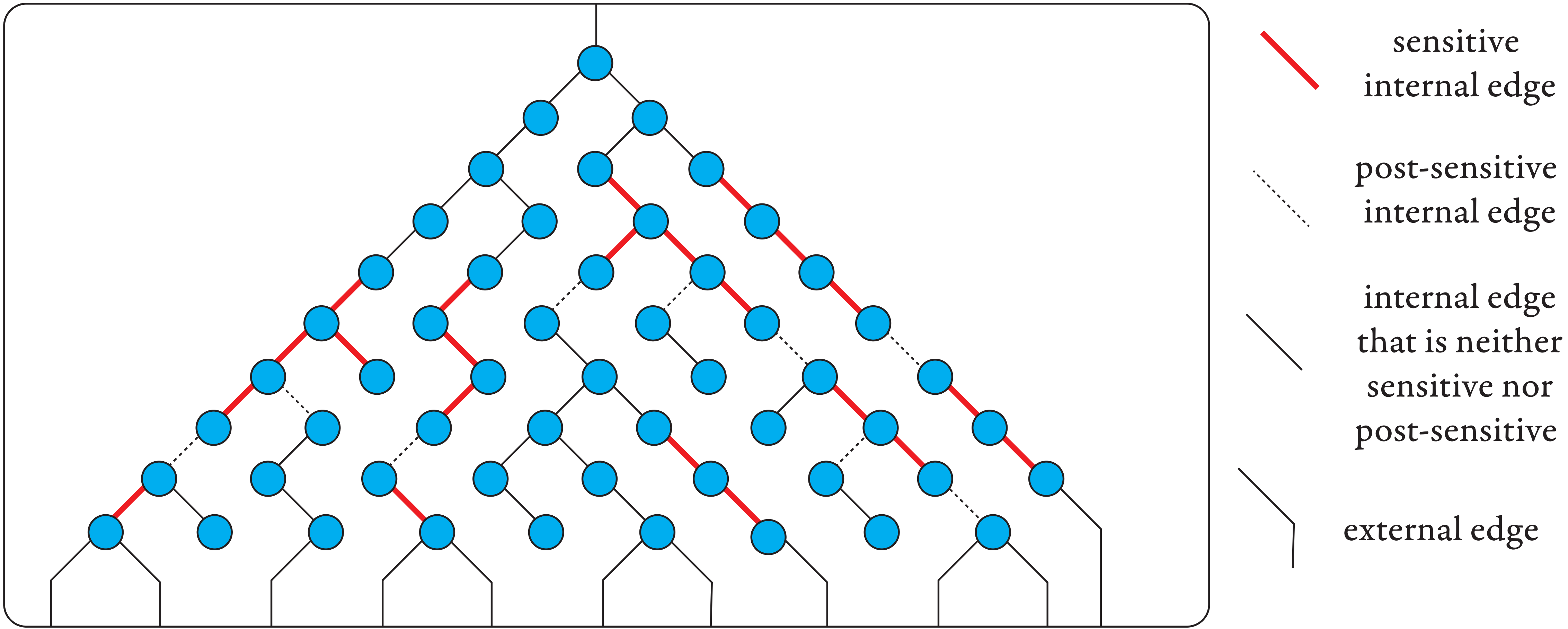

a
x
y
c
a
z
c

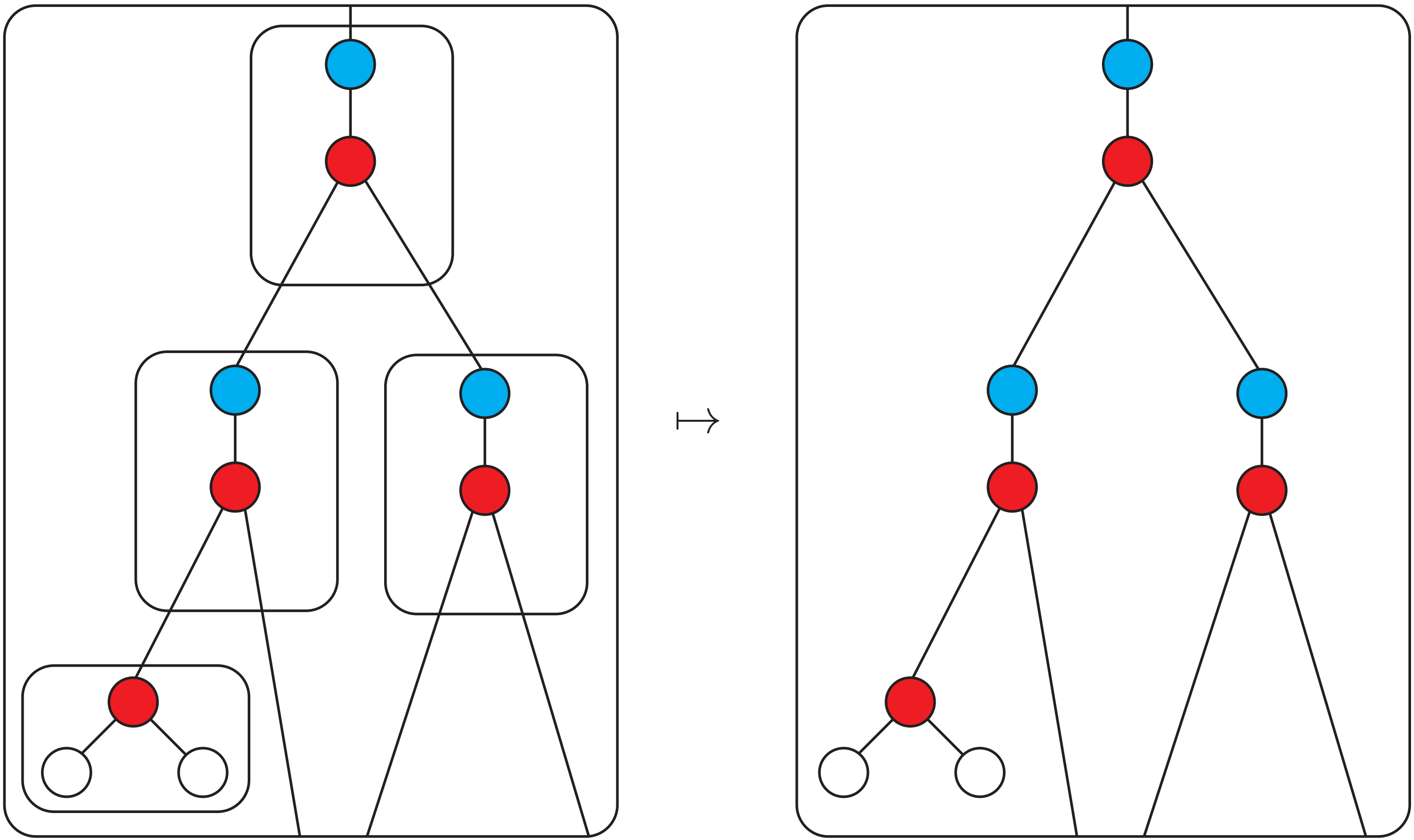


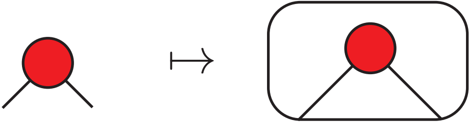
t
a
x
c
y
a
z
c
x
y
z
t1
t2
t3
substitute(t)
a
substitute(t1)
substitute(t2)
c
a
substitute(t3)
c
x

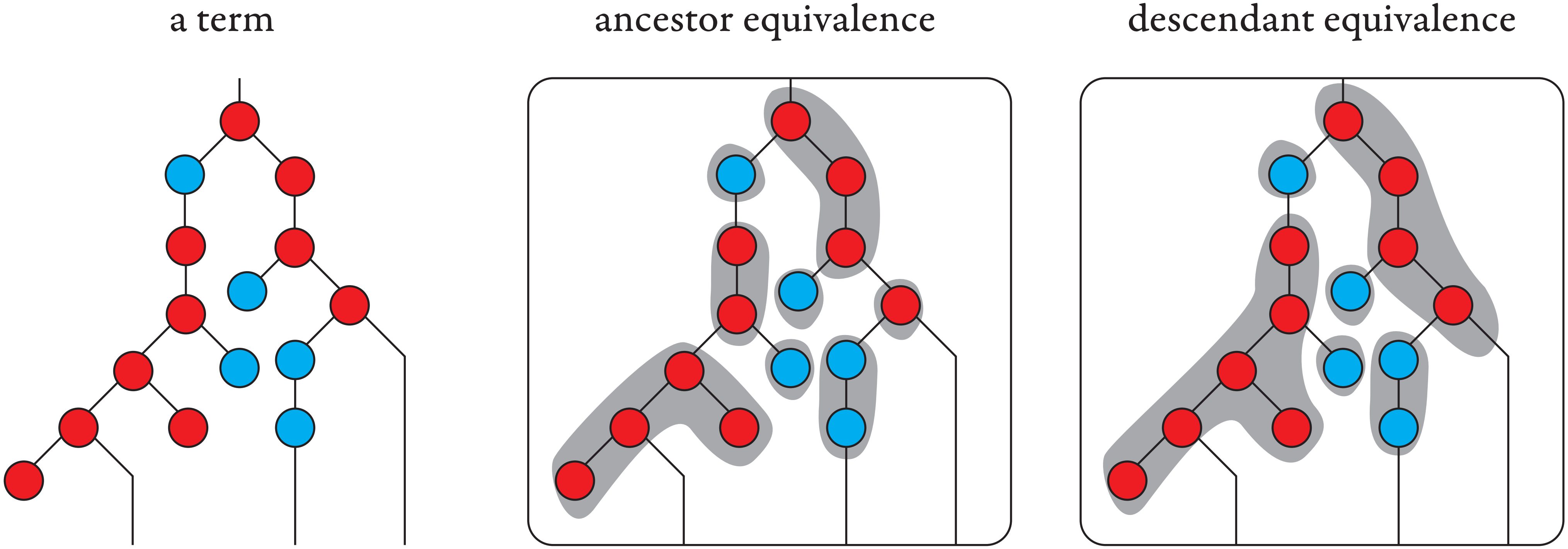

a
(c) for each binary letter from 1,
its left child is a variable
x
c
y
a
c
z
(a) only binary and nullary letters
from 1 are allowed
(b) a nullary letter from 1 is allowed exactly once

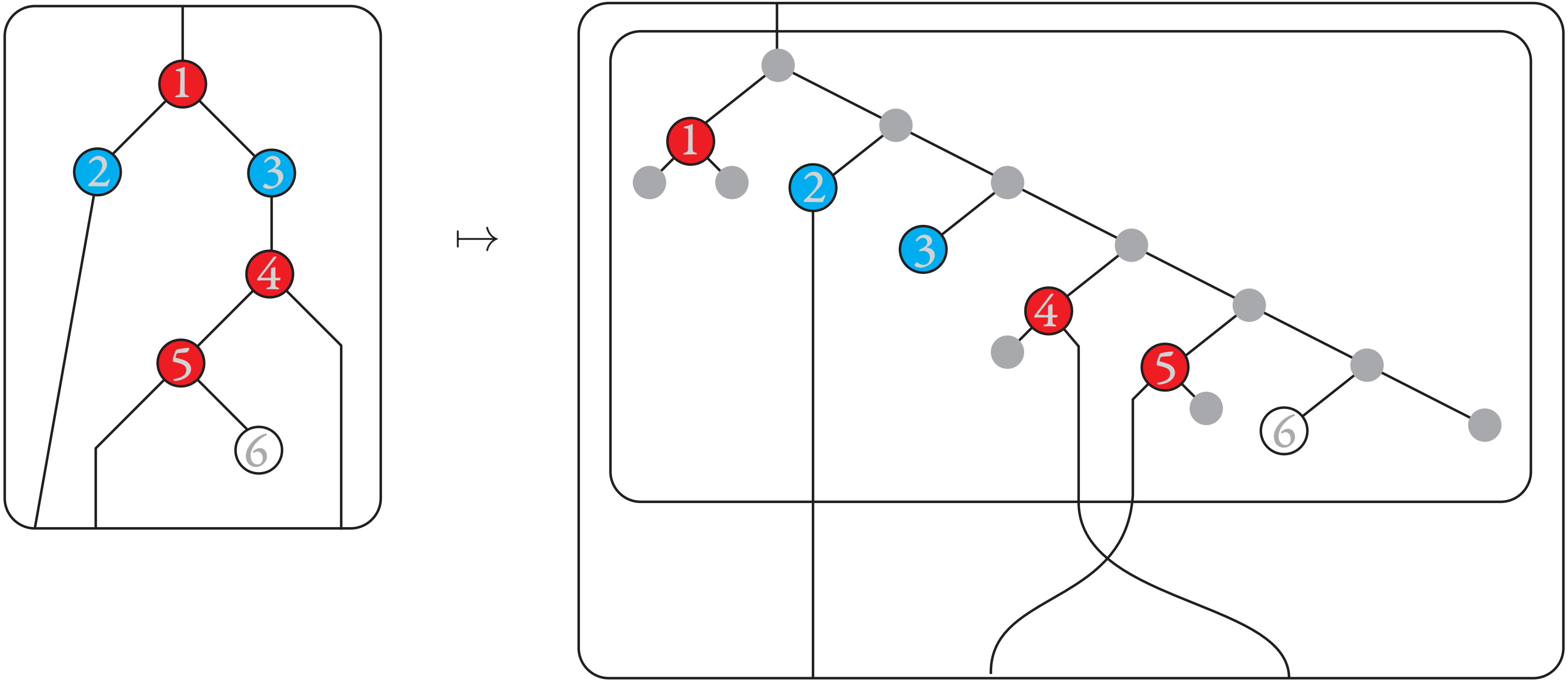

a term of arity 3
x_2
x_1
=
lines leaving at the bottom of the box represent variables
dangling edges represent unused variables


sensitive internal edge
post-sensitive internal edge
internal edge that is neither sensitive nor post-sensitive
external edge

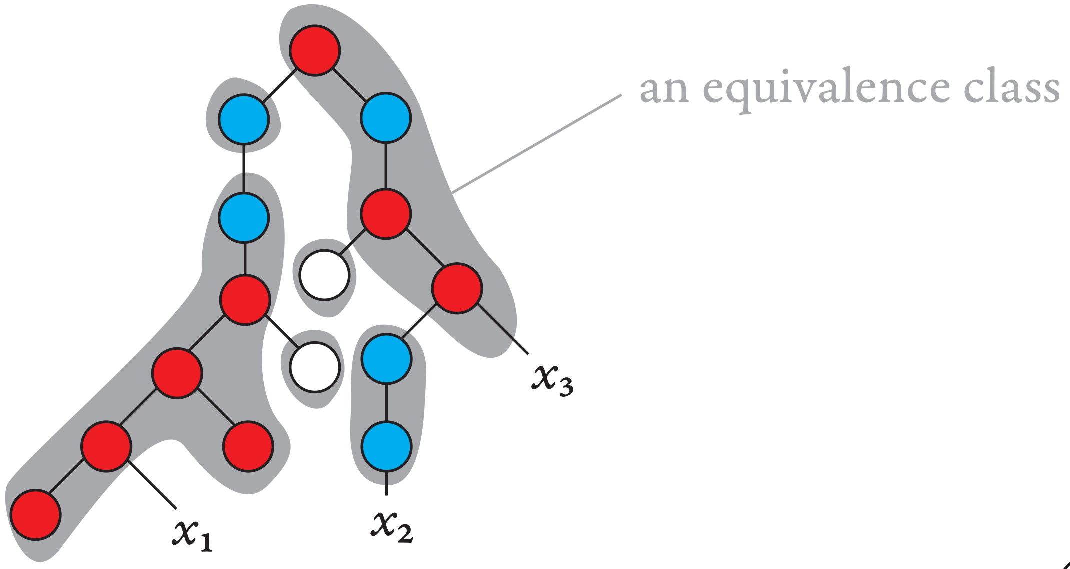


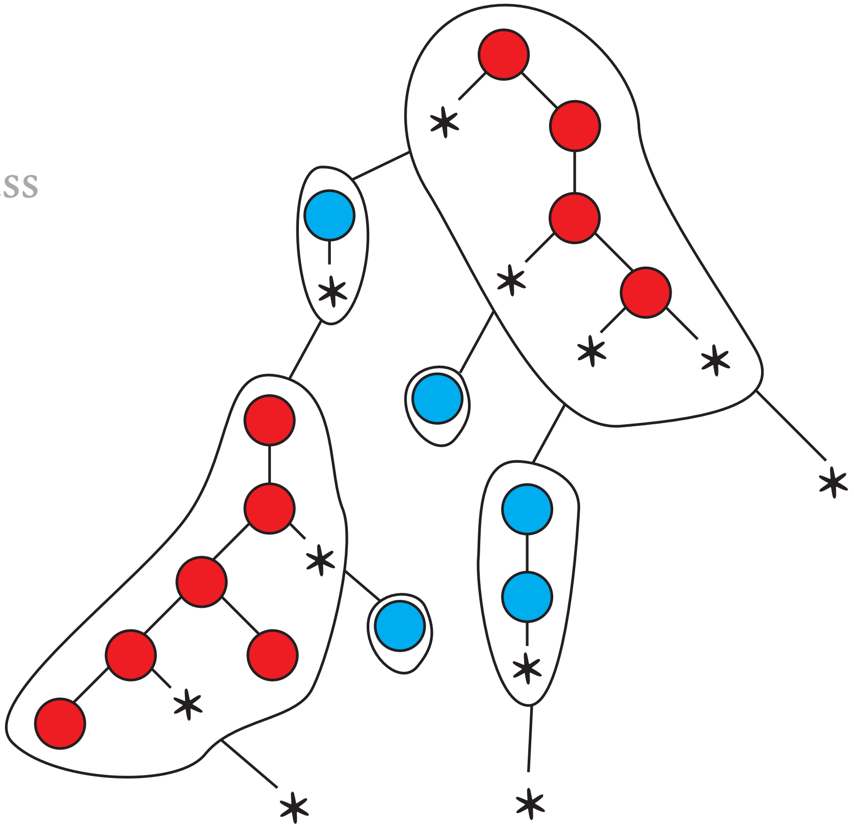



a term
ancestor equivalence
descendant equivalence


1
2
3
4
5
6
$\mapsto$
1
2
3
4
5
6

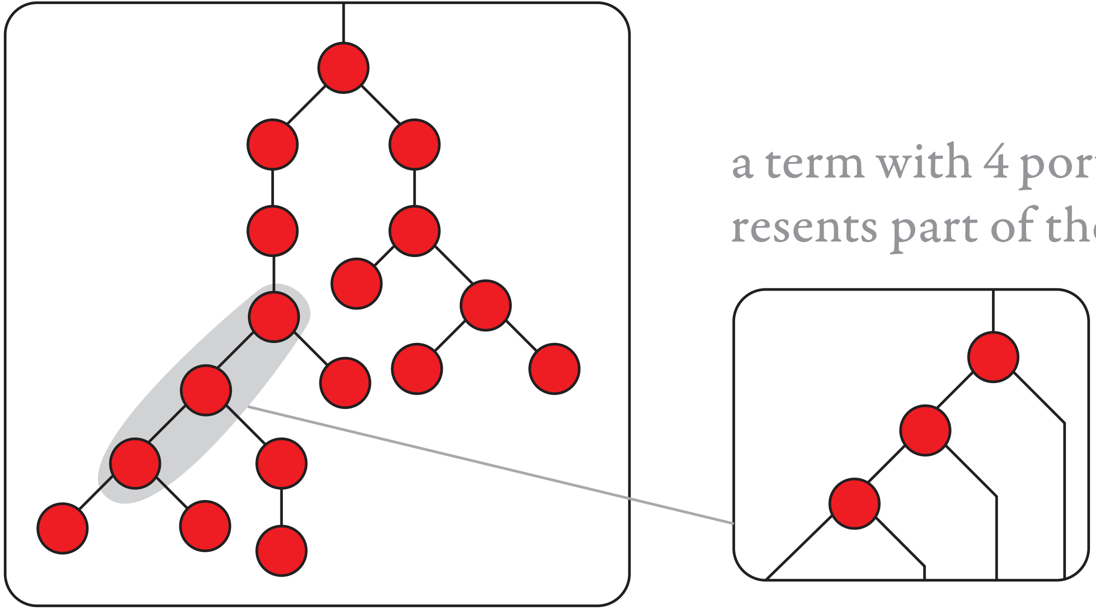
a factorisation equivalence

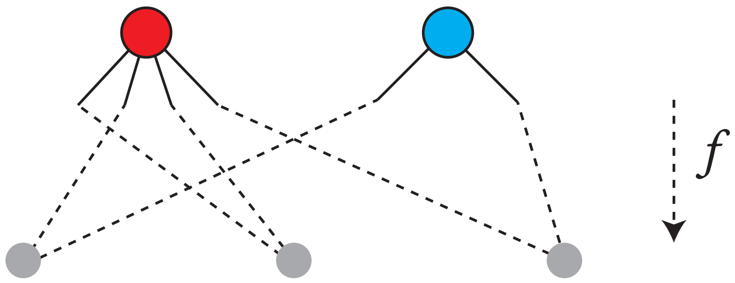




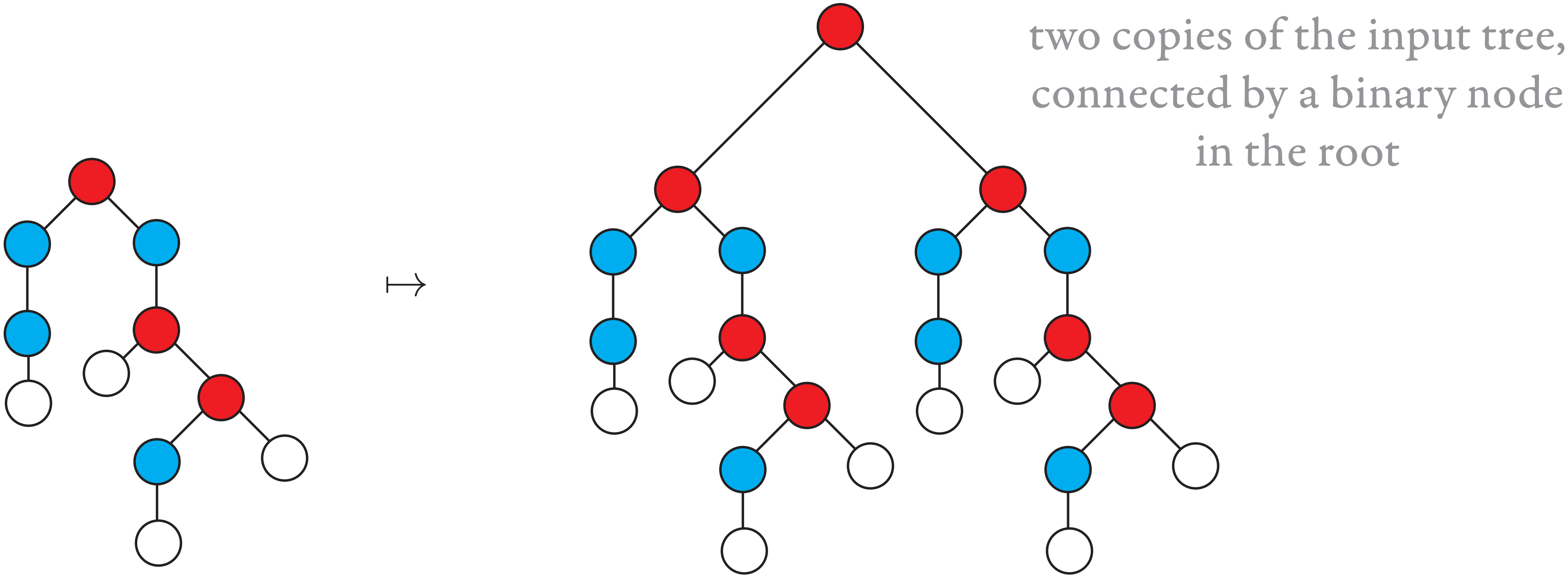
a tree

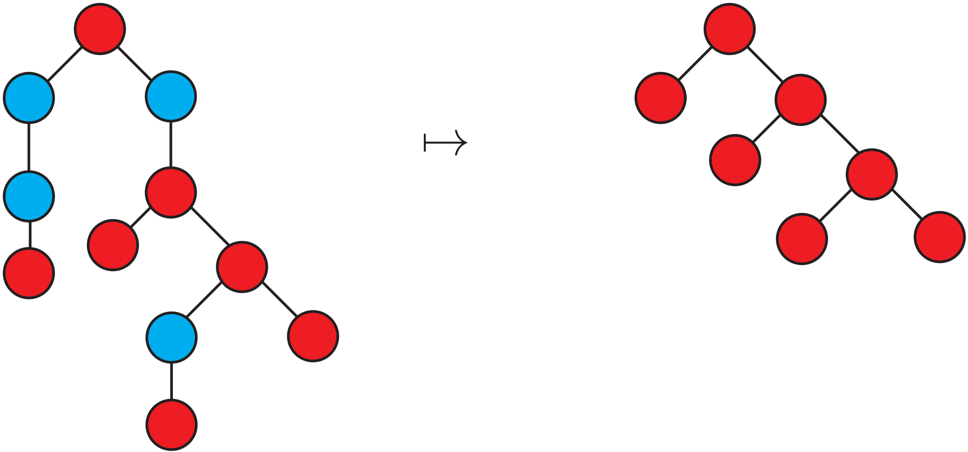
a term with 4 ports that represents part of the tree

