Discrete-time system optimal dynamic traffic assignment (SO-DTA) with partial compliance

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1 Introduction

The goal of this work is to solve the problem of allocating system optimal traffic assignments when only a fraction of the total commuters can be controlled. There is a large amount of literature on the system optimal dynamic traffic assignment (SO-DTA) problem with full compliance and the user equilibrium assignment when there is no control, but very little work on SO-DTA with partial compliance. Partial control situations are essential in corridor management projects where altering route choice is being used as a congestion mitigation strategy, since it is only practical to be able to influence the route choices of a small percentage of the drivers.

We consider a discrete traffic model based on the Godunov discretization of the LWR partial differential equation and a triangular fundamental diagram. The system includes two types of flows. Namely, compliant and non-compliant flows. The non-compliant flow will have fixed turning ratios at junctions and not be controlled by the optimization problem. The compliant flow is the flow that can be controlled to minimize the total travel time of all commuters.

We consider the general network problem with an arbitary number of sources and destinations. The complexity of the optimization problem will depend on the total number of source destination pairs for the compliant flow. Therefore, for computational tractability it is important to restrict the total number of allowed paths to a reasonable number. We will discuss this in detail in the implementation section.

2 Preliminaries

2.1 Notation

Constants

$\triangle t, \triangle x$	Time, space discretization size
v_i	Free flow speed on cell i
w_i	Congestion wave speed on cell i
ρ_i^{jam}	Jam density on cell i
F_i	Max flow leaving mainline of cell i
P_i	Priority factor for cell i

Inputs

ρ_i^0	Initial density on cell i
$D_i(k)$	Input flow on cell i, time step k
$\beta_{ij,c}(k)$	Split ratio for commodity c on cell i to cell j, time step k

Variables

$f_i^{\text{in}}(k)$	Total flow into mainline on cell i, time step k
$f_i^{\text{out}}(k)$	Total flow out of mainline on cell i, time step k
$f_{i,c}^{\text{in}}(k)$	Flow of commodity c into mainline on cell i, time step k
$f_{i,c}^{\text{out}}(k)$	Flow of commodity c out of mainline on cell i, time step k
$\rho_i(k)$	Density on cell i, time step k
$\rho_{i,c}(k)$	Density contribution of commodity c on cell i, time step k
$\sigma_i(k)$	Supply on cell i, time step k
$\delta_i(k)$	Demand on cell i, time step k
$\delta_i(k)$	Demand on cell i, time step k
$d_i(k)$	Boundary demand on cell i, time step k
$d_{i,c}(k)$	Boundary demand of commodity c on cell i, time step k
$l_i(k)$	Boundary queue at the boundary of cell i, time step k
$l_{i,c}(k)$	Boundary queue of commodity c at the boundary of cell i, time step k
$l_i^b(k)$	Queue at the b^{th} buffer at the boundary of cell i, time step k
$l_{i,c}^b(k)$	Queue of commodity c at the b^{th} buffer at the boundary of cell i, time step k
α	Ratio of compliant vehicles

uncModelComp

Model type	Flow	Priority	Blocking
$1 \times n$ diverge	max	N/A	no
$m \times 1$ merge	max	soft	no
$n \times M$ with max flow	max	no	yes
$n \times M$ with exact priority	subject to P	hard	no
$n \times M$ with soft priority	multi objective	soft	no

Table 5: Comparison of the different junction models

α_c	Ratio of compliant vehicles of commodity c

Sets

$\mathcal{J}^{in}$	Incoming links to junction z
$\mathcal{J}^{out}$	Outgoing links to junction z

2.2 System description

The road network is divided into N cells, indexed by $i \in \{1, \ldots, N\}$. The density on cell i at time step k is given by $\rho_i(k)$. The incoming (respectively outgoing) flux to cell i at time step k is given by $f_i^{\text{in}}(k)$ (respectively $f_i^{\text{out}}(k)$). There are C commodities that are transported in the network, indexed by c. The incoming (respectively outgoing) flux of commodity c to cell i at time step k is given by $f_{i,c}^{\text{in}}(k)$ (respectively $f_{i,c}^{\text{out}}(k)$).

We add a ghost cell at the entrance of the network, cell $i = 0$, to impose the boundary flow, or flow demand, given at time step k by $D_0(k)$. Each cell $i \in \{1, \ldots, N-1\}$ is followed by a junction, indexed by $z \in \mathcal{J}$, that connects the set of incoming links J_z^{in} to a set of outgoing link J_z^{out}.

3 Junction model

unctionModel

3.1 Junction models and solvers

The multi-commodity junction problem is essentially equivalent to the source destination model (SDM) in Piccoli [?]. The SDM contains a commodity per source destination pair in the network. Similarly, in our problem there are two commodities for each source destination pair. One corresponding to compliant flow and another corresponding to non-compliant flow. As in the single commodity model and the SDM, a priority vector is needed when the number of incoming links at the junction is greater than the number of outgoing links.

