Continuous, Junction-based Model for Ramp Metering

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Plan

1. Motivation
2. Other highway/control models
3. Piccoli Model
4. Modified 2x2 Piccoli model with Buffers
5. Mathematical Results on Modified Piccoli Model

Why ramp metering?

- High level goal: develop a general optimization framework for many highway problems
  - Partial Rerouting
  - Variable Speed Limit
  - State estimation [4]
  - **Ramp Metering (current application)**
- Consistent physical model
  - Each control scheme uses same forward simulator
  - All control schemes can be modules in the simulation engine

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    - Allows for discrete models to incorporate mid-timestep events, such as a queue emptying (next week).
- Analytical properties, such as uniqueness/existence, shockwave solutions, derived from continuous model
- Prioritize model accuracy over linearity/simplicity of discretized formulation.
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- Only first order approach
- "Black-box" approach, does not address non-linearities at all.
- Currently in open-loop formulation, which loses usefulness as input data becomes noisier
- Geometric constraints are difficult [3]
  - handles equality constraints well, but not inequality constraints
  - barrier functions
  - projected-gradient descent

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- Network CTM [1]
  - Exact Godunov discretization of LWR with triangular fundamental diagram
  - only specifies 1×2, 2×1 junctions
- Link-node CTM with ramp metering problem as LP [5]
  - For $n \times m$ junctions
  - Priorities are proportional to demand
  - In the limit, the junction solver is not "self-similar". More on this later.
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   - Classical Riemann Problem
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Classical Riemann Problem

A classical Riemann Problem is a PDE with a particular choice of initial data:

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, \\ \rho(0,x) = \begin{cases} \rho_L & \text{if } x < 0, \\ \rho_R & \text{if } x > 0. \end{cases} \end{cases} \tag{1}$$

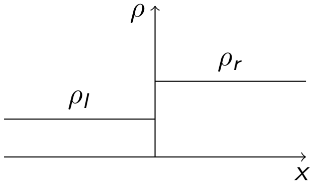

Figure : Initial data for the $\mathcal{RP}$.

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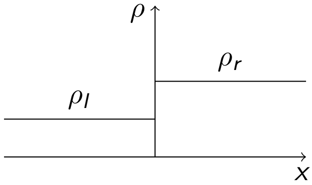

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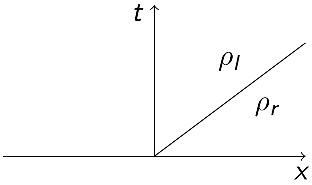


Figure : Solution of the $\mathcal{RP}$.

LWR on a 2x2 junction[1]

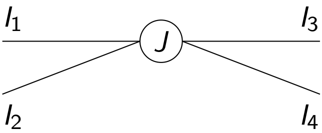


Figure : Generic 2x2 junction.

- $I_i = [a_i, b_i]$ for $i = 1, 2$ and $I_j = [a_j, b_j]$ for $j = 3, 4$.
- $\partial_t \rho + \partial_x f(\rho) = 0$ holds on each I_i and I_j.
- Flux function strictly concave.
- $f(0) = f(1) = 0$.
- Flux function has a unique maximum at $\rho^{cr} \in]0, 1[$.

[1] M.Garavello and B.Piccoli, *Traffic Flows on Networks:Conservation Laws Model*. AIMS Series on Applied Mathematics. American Institute of Mathematical Sciences, 2006

Junction assumptions

1. A traffic distribution matrix is given

$$A = \begin{bmatrix} \alpha_{3,1} & \alpha_{3,2} \\ \alpha_{4,1} & \alpha_{4,2} \end{bmatrix}$$

Remark: $\alpha_{j,1} \neq \alpha_{j,2} \quad \forall j \in \{3, 4\}$.

