Traffic flow modeling: a highway ramp model.

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1 Introduction

We consider a junction with one incoming mainline modeled by the real interval $]-\infty, 0]$ and an outgoing mainline modeled by the interval $[0, +\infty[$, one onramp and one offramp, see Figure 1.

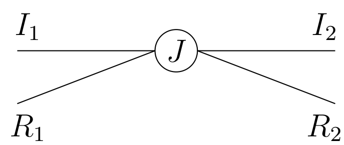

fig:junction

Figure 1: Junction taken into consideration.

From a macroscopic point of view this means that on each mainline road $I_1 =]-\infty, 0[$ and $I_2 =]0, +\infty[$ we consider the equation

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.1}$$ eq:LWR

where $\rho = \rho(t, x) \in [0, \rho_{max}]$ is the density of the cars on each mainline, $v \geq 0$ is the mean traffic speed in free flow, $w \leq 0$ is the mean traffic speed in congested flow, i.e. the slope of the line representing the congested flow in the fundamental diagram and

$f(\rho) = \begin{cases} v\rho & \text{if } \rho \leq \rho^{cr}, \\ w(\rho - \rho_{max}) & \text{otherwise,} \end{cases}$ is the flux.

We assume the following:

(A1) $\rho_{max} = 1$;

(A2) $f(0) = f(1) = 0$;

(A3) the flux is a strictly concave function.

Moreover we consider a triangular fundamental diagram, see Figure 2.

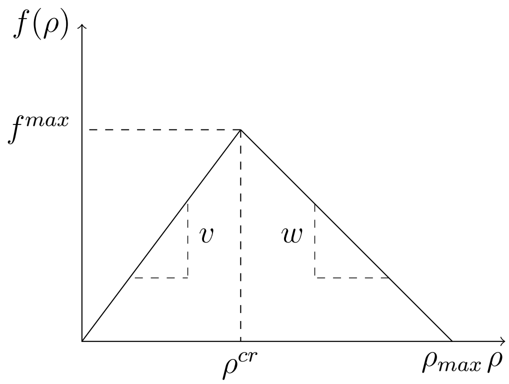

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Figure 2: Triangular fundamental diagram

On the onramp R_1 we consider the presence of a buffer modeled with the following ODE:

$$\frac{dl(t)}{dt} = F_{in}(t) - \gamma_{r1}(t), \quad t \in \mathbb{R}^+, \tag{1.2}$$ eq:ODEonramp

where $l(t) \in [0, +\infty[$ is the load on the buffer, $F_{in}(t)$ is the flux that enters the onramp and $\gamma_{r1}(t)$ is the flux that exits from the onramp.

This particular choice was made to avoid backward waves on the onramp boundary which happens in the case of horizontal queues.

The Cauchy problem to solve is then:

$$\begin{cases} \partial_t \rho_i + \partial_x f(\rho_i) = 0, & (t, x) \in \mathbb{R}^+ \times I_i, \\ \dfrac{dl(t)}{dt} = F_{in}(t) - \gamma_{r1}(t), & t \in \mathbb{R}^+, \\ \rho_i(0, x) = \rho_{i,0}(x), & \text{on } I_i \\ l(0) = l_0, \end{cases} \tag{1.3}$$ eq:CP

where $l_0 \in [0, +\infty[$ is the initial load of the buffer. We consider the offramp as an infinite sink that accept all the flux that comes to it from the mainline I_1.

This will be coupled with the following problem at the junction which will give the distribution of the traffic among the roads:

$$
\begin{aligned}
d(t) &= \begin{cases} \gamma_{r1}^{max} & \text{if } l(t) > 0, \\ \min\left(F_{in}(t), \gamma_{r1}^{max}\right) & \text{if } l(t) = 0, \end{cases} \\
\delta(t) &= \min\left(f^{max}, v\rho_1\right), \\
\sigma(t) &= \min\left(f^{max}, w(\rho_2 - \rho_{max})\right), \\
\gamma_{r2}(t) &= \beta f_1(\rho(t, 0-)), \\
f_2(t, 0+) &= \min\left((1-\beta)\delta(t) + d(t), \sigma(t)\right),
\end{aligned} \tag{1.4}
$$