The junction solution should satisfy the following properties:

1. The solution must satisfy the FIFO principle. That is to say, for any incoming link i, the distribution of its flow out across the different commodities must be in proportion the the ratio of vehicles of each commodity at the link.

$$f_{i,c_i}^{\text{out}}(k) = f_i^{\text{out}}(k) \frac{\rho_{i,c_i}(k)}{\sum_{j=1}^{C} \rho_{i,c_j}(k)}$$

2. Flow maximization across the junction subject to split ratio and priority constraints.

3. Satisfy split ratios and inflow priority constraints.

   - The priority constraint can be a soft constraint in certain cases.

3.1.1 Diverge solver

ivergeSolver

We consider a diverging junction with one incoming link i and m outgoing links. There are C commodities that flow through the network each with their own time-varying split ratio $\beta_{ij,c}(k)$.

- The first step is to compute the aggregate split ratio from i to each outgoing link j:

$$\begin{aligned}\beta_{ij}(k) &= \sum_{c=1}^{C} \frac{\rho_{i,c}(k)}{\rho_i(k)} \beta_{ij,c}(k) \\ &= \frac{1}{\rho_i(k)} \sum_{c=1}^{C} \rho_{i,c}(k)\, \beta_{ij,c}(k)\end{aligned}$$

  TODO: Explain why this is correct.

- Once the aggregate split ratio is found we can solve for the total junction flow as follows:

$$\begin{aligned}&\max f_i^{\text{out}}(k) \\ &\text{subject to} \\ &\quad \beta_{ij}(k)\, f_i^{\text{out}}(k) \leq \sigma_j(k) \qquad \forall j : (i,j) \in A \\ &\quad f_i^{\text{out}}(k) \leq \delta_i(k)\end{aligned}$$

  Or equivalently by solving the feasibility problem:

$$\begin{aligned}&\max 1 \\ &\text{subject to} \\ &\quad f_i^{\text{out}}(k) = \min\left(\frac{\sigma_j(k)}{\beta_{ij}(k)} \quad \forall j : (i,j) \in A, \delta_i(k)\right)\end{aligned}$$

- The total outflow $f_i^{\text{out}}(k)$ for each incoming link i is then divided among the commodities according to the FIFO law.

$$f_{i,c}^{\text{out}}(k) = \frac{\rho_{i,c}(k)}{\rho_i(k)} f_i^{\text{out}}(k)$$

  The commodity flows are split among the outgoing links according to the split ratios.

$$f_{j,c}^{\text{in}}(k) = \sum_{i:(i,j)\in A} \beta_{ij,c}(k)\, f_{i,c}^{\text{out}}(k)$$

Existance and uniqueness of solution A non-zero solution exists if the none of the constraints of the optimization/feasibility problem impose a zero flow. In other words, as long as the demand is non-zero and none of the outgoing links with positive demand ($\beta_{ij}(k) > 0$) have non-zero supply, the non-zero solution will exist. Since, the solution to the maximum junction flow is the flow out of the incoming link (unique maximum of a scalar value), and the outflows are uniquely determined by the split ratios, the solution is unique.

3.1.2 Merge solver

:mergeSolver

We consider a merging junction with n incoming links and one outgoing link j exiting it. There are C commodities on each link. A priority vector P (s.t. $\sum p_i = 1$) prescribes the priorities at which the outgoing link accepts flows from the n incoming links when the junction is supply constrained.

- If the problem is demand constrained (i.e. $\sum_{i:(i,j)\in A}\delta_i(k)\le\sigma_j(k)$), then the solution is given by:

$$f_i^{\text{out}}(k)=\delta_i(k)\quad\forall i:(i,j)\in A$$

- If the problem is supply constrained, then the solution to the junction problem is given by solving the following quadratic optimization problem that finds the flow maximizing solution with the smallest violation of the priority vector, where the violation is measured using the $L2$ distance:

$$\begin{aligned}&\min_{t,f_i^{\text{out}}(k)\ \forall i:(i,j)\in A}\ \sum_{i:(i,j)\in A}\left(f_i^{\text{out}}(k)-t\cdot p_i\right)^2\\ &\text{subject to}\\ &\qquad\sum_{i:(i,j)\in A}f_i^{\text{out}}(k)=\sigma_i(k)\\ &\qquad f_i^{\text{out}}(k)\le\delta_i(k)\end{aligned}$$

- The total outflow $f_i^{\text{out}}(k)$ for each incoming link i is then divided among the commodities according to the FIFO law.

$$f_{i,c}^{\text{out}}(k)=\frac{\rho_{i,c}(k)}{\rho_i(k)}f_i^{\text{out}}(k)$$

Remark 1. *The priorities are satisfied exactly when the intersection of the maximum flow isoline and the priority constraint are feasible. When this point is outside the feasible set, the flow maximizing feasible point that is closest to the priority constraint (in euclidean distance) is chosen.*