2. Rankine-Hugoniot condition is satisfied, i.e.,

$$\sum_{i=1}^{2} f_i(\rho(t, b_i-)) = \sum_{j=3}^{4} f_j(\rho(t, a_j+)).$$

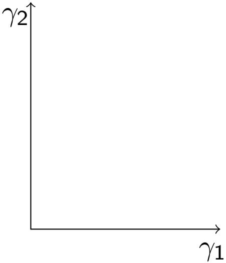
3. Each $f_j(\rho(t, a_j+)) = \sum_{i=1}^{2} \alpha_{j,i} f_i(\rho(t, b_i-))$ for every $j = 3, 4$.

4. $\sum_{i=1}^{2} f_i(\rho(t, b_i-))$ is maximum subject to 3.

- Define initial data $\rho_{i,0}, \rho_{j,0} \in]0,1[$ for every $i = 1,2$ and $j = 3,4$.
- The Riemann Problem at J is the LWR where the initial conditions are given by $\rho_{i,0}(x) \equiv \rho_{i,0}$ in I_i for every $i = 1,2$ and $\rho_{j,0}(x) \equiv \rho_{j,0}$ in I_j for every $j = 3,4$.
- The Riemann Solver $\mathcal{RS}$ is the right continuous map $(t,x) \to \mathcal{RS}(\rho_l, \rho_r)(\frac{x}{t})$ that is the standard weak entropy solution to (1).

To find a solution of the problem we will take the following steps.

- Define $\gamma_1 = f_1(\rho(t, b_1-))$ and $\gamma_2 = f_2(\rho(t, b_2-))$.
- Define the space (γ_1, γ_2).



Riemann Solver

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- Define $\gamma_i^{\max}(\rho_{i,0})$ and $\gamma_j^{\max}(\rho_{j,0})$ for each $i = 1, 2$ and $j = 3, 4$ as the maximum flux that can be obtained by a simple wave solution on each road.

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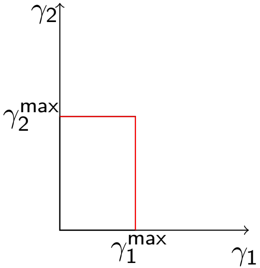
- Define the sets $\Omega_i = [0, \gamma_i^{\max}(\rho_{i,0})] \quad i = 1, 2.$



Riemann Solver II

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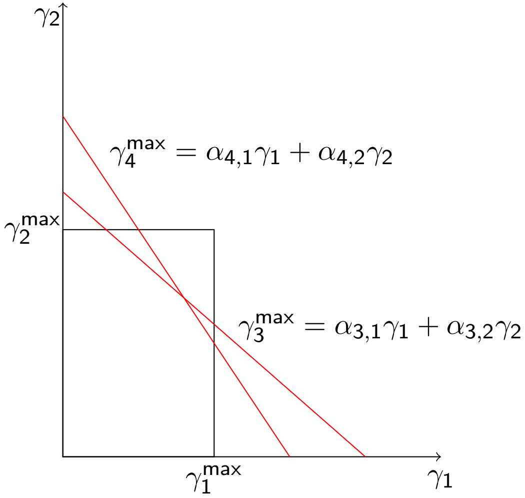

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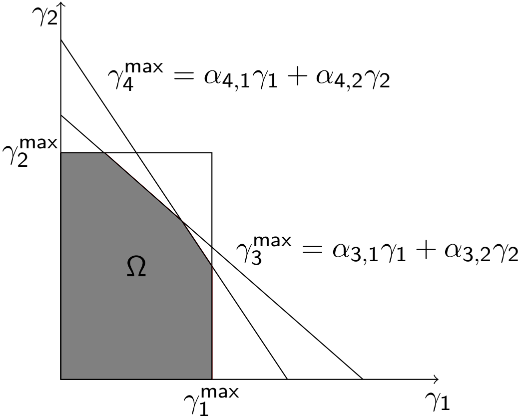
- Define the set $\Omega = \left\{(\gamma_1, \gamma_2) \in \Omega_1 \times \Omega_2 \mid A \cdot (\gamma_1, \gamma_2)^T \in \Omega_3 \times \Omega_4\right\}.$



