Adjoint-based Control of Traffic Systems Application to Ramp Metering

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Outline

1. Introduction to the adjoint method
    - Optimization of a PDE-constrained system
    - Example: linear system
    - Solving the original problem
2. Discretized system dynamics
    - Forward System
    - Lower triangular forward system
    - Exploiting system structure
3. Solving the optimization problem via the adjoint method
    - Overview of procedure
    - A few details
4. Numercial results

Outline

Optimization of a PDE-constrained system

Optimization problem

$$\begin{aligned} \text{minimize}_{u \in \mathcal{U}} \quad & J(x,u) \\ \text{subject to} \quad & H(x,u) = 0 \end{aligned}$$

- $x \in \mathcal{X} \subseteq \mathbb{R}^n$: state variables
- $u \in \mathcal{U} \subseteq \mathbb{R}^m$: control variables

$$\begin{aligned} J : \mathcal{X} \times \mathcal{U} &\to \mathbb{R} \\ (x,u) &\mapsto J(x,u) \end{aligned}$$

$$\begin{aligned} H : \mathcal{X} \times \mathcal{U} &\to \mathbb{R}^{n_H} \\ (x,u) &\mapsto H(x,u) \end{aligned}$$

Want to do gradient descent. How to compute the gradient?

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Example: linear system

Discrete linear dynamics

$$x_{t+1} = Ax_t + Bu_t, \; t \in \{0, \ldots, T-1\}$$

with initial condition x_0.

Let

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_T \end{bmatrix} \qquad u = \begin{bmatrix} u_0 \\ \vdots \\ u_{T-1} \end{bmatrix}$$

Example: linear system

$$x = \begin{bmatrix} Ax_0 + Bu_0 \\ Ax_1 + Bu_1 \\ \vdots \\ Ax_{T-1} + Bu_{T-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & \\ A & \ddots & & \\ & \ddots & \ddots & \\ & & A & 0 \end{bmatrix} x + \begin{bmatrix} B & & & \\ & \ddots & & \\ & & \ddots & \\ & & & B \end{bmatrix} u + \begin{bmatrix} Ax_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Can be written as

$$(\tilde{A} - I)x + \tilde{B}u + c = 0$$

Note: $(\tilde{A} - I)$ is invertible (lower triangular, with -1 on diagonal). Good: system is deterministic!

Example: linear system

Linear system

$$H_x x + H_u u + c = 0$$

- $x \in \mathbb{R}^n$ state
- $u \in \mathbb{R}^m$ control, with $m \leq n$
- $H_x \in \mathbb{R}^{n\times n}$, assume invertible
- $H_u \in \mathbb{R}^{n\times m}$
- $c \in \mathbb{R}^n$

want to minimize linear cost function

$$\begin{aligned} \text{minimize}_{u\in\mathcal{U}} \quad & J_x x + J_u u \\ \text{subject to} \quad & H_x x + H_u u + c = 0 \end{aligned}$$

$J_x \in \mathbb{R}^{1\times n}$ and $J_u \in \mathbb{R}^{1\times m}$ are given row vectors.

Example: linear system

Optimization problem

$$\begin{aligned} \text{minimize}_{u \in \mathcal{U}} \quad & J_x x + J_u u \\ \text{subject to} \quad & H_x x + H_u u + c = 0 \end{aligned}$$

An equivalent problem is

$$\text{minimize}_{u \in \mathcal{U}} - J_x H_x^{-1} (H_u u + c) + J_u u$$

and the gradient is

Gradient

$$\nabla_u J = {\color{red} -J_x H_x^{-1} H_u} + J_u$$

Example: linear system

Gradient

$$\nabla_u J = -J_x H_x^{-1} H_u + J_u$$

Two ways to compute the first term

Forward

$$J_x M$$
$$H_x M = -H_u$$

Solve for $M \in \mathbb{R}^{n \times m}$: m inversions

$$H_x \left[\begin{array}{c|c|c} M_1 & \dots & M_m \end{array}\right] = \left[\begin{array}{c|c|c} H_{u_1} & \dots & H_{u_m} \end{array}\right]$$

Cost $O(mn^2)$.
Then product $1 \times n$ times $n \times m$: $O(nm)$

Adjoint

$$\lambda^T H_u$$
$$\lambda^T H_x = -J_x$$

Solve for $\lambda \in \mathbb{R}^n$: 1 inversion

$$H_x^T \lambda = J_x^T$$

Cost $O(n^2)$.
Then product $1 \times n$ times $n \times m$: $O(nm)$

Outline

Optimization of a PDE-constrained system

General problem

$$\begin{array}{ll} \text{minimize}_{u \in \mathcal{U}} & J(x,u) \\ \text{subject to} & H(x,u) = 0 \end{array}$$

$$\nabla_u J = \frac{\partial J}{\partial x} \nabla_u x + \frac{\partial J}{\partial u}$$