Remark 2. *The solution violates the priority rule only in the case where the demand for one or more of the incoming links is less than what its flow maximizing allocation is based on the priority vector. In other words, the priority rule is only violated when an incoming link doesn't have enough flow to satisfy it's priority based allocation. It is reasonable in the physical sense to maximize flow and only violate the priority when it's a lack of demand that causes the violation. The model is not denying any vehicles with priority the ability to pass through the junction. This is an important property to note, because it avoids having to solve a multi-objective optimization problem to come up with a physically meaningful set of flows through the junction. This is in contrast to the junction model we will consider in a general $n\times m$ junction.*

Existence and uniqueness of solution

- *Demand constrained case*: In the demand constrained case, existence and uniqueness is trivial.
- *General case*: In the general case, we are minimizing the euclidean distance from the solution (a point) to the priority vector (a line) subject to the solution being in the feasible set that is defined by the intersection of a n dimensional hyperplane (supply constraint: $\sum_{i:(i,j)\in A}f_i^{\text{out}}(k)\le\sigma_i(k)$) with a n dimensional hyperrectangle (demand constraints: $f_i^{\text{out}}(k)\ \le\ \delta_i(k)$).
  - A solution exists when the feasible set is non-empty and the feasible set will always be non-empty if the supply constraint is greater than zero. This proves the existence of a solution in all non-degenerate (zero supply) cases.
  - The supply constraint hyperplane intersects each coordinate axis at $x_i=\sigma_i(k)$ and therefore can not be parallel to the priority constraint P which is a link that goes through the origin. Furthermore, the feasible set is the intersection of a n dimensional hyperplane (supply constraint) with a n dimensional hyperrectangle (demand constraints), which is a $n-1$ dimensional convex set. Since this is a subset of the supply constraint set, this is also not parallel to P. Finally, the point on a convex set with the minimum euclidean distance to a line that is not parallel to the set is unique. This concludes the proof.

3.1.3 Merge and diverge solver with $n \leq m$ (Flow Maximizing Model)

We consider a junction with n incoming links and m outgoing links where $n \leq m$. There are C commodities on each link.

Remark 3. *The Piccoli model maximizes flow, but has the limitation that some incoming links might have zero flux when the junction is supply limited and the other incoming links allow for more flux through the junction due to their split ratios.*

TODO: Give a simple example.

Let J^{in} and J^{out} be the sets of incoming and outgoing links at the junction.

- The first step is to compute the aggregate split ratio from each incoming link i to each outgoing link j:

$$\begin{aligned} \beta_{ij}(k) &= \sum_{c=1}^{C} \frac{\rho_{i,c}(k)}{\rho_i(k)} \beta_{ij,c}(k) \\ &= \frac{1}{\rho_i(k)} \sum_{c=1}^{C} \rho_{i,c}(k)\, \beta_{ij,c}(k) \end{aligned}$$

- Once the aggregate split ratio is found we can solve for the total junction flow as follows:

$$\begin{aligned} &\max \sum_{i \in J^{in}} f_i^{\text{out}}(k) \\ &\text{subject to} \\ &\quad \sum_{i \in J^{in}} \beta_{ij}(k)\, f_i^{\text{out}}(k) \leq \sigma_j(k) \qquad \forall j \in J^{out} \\ &\quad f_i^{\text{out}}(k) \leq \delta_i(k) \qquad \forall i \in J^{in} \end{aligned}$$

- The total outflow $f_i^{\text{out}}(k)$ for each incoming link i is then divided among the commodities according to the FIFO law.

$$f_{i,c}^{\text{out}}(k) = \frac{\rho_{i,c}(k)}{\rho_i(k)} f_i^{\text{out}}(k)$$

  The commodity flows are split among the outgoing links according to the split ratios.

$$f_{j,c}^{\text{in}}(k) = \sum_{i:(i,j)\in A} \beta_{ij,c}(k)\, f_{i,c}^{\text{out}}(k)$$

Existence and uniqueness of solution

3.1.4 Merge and diverge solver with $n \leq m$ (Soft Priority Model)

We consider a junction with n incoming links and m outgoing links where $n \leq m$. There are C commodities on each link. Let the incoming links be indexed by $\{1, \ldots, n\}$ and the outgoing links be indexed by $\{n+1, \ldots, n+m\}$ A priority vector P prescribes the priorities at which the m outgoing links accept flows from the n incoming links. P_{ij} is the priority allocation for flow from incoming link i to outgoing link j such that $\sum_{j \in \{n+1,\ldots,n+m\}} p_{ij} = 1 \;\; \forall i \in \{1, \ldots, n\}$. A multi-objective optimization problem is used with the cost function weighing the two objectives of flow maximization and satisfying the priority vector with the scaling terms $R1$, $R2$.