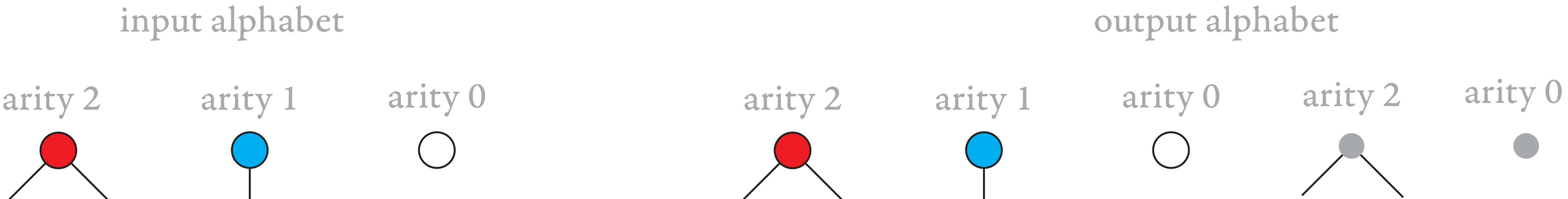



f

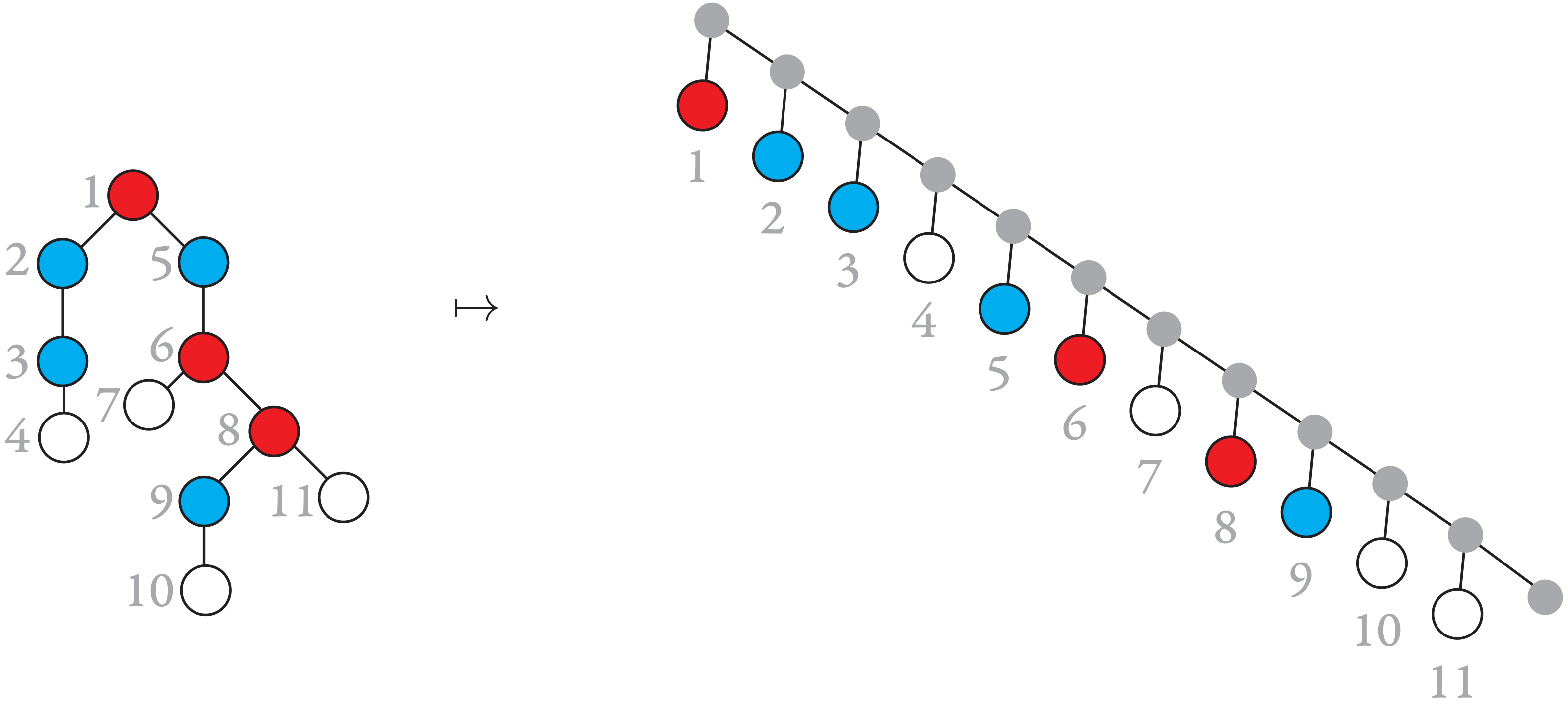

input alphabet
arity 2
arity 1
arity 0
output alphabet
arity 2
arity 0


two copies of the input tree,
connected by a binary node
in the root

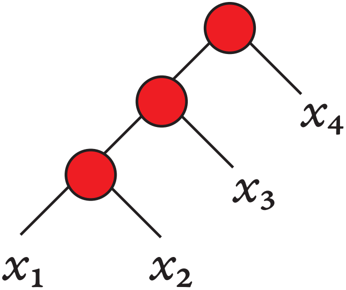




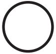
input alphabet
arity 2
arity 1
arity 0
output alphabet
arity 2
arity 1
arity 0
arity 2
arity 0

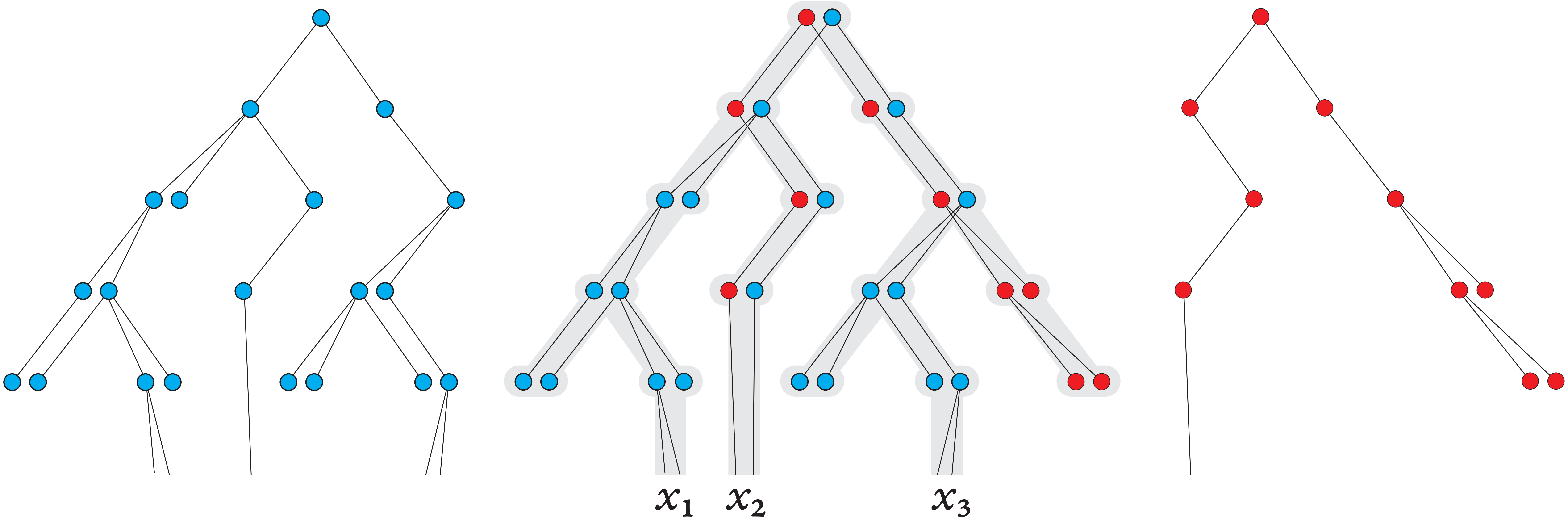

1
2
5
3
6
7
4
8
9
11
10
$\mapsto$
1
2
3
4
5
6
7
8
9
10
11

a term of arity 4

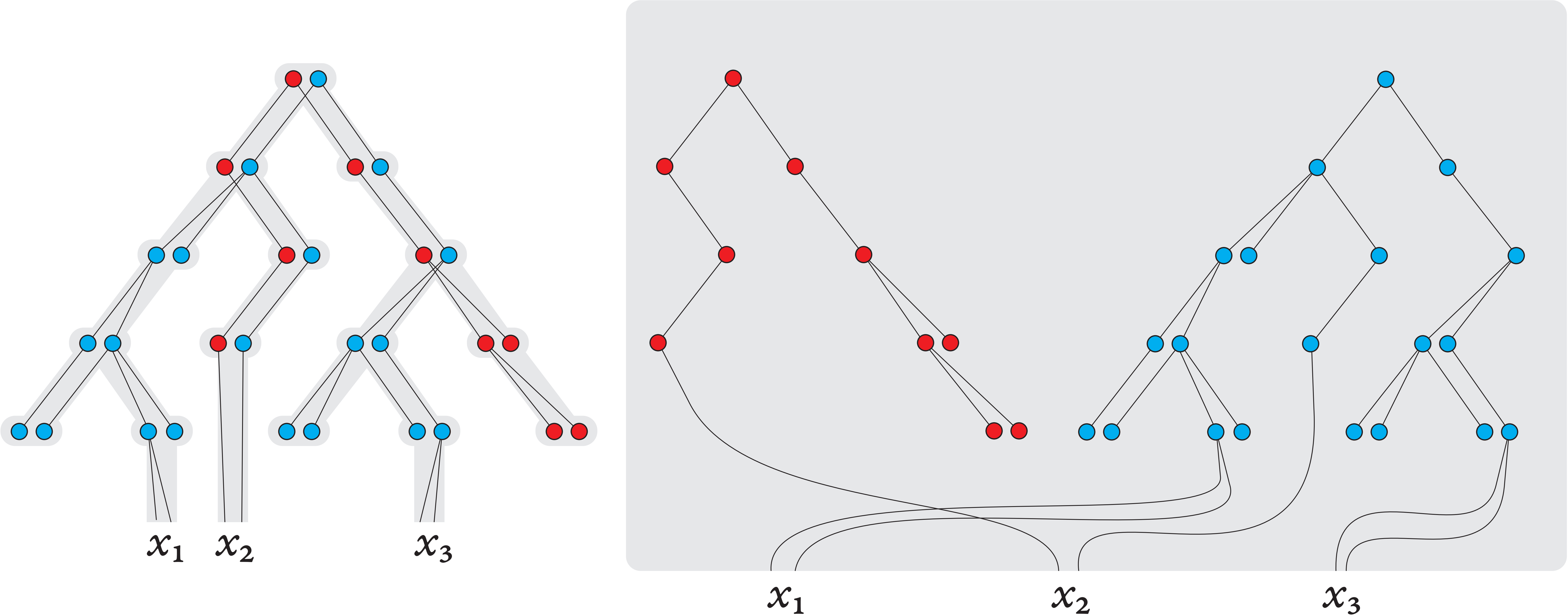


a term of arity 0

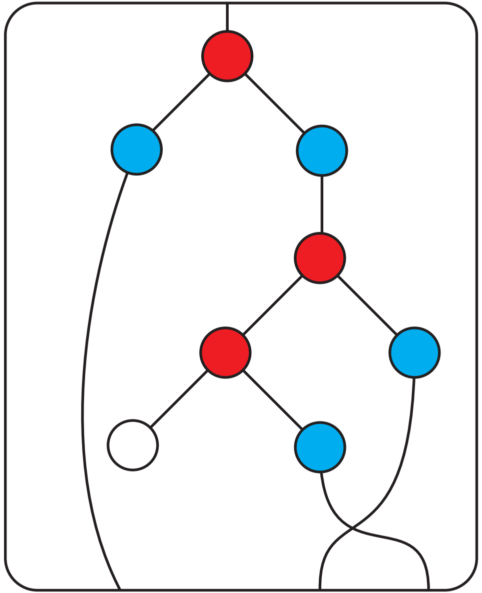




x_1 x_2 x_3



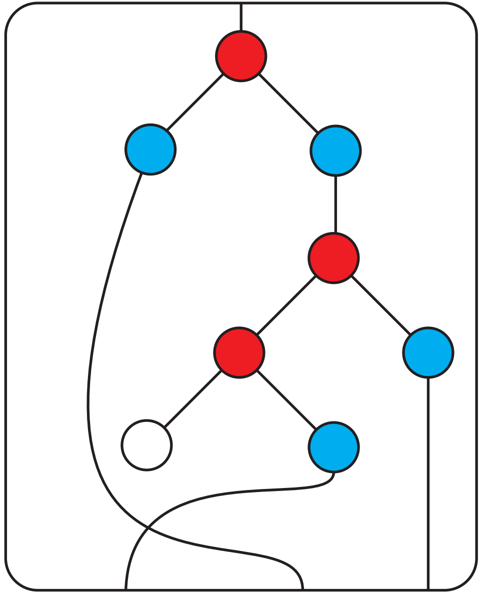
x_1
x_2
x_3
x_1
x_2
x_3



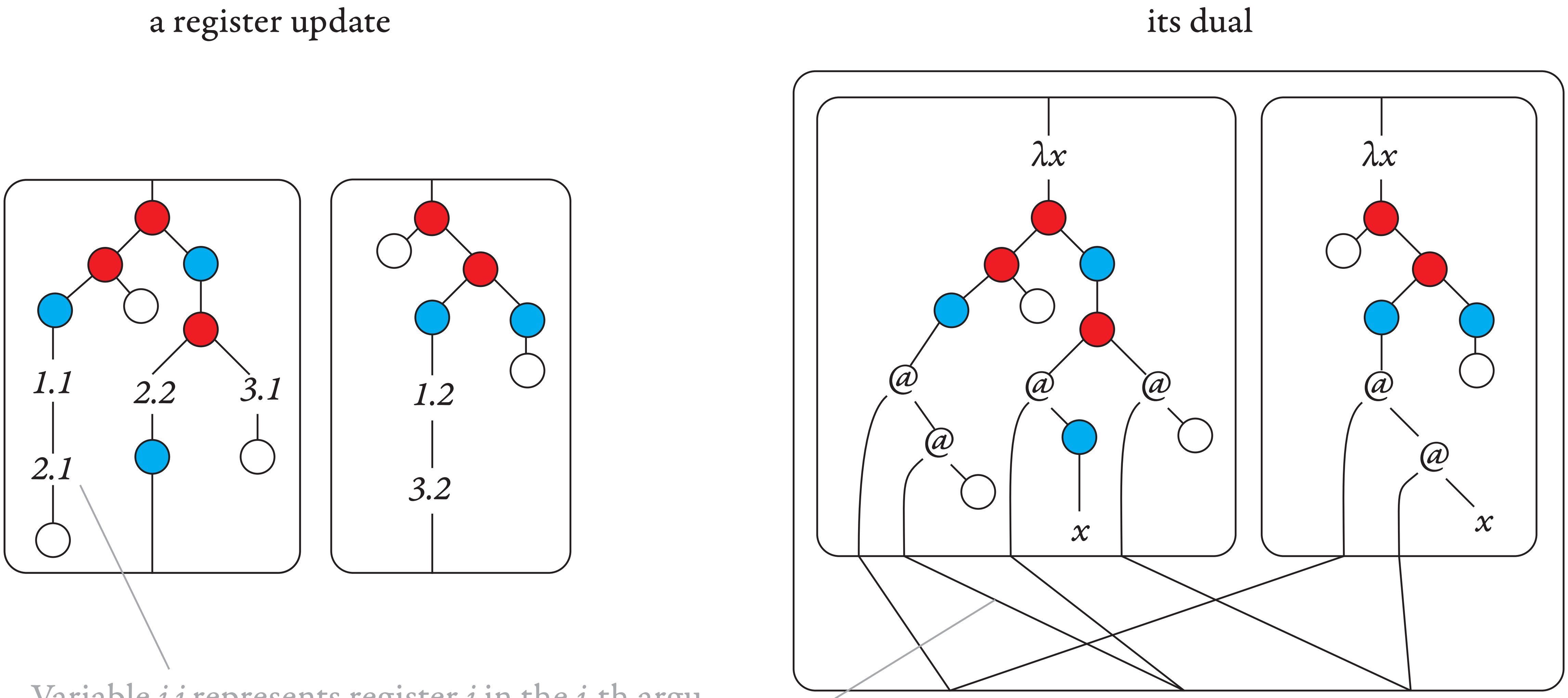
satisfies (*)

(*)

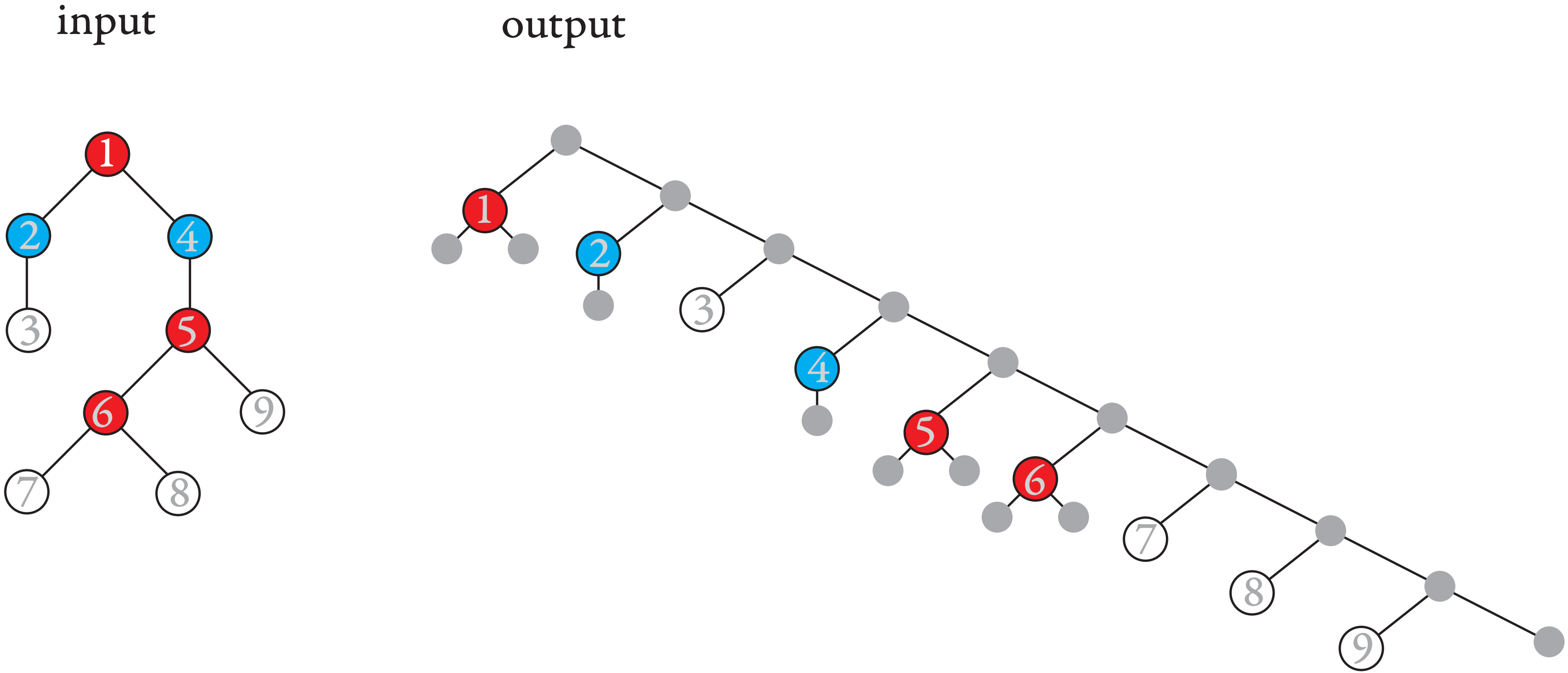
If the root has arity n, and $1 \leq i < j \leq n$, then all ports of the j-th subterm of the root are after all ports of the i-th subterm of the root

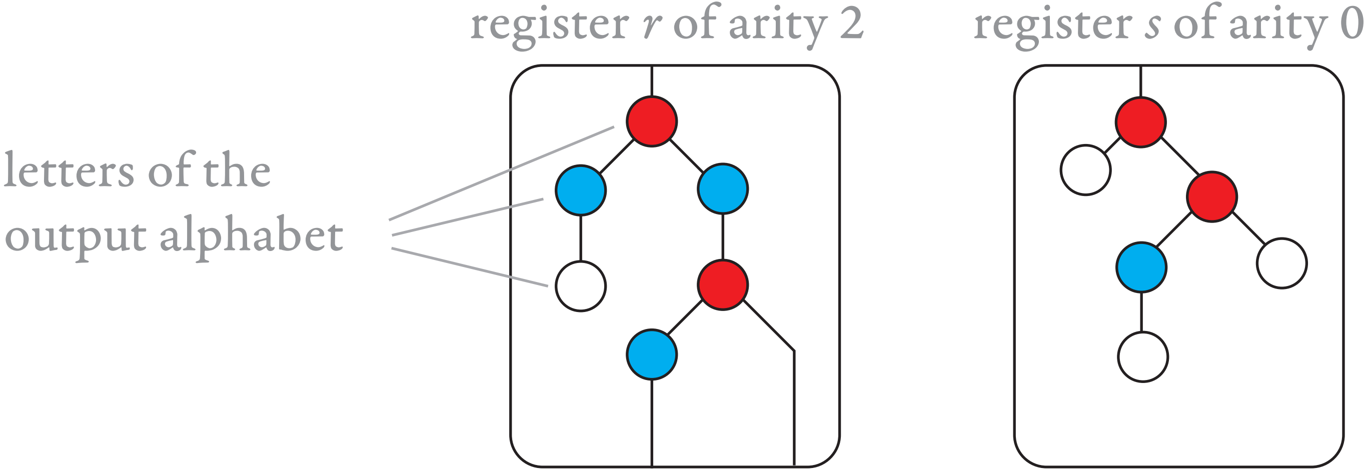


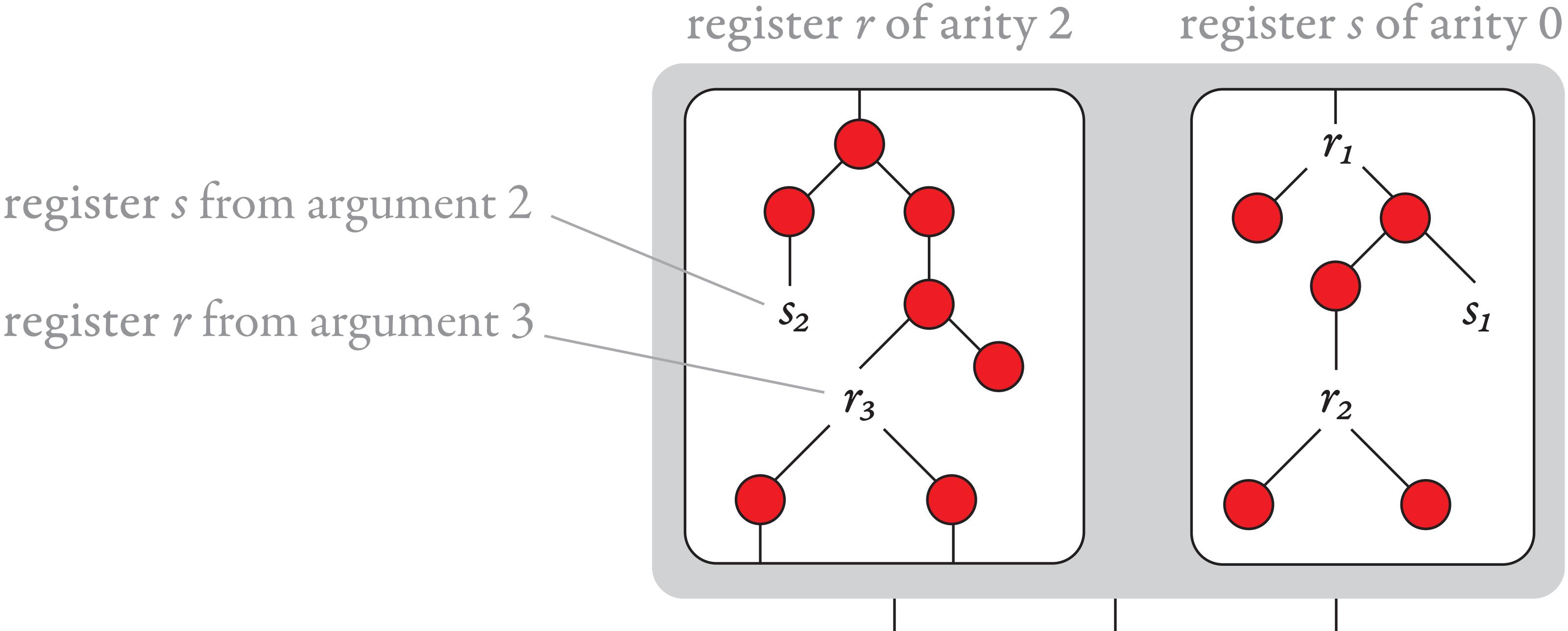
violates (*)



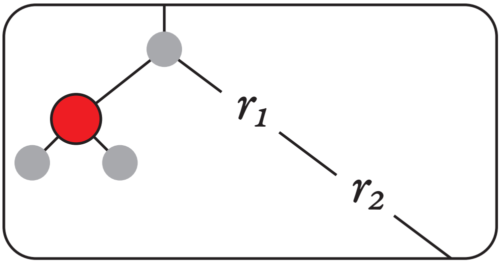
Variable *i.j* represents register *i* in the *j*-th argument of the reigster update.
In the dual, this variable is mapped to the *i*-th edge which enters the *j*-th port of the reducer.