On trajectories, $H(x,u) = 0$ constant, thus $\nabla_u H = 0$

$$\frac{\partial H}{\partial x} \nabla_u x + \frac{\partial H}{\partial u} = 0$$

Linear system

$$\begin{array}{ll} \text{minimize}_{u \in \mathcal{U}} & J_x x + J_u u \\ \text{subject to} & H_x x + H_u u + c = 0 \end{array}$$

$$\nabla_u J = J_x M + J_u$$

$$H_x M = -H_u$$

Optimization of a PDE-constrained system

General problem

$$\begin{array}{ll} \text{minimize}_{u \in \mathcal{U}} & J(x,u) \\ \text{subject to} & H(x,u) = 0 \end{array}$$

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Linear system

$$\begin{array}{ll} \text{minimize}_{u \in \mathcal{U}} & J_x x + J_u u \\ \text{subject to} & H_x x + H_u u + c = 0 \end{array}$$

$$\nabla_u J = J_x M + J_u$$

$$H_x M = -H_u$$

Instead, solve for $\lambda \in \mathbb{R}^n$

Adjoint

$$H_x^T \lambda = J_x^T$$

then

$$\nabla_u J = \lambda^T H_u + J_u$$

Optimization of a PDE-constrained system

General problem

$$\begin{array}{ll} \text{minimize}_{u\in\mathcal{U}} & J(x,u) \\ \text{subject to} & H(x,u)=0 \end{array}$$

$$\nabla_u J = \frac{\partial J}{\partial x}\nabla_u x + \frac{\partial J}{\partial u}$$

On trajectories, $H(x,u) = 0$ constant, thus $\nabla_u H = 0$

$$\frac{\partial H}{\partial x}\nabla_u x + \frac{\partial H}{\partial u} = 0$$

Adjoint

$$\frac{\partial H}{\partial x}^T \lambda = \frac{\partial J}{\partial x}$$

then

$$\nabla_u J = \lambda^T \frac{\partial H}{\partial u} + \frac{\partial J}{\partial u}$$

Linear system

$$\begin{array}{ll} \text{minimize}_{u\in\mathcal{U}} & J_x x + J_u u \\ \text{subject to} & H_x x + H_u u + c = 0 \end{array}$$

$$\nabla_u J = J_x M + J_u$$

$$H_x M = -H_u$$

Instead, solve for $\lambda \in \mathbb{R}^n$

Adjoint

$$H_x^T \lambda = J_x^T$$

then

$$\nabla_u J = \lambda^T H_u + J_u$$

Computing $\nabla_u J(x, u)$

Want to evaluate

$$\frac{\partial J}{\partial x} \nabla_u x$$

$$\text{where } \frac{\partial H}{\partial x} \nabla_u x + \frac{\partial H}{\partial u} = 0$$

Computing $\nabla_u J(x, u)$

Want to evaluate

$$\frac{\partial J}{\partial x} \nabla_u x$$

$$\text{where } \frac{\partial H}{\partial x} \nabla_u x + \frac{\partial H}{\partial u} = 0$$

If λ is solution to the adjoint equation

$$\frac{\partial J}{\partial x} + \lambda^T \frac{\partial H}{\partial x} = 0$$

Computing $\nabla_u J(x, u)$

Want to evaluate

$$\frac{\partial J}{\partial x} \nabla_u x$$
$$\text{where } \frac{\partial H}{\partial x} \nabla_u x + \frac{\partial H}{\partial u} = 0$$

If λ is solution to the adjoint equation

$$\frac{\partial J}{\partial x} + \lambda^T \frac{\partial H}{\partial x} = 0$$

Then

$$\frac{\partial J}{\partial x} \nabla_u x = -\lambda^T \frac{\partial H}{\partial x} \nabla_u x = \lambda^T \frac{\partial H}{\partial u}$$

Adjoint solution λ

$$\nabla_u J = \lambda^T \frac{\partial H}{\partial u} + \frac{\partial J}{\partial u}$$

Also useful for sensitivity analysis.