Remark 4. *The Piccoli model maximizes flow at the expense of possibly blocking out certain incoming links completely, and the exact priority model satisfies priorities at the expense of unsatisfied supply. The Piccoli model can entirely block out some incoming flows, which is physically unrealistic in merges where each merging road has dedicated merge lanes. The priority model is more realistic, since the priorities are generally picked based on physical properties such as the number of dedicated merging lanes associated with each incoming link. However, we might want to consider junctions where the priority law can be violated.*

Note: The physical argument seems to suggest that fixed priority is the correct approach. We need to explore this more.

Remark 5. *This model doesn't have the nice property from section 3.1.2 where the priority rule is only violated when the incoming link doesn't have enough demand to satisfy the priority allocation.*

Let J^{in} and J^{out} be the sets of incoming and outgoing links at the junction.

- The first step is to compute the aggregate split ratio from each incoming link i to each outgoing link j:

$$\begin{aligned}\beta_{ij}(k) &= \sum_{c=1}^{C} \frac{\rho_{i,c}(k)}{\rho_i(k)} \beta_{ij,c}(k) \\ &= \frac{1}{\rho_i(k)} \sum_{c=1}^{C} \rho_{i,c}(k)\, \beta_{ij,c}(k)\end{aligned}$$

- Once the aggregate split ratio is found we can solve for the total junction flow as follows:

$$\max_{t, f_i^{\text{out}}(k)\ \forall i:(i,j)\in A} R1\left(\sum_{i\in J^{in}} f_i^{\text{out}}(k)\right) - R2\left(\sum_{i:(i,j)\in A} \left(f_i^{\text{out}}(k) - t\cdot p_i\right)^2\right)$$

subject to

$$\sum_{i\in J^{in}} \beta_{ij}(k)\, f_i^{\text{out}}(k) \le \sigma_j(k) \qquad \forall j \in J^{out}$$

$$f_i^{\text{out}}(k) \le \delta_i(k) \qquad \forall i \in J^{in}$$

- The total outflow $f_i^{\text{out}}(k)$ for each incoming link i is then divided among the commodities according to the FIFO law.

$$f_{i,c}^{\text{out}}(k) = \frac{\rho_{i,c}(k)}{\rho_i(k)} f_i^{\text{out}}(k)$$

The commodity flows are split among the outgoing links according to the split ratios.

$$f_{j,c}^{\text{in}}(k) = \sum_{i:(i,j)\in A} \beta_{ij,c}(k)\, f_{i,c}^{\text{out}}(k)$$

Existence and uniqueness of solution

3.1.5 Merge and diverge solver with $n > m$

Existence and uniqueness of solution

3.1.6 Merge and diverge solver with $n \leq m$ (Exact Priority Model)

We consider a junction with n incoming links and m outgoing links where $n \leq m$. There are C commodities on each link. Let the incoming links be indexed by $J^{in} = \{1, \ldots, n\}$ and the outgoing links be indexed by $J^{out} = \{n+1, \ldots, n+m\}$ A priority vector P prescribes the priorities at which the m outgoing links accept flows from the n incoming links. P_{ij} is the priority allocation for flow from incoming link i to outgoing link j such that $\sum_{i \in J^{in}} p_{ij} = 1 \ \ \forall j \in J^{out}$.

Remark 6. *The exact priority model finds the flow allocation that exactly satisfies the priority vector and maximizes flow subject to this constraint.*

TODO: Give a simple example.

Remark 7. *We assume for now that the priority vectors for all outgoing links are identical. The notation allows for different priority vectors, but the analysis will ignore that for now. Different priorities can make the exact priority model infeasible in most cases.*

Let J^{in} and J^{out} be the sets of incoming and outgoing links at the junction.

- The first step is to compute the aggregate split ratio from each incoming link i to each outgoing link j:

$$
\begin{aligned}
\beta_{ij}(k) &= \sum_{c=1}^{C} \frac{\rho_{i,c}(k)}{\rho_i(k)} \beta_{ij,c}(k) \\
&= \frac{1}{\rho_i(k)} \sum_{c=1}^{C} \rho_{i,c}(k)\, \beta_{ij,c}(k)
\end{aligned}
$$

- Once the aggregate split ratio is found we can solve for the total junction flow as follows:

$$
\begin{aligned}
&\max \sum_{i \in J^{in}} f_i^{\text{out}}(k) \\
&\text{subject to} \\
&\quad \sum_{i \in J^{in}} \beta_{ij}(k)\, f_i^{\text{out}}(k) \leq \sigma_j(k) \qquad \forall j \in J^{out} \\
&\quad f_i^{\text{out}}(k) \leq \delta_i(k) \qquad \forall i \in J^{in} \\
&\quad \beta_{ij}(k)\, f_i^{\text{out}}(k) = P_{ij} f_j^{\text{in}}(k) \qquad \forall (i,j) : i \in J^{in}, j \in J^{out}
\end{aligned}
$$