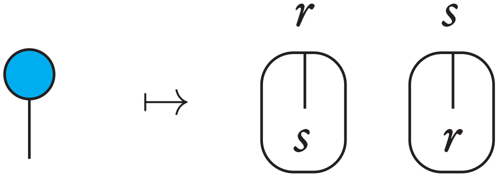

input
output
1
2
4
3
5
6
9
7
8
1
2
3
4
5
6
7
8
9

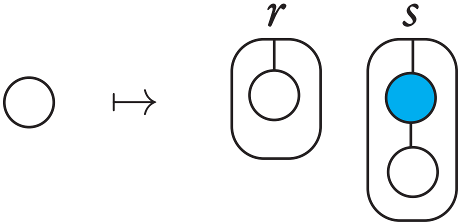

register r of arity 2
register s of arity 0
letters of the
output alphabet

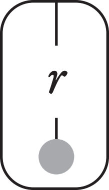

register r of arity 2
register s of arity 0
register s from argument 2
register r from argument 3
s_2
r_3
r_1
s_1
r_2

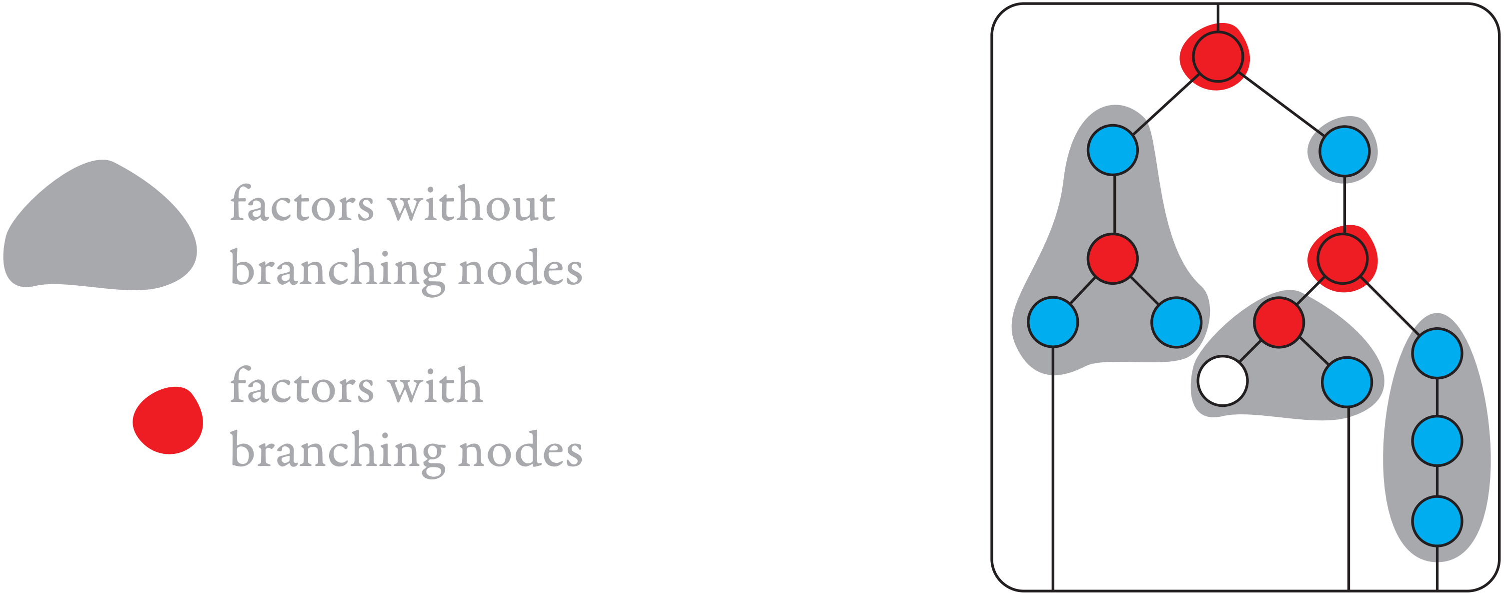

r
r
r
s

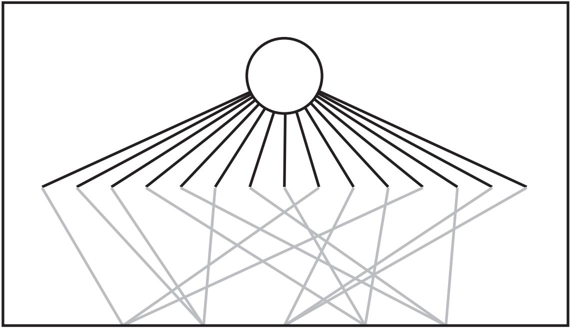

r_1
r_2


r
s
s
r

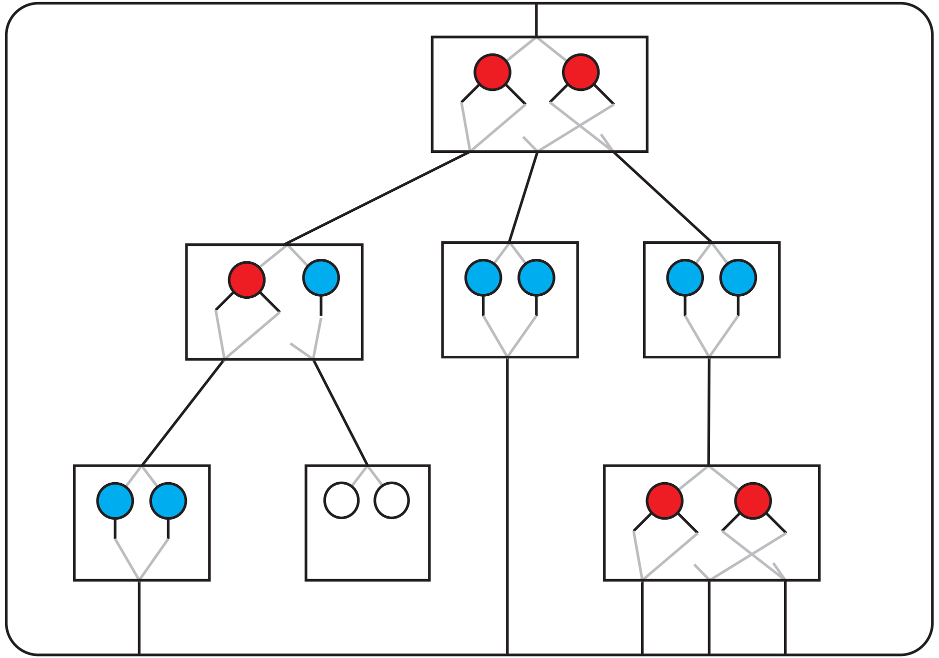

r
s


r

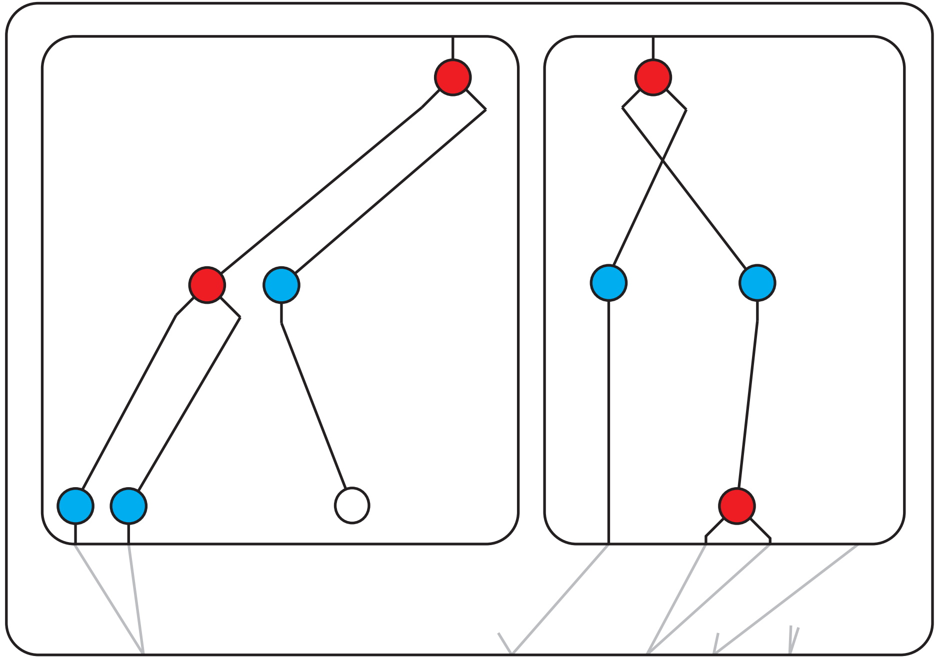

factors without
branching nodes
factors with
branching nodes

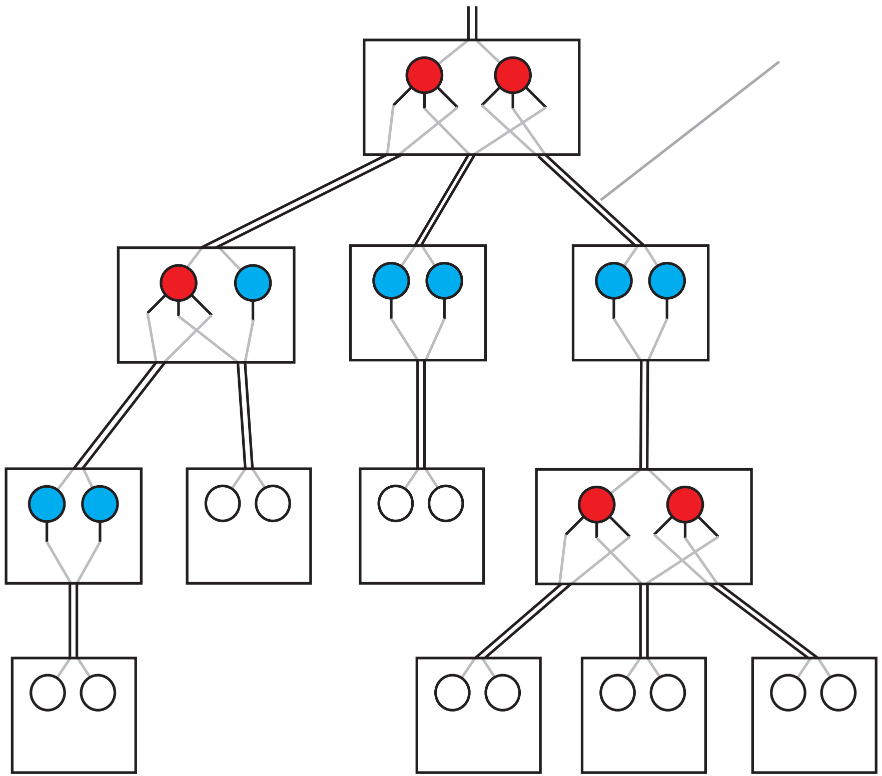


a term of matrix powers

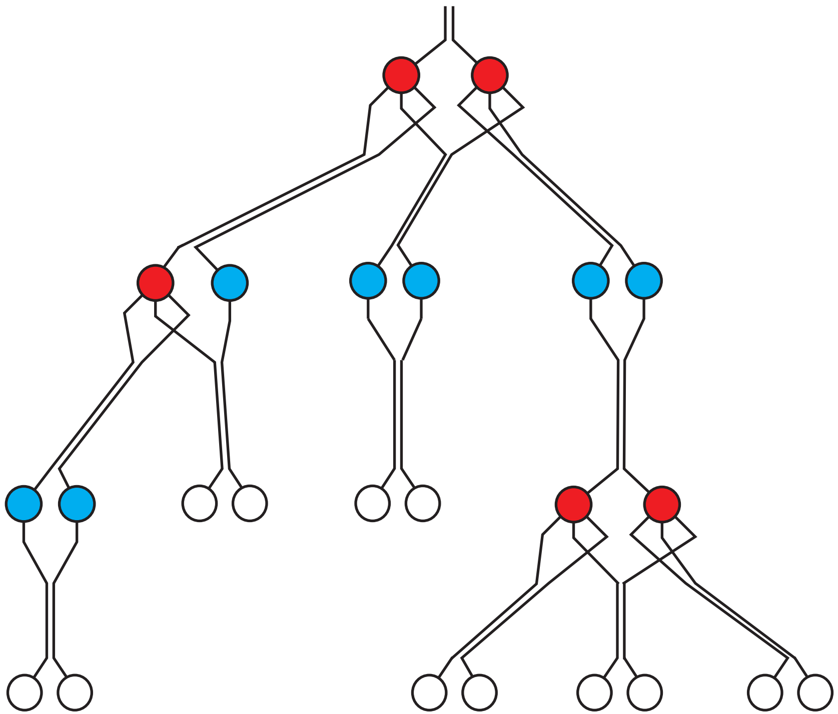


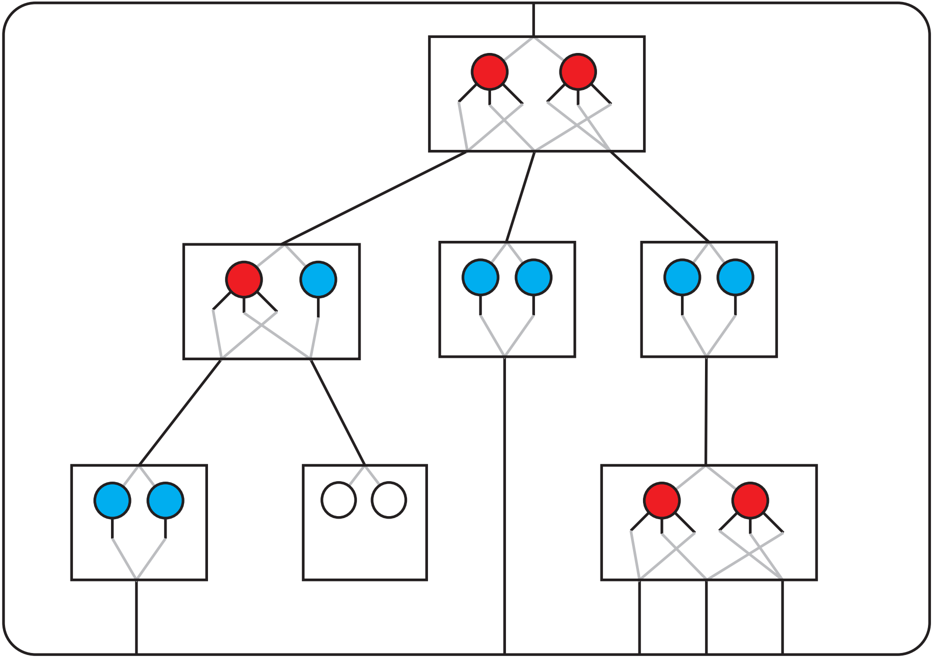
its term unfolding

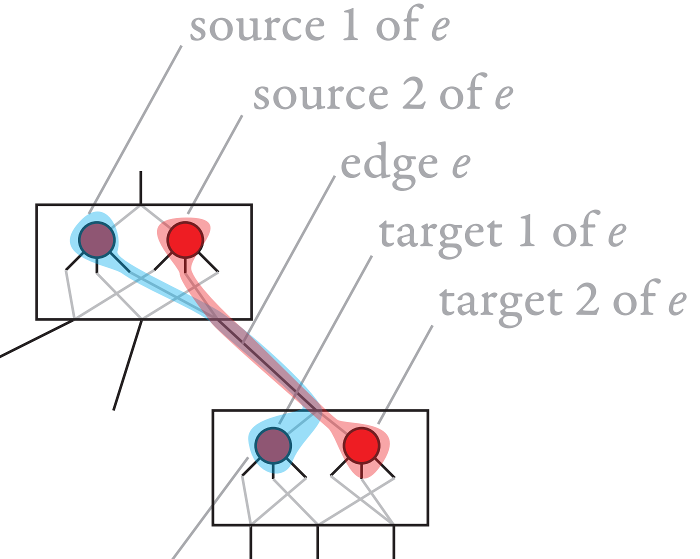


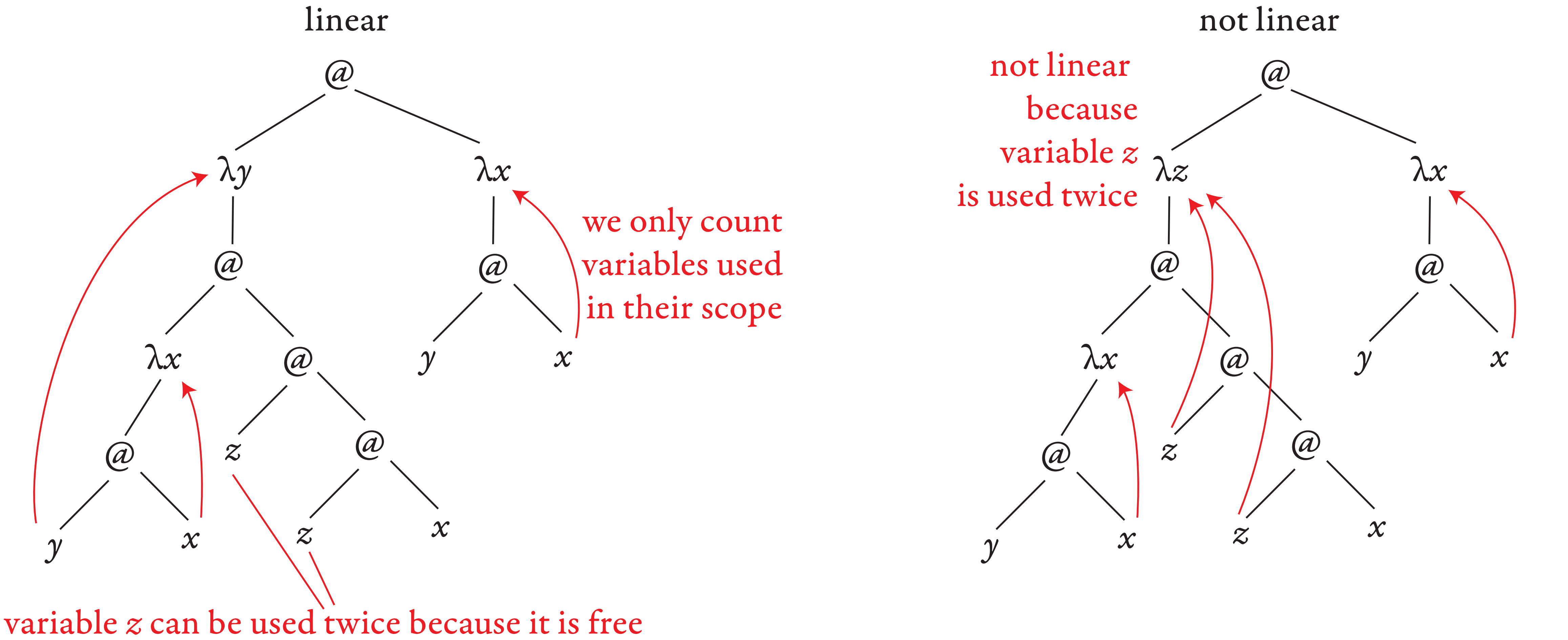


$\mapsto$



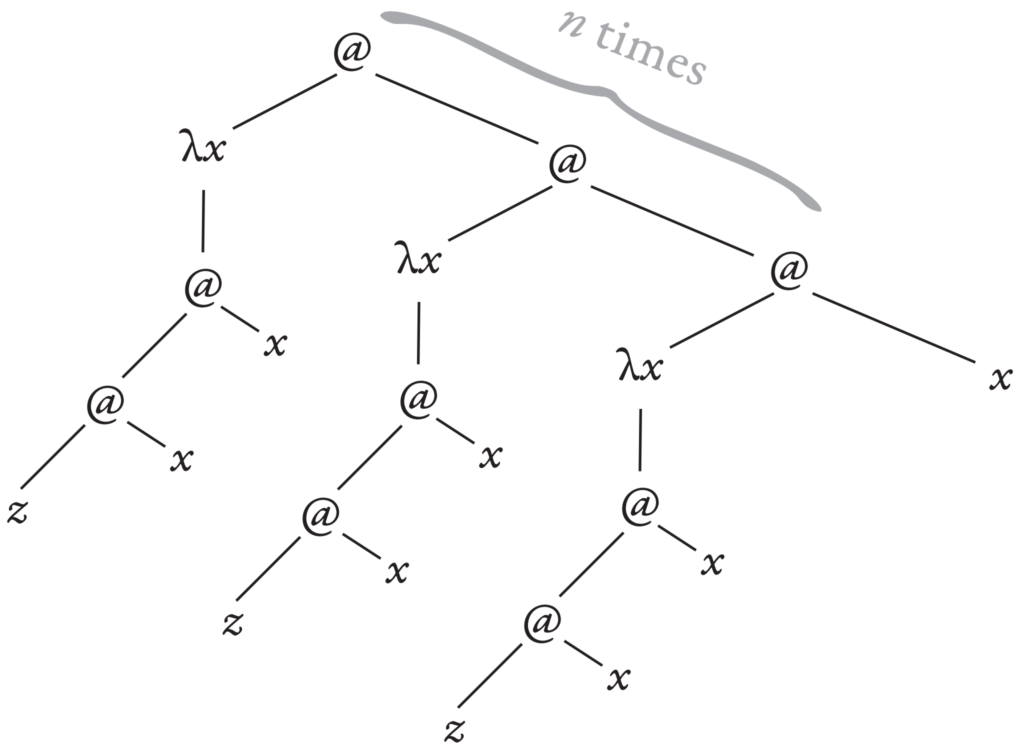



source 1 of e
source 2 of e
edge e
target 1 of e
target 2 of e

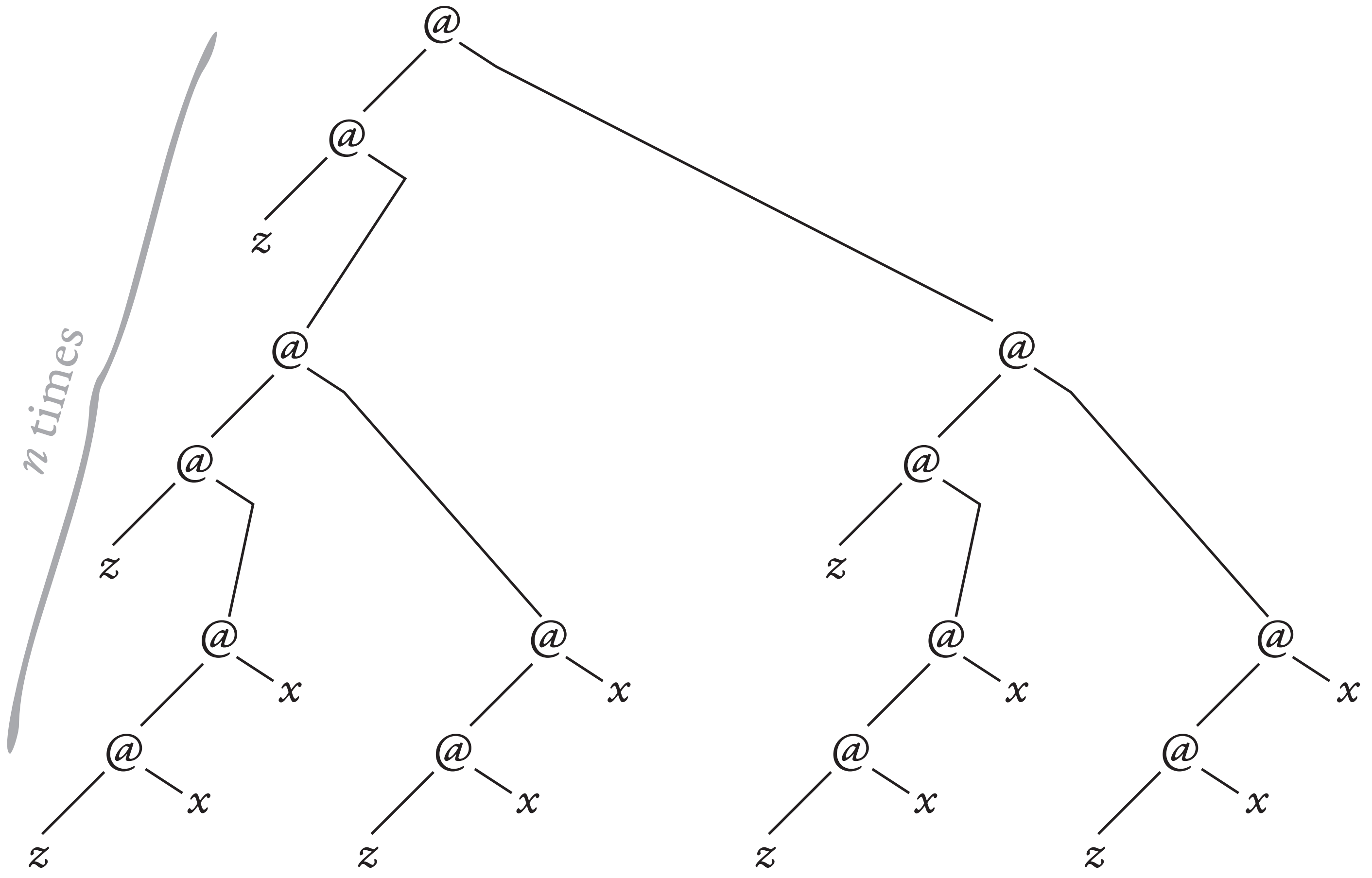


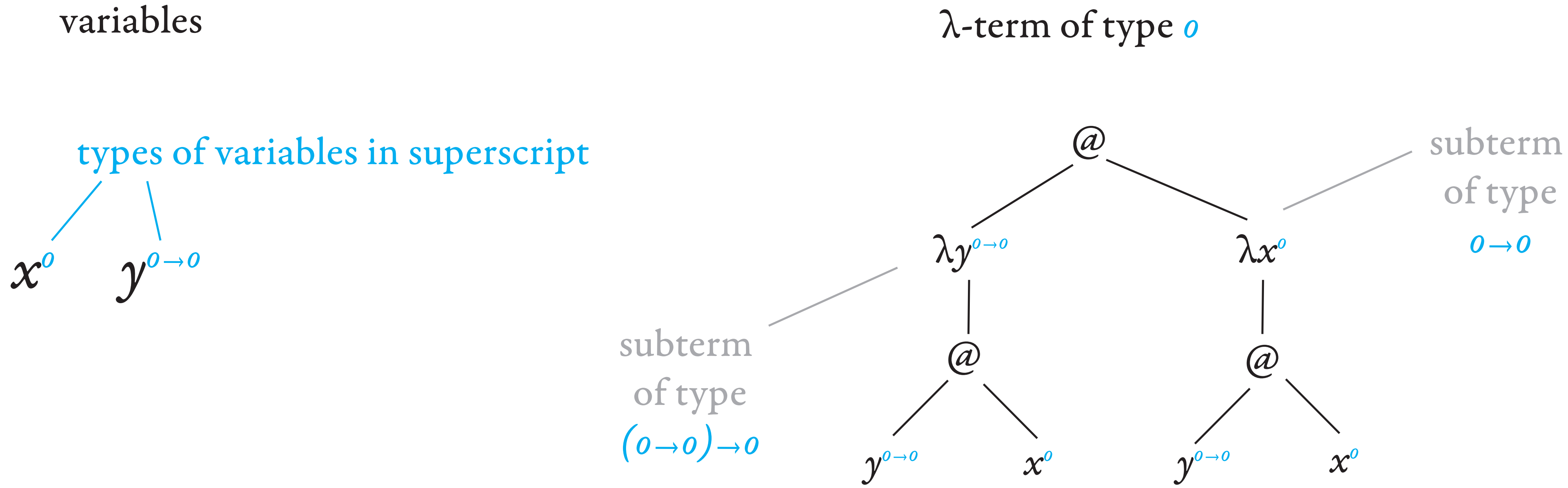
linear
not linear
we only count variables used in their scope
variable z can be used twice because it is free
not linear because variable z is used twice

a λ-term of size $O(n)$

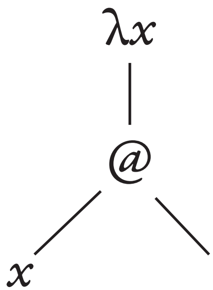


its normal form of size $O(2^n)$

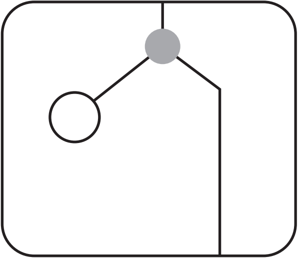


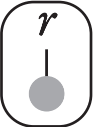

variables
λ-term of type o
types of variables in superscript
x^o y^{o→o}
@
λy^{o→o}
λx^o
subterm of type (o→o)→o
subterm of type o→o
@
@
y^{o→o} x^o
y^{o→o} x^o

