SO-DTA with partial compliance

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Contents

1 Introduction

2 Junction model

2.1 Notation

Constants

$\triangle t, \triangle x$	Time, space discretization size
v_i	Free flow speed on cell i
w_i	Congestion wave speed on cell i
ρ_i^{jam}	Jam density on cell i
F_i	Max flow leaving mainline of cell i
ρ_i	Priority factor for cell i

Inputs

ρ_i^0	Initial density on cell i
$D_i(k)$	Input flow on cell i, time step k
$\beta_{ij,k}(c)$	Split ratio for commodity c on cell i to cell j, time step k

Variables

$f_i^{\text{in}}(k)$	Total flow into mainline on cell i, time step k
$f_i^{\text{out}}(k)$	Total flow out of mainline on cell i, time step k
$f_{i,c}^{\text{in}}(k)$	Flow of commodity c into mainline on cell i, time step k
$f_{i,c}^{\text{out}}(k)$	Flow of commodity c out of mainline on cell i, time step k
$\rho_i(k)$	Density on cell i, time step k

$\rho_{i,c}(k)$	Density contribution of commodity c on cell i, time step k
$\sigma_i(k)$	Supply on cell i, time step k
$\delta_i(k)$	Demand on cell i, time step k
$\delta_i(k)$	Demand on cell i, time step k

2.2 Some remarks

The multi-commodity junction problem is essentially equivalent to the source destination model (SDM) in Piccoli [**?**]. The SDM contains a commodity per source destination pair in the network. Similarly, in our problem there are two commodities for each source destination pair. One corresponding to compliant flow and another corresponding to non-compliant flow. As in the single commodity model and the SDM, a priority vector is needed when the number of incoming links at the junction is greater than the number of outgoing links.

The junction solution should satisfy the following properties:

1. The solution must satisfy the FIFO principle. That is to say, for any incoming link i, the distribution of its flow out across the different commodities must be in proportion the the ratio of vehicles of each commodity at the link.

$$f_{i,c_i}^{\text{out}}(k) = f_i^{\text{out}}(k)\frac{\rho_{i,c_i}(k)}{\sum_{j=1}^{C}\rho_{i,c_j}(k)}$$

2. Flow maximization across the junction.

3. Satisfy soft inflow priority constraints when the number of incoming links is greater than the number of outgoing links. The priorities are satisfied exactly when the intersection of the maximum flow isoline and the priority constraint are feasible. When this point is outside the feasible set, the flow maximizing feasible point that is closest to the priority constraint (in euclidean distance) is chosen.

2.3 Diverge solver

We consider a diverging junction with one incoming link i and m outgoing links. There are C commodities that flow through the network each with their own time-varying split ratio $\beta_{ij,c}(k)$.

- The first step is to compute the aggregate split ratio from i to each outgoing link j:

$$\begin{aligned}\beta_{ij}(k) &= \sum_{c=1}^{C}\frac{\rho_{i,c}(k)}{\rho_i(k)}\beta_{ij,c}(k)\\ &= \frac{1}{\rho_i(k)}\sum_{c=1}^{C}\rho_{i,c}(k)\,\beta_{ij,c}(k)\end{aligned}$$

  TODO: Explain why this is correct.

- Once the aggregate split ratio is found we can solve for the total junction flow as follows:

$$\begin{aligned}&\max f_i^{\text{out}}(k)\\ &\text{subject to}\\ &\quad \beta_{ij}(k)\, f_i^{\text{out}}(k) \le \sigma_j(k) \qquad \forall j : (i,j)\in A\\ &\quad f_i^{\text{out}}(k) \le \delta_i(k)\end{aligned}$$

Or equivalently by solving the feasibility problem:

$$
\begin{aligned}
&\max 1 \\
&\text{subject to} \\
&\quad f_i^{\text{out}}(k) = \min\left(\frac{\sigma_j(k)}{\beta_{ij}(k)} \quad \forall j : (i,j) \in A, \delta_i(k)\right) \\
&\quad \beta_{ij}(k) f_i^{\text{out}}(k) \leq \sigma_j(k) \qquad \forall j : (i,j) \in A \\
&\quad f_i^{\text{out}}(k) \leq \delta_i(k)
\end{aligned}
$$

- The total outflow $f_i^{\text{out}}(k)$ for each incoming link i is then divided among the commodities according to the FIFO law.

$$
f_{i,c}^{\text{out}}(k) = \frac{\rho_{i,c}(k)}{\rho_i(k)} f_i^{\text{out}}(k)
$$

The commodity flows are split among the outgoing links according to the split ratios.

$$
f_{j,c}^{\text{in}}(k) = \sum_{i:(i,j)\in A} \beta_{ij,c}(k) f_{i,c}^{\text{out}}(k)
$$

2.3.1 Existance and uniqueness of solution

A non-zero solution exists if the none of the constraints of the optimization/feasibility problem impose a zero flow. In other words, as long as the demand is non-zero and none of the outgoing links with positive demand ($\beta_{ij}(k) > 0$) have non-zero supply, the non-zero solution will exist. Since, the solution the maximum of a scalar value it is necessarily unique.