- The total outflow $f_i^{\text{out}}(k)$ for each incoming link i is then divided among the commodities according to the FIFO law.

$$
f_{i,c}^{\text{out}}(k) = \frac{\rho_{i,c}(k)}{\rho_i(k)} f_i^{\text{out}}(k)
$$

  The commodity flows are split among the outgoing links according to the split ratios.

$$
f_{j,c}^{\text{in}}(k) = \sum_{i:(i,j) \in A} \beta_{ij,c}(k)\, f_{i,c}^{\text{out}}(k)
$$

Existence and uniqueness of solution

3.2 Junction model and solver used in the optimization problem

nModelForOpt

We first describe the system equations for the 1×2 diverge solver(based on the general diverge solver from 3.1.1) and then a 2×1 merge solver(based on the general merge solver from 3.1.2) before presenting the equations governing the 2×2 solver we will use. The models and solvers presented in section 3.1 can be generalized to a $n \times m$ junction, but we restrict our solver to 2×2 junctions. The flows can be determined by first computing the demand functions (equation $H^3_{k,i}$) and the supply functions (equation $H^4_{k,i}$).

(1×2) diverge

Let i be the incoming link and $j \in J^{out}$ be the outgoing links.

Aggregate split ratio:

$$\beta_{ij}(k) = \frac{1}{\rho_i(k)} \sum_{c=1}^{C} \rho_{i,c}(k)\, \beta_{ij,c}(k)$$

Flow out of incoming link:

$$f_i^{\text{out}}(k) = \min\left(\frac{\sigma_j(k)}{\beta_{ij}(k)}\ \forall j \in J^{out}, \delta_i(k)\right)$$

Flow out of incoming link by commodity:

$$f_{i,c}^{\text{out}}(k) = \frac{\rho_{i,c}(k)}{\rho_i(k)} f_i^{\text{out}}(k)$$

Flow in to outgoing links by commodity:

$$f_{j,c}^{\text{in}}(k) = \sum_{i:(i,j)\in A} \beta_{ij,c}(k)\, f_{i,c}^{\text{out}}(k)$$

(2×1) merge

Let i_1 be one incoming link, i_2 be the other incoming link and j be the outgoing link.

Flow in to outgoing link:

$$f_j^{\text{in}}(k) = \min\left(\delta_{i_1}(k) + \delta_{i_2}(k), \sigma_j(k)\right)$$

Flow in to outgoing link by commodity:

$$f_{j,c}^{\text{in}}(k) = \frac{\rho_{j,c}(k)}{\rho_j(k)} f_j^{\text{in}}(k)$$

Flow out of incoming links:

$$f_{i_1}^{\text{out}}(k) = \begin{cases} \delta_{i_1}(k) & \text{if } \frac{P_{i_1}}{1-P_{i_1}} > \frac{\delta_{i_1}(k)}{f_j^{\text{in}}(k)-\delta_{i_1}(k)} \\ f_j^{\text{in}}(k) - \delta_{i_2}(k) & \text{if } \frac{P_{i_1}}{1-P_{i_1}} < \frac{f_j^{\text{in}}(k)-\delta_{i_2}(k)}{\delta_{i_2}(k)} \\ P_{i_1} f_j^{\text{in}}(k) & \text{otherwise} \end{cases}$$

$$f_{i_2}^{\text{out}}(k) = f_j^{\text{in}}(k) - f_{i_1}^{\text{out}}(k)$$

Flow out by commodity:

$$f_{i,c}^{\text{out}}(k) = \frac{\rho_{i,c}(k)}{\rho_i(k)} f_i^{\text{out}}(k) \qquad \forall i \in \{i_1, i_2\}$$

(2 × 2) merge and diverge

Let i_1, i_2 be the incoming links and j_1, j_2 be the outgoing links. The priority vectors for outgoing links j_1 and j_2 are assumed to be identical as described in section 3.1.

To simplify the notation, we use the following shorthand:

- drop the time index k
- $\delta_1 = \delta_{i_1}$, $\delta_2 = \delta_{i_2}$
- $\sigma_1 = \sigma_{j_1}$, $\sigma_1 = \sigma_{j_2}$
- $P_1 = P_{i_1}$, $P_2 = P_{i_2}$

  Aggregate split ratio:

$$\beta_{ij}(k) = \frac{1}{\rho_i(k)} \sum_{c=1}^{C} \rho_{i,c}(k)\, \beta_{ij,c}(k) \qquad \forall (i,j) \in \{1,2\} \times \{1,2\}$$