Sensitivity analysis

λ_k is the price of changing H_k

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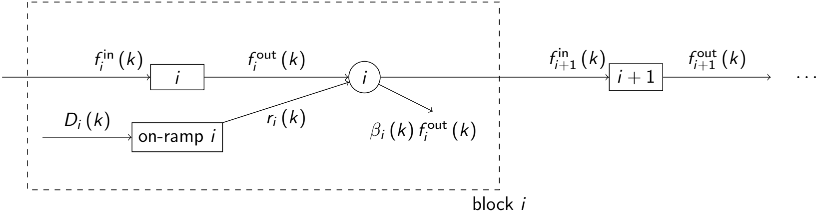
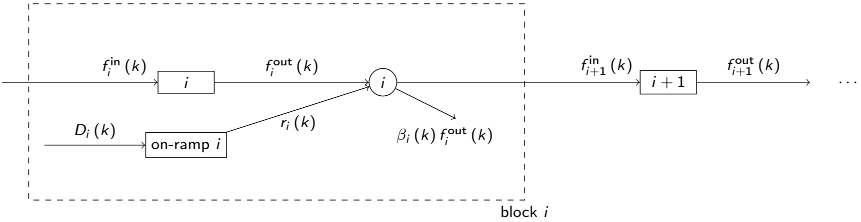
Forward system



- Dynamics based on the discretized LWR PDE
- Piecewise affine system
- Junction solver based on the modified Piccoli model presented a few weeks ago

Mass conservation



Density evolution

$$\rho_i(k) = \rho_i(k-1) + \frac{\triangle t}{\triangle x}\left(f_i^{\text{in}}(k-1) - f_i^{\text{out}}(k-1)\right) \quad \forall i \in \{1,\dots,N-1\}, k \in \{1,\dots,T\} \tag{H1a}$$

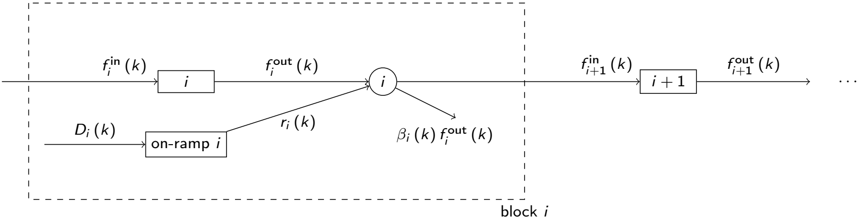
$$\rho_0(k) = \rho_0(k-1) + \frac{\triangle t}{\triangle x}\left(D_0(k-1) - f_0^{\text{out}}(k-1)\right) \quad \forall k \in \{1,\dots,T\} \tag{H1b}$$

$$\rho_N(k) = \rho_N(k-1) + \frac{\triangle t}{\triangle x}\left(f_N^{\text{in}}(k-1) - \delta_N(k-1)\right) \quad \forall k \in \{1,\dots,T\} \tag{H1c}$$

and initial condition

$$\rho_i(0) = \rho_i^0 \quad \forall i \in \{0,\dots,N\} \tag{I1}$$

Ramp buffer



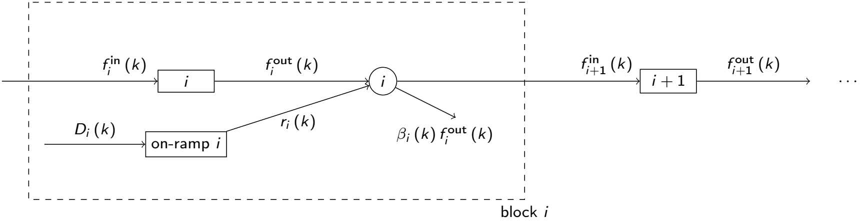
Queue evolution

$$l_i(k) = l_i(k-1) + \triangle t\,(D_i(k-1) - r_i(k-1)) \quad \forall i \in \{1, \dots, N-1\}, k \in \{1, \dots, T\} \tag{H2}$$

and initial condition

$$l_i(0) = l_i^0 \qquad \forall i \in \{1, \dots, N-1\} \tag{I2}$$

Junction supply and demand

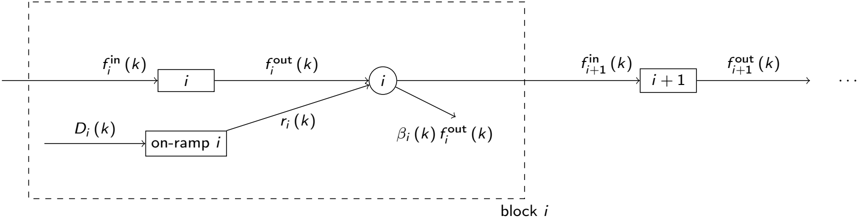


$$\delta_i(k) = \min\left(F_i, v_i\rho_i(k)\right) \qquad \forall i \in \{0,\ldots,N\}, k \in \{0,\ldots,T-1\} \tag{H3}$$