where γ_{r1}^{max} is the maximal flow on the onramp, $\delta(t)$ is the demand function on the mainline, $d(t)$ is the demand of the onramp, f^{max} the maximal flux on I_1 and I_2, $\sigma(t)$ is the supply function on I_2 and $\beta \in [0,1]$ is the split ratio of the offramp.
Denote with $\rho_i : [0, +\infty[\times I_i \to [0,1]$ the density of the cars in the road I_i of the junction. We want ρ_i to be a weak solution on I_i, i.e. for all $\varphi \in \mathcal{C}_c^1(\mathbb{R}^+ \times I_i)$

$$
\int_{\mathbb{R}^+} \int_{\mathbb{R}} \Big(\rho_i \partial_t \varphi + f(\rho_i) \partial_x \varphi\Big) dx dt = 0 \tag{1.5}
$$

eq:weakSolRhoi

def:weakSolRho

Definition 1.1. *Let J be a junction with one incoming road I_1 and one outgoing road I_2. A weak solution at J is a couple of functions $\rho_i : [0, +\infty[\times I_i \to \mathbb{R}$, $i = 1, 2$, such that*

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^+} \int_{I_1} \Big(\rho_1 \partial_t \varphi_1 + f(\rho_1) \partial_x \varphi_1\Big) dx dt\Big) = 0, \\
&\Big(\int_{\mathbb{R}^+} \int_{I_2} \Big(\rho_2 \partial_t \varphi_2 + f(\rho_2) \partial_x \varphi_2\Big) dx dt\Big) = 0,
\end{aligned} \tag{1.6}
$$

for every $\varphi_i \in \mathcal{C}_c^1(\mathbb{R}^+ \times I_i)$.

Remark 1. *The fluxes $f_1(\rho(t, 0-))$, $f_2(\rho(t, 0+))$,$\gamma_{r1}(t)$ and $\gamma_{r2}(t)$ must satisfy the Rankine-Hugoniot condition:*

$$
f_1(\rho(t, 0-)) + \gamma_{r1}(t) = f_2(\rho(t, 0+)) + \gamma_{r2}(t). \tag{1.7}
$$

eq:conservationAt

def:weak_sol

Definition 1.2. *A collection of functions $(\rho_1, \rho_2, l) \in \prod_{i=1}^2 \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L}^1 \cap BV(\mathbb{R})\right) \times \mathbf{W}^{1,\infty}(\mathbb{R}^+; \mathbb{R}^+)$ is a solution at J to (1.3) if* item:i *(i)* ρ_1,ρ_2 *are weak solutions on* I_1, I_2, *i.e., see Definition 1.1;*

item:ii *(ii)* $f_1(\rho(t,0-)) = \dfrac{P}{1-P}\gamma_{r1}$ *with* $P \in]0,1[$ *the priority factor;*

item:iii *(iii)* $f_1(\rho(t,0-)) + \gamma_{r1}(t) = f_2(\rho(t,0+)) + \gamma_{r2}(t)$*;*

item:iv *(iv)* $f_2(\rho(t,0+))$ *is maximum subject to (ii),(iii) and* (1.4)*;*

item:v *(v)* l *is a solution of the ODE* (1.2) *for a.e.* $t \in \mathbb{R}^+$*.*

Remark 2. *The priority factor* P *is introduced to ensure the uniquess of the solution which was not guaranteed only with the maximization of the flow* $f_2(\rho(t,0+))$*.*

2 The Riemann problem at the junction

In this section we construct step by step the Riemann Solver ($\mathcal{RS}$) at the junction. Fix $\rho_{1,0}, \rho_{2,0} \in [0,1]$, $l_0 \in [0,+\infty[$ and a priority factor $P \in]0,1[$. The Riemann problem at J is the Cauchy problem (1.3) where the initial conditions are given by $\rho_{0,i}(x) \equiv \rho_{0,i}$ in I_i for every $i = 1,2$. We define the Riemann Solver by means of a Riemann Solver $\mathcal{RS}_{\bar{l}}$, which depends on the instantaneous load of the buffer $\bar{l}$. For each $\bar{l}$ the Rieman Solver $\mathcal{RS}_{\bar{l}}$ is constructed in the following way.