  Flow out of incoming links:

$$f_1^{\text{out}} = \begin{cases} \delta_1 & \text{if } \frac{1-P_1}{P_1} < \frac{\min\left(\delta_2, \frac{\sigma_1 - \beta_{11}\delta_1}{\beta_{21}}, \frac{\sigma_2 - \beta_{12}\delta_1}{\beta_{22}}\right)}{\delta_1(k)} \\ \min\left(\delta_1, \frac{\sigma_1 - \beta_{21}\delta_2}{\beta_{11}}, \frac{\sigma_2 - \beta_{22}\delta_2}{\beta_{12}}\right) & \text{if } \frac{1-P_1}{P_1} > \frac{\delta_2}{\min\left(\delta_1, \frac{\sigma_1 - \beta_{21}\delta_2}{\beta_{11}}, \frac{\sigma_2 - \beta_{22}\delta_2}{\beta_{12}}\right)} \\ P_1 \min\left(\delta_1, \frac{P_1\sigma_1}{P_1\beta_{11} + (1-P_1)\beta_{21}}, \frac{P_1\sigma_2}{P_1\beta_{12} + (1-P_1)\beta_{22}}\right) & \text{otherwise} \end{cases}$$

  Flow out of incoming links by commodity:

$$f_{i,c}^{\text{out}} = \frac{\rho_{i,k}}{\rho_i} f_i^{\text{out}} \qquad \forall i \in \{1,2\}, c \in C$$

  Flow in to outgoing links by commodity:

$$f_{j,c}^{\text{in}} = \sum_{i:(i,j)\in A} \beta_{ij,k} f_{i,c}^{\text{out}} \qquad \forall j \in \{1,2\}, c \in C$$

4 Boundary conditions

The boundary conditions at each source link of the network dictate the flows the enter the network. Each boundary condition is given as a flow rate at the boundary. Since the inflow to the network is limited by the max flow and density of the immediate downstream link, all of the demand at a given time step might not make it into the network. A source buffer is used to accumulate the flow that can not enter the network to guarantee conservation of boundary flows. In the single commodity case, nothing else is required. However, in the multi commodity case, we also need to make sure that the flow through the boundary respects the FIFO condition.

Let $d_i(k)$ be the total demand on link i at time step k, and the demand per commodity be given by $d_{i,c}(k)$. The FIFO condition dictates that the vehicles entering the boundary buffer at time k must enter link i at the ratio $\frac{d_{i,c}(k)}{d_i(k)}$ for each commodity c.

4.1 Boundary source buffer for multiple commodities

4.1.1 Single buffer model

The simplest solution is to have a single buffer l at the boundary, as in the single commodity case, and keep track of how many vehicles of each commodity are at the buffer. The flow into the boundary cell will be as follows:

$$f_{i,c}^{\text{in}}(k) = \frac{l_{i,c}(k)}{l_i(k)} f_i^{\text{in}}(k)$$

This condition satisfies the FIFO condition assuming that the vehicles in the buffer are uniformly distributed. However, in reality the buffer can accumulate vehicles arriving at the boundary at different time steps with different commodity ratios $\frac{d_{i,c}(k)}{d_i(k)}$. Thus, this model violates the FIFO property across multiple time steps.

4.2 Multi buffer model

A simple extension to this model is to allow for multiple input buffers that are connected to each other. In this model, the commodity ratios are maintained separately for each buffer. This restricts the violation of the FIFO condition across multiple steps to the capacity of a single buffer.

Remark 8. ***FIFO condition is only satisfied approximately on the interior of the network*** *It is important to note that the FIFO condition is violated in its strict sense even within the network. The flow propagation model assumes that all the flow within a cell is uniformly distributed according to the individual commodity ratios regardless of when the vehicles arrived at the cell.*

Example 1. *Consider the following simple example. There are two commodities a, b in cell i with 10 vehicles of each commodity at time k. At time $k+1$, 10 vehicles exit the cell (5 of a and 5 of b by the FIFO rule) and 10 new vehicles (3 of a and 7 of b) enter the cell. The new ratio of vehicles at i is 8 a to 12 b. At time $k+2$, once again 10 vehicles exit the network. According to the cell level FIFO rule, the 10 vehicles will consist of 4 a's and 6 b's. However, the first 10 cars of those currently in cell i came at the ratio of 1:1 and truly satisfying the FIFO rule would require the 10 exiting vehicles to consist of 5 a's and 5 b's.*

The strict FIFO condition is not satisfied in most traffic flow models and it is considered acceptable to limit the FIFO requirement to the cell level FIFO condition.

The size of each buffer is chosen to be equal to the cell sizes on the network such that the FIFO violation at the buffer is no worse than the violation in the interior of the network and we satisfy the cell level FIFO condition. The limitation of this model is that we require $\frac{l^{max}}{L}$ buffers, where l^{max} is the maximum queue length at the boundary and L is the capacity of each buffer. We assume that $f^{\text{in}}_{i,\leq}(k)\, L$.