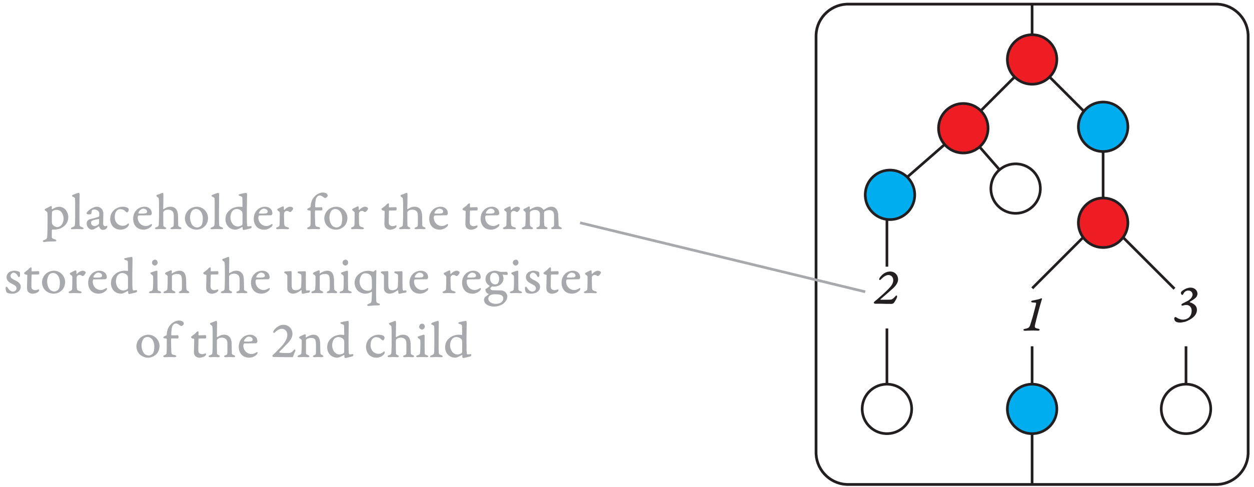
@

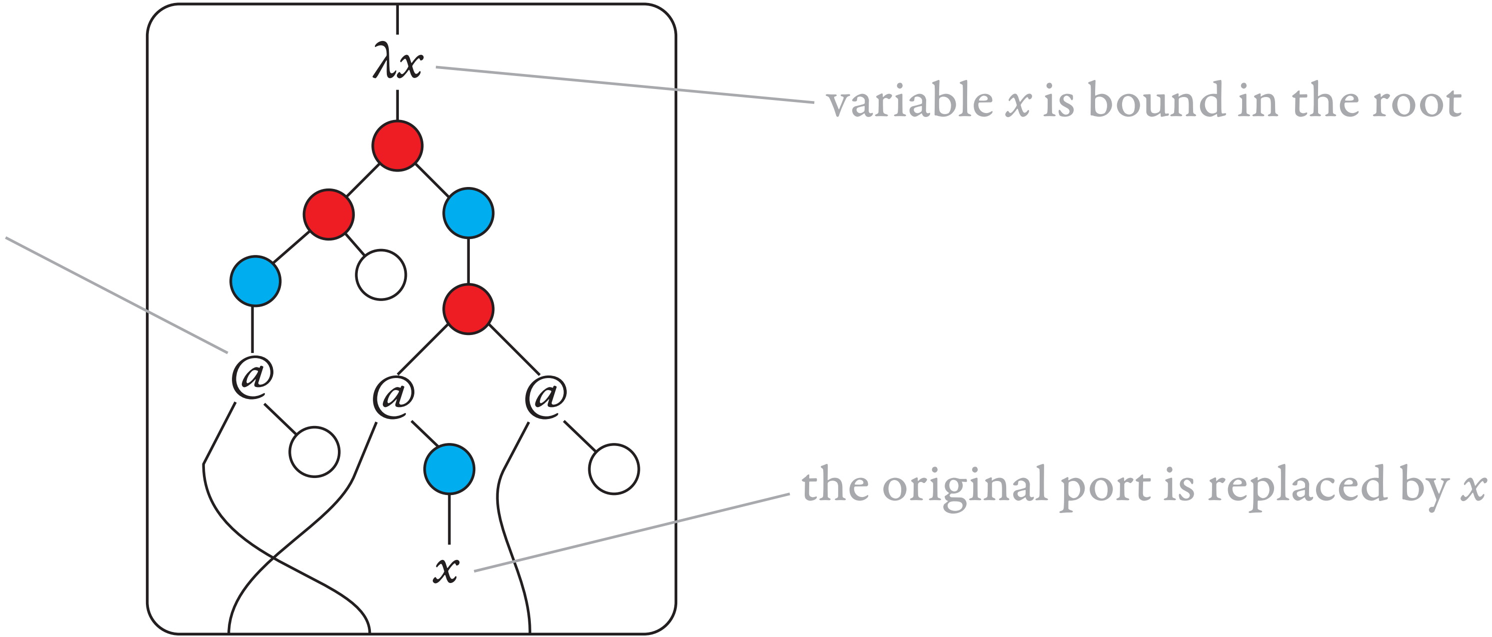


λx.

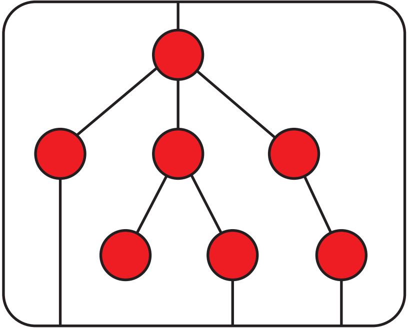



r

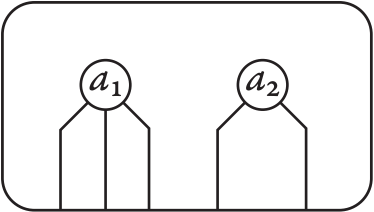

placeholder for the term
stored in the unique register
of the 2nd child
2
1
3

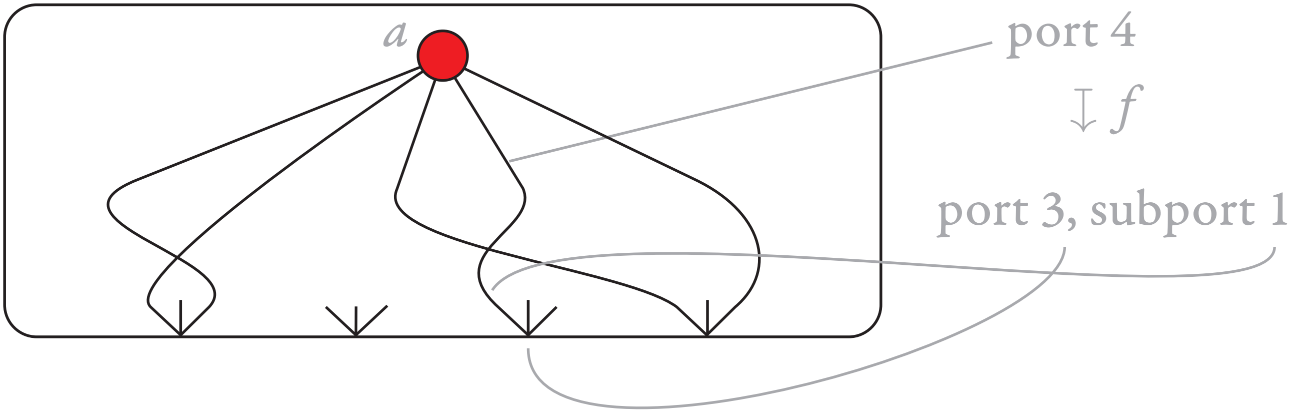


λx
@
@
@
x
variable x is bound in the root
the original port is replaced by x

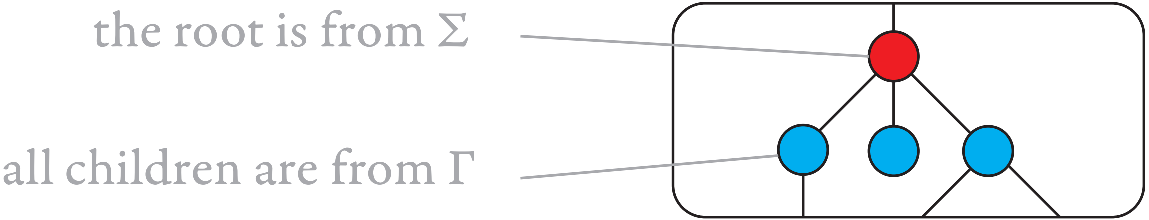


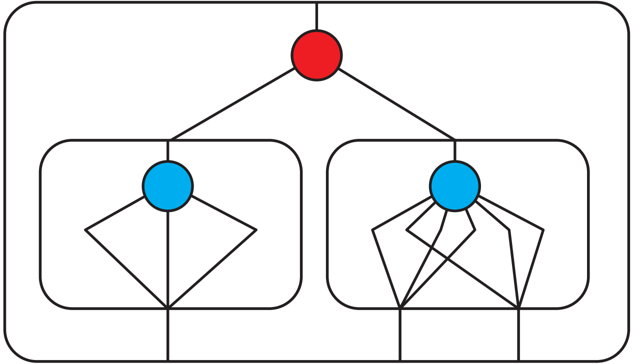
dangling edges
represent ports


a_1
a_2

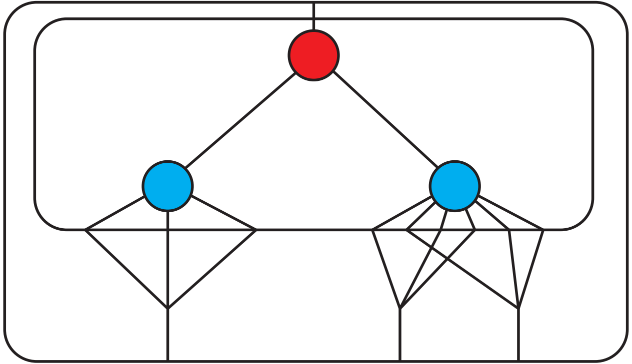


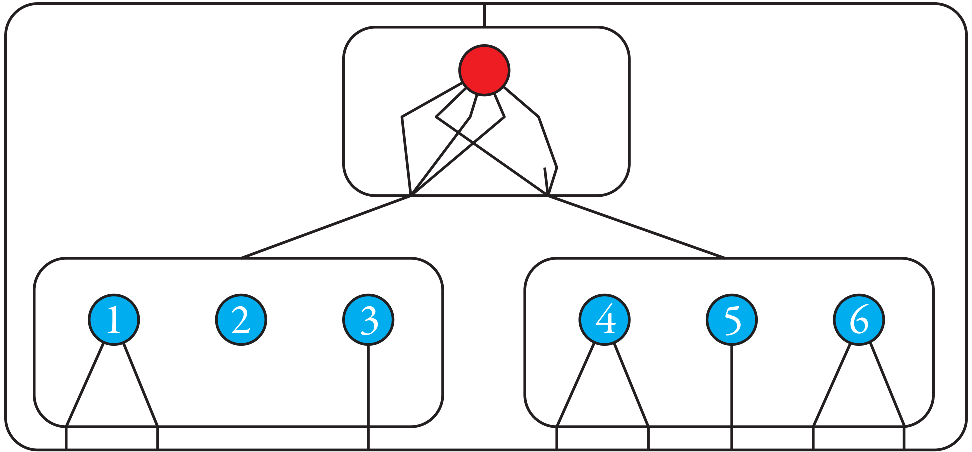
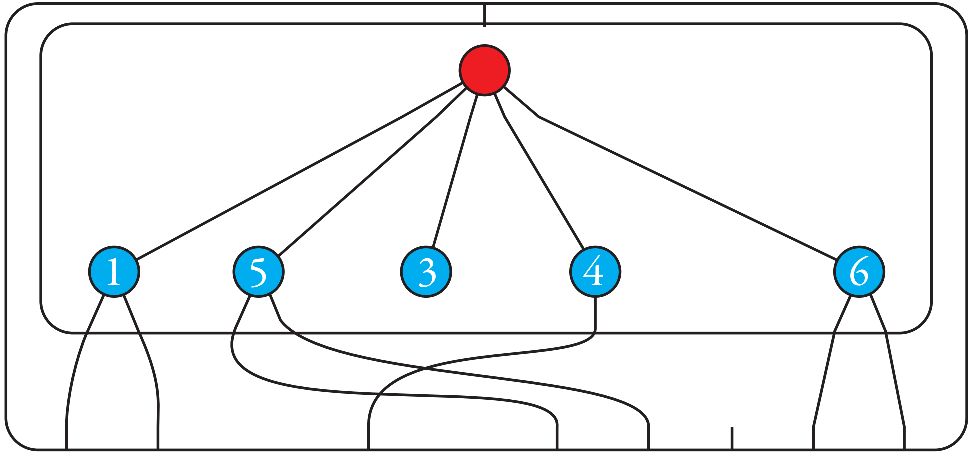
a
port 4
$\downarrow f$
port 3, subport 1

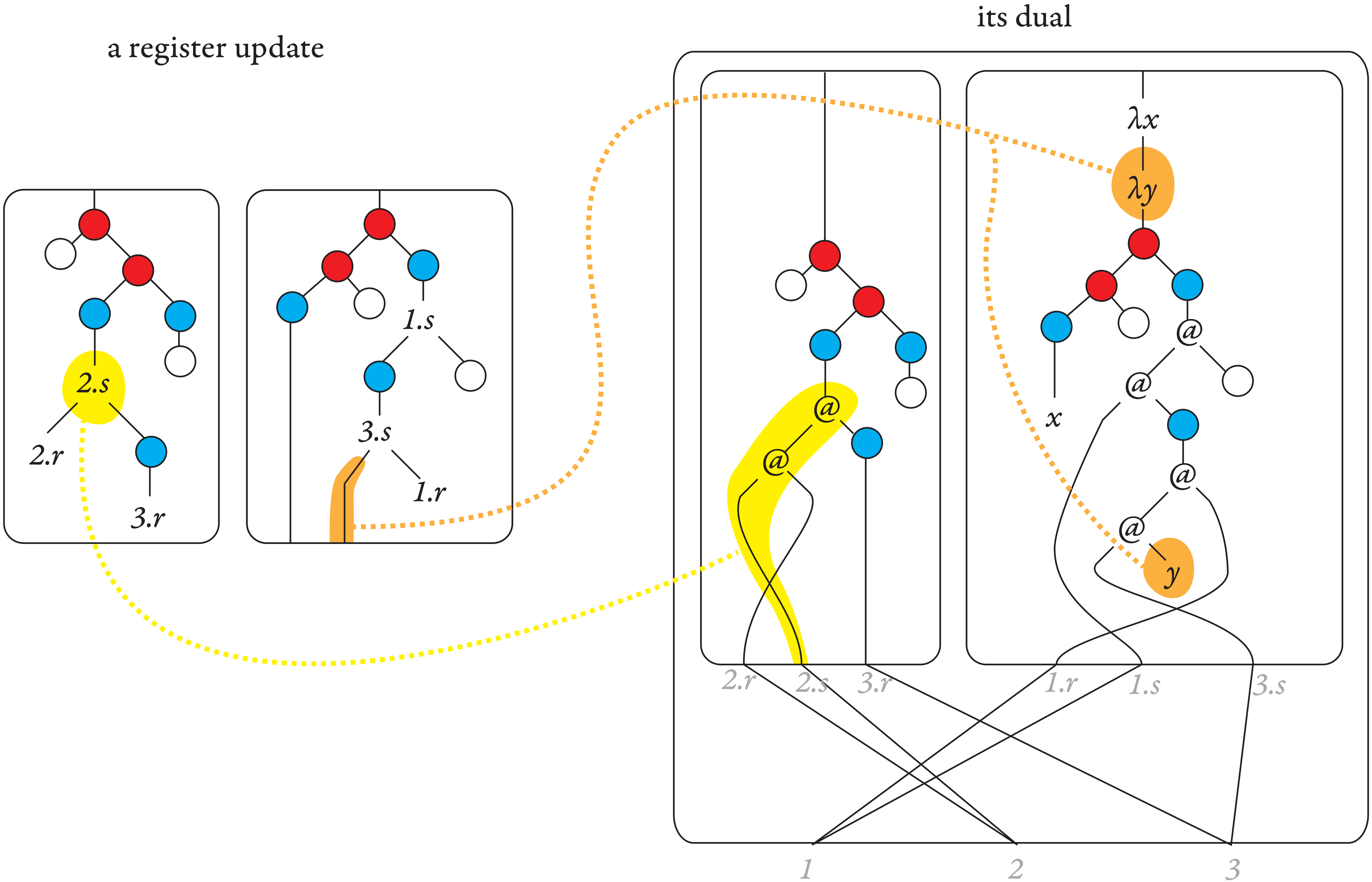

the root is from Σ
all children are from Γ

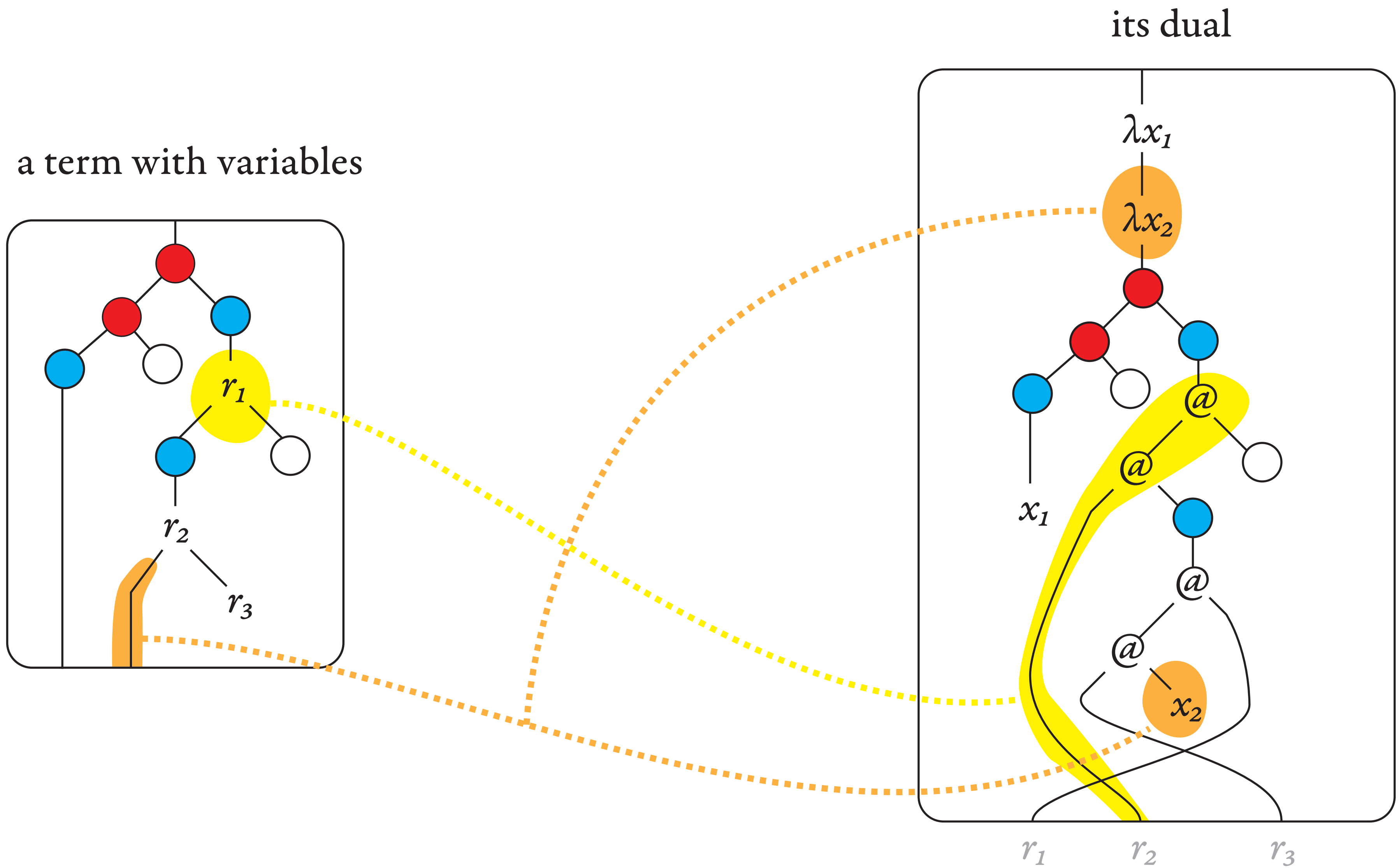


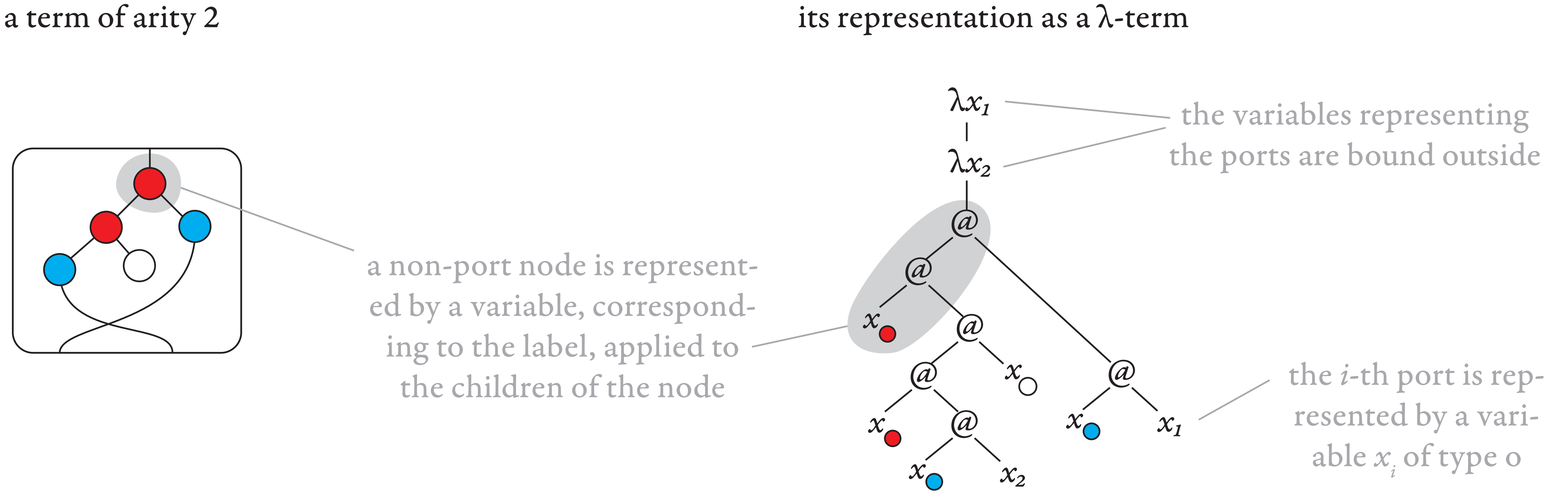
$\mapsto$

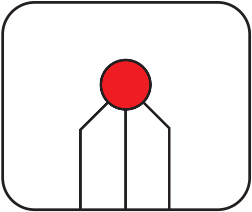



1
2
3
4
5
6

1
5
3
4
6

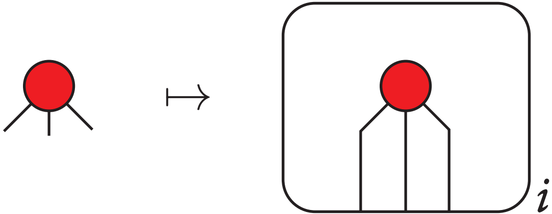

a register update
its dual
2.s
2.r
3.r
1.s
3.s
1.r
λx
λy
x
y
@
2.r
2.s
3.r
1.r
1.s
3.s
1
2
3

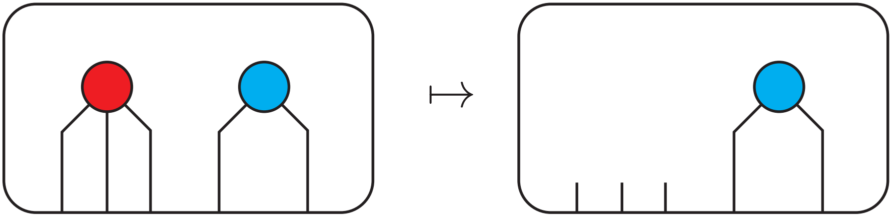

a term with variables
its dual
r_1
r_2
r_3
λx_1
λx_2
@
@
@
@
x_1
x_2
r_1
r_2
r_3

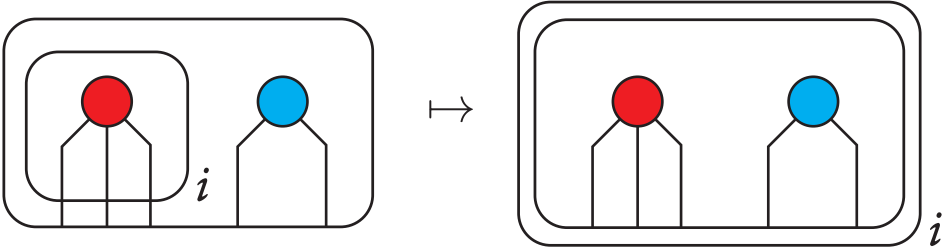

a term of arity 2
its representation as a λ-term
λx1
λx2
the variables representing the ports are bound outside
a non-port node is represented by a variable, corresponding to the label, applied to the children of the node
@
x
x1
x2
the i-th port is represented by a variable xi of type o