$$\sigma_i(k) = \min\left(F_i, w_i\left(\rho_i^{\text{jam}} - \rho_i(k)\right)\right) \qquad \forall i \in \{1,\ldots,N\}, k \in \{0,\ldots,T-1\} \tag{H4}$$

$$d_i(k) = \min\left(l_i(k), u_i(k)\right) \qquad \forall i \in \{1,\ldots,N-1\}, k \in \{0,\ldots,T-1\} \tag{H5}$$

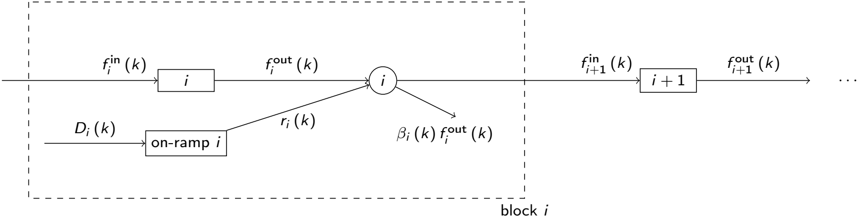
Junction outflow

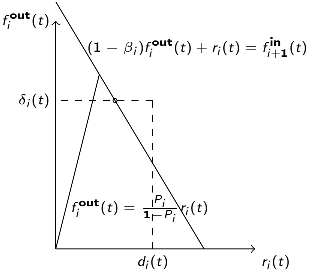


$$f_i^{\text{in}}(k) = \min\left(\delta_{i-1}(k)\left(1-\beta_{i-1}(k)\right) + d_{i-1}(k), \sigma_i(k)\right) \quad \forall i \in \{2, \dots, N\}, k \in \{0, \dots, T-1\} \tag{H6a}$$

$$f_1^{\text{in}}(k) = \min\left(\delta_0(k), \sigma_1(k)\right) \quad \forall k \in \{0, \dots, T-1\} \tag{H6b}$$

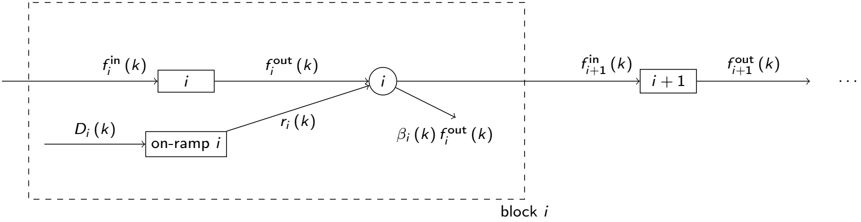
Junction inflow

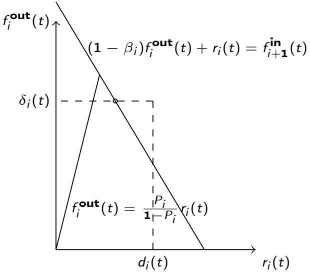




$$f_{i+1}^{\text{in}}(k) = \underbrace{r_i(k)}_{=(1-P)f_{i+1}^{\text{in}}(k)} + \underbrace{f_i^{\text{out}}(k)\,(1-\beta_i(k))}_{=Pf_{i+1}^{\text{in}}(k)}$$

Junction inflow

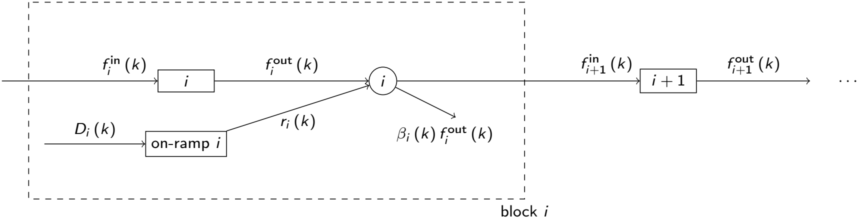


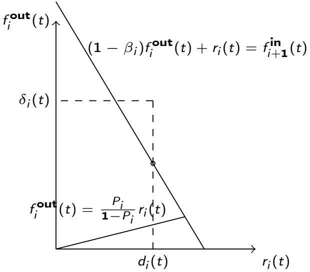


$$f_{i+1}^{in}(k) = \underbrace{r_i(k)}_{=(1-P)f_{i+1}^{\mathbf{in}}(k)} + \underbrace{f_i^{out}(k)(1-\beta_i(k))}_{=Pf_{i+1}^{\mathbf{in}}(k)}$$

$$f_i^{out}(k) = \delta_i(k)$$
$$\text{if } P_i f_{i+1}^{in}(k) > (1-\beta_i(k))\,\delta_i(k)$$