1. Define $\Gamma_1 = f_1(\rho_1(t,0-))$, $\Gamma_2 = f_2(\rho_2(t,0+))$, $\Gamma_3 = \gamma_{r1}(t)$;
2. Consider the space (Γ_1,Γ_3) and the sets $\mathcal{O}_1 = [0,\delta(t)]$, $\mathcal{O}_3 = [0,d(t)]$;
3. Trace the lines $(1-\beta)\Gamma_1 + \Gamma_3 = \Gamma_2$ and $\Gamma_1 = \frac{P}{1-P}\Gamma_3$;
4. Define Q to be the point of intersection of the two lines;
5. Consider the region

$$\Omega = \Big\{(\Gamma_1,\Gamma_3) \in \mathcal{O}_1 \times \mathcal{O}_3 : (1-\beta)\Gamma_1 + \Gamma_3 \in [0,\Gamma_2]\Big\}.$$

Two situations can occur at this point:

- Q belongs to Ω,

- Q is outside Ω.

In the first case we set $(\hat{\Gamma}_1,\hat{\Gamma}_3)=Q$, while in the second case we set $(\hat{\Gamma}_1,\hat{\Gamma}_3)=S$ where S is the point of the segment $\Omega \cap (\Gamma_1, \Gamma_3) : (1-\beta)\Gamma_1 + \Gamma_3 = \Gamma_2$ closest to the line $\Gamma_1 = \frac{P}{1-P}\Gamma_3$. We show in Figure 3 the different cases that can occur.

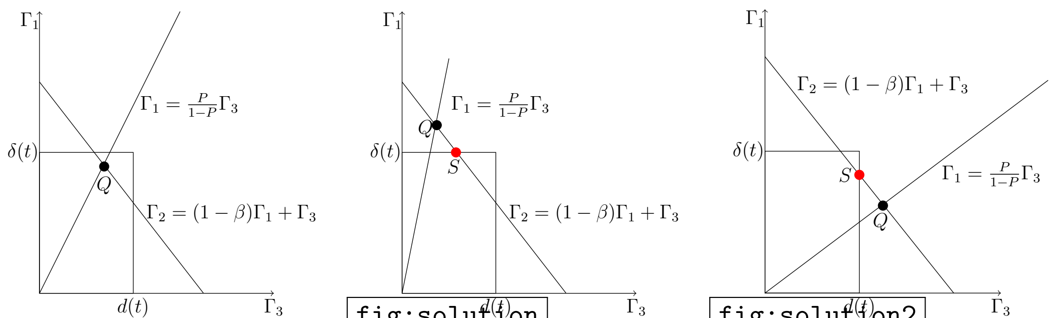

fig:solution fig:solution2 fig:solution3

(a) Intersection inside Ω (b) Intersection outside Ω (c) Intersection outside Ω

fig:junctionRS

Figure 3: Solutions Riemann Solver at the junction

Note that in the second case it is not possible to respect in an exact way the priority given by the parameter P if we want to maximize also the flux. So we solve in this case the minimization problem

$$\begin{aligned} \text{minimize} \quad & \left\| \begin{pmatrix} \gamma_{r1}(t) \\ f_1(\rho_1(t,0-)) \end{pmatrix} - \begin{pmatrix} \gamma_{r1}(t) \\ f_1(\rho_1(t,0-)) \end{pmatrix} \cdot \alpha^P \alpha^P \right\|_2^2 \\ \text{subject to} \quad & f_2(\rho_2(t,0+)) = (1-\beta) f_1(\rho_1(t,0-)) + \gamma_{r1}(t), \\ & \gamma_{r1}(t) \leq d(t), \\ & f_1(\rho_1(t,0+)) \leq \delta(t), \end{aligned} \tag{2.1}$$

eq:minimizationP

where α^P is the normalized vector $\alpha^P = \frac{1}{\sqrt{P^2 + (1-P)^2}} \begin{pmatrix} P \\ 1-P \end{pmatrix}$ with P the priority factor.
Once we have determined $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ we can determine in an unique way $\hat{\rho}_1$, $\hat{\rho}_2$ as follows.
We recall that $\rho = \rho^{cr} \in]0,1[$ is the unique point of maximum of the flux and we define the function τ as follows.