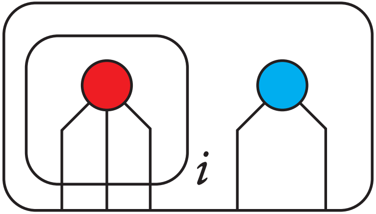


i

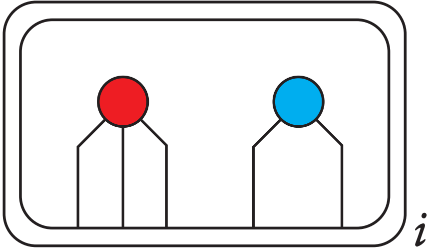

i

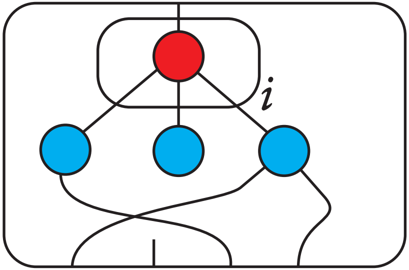



i
i

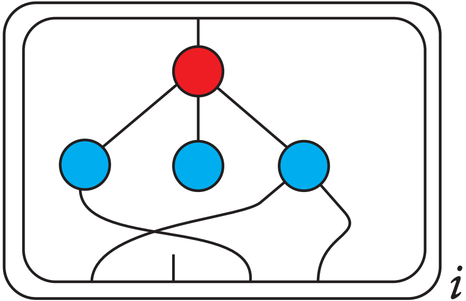


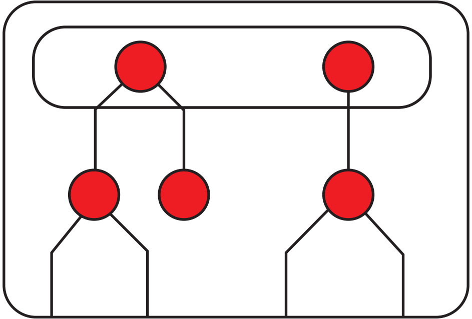
$\mapsto$



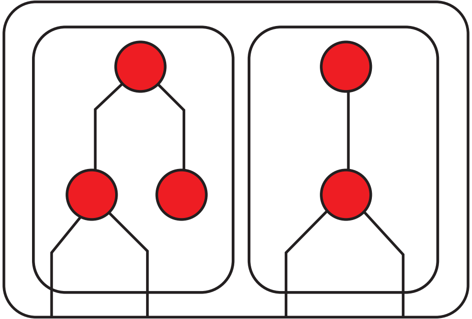


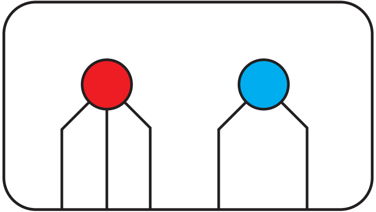
$\mapsto$



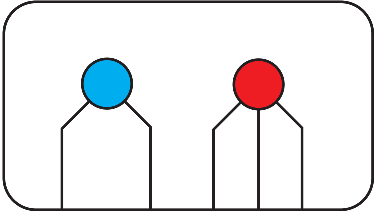


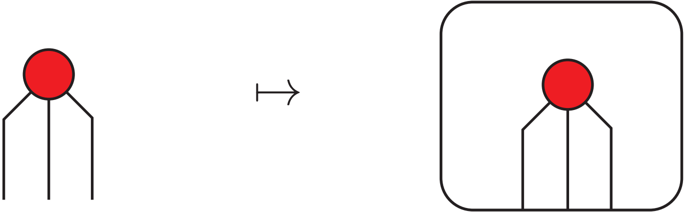
$\mapsto$

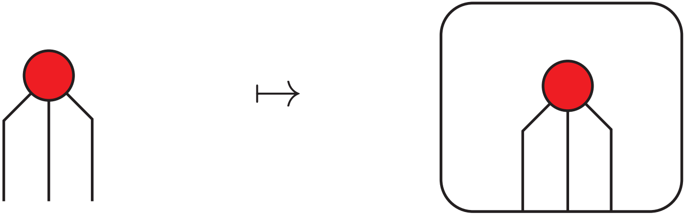


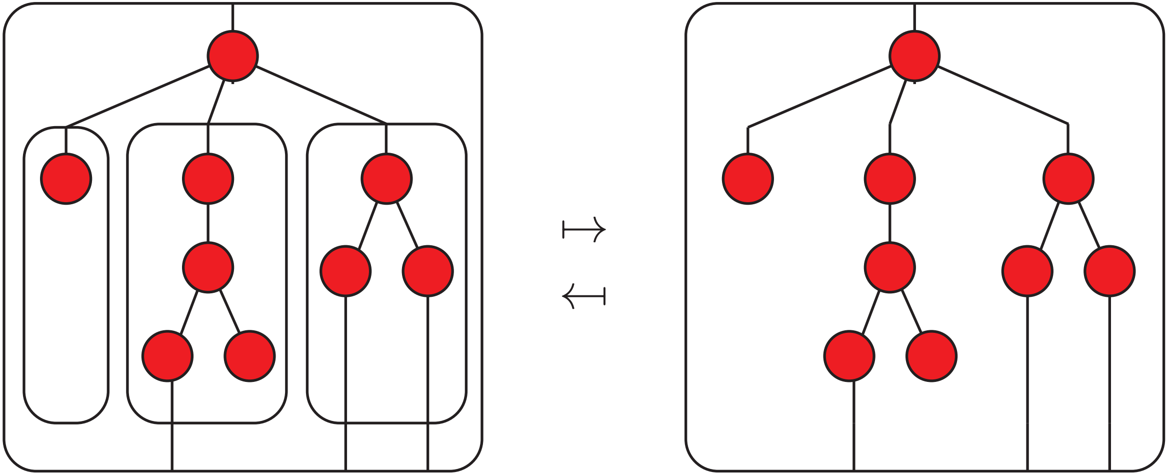


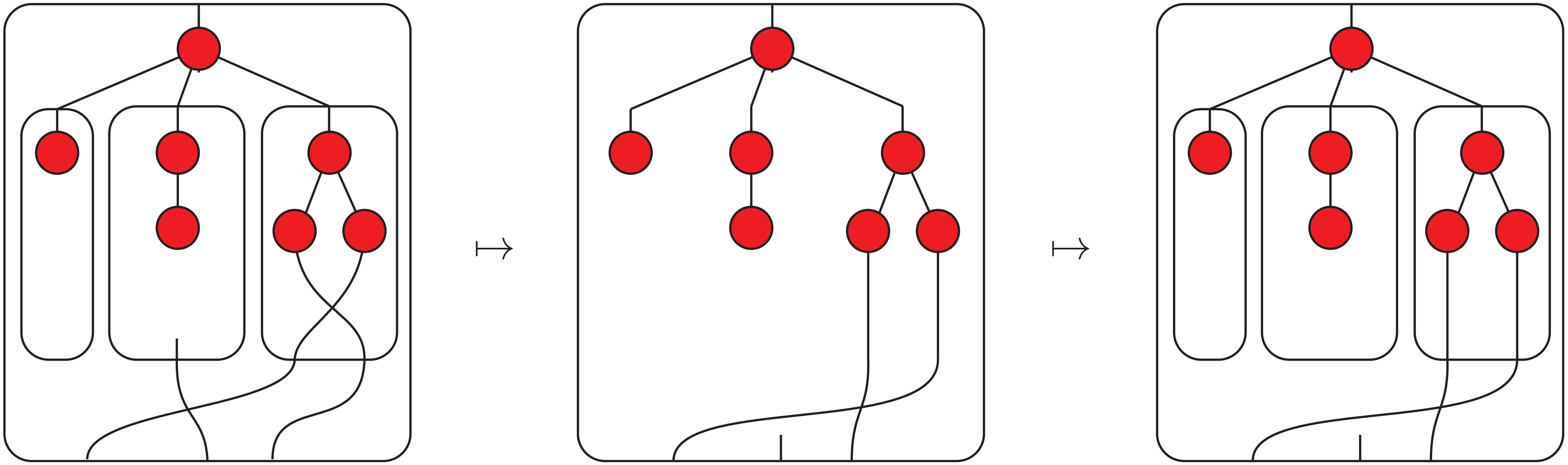
$\mapsto$

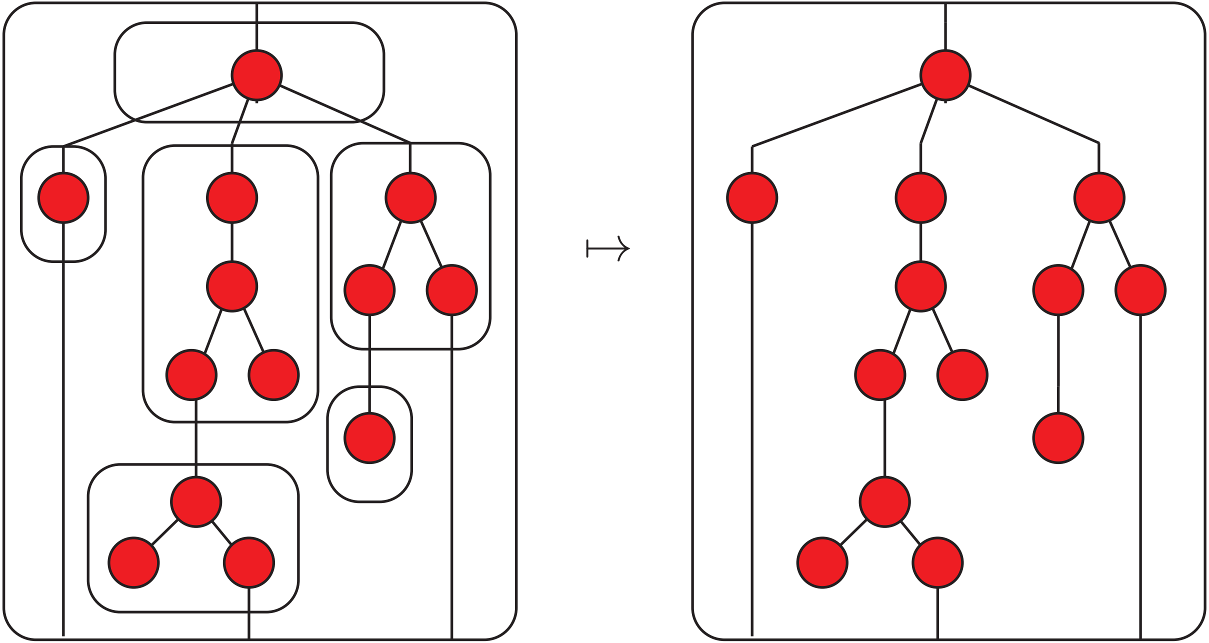


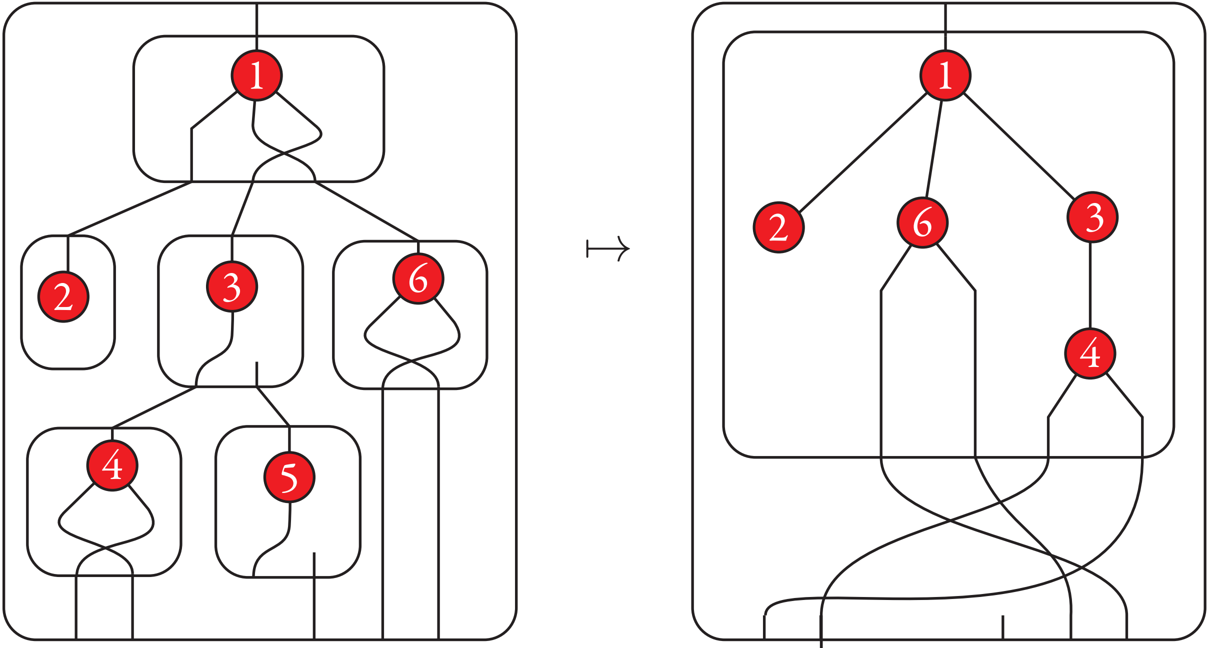


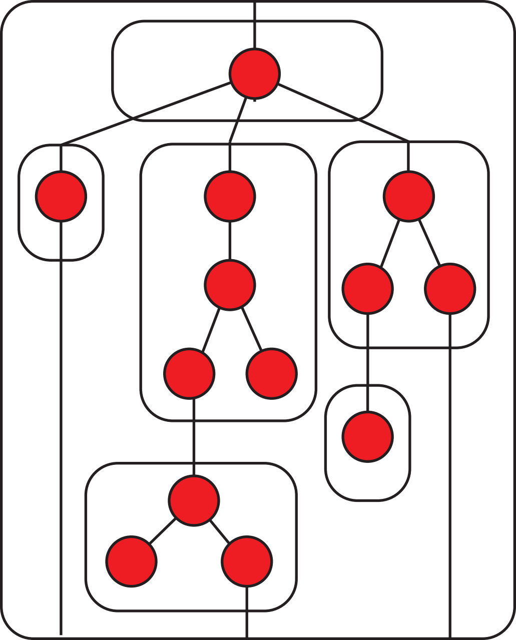


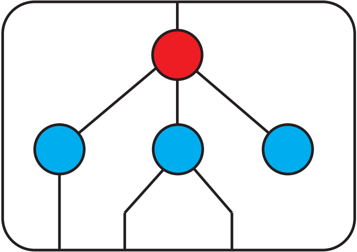


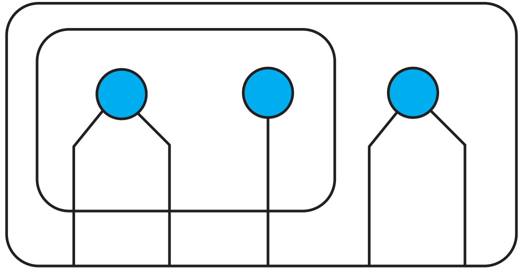


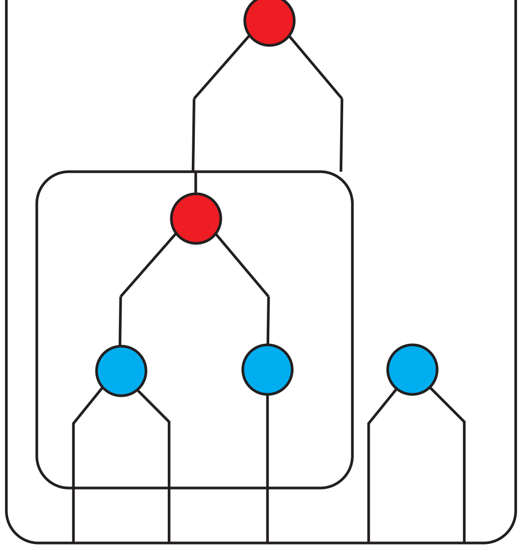


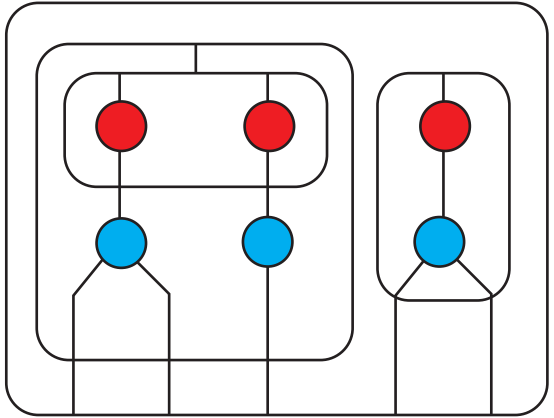

1
2
3
6
4
5
↦
1
2
6
3
4

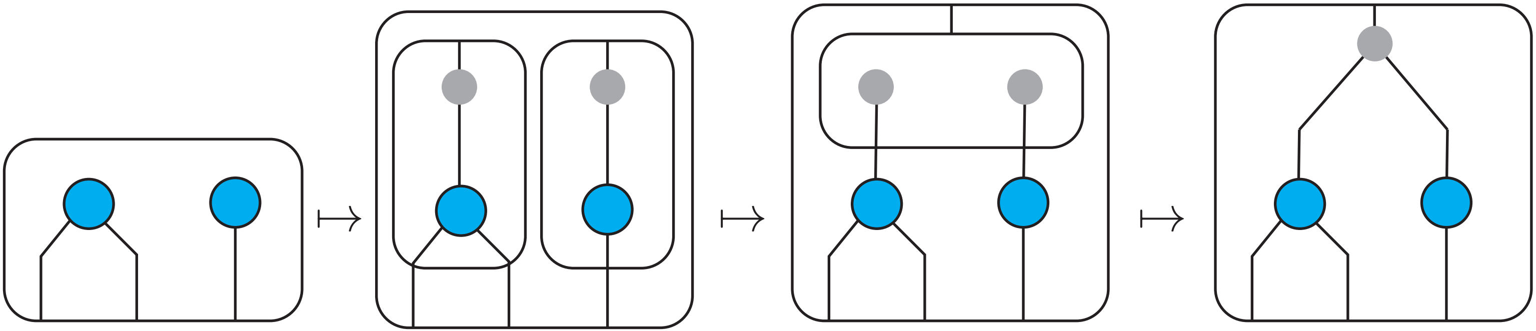


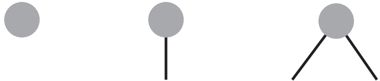


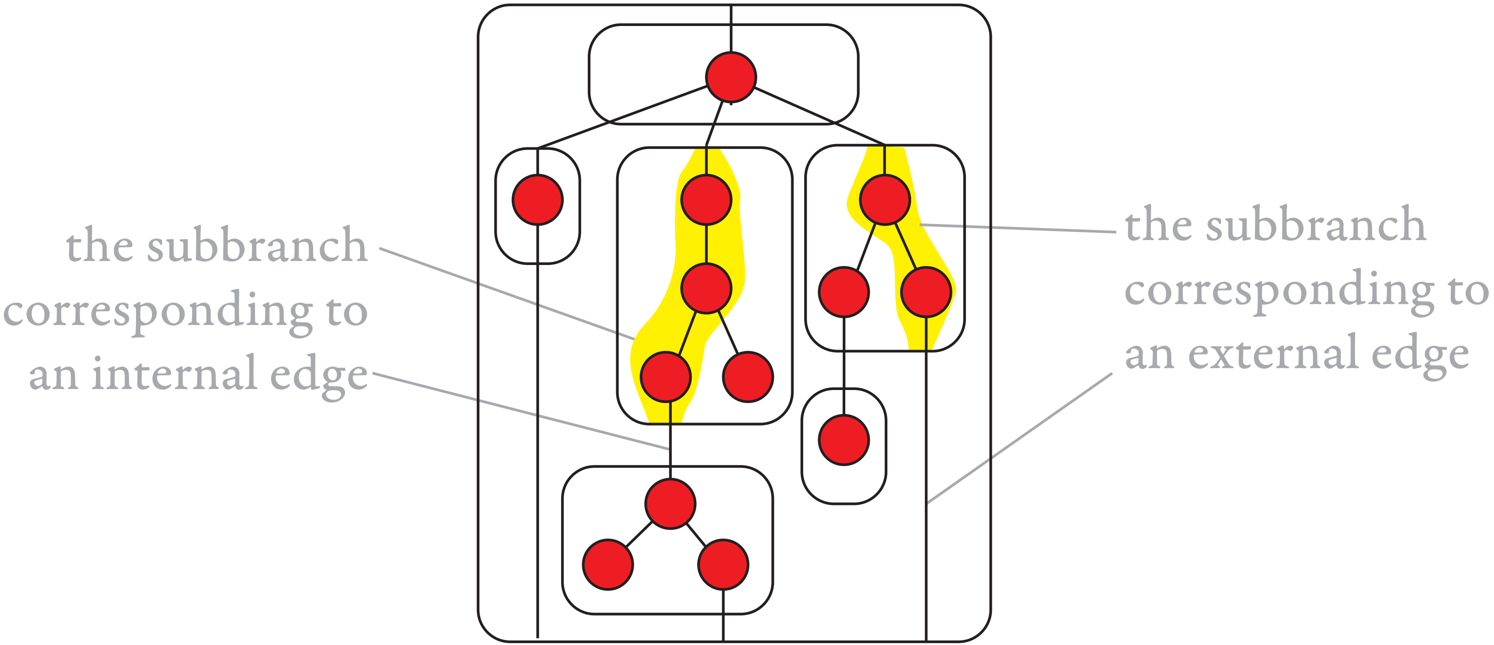


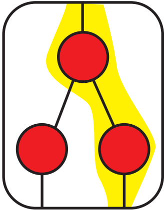




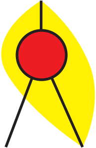


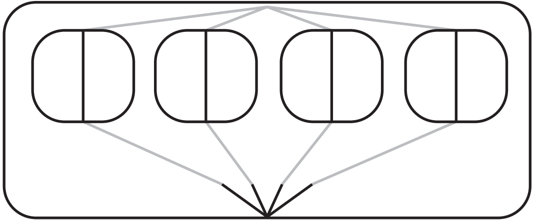



the subbranch
corresponding to
an internal edge
the subbranch
corresponding to
an external edge

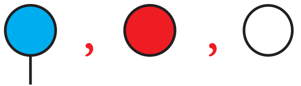


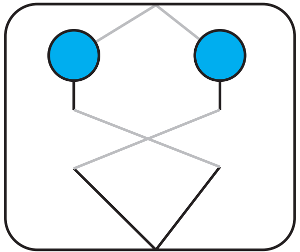
a branch can be visualised as
a term with a distinguished
root-to-port path