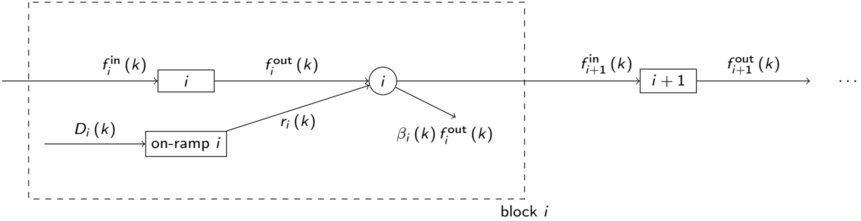
Junction inflow

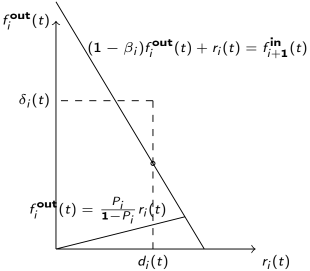




$$f_{i+1}^{in}(k) = \underbrace{r_i(k)}_{=(1-P)f_{i+\mathbf{1}}^{\mathbf{in}}(k)} + \underbrace{f_i^{out}(k)\left(1-\beta_i(k)\right)}_{=Pf_{i+\mathbf{1}}^{\mathbf{in}}(k)}$$

Junction inflow



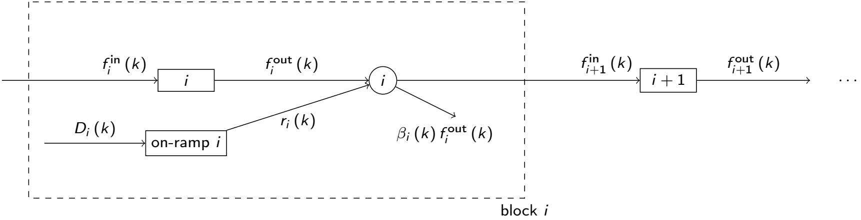


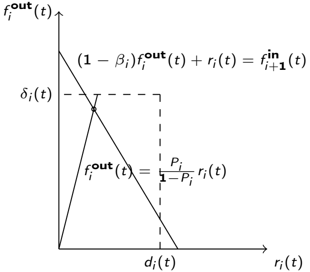
$$f_{i+1}^{\text{in}}(k) = \underbrace{r_i(k)}_{=(\mathbf{1}-P)f_{i+\mathbf{1}}^{\mathbf{in}}(k)} + \underbrace{f_i^{\text{out}}(k)\left(1-\beta_i(k)\right)}_{=Pf_{i+\mathbf{1}}^{\mathbf{in}}(k)}$$

$$f_i^{\text{out}}(k) = \frac{f_{i+1}^{\text{in}}(k) - d_i(k)}{1-\beta_i(k)}$$

$$\text{if } \left(1-P_i\right) f_{i+1}^{\text{in}}(k) > d_i(k)$$

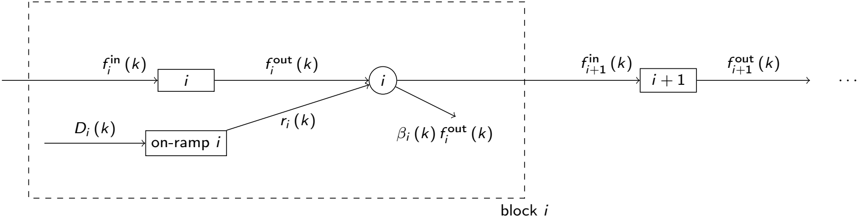
Junction inflow

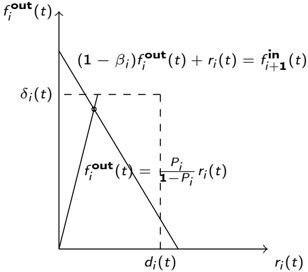




$$f_{i+1}^{in}(k) = \underbrace{r_i(k)}_{=(1-P)f_{i+1}^{in}(k)} + \underbrace{f_i^{out}(k)\,(1-\beta_i(k))}_{=Pf_{i+1}^{in}(k)}$$