The buffers are updated as follows:

- First move flow out of the initial buffer

$$f^{\text{in}}_{i,c}(k) = \frac{l_{i,c}(k)}{l_i(k)} f^{\text{in}}_i(k)$$

- Let n be the number of buffers in use. Iterate through the buffers and push flow upstream using the following algorithm.

For $b = 2$ to n
    $\Delta l^{b-1} = L - l^{b-1}_i(k)$
    $l^{b-1}_{i,c}(k) = l^{b-1}_{i,c}(k) + \frac{l^b_{i,c}(k)}{l^b_i(k)} \Delta l \quad \forall c \in C$
$\Delta l^n = L - l^n_i(k)$
If $\Delta l^n \leq d_i(k)$
    $l^n_{i,c}(k) = l^n_{i,c}(k) + \frac{d_{i,c}(k)}{d_i(k)} d_i(k) \quad \forall c \in C$
else
    $l^n_{i,c}(k) = l^n_{i,c}(k) + \frac{d_{i,c}(k)}{d_i(k)} \Delta l \quad \forall c \in C$
    $l^{n+1}_{i,c}(k) = l^{n+1}_{i,c}(k) + \frac{d_{i,c}(k)}{d_i(k)} \left(d_i(k) - \Delta l\right) \quad \forall c \in C$

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5 Optimization problem

5.1 Optimization problem formulation

We wish to minimize the total travel time of the system (system optimal) subject to the boundary conditions of the network, the flow propagation model constraints, junction model constraints and the partial controllability of vehicle route choice. A fixed percentage α_c of vehicles can be controlled for each commodity c.

The system equations are written formally in the form $H(x,u)=0$. The discretized system can be described using eight types of constraints, given by $H^x_{k,i}=0$ for $x\in\{1,\dots,8\}$, where we index each equality constraint by time index k, and cell index i. We now give the system equations.

$$\begin{aligned}
&\min \sum_{i\in\mathcal{I},k\in\mathcal{T}} \rho_i\left(k\right)\\
&\text{subject to}\\
&\quad \textit{mass conservation constraints}\\
&\quad \textit{boundary constraints}\\
&\quad \textit{flow propagation constraints}\\
&\quad \textit{junction constraints}
\end{aligned}$$

We assume that the FIFO condition at the source boundaries can be approximated by a single large buffer.

Mass conservation

$$H^1_{k,i,c}:\quad \rho_{i,c}\left(k\right)=\rho_{i,c}\left(k-1\right)+\frac{\triangle t}{\triangle x}\left(f^{\text{in}}_{i,c}\left(k-1\right)-f^{\text{out}}_{i,c}\left(k-1\right)\right)\qquad \forall i\in\left\{\mathcal{A}-\mathcal{S}\cup\mathcal{T}\right\},k\in\left\{1,\dots,T\right\},\forall c\in C \tag{H1a}$$

eq:conservat

$$H^1_{k,i,c}:\quad \rho_{N,c}\left(k\right)=\rho_{N,c}\left(k-1\right)+\frac{\triangle t}{\triangle x}\left(f^{\text{in}}_{N,c}\left(k-1\right)-f^{\text{out}}_{N,c}\left(k-1\right)\right)\qquad \forall i\in\left\{\mathcal{T}\right\},\forall k\in\left\{1,\dots,T\right\},\forall c\in C \tag{H1b}$$

eq:conservat

conservation

with initial condition

$$I^1_{i,c}:\rho_{i,c}\left(0\right)=\rho^0_{i,c}\qquad \forall i\in\left\{\mathcal{A}-\mathcal{S}\right\},\forall c\in C \tag{I1}$$

eq:densityIn

Boundary conditions

$$H^2_{k,i,c}:\quad l_{i,c}\left(k\right)=l_{i,c}\left(k-1\right)+\triangle t\left(D_{i,c}\left(k-1\right)-f^{\text{out}}_{i,c}\left(k-1\right)\right)\quad \forall i\in\mathcal{S},k\in\left\{1,\dots,T\right\},\forall c\in C \tag{H2}$$

eq:sourceCon

with initial condition

$$I^2_{i,c}:l_{i,c}\left(0\right)=l_{i,c}\left(0\right)\qquad \forall i\in\mathcal{S},\forall c\in C \tag{I2}$$

eq:sourceIni

Flow propagation

Let, $\rho_i\left(k\right)=\sum_{c=1}^{C}\rho_{i,c}\left(k\right)$

$$H^3_{k,i}:\qquad \delta_i\left(k\right)=\min\left(F_i,v_i\rho_i\left(k\right)\right)\qquad \forall i\in\left\{\mathcal{A}-\mathcal{S}\cup\mathcal{T}\right\},k\in\left\{0,\dots,T-1\right\} \tag{H3a}$$