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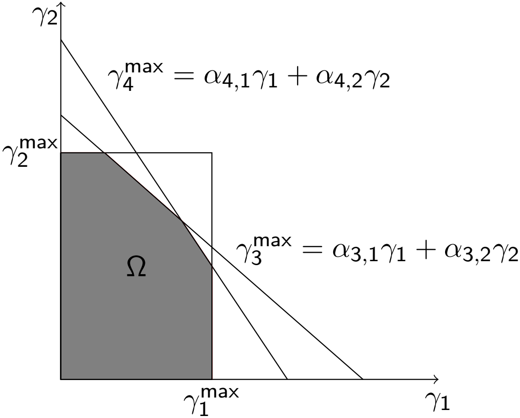

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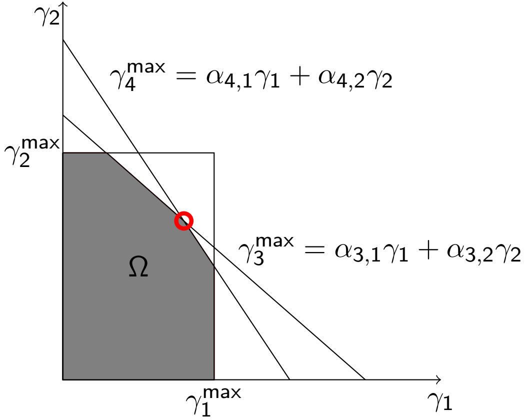
- The set Ω is closed, convex and not empty.
- Unique solution:



2x1 Junction Solution

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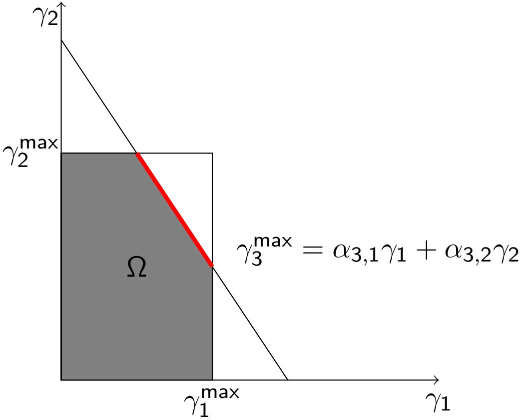

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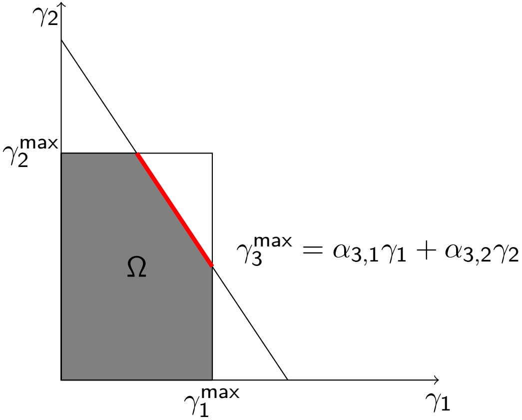


The solution is not unique!

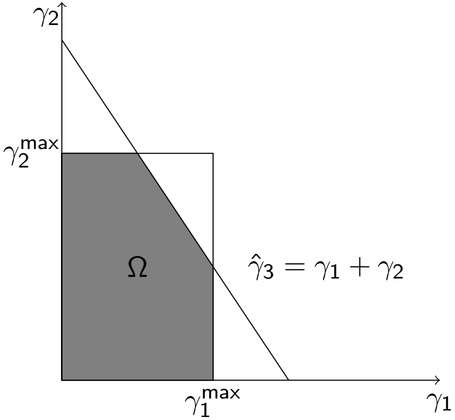
Assumptions

- Introduce a priority parameter $P \in]0,1[$.
- In the outgoing road, $Pf_1(\rho_1(t,b_1+))$ will be the portion of flux coming from the first incoming road, and $(1-P)f_2(\rho_2(t,b_2+))$, the one coming from the second.
- To maximize the traffic going through, we set

$$\hat{\gamma}_3 = \min\left\{\gamma_1^{\max}(\rho_{1,0}) + \gamma_2^{\max}(\rho_{2,0}), \gamma_3^{\max}(\rho_{3,0})\right\}.$$