Junction inflow

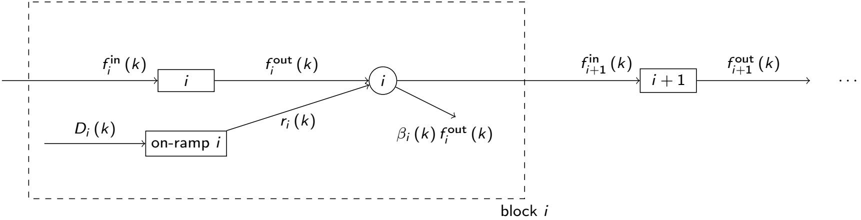




$$f_{i+1}^{in}(k) = \underbrace{r_i(k)}_{=(1-P)f_{i+\mathbf{1}}^{\mathbf{in}}(k)} + \underbrace{f_i^{out}(k)\,(1-\beta_i(k))}_{=Pf_{i+\mathbf{1}}^{\mathbf{in}}(k)}$$

$$f_i^{out}(k) = \frac{P_i f_{i+1}^{in}(k)}{1-\beta_i(k)} \quad \text{otherwise}$$

Junction inflow

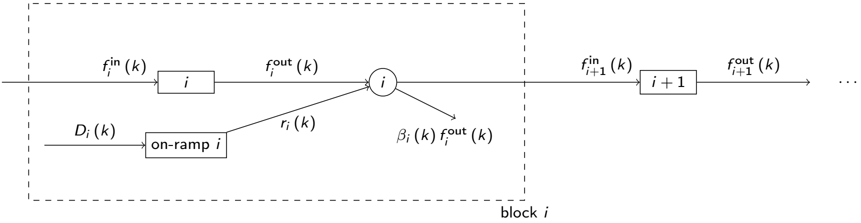


$$f_i^{\text{out}}(k) = \begin{cases} \delta_i(k) & \text{if } (R1_{k,i}) : P_i f_{i+1}^{\text{in}}(k) > (1-\beta_i(k))\,\delta_i(k) \\ \frac{f_{i+1}^{\text{in}}(k) - d_i(k)}{1-\beta_i(k)} & \text{if } (R2_{k,i}) : (1-P_i)\, f_{i+1}^{\text{in}}(k) > d_i(k) \\ \frac{P_i f_{i+1}^{\text{in}}(k)}{1-\beta_i(k)} & \text{otherwise } (R3_{k,i}) \end{cases}$$

$$\forall i \in \{1,\ldots,N-1\},\, k \in \{0,\ldots,T-1\} \quad \text{(H7a)}$$

$$f_0^{\text{out}}(k) = f_1^{\text{in}}(k) \qquad \forall k \in \{0,\ldots,T-1\} \qquad \text{(H7b)}$$

Ramp flow



$$r_i(k) = f_{i+1}^{\text{in}}(k) - f_i^{\text{out}}(k)\,(1-\beta_i(k)) \quad \forall i \in \{1,\dots,N-1\}\,, k \in \{0,\dots,T-1\} \qquad \text{(H8)}$$

Outline

Lower triangular forward system

The forward system H has C constraints that need to be satisfied over T time steps for N cells.

1. The causality of the system implies that the state at time t only depends on the state at times $t' \leq t$, so we first iterate over time.
2. At each time step, the nature of our system also allows a topological ordering of the variables (no loops in the dependency graph!)
3. Finally we iterate over the cells, as there is no equation that couples the same variable at a given time step for two seperate cells.

Lower triangular forward system

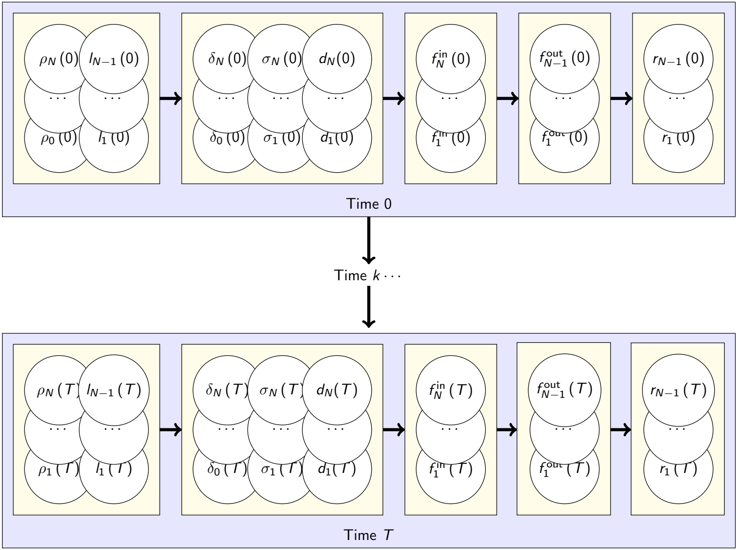


Figure: Dependency diagram of the variables in the system.