eq:junctionD

$$H^3_{k,i}:\qquad \delta_i\left(k\right)=\min\left(F_i,l_i\left(k\right)\right)\qquad \forall i\in\left\{\mathcal{S}\right\},k\in\left\{0,\dots,T-1\right\} \tag{H3b}$$

eq:junctionD

$$H^4_{k,i}:\qquad \sigma_i\left(k\right)=\min\left(F_i,w_i\left(\rho^{\text{jam}}_i-\rho_i\left(k\right)\right)\right)\qquad \forall i\in\left\{\mathcal{A}-\mathcal{S}\cup\mathcal{T}\right\},k\in\left\{0,\dots,T-1\right\} \tag{H4a}$$

eq:junctionS

$$H^4_{k,i}:\qquad \sigma_i\left(k\right)=F_i\qquad \forall i\in\left\{\mathcal{T}\right\},k\in\left\{0,\dots,T-1\right\} \tag{H4b}$$

eq:junctionS

Junction solution

At junctions, the flows are given by the solver described in section 3.2.

To simplify the notation, we use the following shorthand:

- drop the time index k
- $\delta_1 = \delta_{i_1}$, $\delta_2 = \delta_{i_2}$
- $\sigma_1 = \sigma_{j_1}$, $\sigma_1 = \sigma_{j_2}$
- $P_1 = P_{i_1}$, $P_2 = P_{i_2}$

Let the aggregate split ratio,

$$\beta_{ij} = \frac{1}{\rho_i} \sum_{c=1}^{C} \rho_{i,k} \beta_{ij,k}$$

Flow out of incoming links by commodity:

$$H^5_{k,i,c}: \quad f^{\text{out}}_{i,c} = \frac{\rho_{1,k}}{\rho_1} \min\left(\frac{\sigma_1}{\beta_{11}}, \frac{\sigma_2}{\beta_{12}}, \delta_1\right) \qquad \forall i \in \mathcal{J}^{in}_{1\times 2} \tag{H5a}$$

`eq:junctionF`

$$H^5_{k,i,c}: \quad f^{\text{out}}_{i,c} = \frac{\rho_{1,k}}{\rho_1} \begin{cases} \delta_1 & \text{if } \frac{1-P_1}{P_1} < \frac{\min\left(\delta_2, \frac{\sigma_1 - \beta_{11}\delta_1}{\beta_{21}}\right)}{\delta_1(k)} \\ \min\left(\delta_1, \frac{\sigma_1 - \beta_{21}\delta_2}{\beta_{11}}\right) & \text{if } \frac{1-P_1}{P_1} > \frac{\delta_2}{\min\left(\delta_1, \frac{\sigma_1 - \beta_{21}\delta_2}{\beta_{11}}\right)} \\ P_1 \min\left(\delta_1, \frac{P_1\sigma_1}{P_1\beta_{11} + (1-P_1)\beta_{21}}\right) & \text{otherwise} \end{cases} \qquad \forall i \in \mathcal{J}^{in}_{2\times 1} \tag{H5b}$$

`eq:junctionF`

$$H^5_{k,i,c}: \quad f^{\text{out}}_{i,c} = \frac{\rho_{1,k}}{\rho_1} \begin{cases} \delta_1 & \text{if } \frac{1-P_1}{P_1} < \frac{\min\left(\delta_2, \frac{\sigma_1 - \beta_{11}\delta_1}{\beta_{21}}, \frac{\sigma_2 - \beta_{12}\delta_1}{\beta_{22}}\right)}{\delta_1(k)} \\ \min\left(\delta_1, \frac{\sigma_1 - \beta_{21}\delta_2}{\beta_{11}}, \frac{\sigma_2 - \beta_{22}\delta_2}{\beta_{12}}\right) & \text{if } \frac{1-P_1}{P_1} > \frac{\delta_2}{\min\left(\delta_1, \frac{\sigma_1 - \beta_{21}\delta_2}{\beta_{11}}, \frac{\sigma_2 - \beta_{22}\delta_2}{\beta_{12}}\right)} \\ P_1 \min\left(\delta_1, \frac{P_1\sigma_1}{P_1\beta_{11} + (1-P_1)\beta_{21}}, \frac{P_1\sigma_2}{P_1\beta_{12} + (1-P_1)\beta_{22}}\right) & \text{otherwise} \end{cases} \qquad \forall i \in \mathcal{J}^{in}_{2\times 2} \tag{H5c}$$

`eq:junctionF`

Flow in to outgoing links by commodity:

$$f^{\text{in}}_{j,c} = \sum_{i:(i,j)\in A} \beta_{ij,k} f^{\text{out}}_{i,c} \qquad \forall j \in \{1,2\}, c \in C$$

5.2 Regularization

5.3 Gradient decent

6 Forward simulation

7 Adjoint system

7.1 Adjoint method in highway control

7.2 How to use the adjoint method

7.3 Adjoint formulation of the flow reroute problem

8 Implementation

9 Results