Riemann Solver

- We follow all the steps as in the other case and what we obtain is $\Omega = \{(\gamma_1, \gamma_2) : 0 \leq \gamma_i \leq \gamma_i^{max}(\rho_{i,0}), 0 \leq \gamma_1 + \gamma_2 \leq \hat{\gamma}_3\}$.



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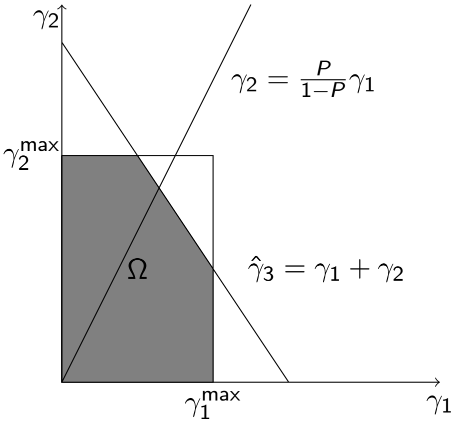

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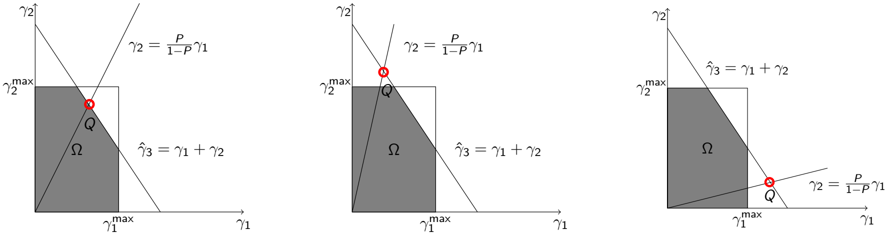
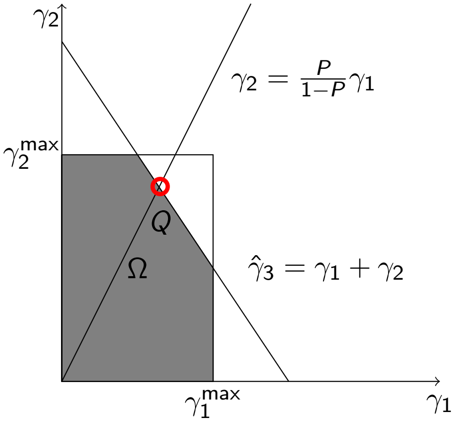
- We draw the line $\gamma_2 = \dfrac{P}{1-P}\gamma_1$.
- Three cases:



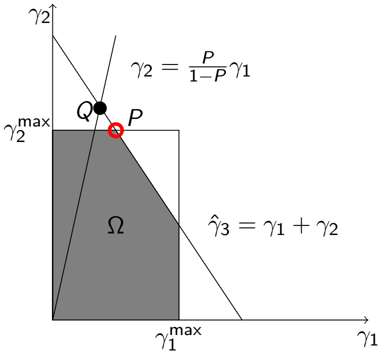
Riemann Solver II

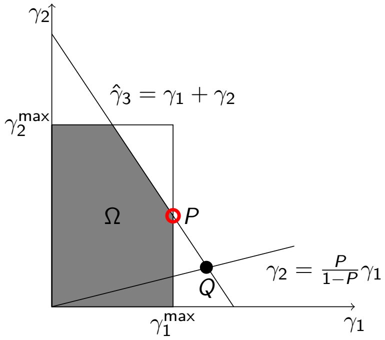
- If $Q \in \Omega$ then Q is the solution.