Lower triangular forward system

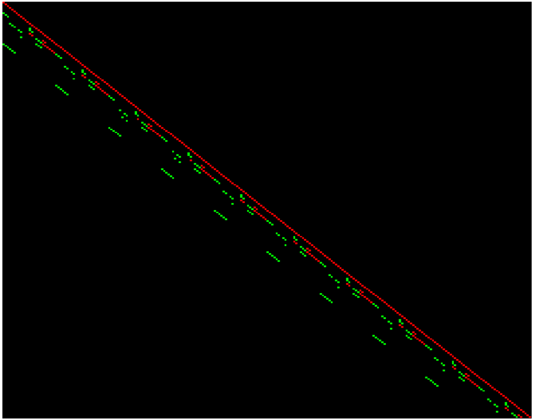
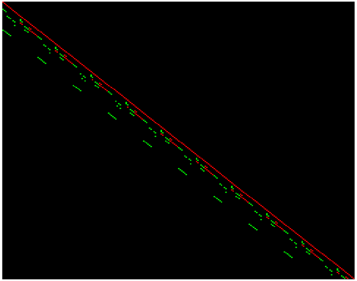


Figure: $\frac{dH}{dX}$ matrix

Lower triangular forward system



$$\rho_i(k) = \rho_i(k-1) + \frac{\triangle t}{\triangle x}\left(f_i^{\mathbf{in}}(k-1) - f_i^{\mathbf{out}}(k-1)\right)$$

$$l_i(k) = l_i(k-1) + \triangle t\,(D_i(k-1) - r_i(k-1))$$

$$\delta_i(k) = \min(F_i, v_i\rho_i(k))$$

$$\sigma_i(k) = \min\left(F_i, w_i\left(\rho_i^{\mathbf{jam}} - \rho_i(k)\right)\right)$$

$$d_i(k) = \min(l_i(k), u_i(k))$$

$$f_i^{\mathbf{in}}(k) = \min(\delta_{i-1}(k)(1-\beta_{i-1}(k)) + d_{i-1}(k), \sigma_i(k))$$

$$f_i^{\mathbf{out}}(k) = \begin{cases} \delta_i(k) & \text{if } P_i f_{i+1}^{\mathbf{in}}(k) > (1-\beta_i(k))\,\delta_i(k) \\ \frac{f_{i+1}^{\mathbf{in}}(k) - d_i(k)}{1-\beta_i(k)} & \text{if } : (1-P_i)\, f_{i+1}^{\mathbf{in}}(k) > d_i(k) \\ \frac{P_i f_{i+1}^{\mathbf{in}}(k)}{1-\beta_i(k)} & \text{otherwise} \end{cases}$$

- H is piecewise affine
- Solving the forward system gives H(x) as a linear system
- $\left.\frac{dH}{dX}\right|_{X=x}$ is computed simultaneously

$$r_i(k) = f_{i+1}^{\mathbf{in}}(k) - f_i^{\mathbf{out}}(k)(1-\beta_i(k))$$

Outline

Exploiting system structure

The structure of the forward system influences the efficiency of computing the adjoint system

Solving for λ

$$\frac{\partial H}{\partial x}^T \lambda = -\frac{\partial J}{\partial x}$$

Exploiting system structure

The structure of the forward system influences the efficiency of computing the adjoint system

Solving for λ

$$\frac{\partial H}{\partial x}^T \lambda = -\frac{\partial J}{\partial x}$$

Since $\frac{\partial H}{\partial x}$ is lower triangular, $\frac{\partial H}{\partial x}^T$ is an upper triangular matrix

$\Longrightarrow$ The adjoint system can be solved effciently using backwards substitution

Outline

Optimization algorithm

Algorithm 1 Gradient descent loop

Pick initial control u^{init}

while not converged **do**

$x = forwardSim(u, IC, BC)$ solve for state trajectory (forward system)

$\lambda = adjointSln(x, u)$ solve for adjoint parameters (adjoint system)

$\Delta u = \nabla_u J = \lambda^T \frac{\partial H}{\partial u} + \frac{\partial J}{\partial u}$ Compute the gradient (search direction)

$u \leftarrow u + t\Delta u$ update u using line search along Δu

end while

Line search

Example 1: decreasing step size

$$t^{(k)} = t^{(1)}/k$$

Example 2: backtracking line-search

- fix parameters $0 < \alpha < 0.1$ and $0 < \beta < 1$
- given search direction Δu

Algorithm 2 Backtracking line search

while $J(u + t\Delta u) - J(u) > \alpha(\nabla_u J)^T(t\Delta u)$ **do**
  $t \leftarrow \beta t$
end while

Objective function

The objective function is the total travel time, given by

$$J(x,u) = \sum_{k=0}^{T} \left(\sum_{i=0}^{N} \Delta x \rho_i(k) + \sum_{i=1}^{N-1} l_i(k) \right)$$

Outline

Solving the optimization problem: Constrained optimization?