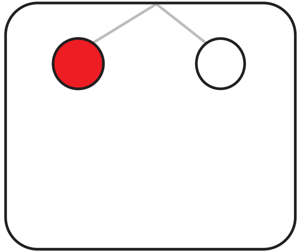


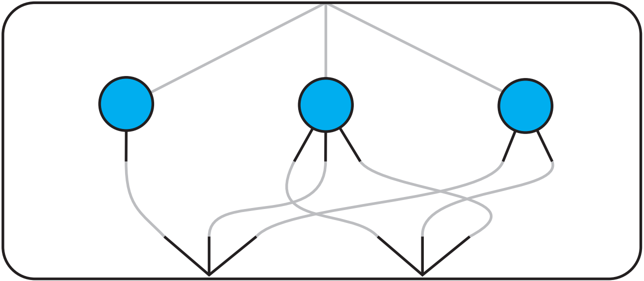


a tuple of *k* identity terms with all their ports folded into one

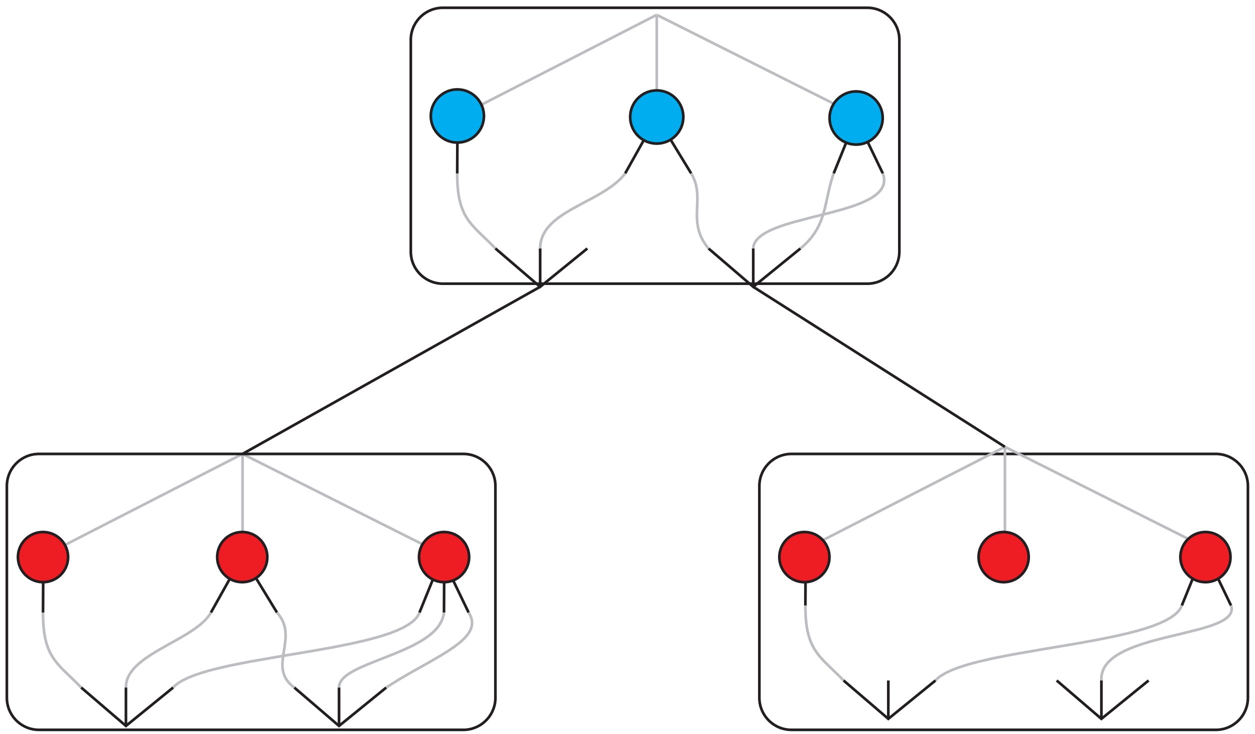
$\Sigma = \{$  $\}$

$a \in \Sigma^{[2]}$ 

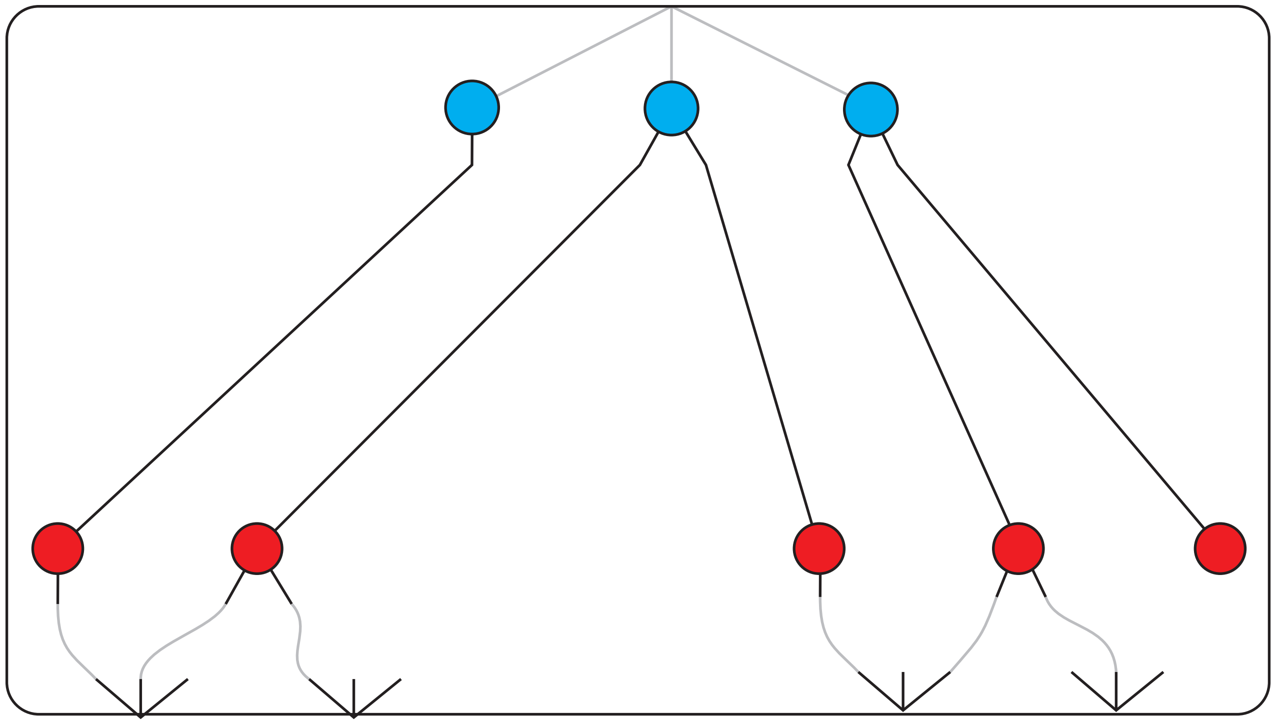
$b \in \Sigma^{[2]}$ 

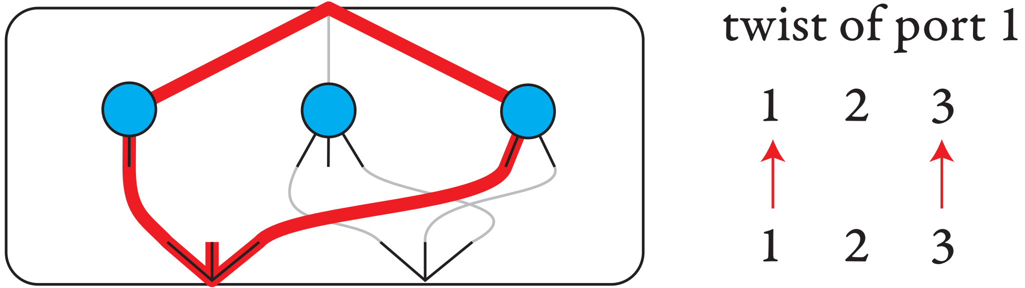


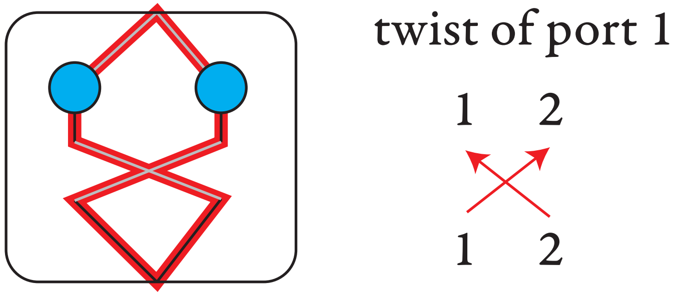
a shallow term of matrix powers

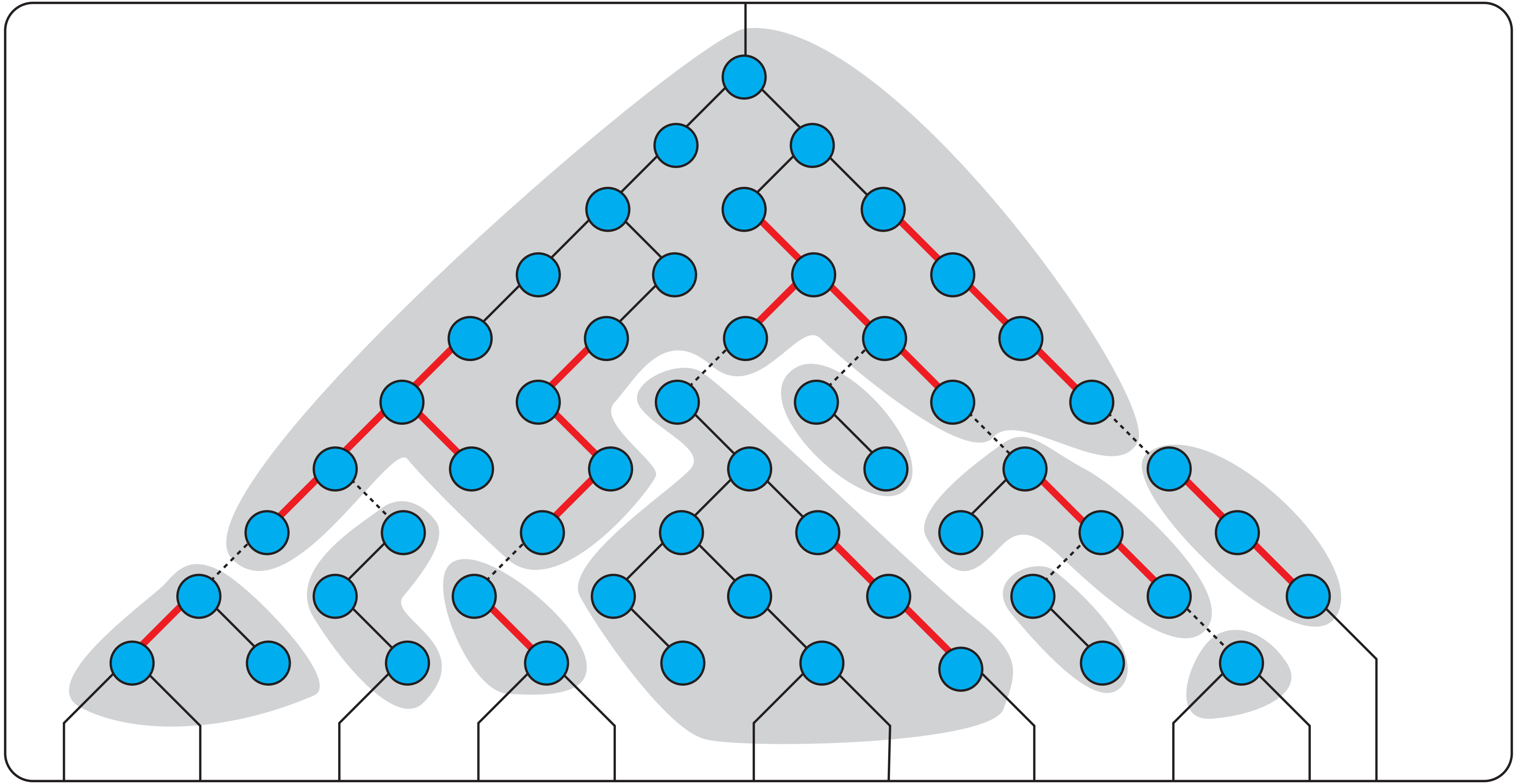


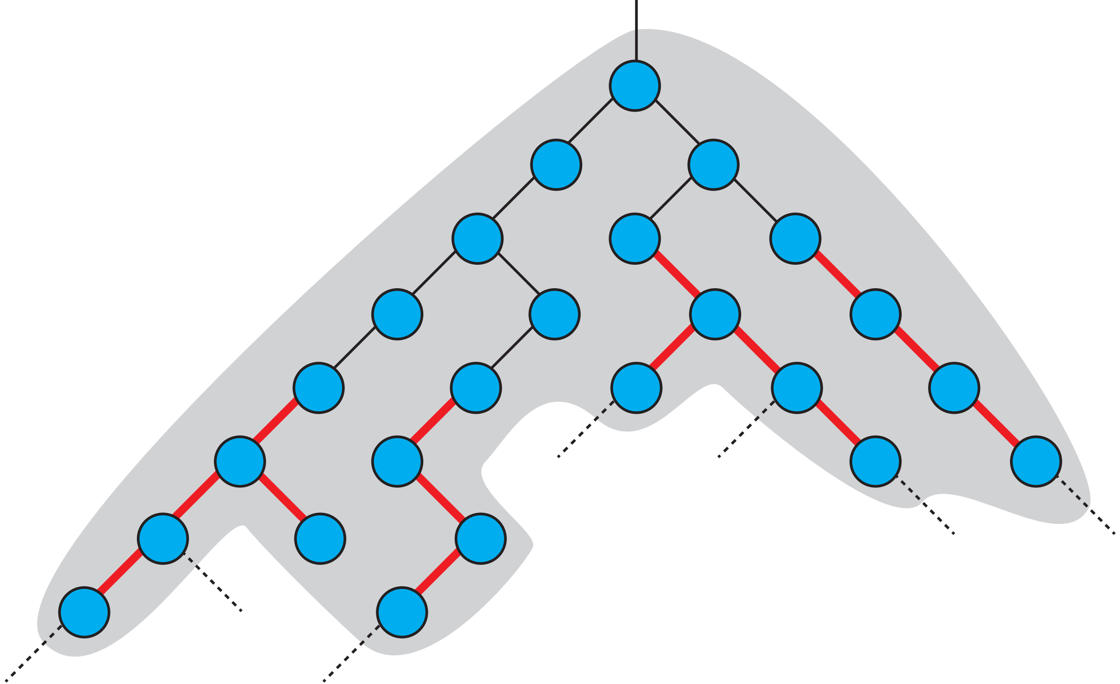
its shallow unfolding

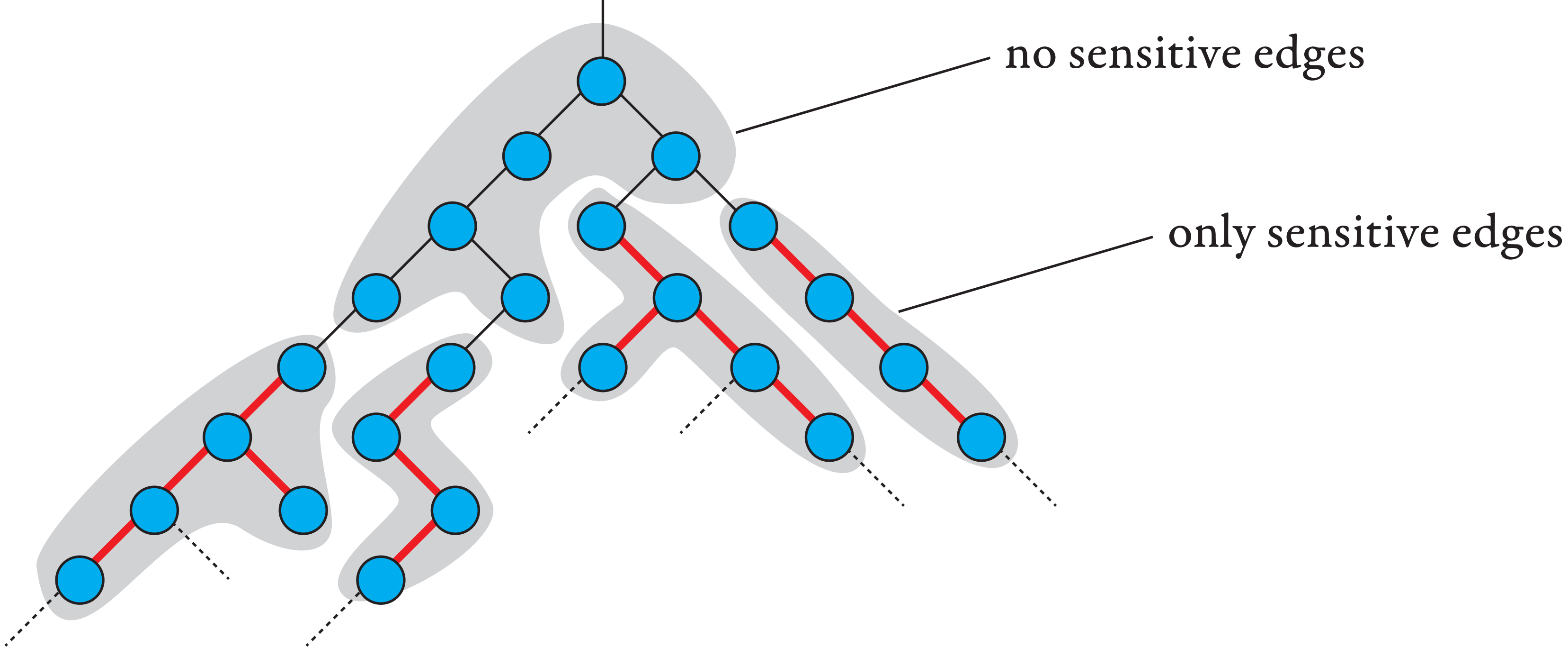


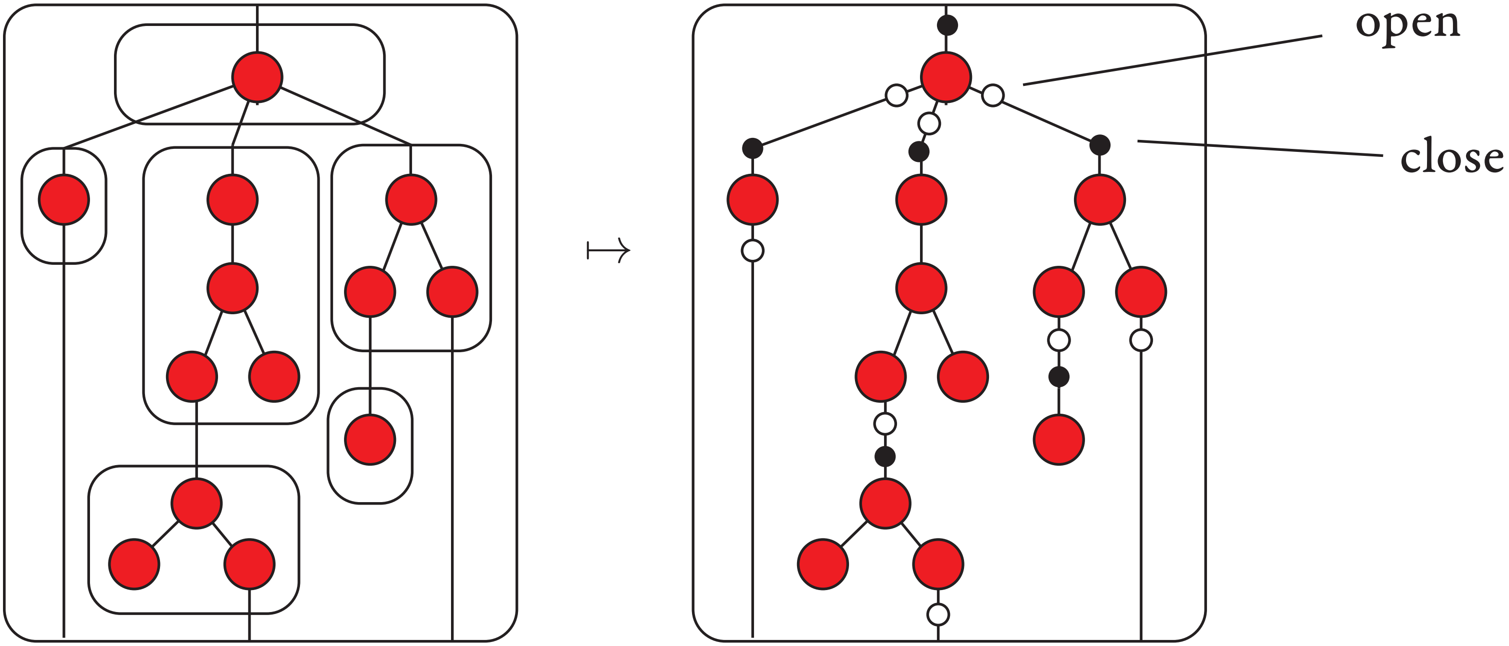

twist of port 1
1 2 3
1 2 3

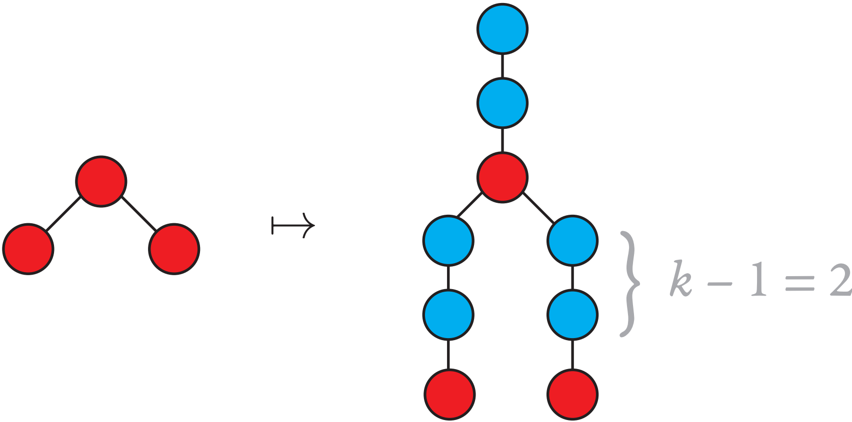

twist of port 1
1 2
1 2

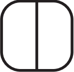


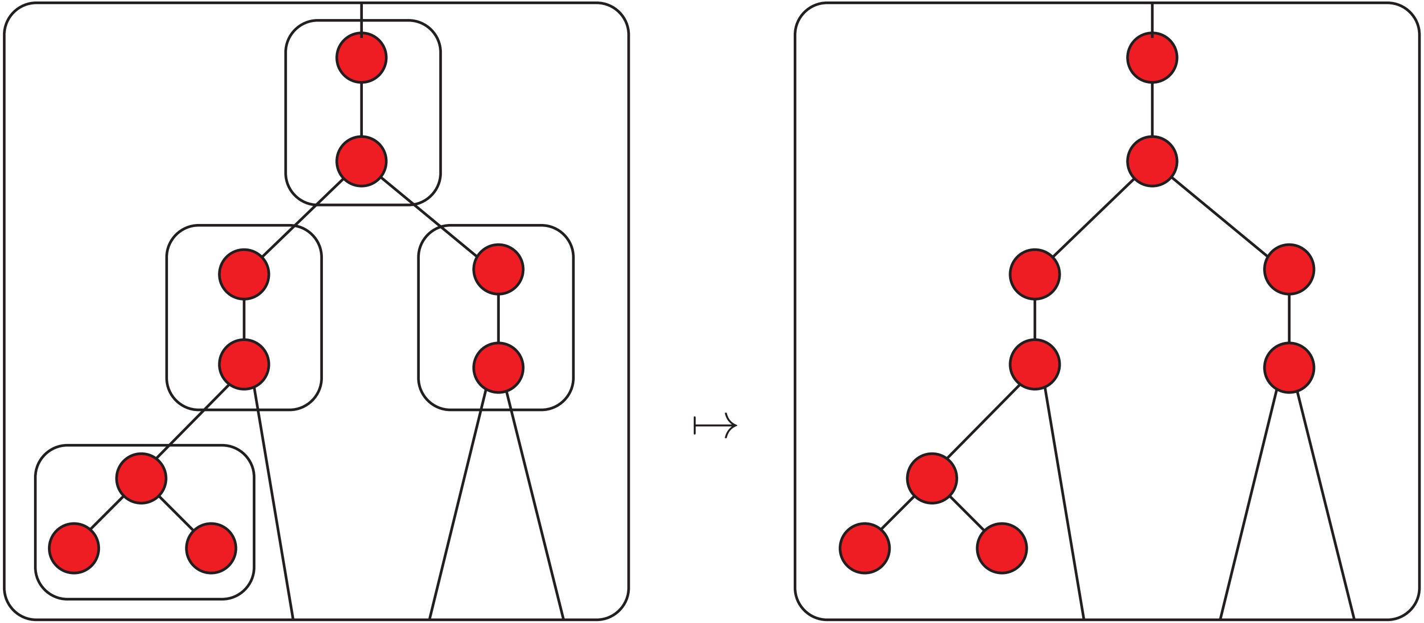


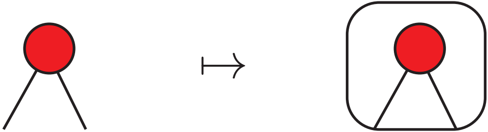

no sensitive edges
only sensitive edges

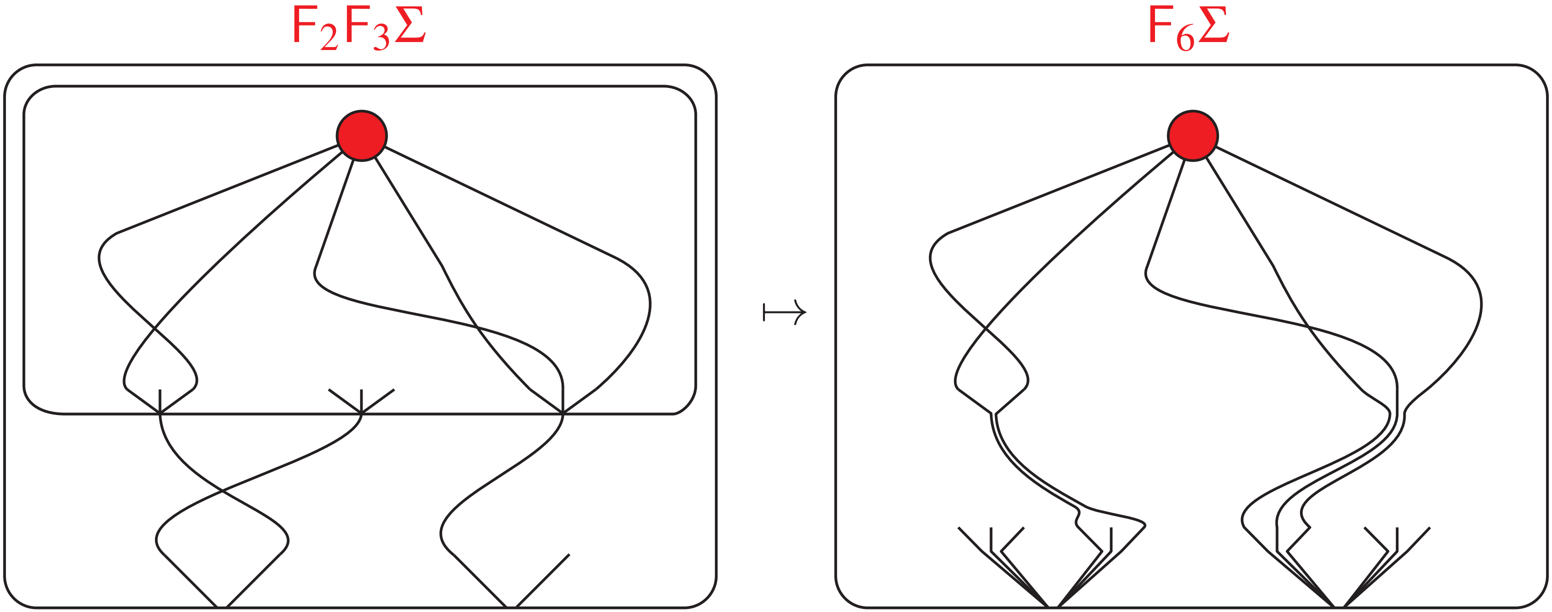

open
close

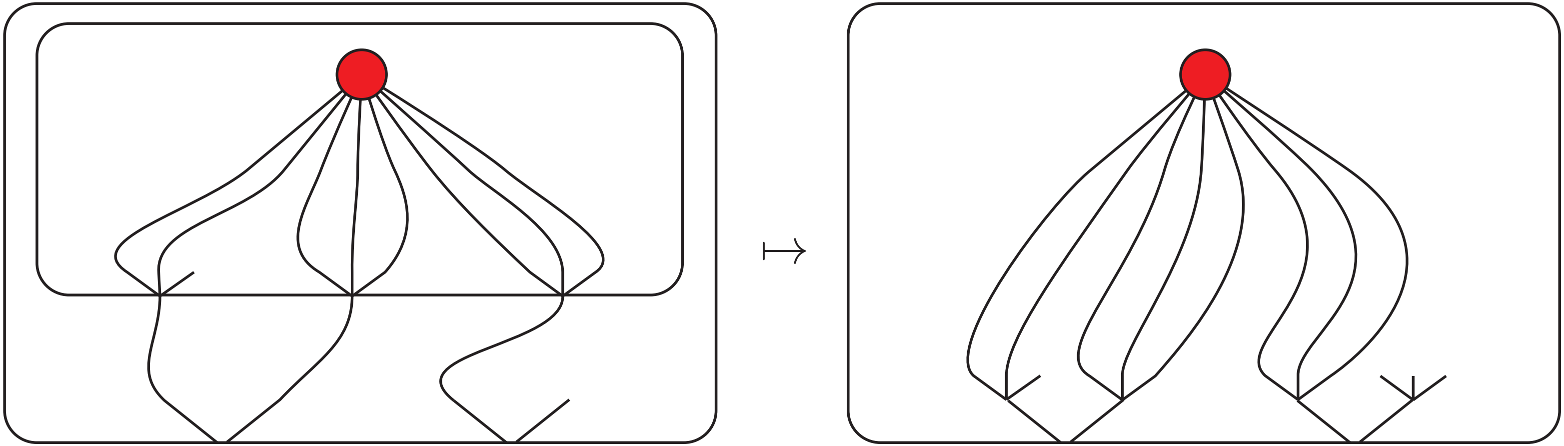

$k - 1 = 2$