2.4 Merge solver

We consider a merging junction with n incoming links and one outgoing link j exiting it. There are C commodities on each link. A priority vector P (s.t. $\sum p_i = 1$) prescribes the priorities at which the outgoing link accepts flows from the n incoming links when the junction is supply constrained.

- If the problem is demand constrained (i.e. $\sum_{i:(i,j)\in A} \delta_i(k) \leq \sigma_j(k)$), then the solution is given by:

$$
f_i^{\text{out}}(k) = \delta_i(k) \quad \forall i : (i,j) \in A
$$

- If the problem is supply constrained, then the solution to the junction problem is given by solving the following quadratic optimization problem that finds the flow maximizing solution with the smallest violation of the priority vector, where the violation is measured using the $L2$ distance:

$$
\begin{aligned}
&\min_{t, f_i^{\text{out}}(k)\ \forall i:(i,j)\in A} \sum_{i:(i,j)\in A} \left(f_i^{\text{out}}(k) - t \cdot p_i\right)^2 \\
&\text{subject to} \\
&\quad \sum_{i:(i,j)\in A} f_i^{\text{out}}(k) = \sigma_i(k) \\
&\quad f_i^{\text{out}}(k) \leq \delta_i(k)
\end{aligned}
$$

2.4.1 Existence and uniqueness of solution

- *Demand constrained case*: In the demand constrained case, existence and uniqueness is trivial.
- *Supply constrained case*: In the supply constrained case, we are minimizing the euclidean distance from the solution (a point) to the priority vector (a line) subject to the solution being in the feasible set that is defined by the intersection of a $n-1$ dimensional surface with a n dimensional hypercube.
  - A solution exists when the feasible set is non-empty and the feasible set will always be non-empty if the supply constraint is greater than zero. This proves the existence of a solution in all non-degenerate (zero supply) cases.
  - The intersection of a $n-1$ dimensional surface with a n dimensional hypercube results in a convex $n-1$ dimensional surface. We also know that this surface is at a fixed distance $\sigma_i(k)$ from the origin. The priority line P extends from the origin. Therefore, the priority line and the feasible set can not be parallel to each other in any of the dimensions. The point on a convex set with the minimum euclidean distance to a line that is not parallel to the set is unique. This concludes the proof.

2.5 Merge and diverge solver with $n \leq m$ (Piccoli Model)

Remark 1. *The Piccoli model maximizes flow, but has the limitation that some incoming links might have zero flux when the junction is supply limited and the other incoming links allow for more flux through the junction due to their split ratios.*

Let J^{in} and J^{out} be the sets of incoming and outgoing links at the junction.

- The first step is to compute the aggregate split ratio from each incoming link i to each outgoing link j:

$$\begin{aligned}\beta_{ij}(k) &= \sum_{c=1}^{C} \frac{\rho_{i,c}(k)}{\rho_i(k)} \beta_{ij,c}(k) \\ &= \frac{1}{\rho_i(k)} \sum_{c=1}^{C} \rho_{i,c}(k)\, \beta_{ij,c}(k)\end{aligned}$$

- Once the aggregate split ratio is found we can solve for the total junction flow as follows:

$$\begin{aligned}&\max \sum_{i \in J^{in}} f_i^{\text{out}}(k) \\ &\text{subject to} \\ &\qquad \sum_{i \in J^{in}} \beta_{ij}(k)\, f_i^{\text{out}}(k) \leq \sigma_j(k) \qquad \forall j \in J^{out} \\ &\qquad f_i^{\text{out}}(k) \leq \delta_i(k) \qquad \forall i \in J^{in}\end{aligned}$$

- The total outflow $f_i^{\text{out}}(k)$ for each incoming link i is then divided among the commodities according to the FIFO law.

$$f_{i,c}^{\text{out}}(k) = \frac{\rho_{i,c}(k)}{\rho_i(k)} f_i^{\text{out}}(k)$$

  The commodity flows are split among the outgoing links according to the split ratios.

$$f_{j,c}^{\text{in}}(k) = \sum_{i:(i,j) \in A} \beta_{ij,c}(k)\, f_{i,c}^{\text{out}}(k)$$

2.5.1 Existence and uniqueness of solution

A non-zero solution exists if the none of the constraints of the optimization/feasibility problem impose a zero flow. In other words, as long as the demand is non-zero and none of the outgoing links with positive demand ($\beta_{ij}(k) > 0$) have non-zero supply, the non-zero solution will exist.

2.6 Merge and diverge solver with $n > m$

2.6.1 Existence and uniqueness of solution