Riemann Solver II

- If $Q \in \Omega$ then Q is the solution.
- If $Q \notin \Omega$ we set a point P such that $P \in \Omega \cap \{(\gamma_1, \gamma_2) : \gamma_1 + \gamma_2 = \hat{\gamma}_3\}$ closest to priority line.





Riemann Solver III

The solutions we found are in the flux domain. How to go back to the density domain?

Definition

Let $\tau : [0,1] \to [0,1]$ be the map such that:

- $f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0,1]$;
- $\tau(\rho) \neq \rho$ for every $\rho \in [0,1] \setminus \{\rho^{cr}\}$.

The function $\tau(\rho)$ is continuous and well defined. Moreover, it satisfies

$$0 \leq \rho \leq \rho^{cr} \Longleftrightarrow \rho^{cr} \leq \tau(\rho) \leq 1 \qquad 0 \leq \tau(\rho) \leq \rho^{cr} \Longleftrightarrow \rho^{cr} \leq \rho \leq 1.$$

Riemann Solver IV

Given $\rho_i(0,\cdot), \rho_j(0,\cdot)$ for every i, j there exists a $n+m$-tuple $(\hat{\rho}_i, \hat{\rho}_j) \in [0,1]^{n+m}$ such that

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1] & \text{if } 0 \leq \rho_{i,0} \leq \rho^{cr}, \\ [\rho^{cr}, 1] & \text{if } \rho^{cr} \leq \rho_{i,0} \leq 1; \end{cases} \quad \text{...} \tag{2}$$

and

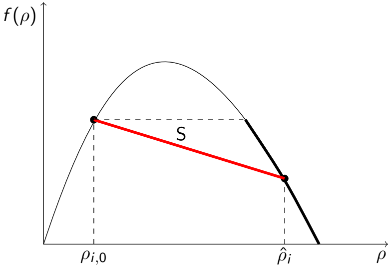
$$\hat{\rho}_j \in \begin{cases} [0, \rho^{cr}] & \text{if } 0 \leq \rho_{j,0} \leq \rho^{cr}, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[& \text{if } \rho^{cr} \leq \rho_{j,0} \leq 1; \end{cases} \quad \text{...} \tag{3}$$

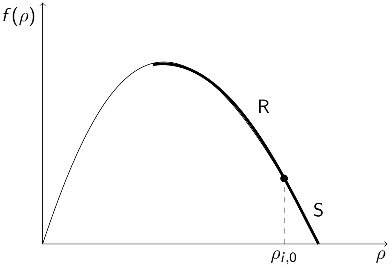
and for the incoming roads the solutions are given by the waves $(\rho_{i,0}, \hat{\rho}_i)$, while for the outgoing road the solutions are given by the waves $(\hat{\rho}_j, \rho_{j,0})$ for every i, j.

Theorem

Riemann Solver V

For the incoming roads

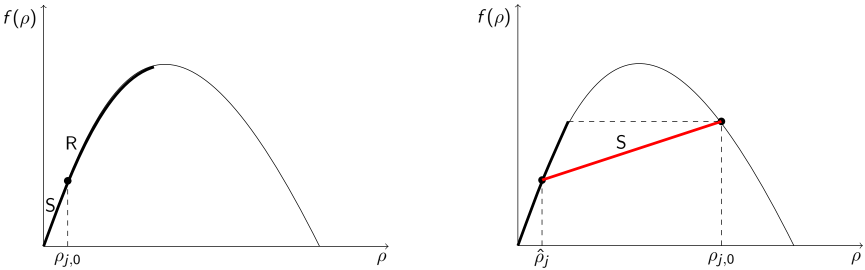




Back

Riemann Solver V

For the outgoing roads



Back

Self-similarity

Theorem

Consider a junction J and a priority parameter $P \in]0,1[$ or a distribution matrix A. Then there exists a unique Riemann Solver $\mathcal{RS}$ such that for a.e. $t > 0$ it holds

$$(\rho_i(t, b_i-), \rho_j(t, a_j+)) = \mathcal{RS}(\rho_i(t, b_i-), \rho_j(t, a_j+)) \quad \forall i, j.$$