Definition 2.1. *Let $\tau : [0,1] \to [0,1]$ be the map such that:*

- $f(\tau(\rho)) = f(\rho)$ *for every* $\rho \in [0,1]$*;*
- $\tau(\rho) \neq \rho$ *for every* $\rho \in [0,1] \setminus \{\rho^{cr}\}$

For the function τ in this context is valid [1, Proposition 4.3.2.].
Given

$$\rho_1(0,\cdot) \equiv \rho_{1,0}\ ,\ \rho_2(0,\cdot) \equiv \rho_{2,0}$$

there exists a unique couple $(\hat{\rho}_1, \hat{\rho}_2) \in [0,1]^2$ such that

$$\hat{\rho}_1 \in \begin{cases} \{\rho_{1,0}\} \cup]\tau(\rho_{1,0}), 1] & \text{if } 0 \leq \rho_{1,0} \leq \rho^{cr}, \\ [\rho^{cr}, 1] & \text{if } \rho^{cr} \leq \rho_{1,0} \leq 1; \end{cases} \tag{2.2}$$

eq:rho_1

and

$$\hat{\rho}_2 \in \begin{cases} [0, \rho^{cr}] & \text{if } 0 \leq \rho_{2,0} \leq \rho^{cr}, \\ \{\rho_{2,0}\} \cup [0, \tau(\rho_{2,0})[& \text{if } \rho^{cr} \leq \rho_{1,0} \leq 1; \end{cases} \tag{2.3}$$

eq:rho_2

and for the incoming road the solution is given by the wave $(\rho_{1,0}, \hat{\rho}_1)$, while for the outgoing road the solution is given by the wave $(\hat{\rho}_2, \rho_{2,0})$. In this setting, given the initial data $\rho_{1,0}$, $\rho_{2,0}$ we can define $\mathcal{RS}_{\bar{l}} : [0,1]^2 \to [0,1]^2$ by

$$\mathcal{RS}_{\bar{l}}(\rho_{1,0}, \rho_{2,0}) = (\hat{\rho}_1, \hat{\rho}_2, \hat{\Gamma}_3). \tag{2.4}$$

eq:Rieml

Now given the initial load of the buffer $l_0 = \bar{l}$, the function $l(t)$ at time $t > 0$ is given by

$$l(t) = \begin{cases} l_0 + (F_{in} - \hat{\Gamma}_3)t & \text{if } 0 < t < -\frac{l_0}{F_{in} - \hat{\Gamma}_3}, \\ 0 & \text{if } t > -\frac{l_0}{F_{in} - \hat{\Gamma}_3}, \end{cases} \tag{2.5}$$

eq:solODE

Remark 3. *The presence of the buffer can create waves at time* $\bar{t} > 0$*. In particular, we have a new wave if* $F_{in} < \hat{\Gamma}_3$ *and the buffer empties, see Figure 4.*
No waves are created instead in the other case due to the infinity capacity of the buffer.

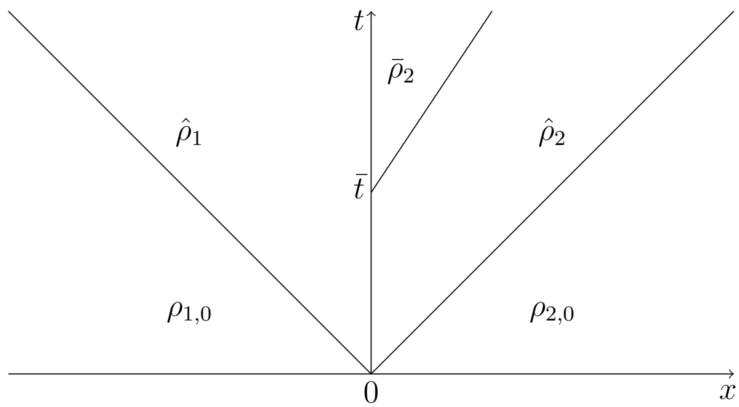

fig:RiemProb

Figure 4: Solution of the Riemann Problem

The following theorem holds.

th:existenceSol

Theorem 1. *Consider a junction J and fix a priority parameter $P \in]0,1[$. For every $\rho_{1,0}$,$\rho_{2,0} \in [0,1]$ and $l_0 \in [0,+\infty]$, there exists a unique admissible solution $(\rho_1(t,x), \rho_2(t,x), l(t))$ in the sense of Definition 1.2 such that for a.e. $t > 0$ it holds*

$$(\rho_1(t,0-), \rho_2(t,0+)) = \mathcal{RS}_{l(t)}(\rho_1(t,0-), \rho_2(t,0+))$$

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Proof. Self-similarity $\square$

References

avello2006traffic

[1] M. Garavello and B. Piccoli. *Traffic Flow on Networks: Conservation Laws Model*. AIMS Series on Applied Mathematics. American Institute of Mathematical Sciences, 2006.