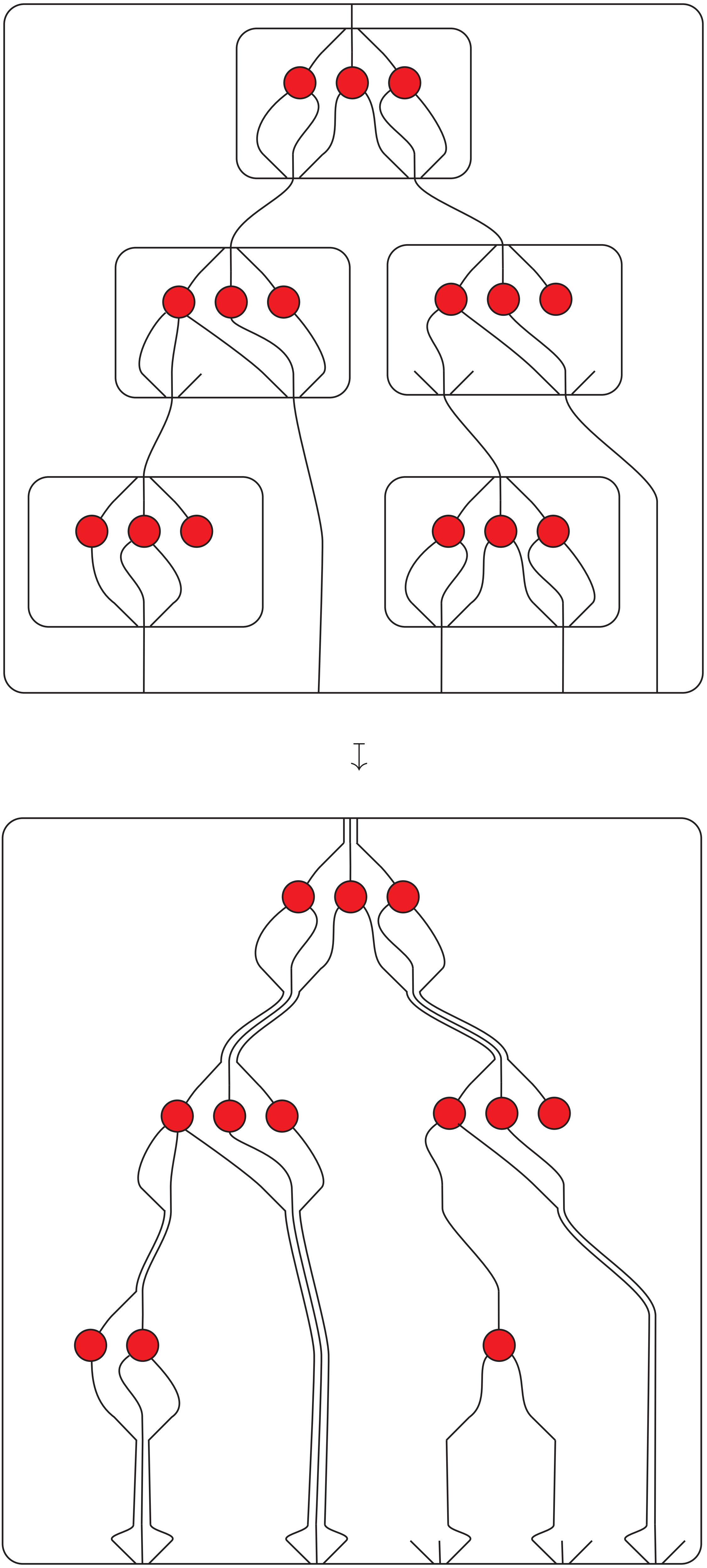


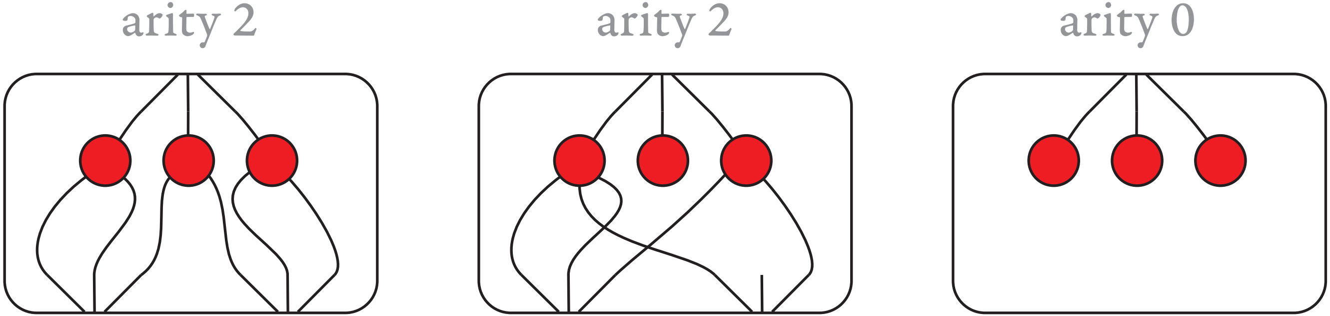


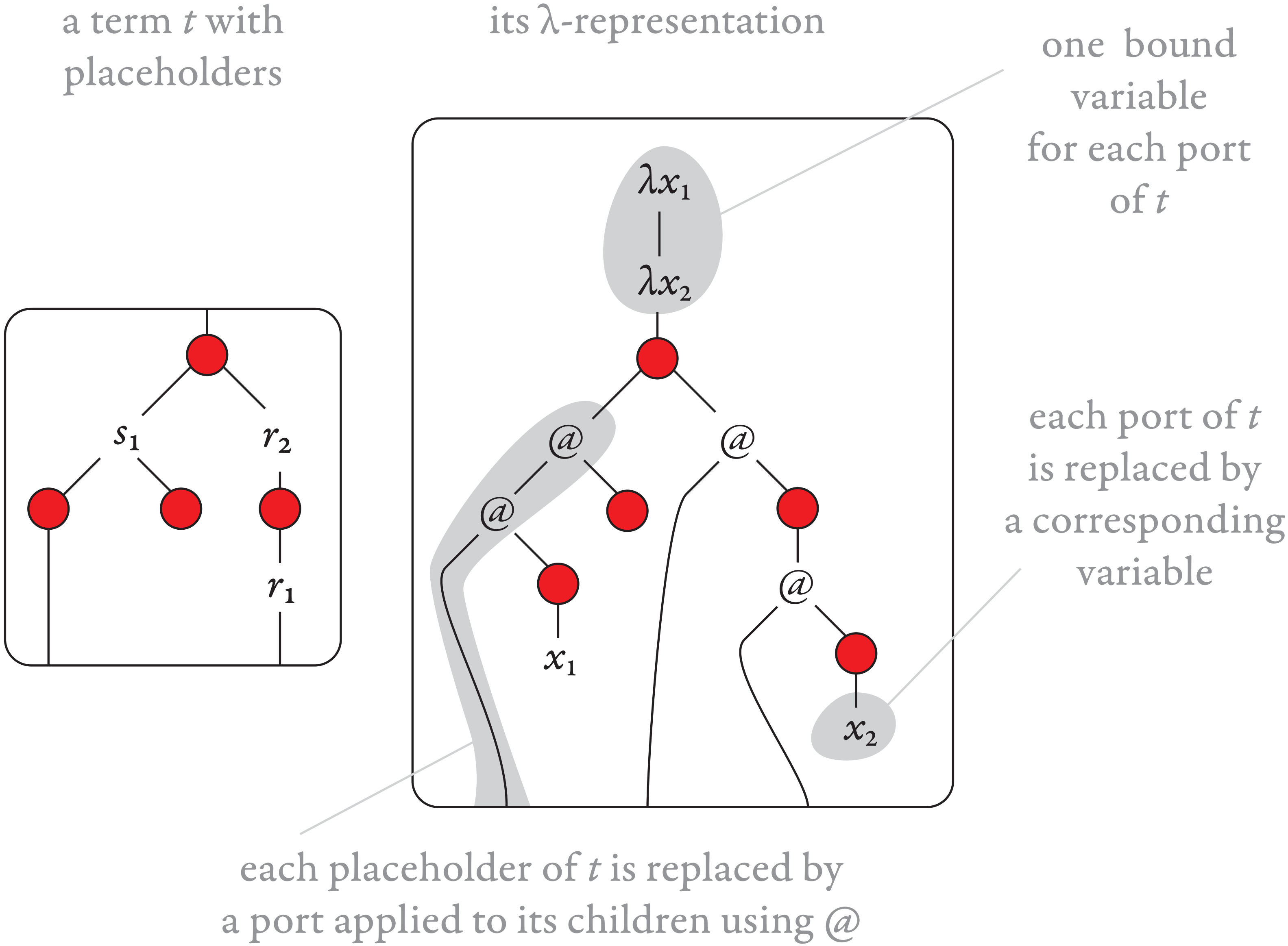


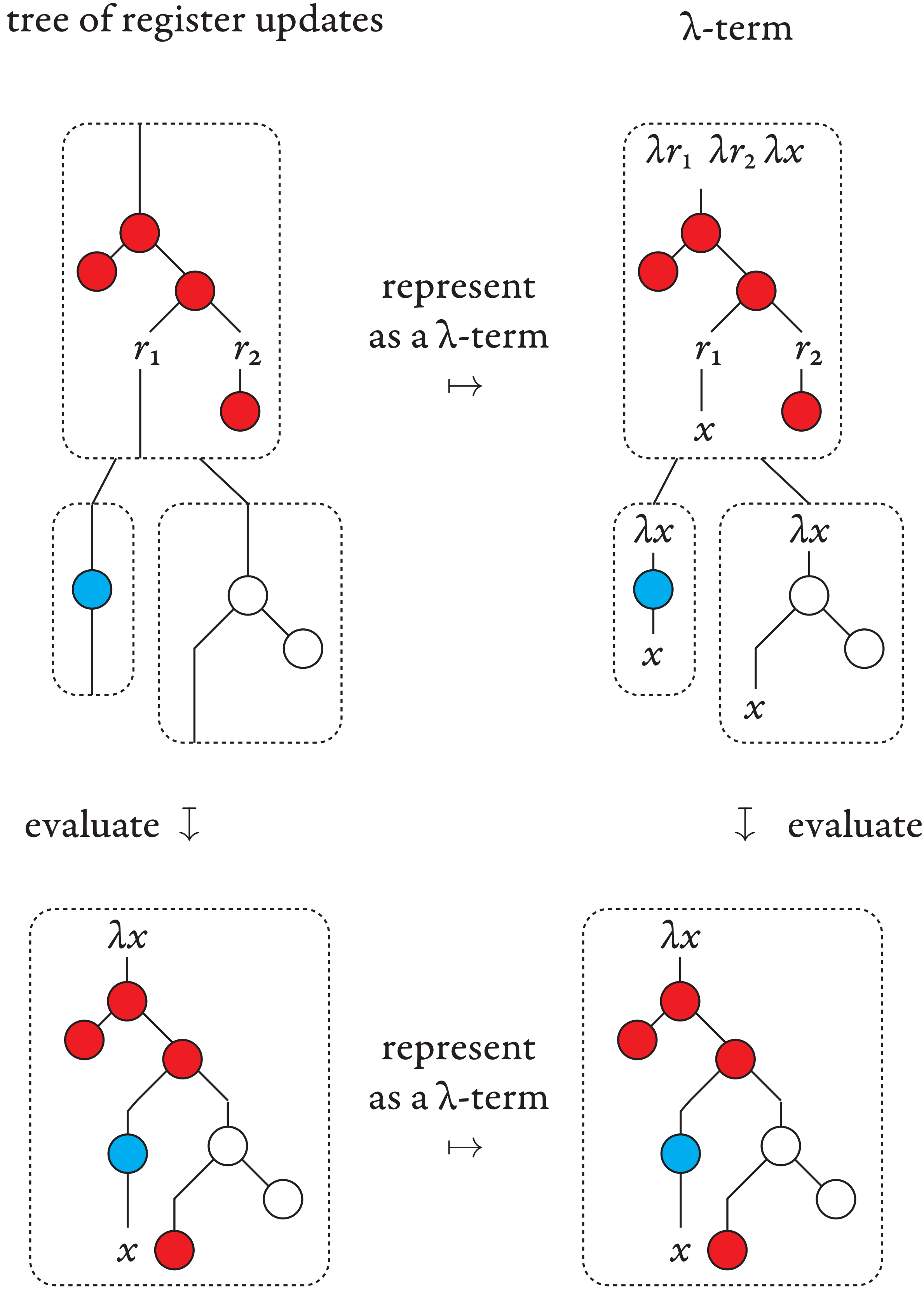

$F_2F_3\Sigma$
$\mapsto$
$F_6\Sigma$



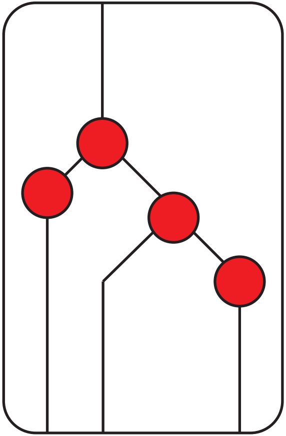



arity 2
arity 2
arity 0

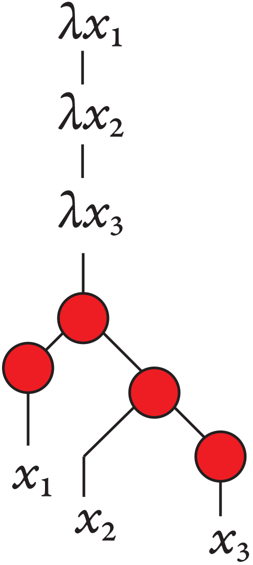

a term t with placeholders
its λ-representation
one bound variable for each port of t
s_1
r_2
r_1
λx_1
λx_2
@
@
@
@
x_1
x_2
each port of t is replaced by a corresponding variable
each placeholder of t is replaced by a port applied to its children using @

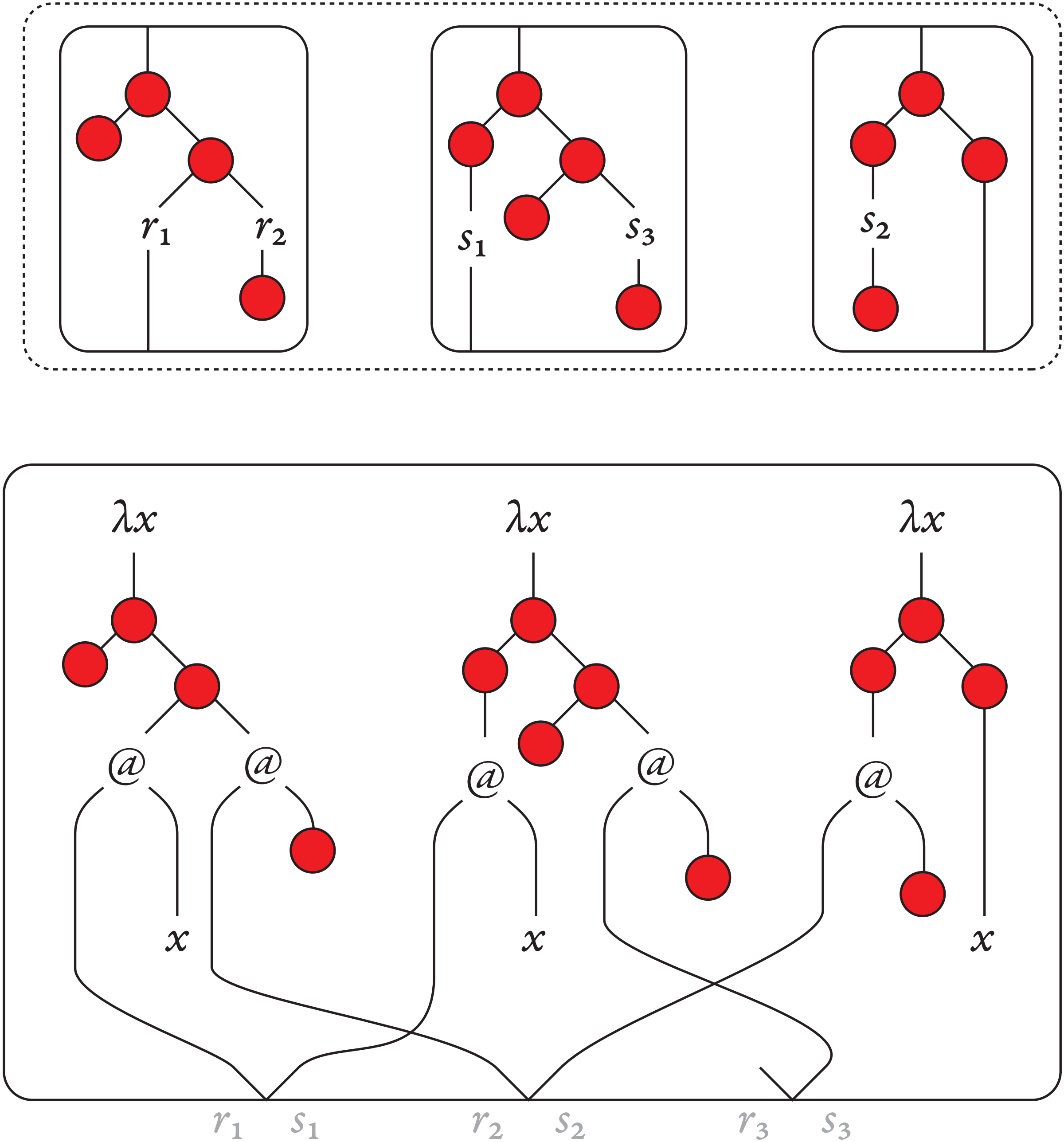

tree of register updates
λ-term
λr1 λr2 λx
r1
r2
represent as a λ-term ↦
x
λx
x
λx
x
evaluate ↧
↧ evaluate
λx
x
represent as a λ-term ↦
λx
x

a term



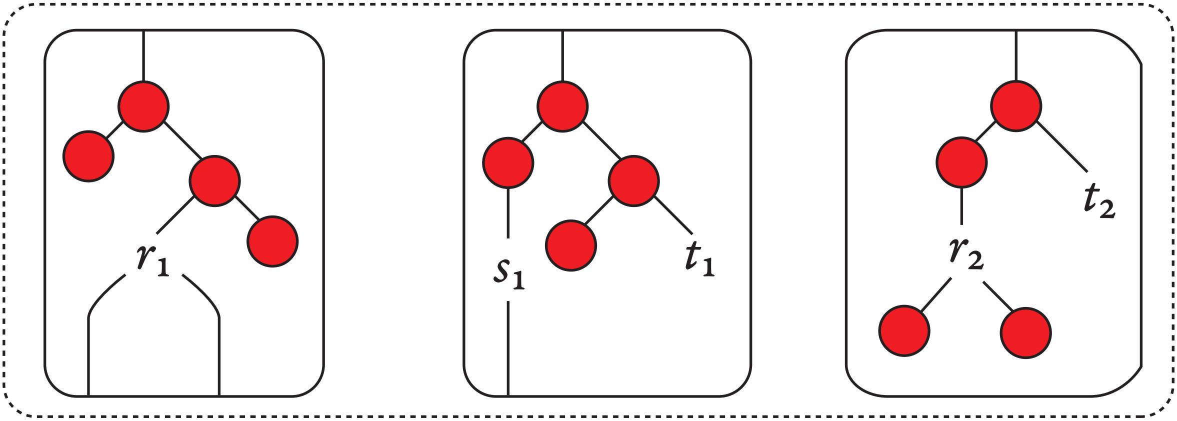
its λ-representation



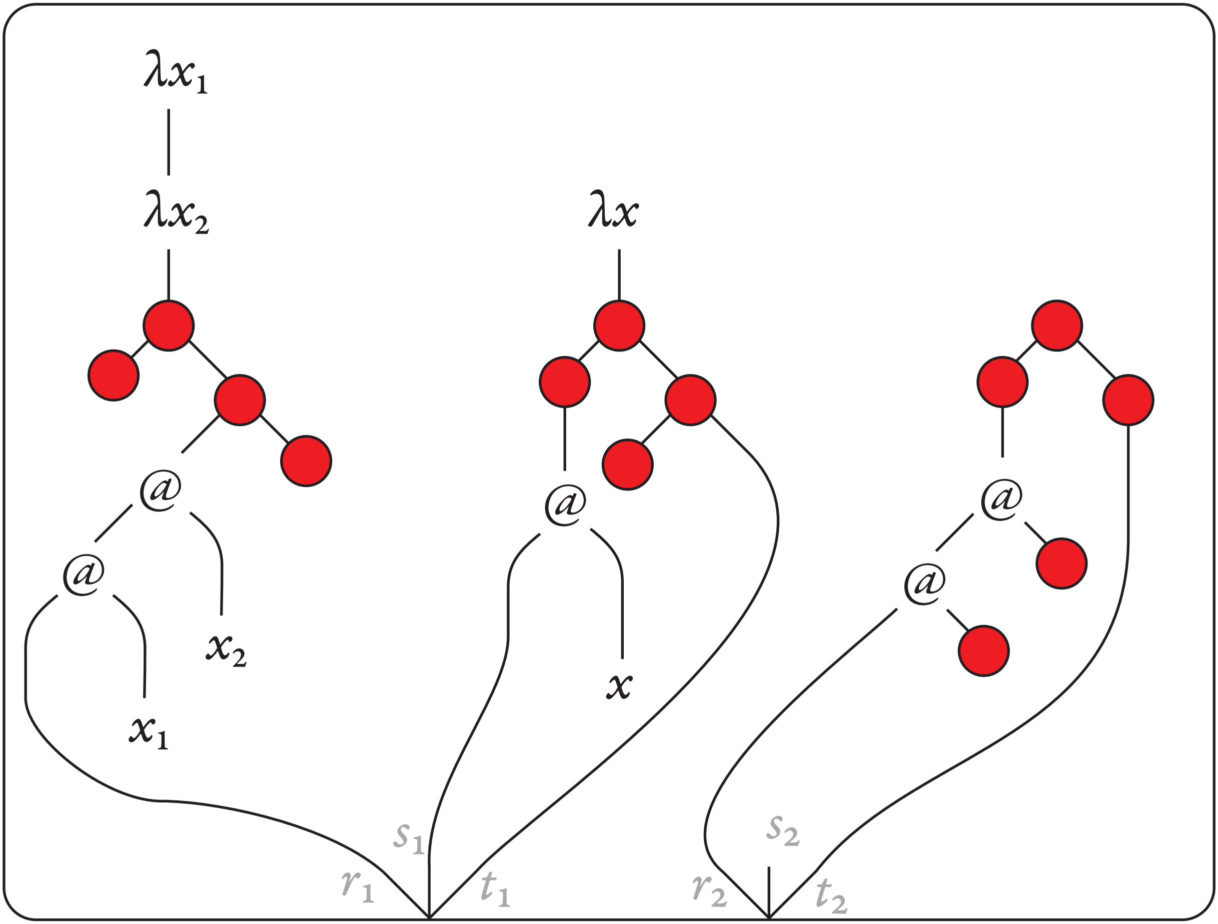


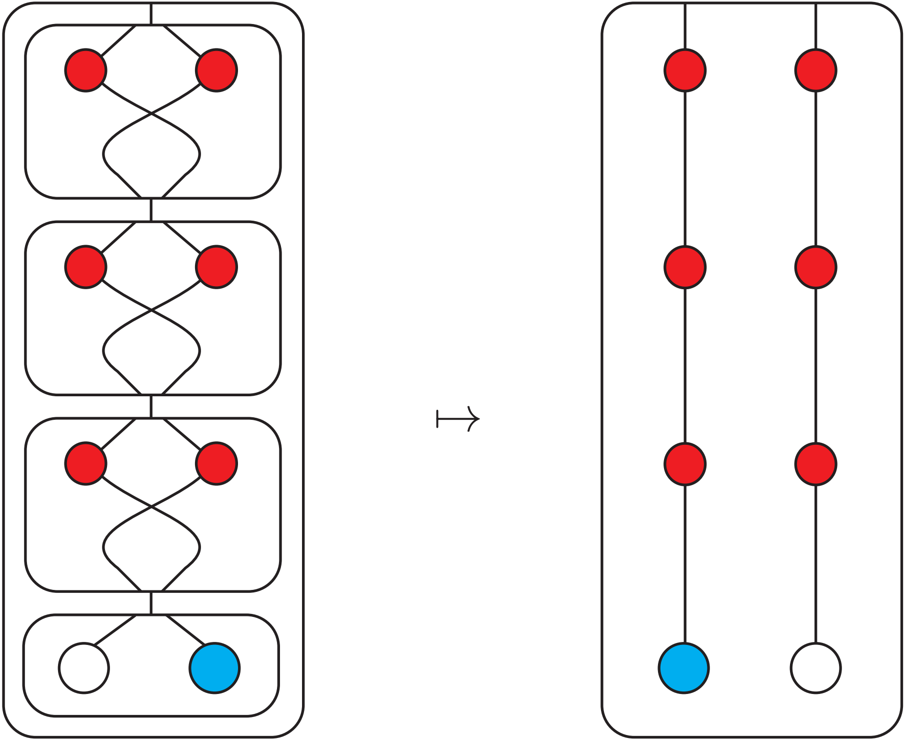
r_1
r_2
s_1
s_3
s_2
λx
λx
λx
@
@
@
@
@
x
x
x
r_1
s_1
r_2
s_2
r_3
s_3