Original optimization problem

$$\begin{aligned} \text{minimize}_{u \in \mathcal{U}} \quad & J(x, u) \\ \text{subject to} \quad & H(x, u) = 0 \end{aligned}$$

Example: may want to impose minimal metering rate $u \geq u^{\text{min}}$

Modified optimization problem: log barrier function

Fix $t > 0$.

$$\begin{aligned} \text{minimize}_{u \in \mathbb{R}^m} \quad & tJ(x, u) - \sum_{i=1}^{m} \log\left(u_i - u_i^{\text{min}}\right) \\ \text{subject to} \quad & H(x, u) = 0 \end{aligned}$$

As $t \to \infty$, the solution converges to the solution of the original problem.

Solving the optimization problem: Zero gradient?

$\nabla_u J = \lambda^T \frac{\partial H}{\partial u} + \frac{\partial J}{\partial u}$. Only non-zero terms in $\frac{\partial H}{\partial u}$ are

$$H^5_{k,i}: \quad d_i(k) = \min(l_i(k), u_i(k))$$

$\frac{\partial H}{\partial u}$

$$\frac{\partial H^5_{i,k}}{\partial u_i(k)} = \begin{cases} 0 & \text{if } u_i(k) > l_i(k) \\ 1 & \text{if } u_i(k) < l_i(k) \end{cases}$$

What if we start at $u > l$? Then gradient is zero. Need to add a penalty term

Penalized problem

$$J(x,u) \leftarrow J(x,u) + h(l-u)$$

where h is a penalty function, e.g.

- $h(l-u) = c((u-l)^+)^r$ will push the solution to $l - u \geq 0$
- $h(l-u) = -\frac{1}{t}\log(l-u)$ will keep the solution inside the region $l - u > 0$

Notes on oscillation, slow convergence, and second order methods

- System is piecewise affine.
  Gradient descent sometimes oscillates around boundary of two (affine) regions.
- Seems to be improved by using a second order method:
  computes an approximate Hessian (note the real Hessian is zero!)

Numerical Results

Numerical Results

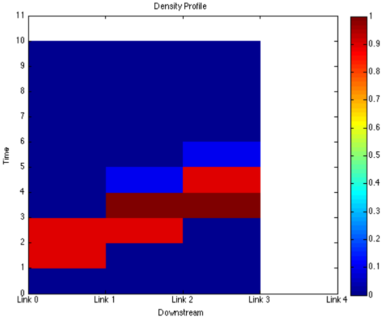


Figure: No ramp metering

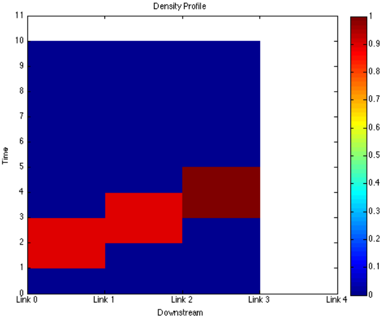
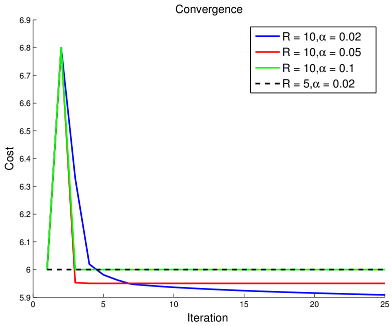
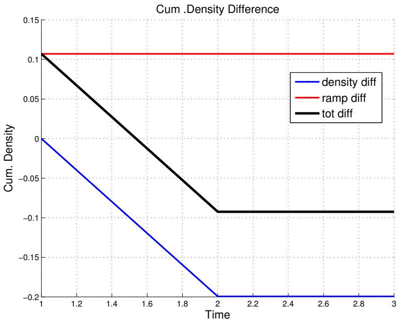


Figure: Optimal ramp metering

Numerical Results





Thank you.