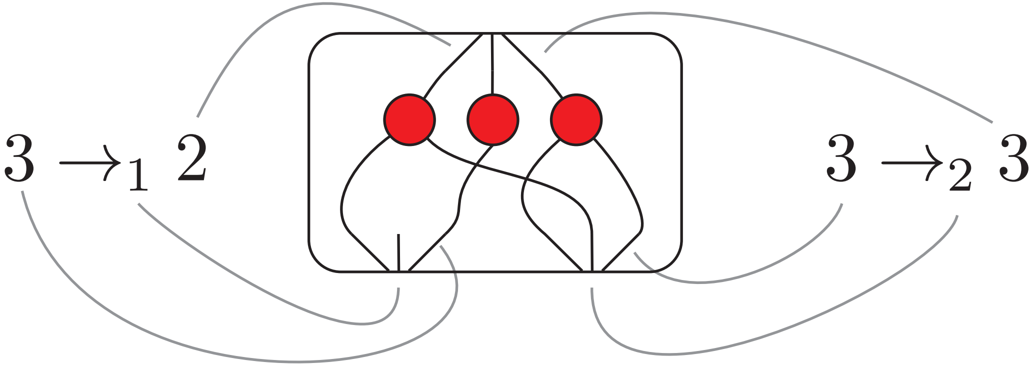
a register update

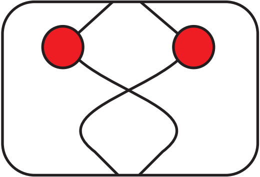


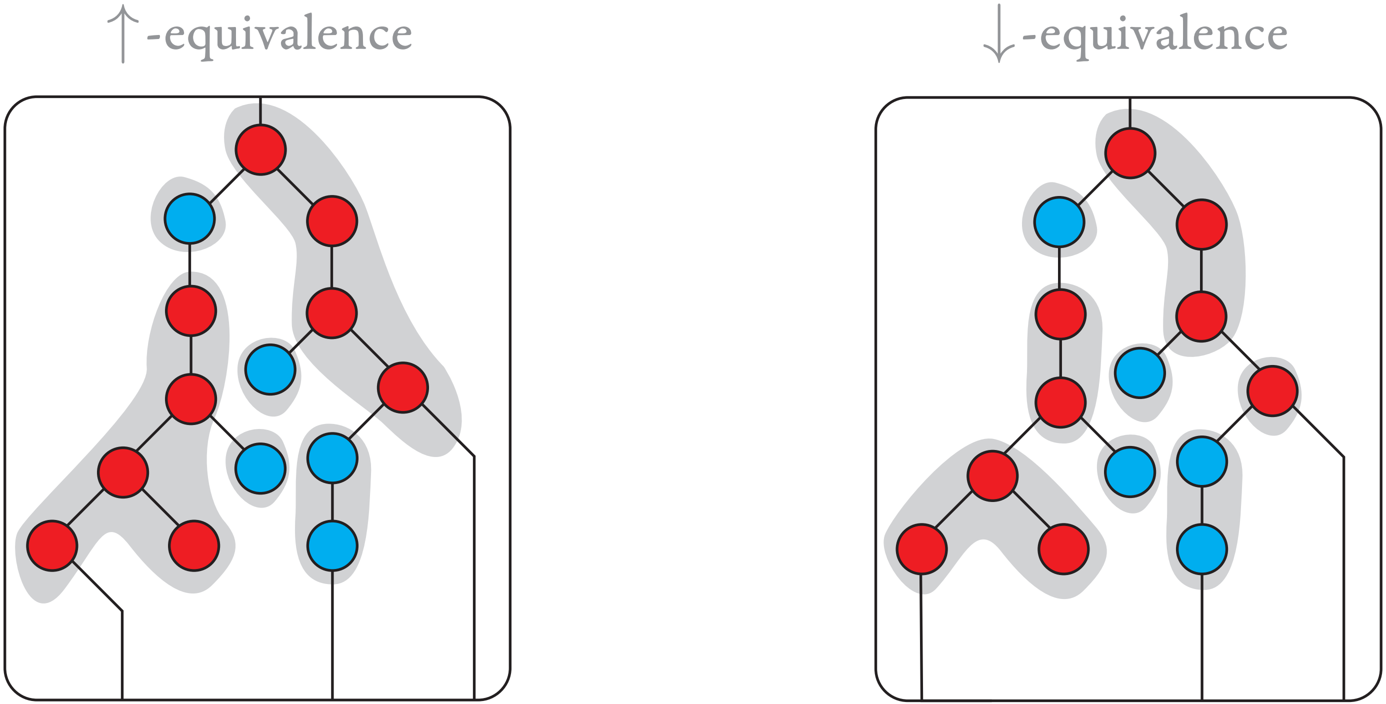
its λ-representation

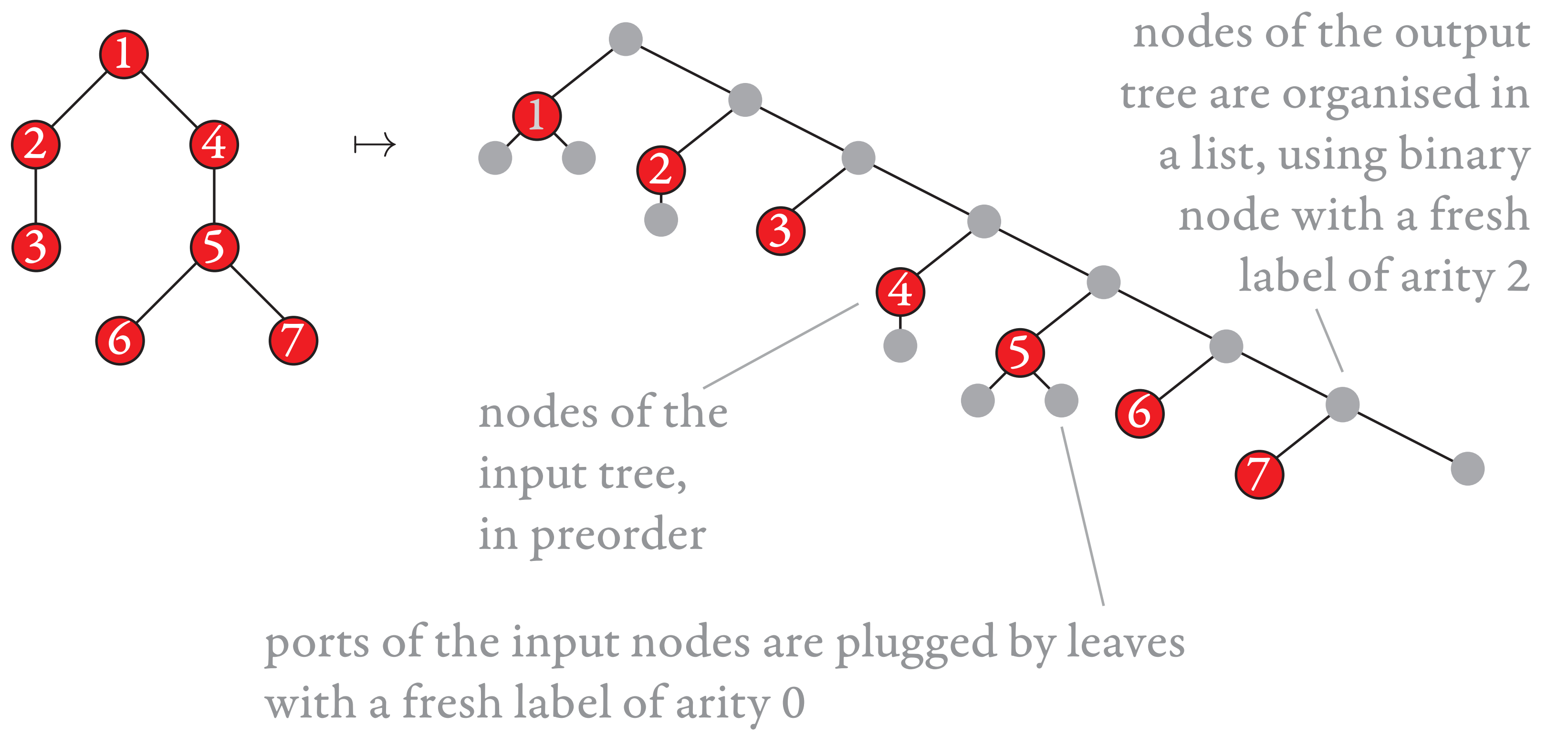




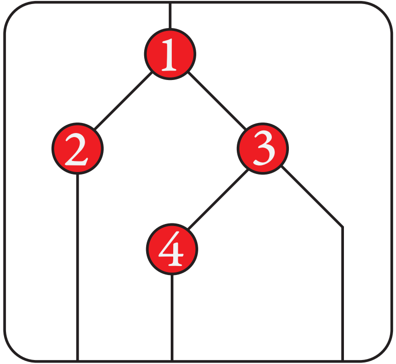

$3 \to_1 2$
$3 \to_2 3$



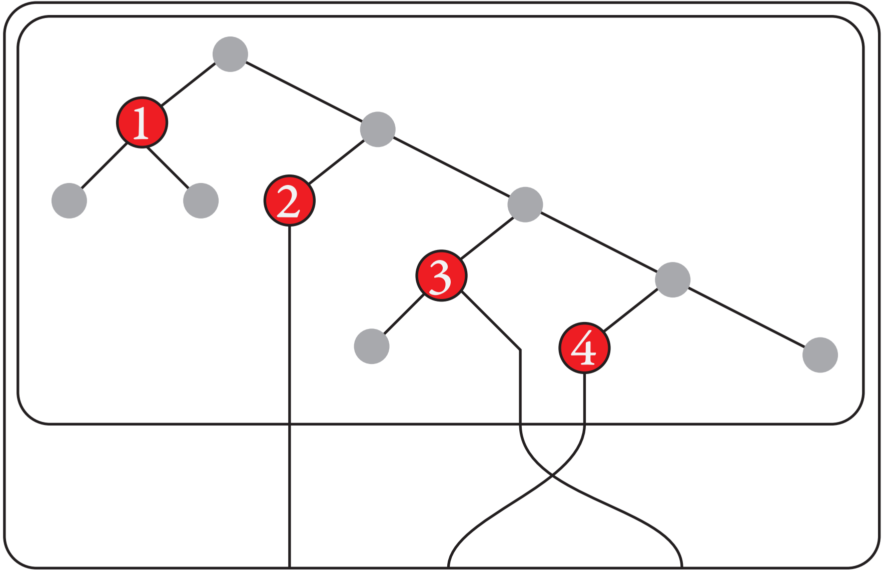

$\uparrow$-equivalence
$\downarrow$-equivalence



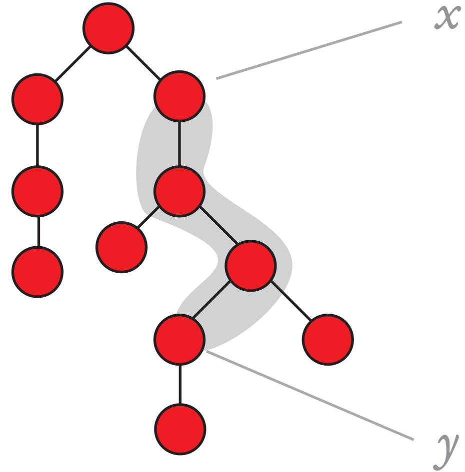
1
2
4
3
5
6
7
↦
1
2
3
4
5
6
7
nodes of the output tree are organised in a list, using binary node with a fresh label of arity 2
nodes of the input tree, in preorder
ports of the input nodes are plugged by leaves with a fresh label of arity 0



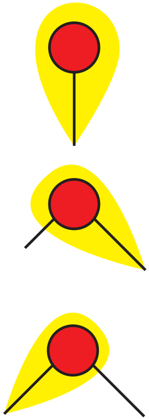
$\mapsto$

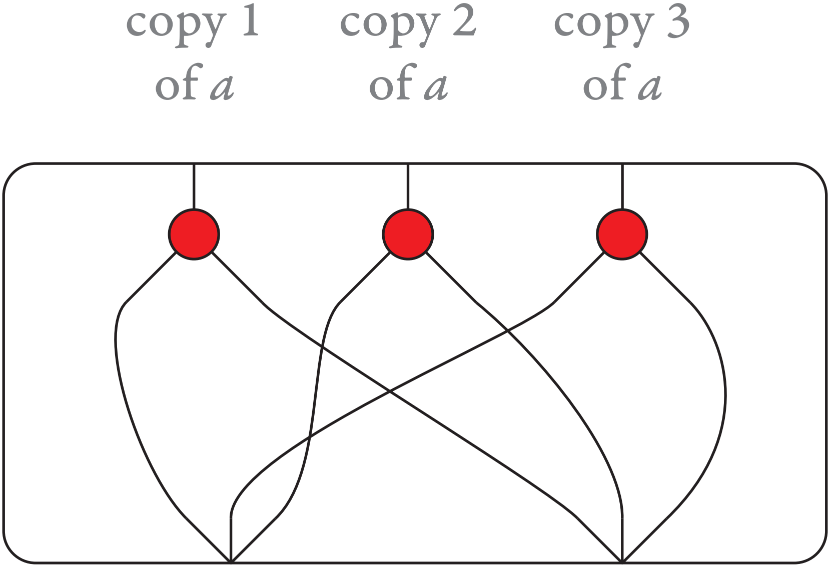


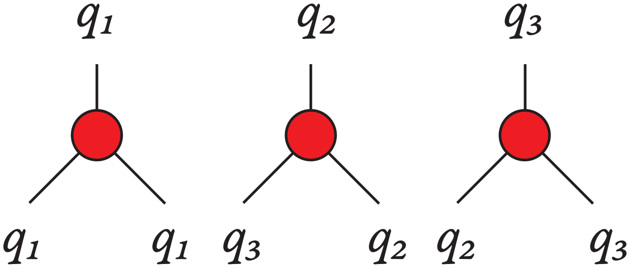
tree with a path from x to y



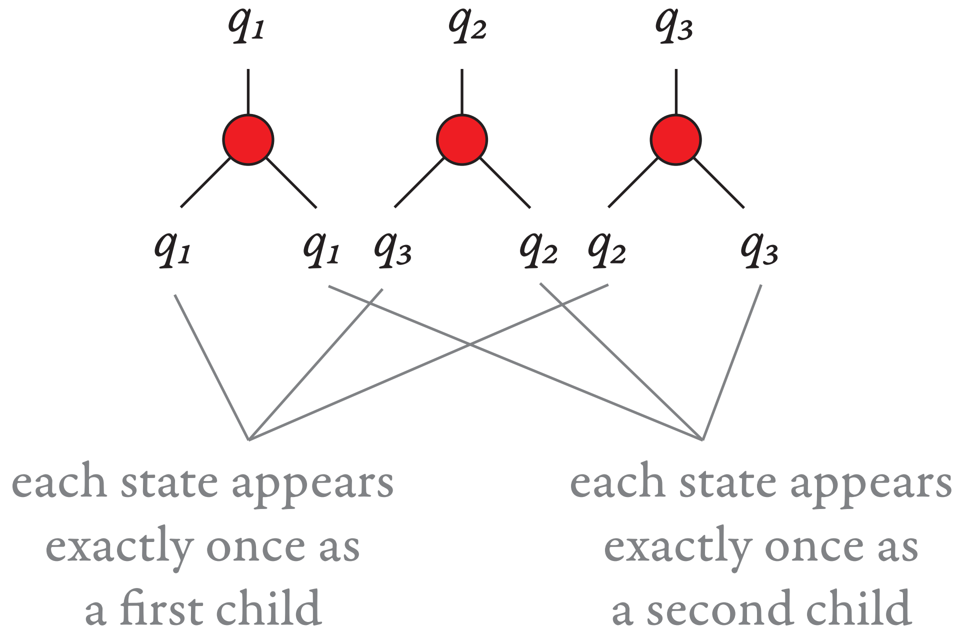
word corresponding to the path

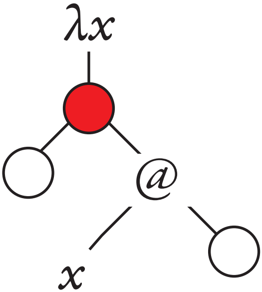


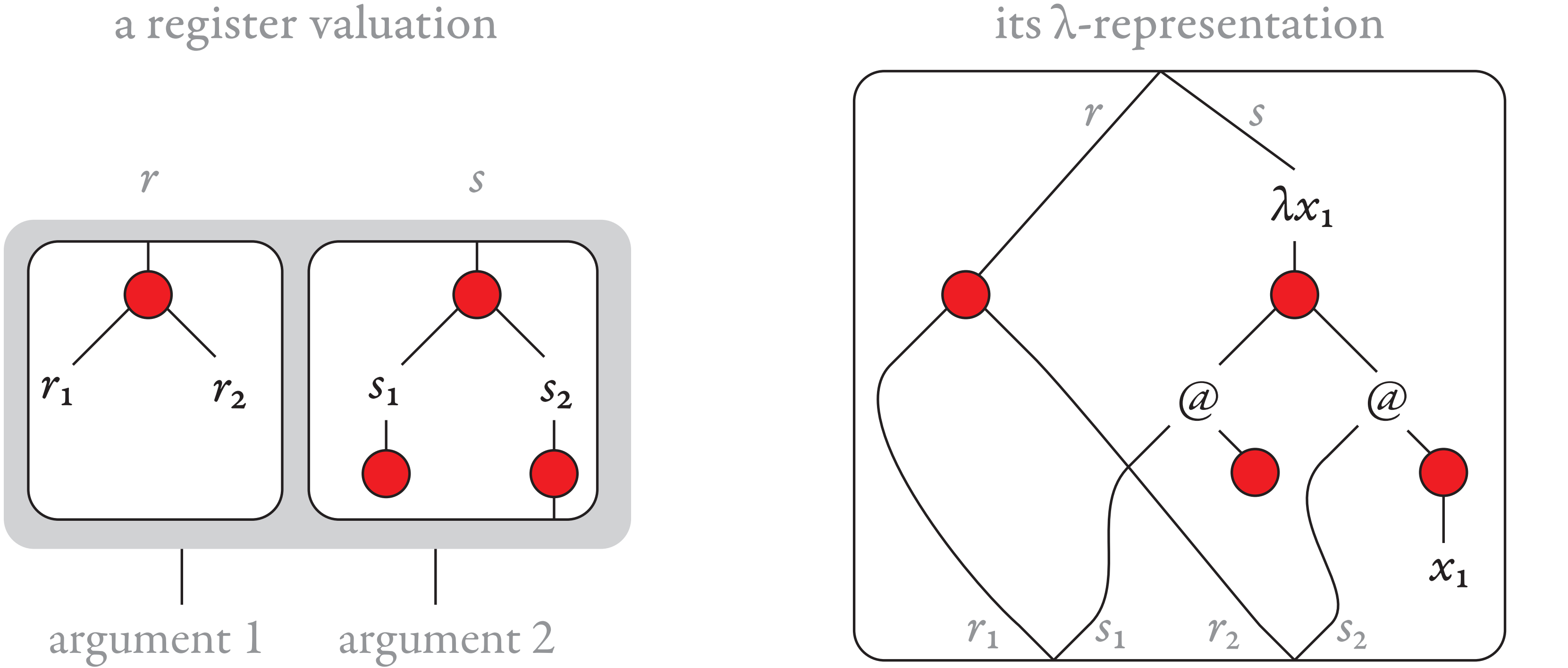

copy 1
of *a*
copy 2
of *a*
copy 3
of *a*



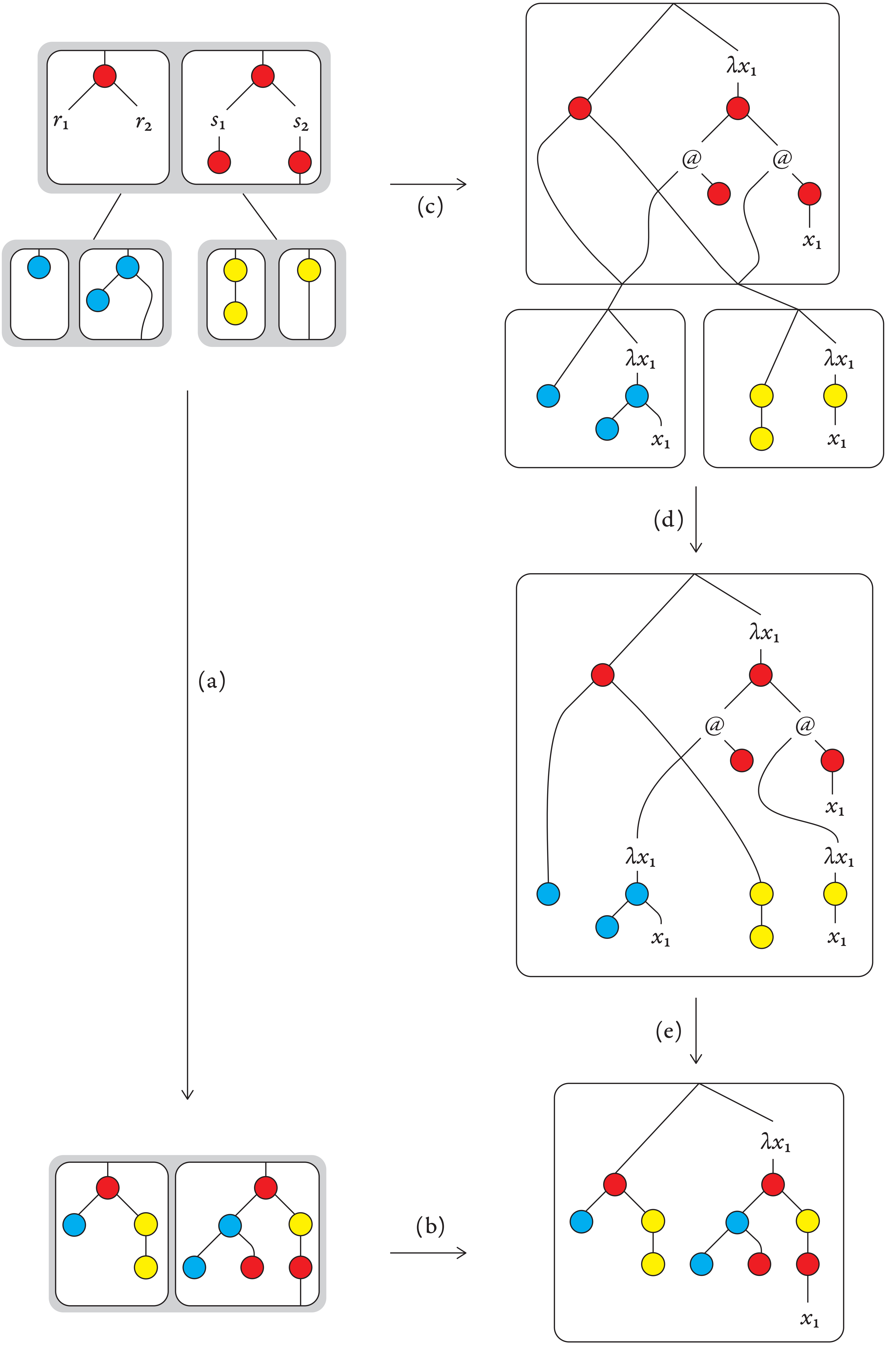
q1
q1
q1
q2
q3
q2
q3
q2
q3


q_1
q_2
q_3
q_1
q_1
q_3
q_2
q_2
q_3
each state appears exactly once as a first child
each state appears exactly once as a second child

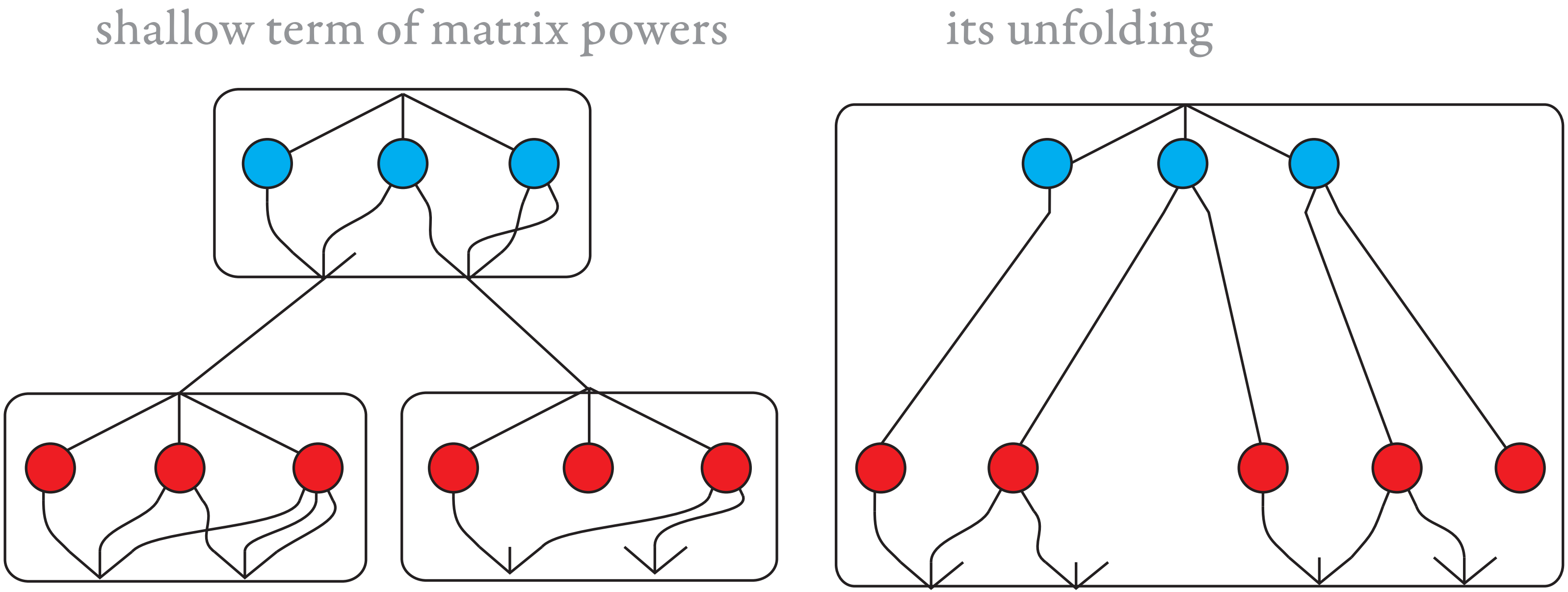

λx
@
x



a register valuation
r
s
r1
r2
s1
s2
argument 1
argument 2
its λ-representation
r
s
λx1
@
@
x1
r1
s1
r2
s2



r_1
r_2
s_1
s_2
(a)
(b)
(c)
(d)
(e)
λx_1
@
x_1



shallow term of matrix powers
its unfolding