Self-similarity and LN-CTM [5]

- Self-similarity: guarantees junction solution only generates one shock
- Subsequent solutions of junction should be stationary, assuming Riemann conditions (see Figure 4)

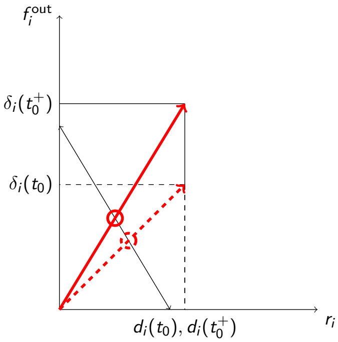


Figure : LN-CTM is not self-similar when considering the limit of the discrete model

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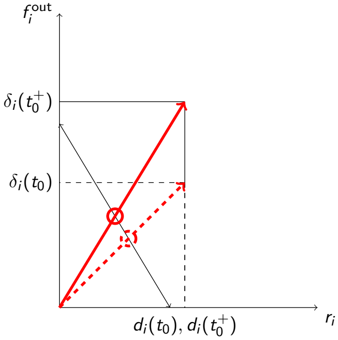

Figure : LN-CTM is not self-similar when considering the limit of the discrete model

Plan

Generalized mainline junction

- Two incoming links
  - Upstream mainline
  - Onramp
- Two outgoing links
  - Downstream mainline
  - Offramp

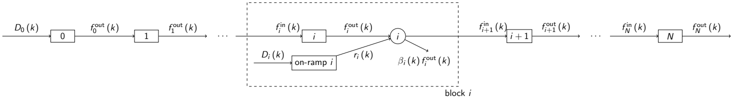

Figure : Mainline model with generalized 2×2 junction.

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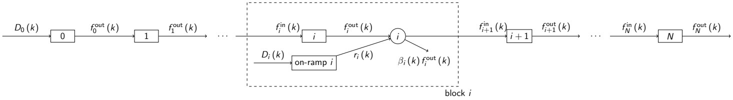

Figure : Mainline model with generalized 2×2 junction.

Weak boundary conditions

- For continuous models, boundary conditions are typically specified as densities
  - Theoretically, one can apply inverse flux map to obtain densities from flux demands.
- Solved as 1×1 junctions.
- Boundary condition **DOES NOT** always apply (see Figure 7). Information from this time step is "lost". OK for estimation, but demand not satisfied for control schemes.

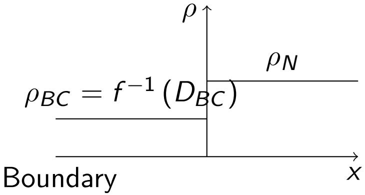

Figure : Riemann problem at boundary with low boundary density ρ_{BC} and high network density ρ_N.

Weak boundary conditions

- For continuous models, boundary conditions are typically specified as densities
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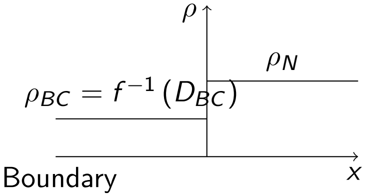

Figure : Riemann problem at boundary with low boundary density ρ_{BC} and high network density ρ_N.

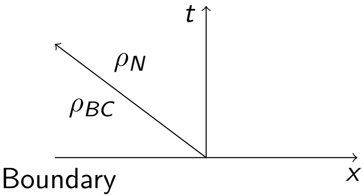

Figure : Boundary density condition does not enter network due to congestion.

Solution to demand conservation: Buffers

Introduce a point queue buffer, $dl_i(t)$, for cell i at time t as the model for onramps:

$$\frac{dl_i(t)}{dt} = \bar{D}_i(t) - r_i(t) \tag{4}$$

where the ramp's demand at the junction is given by:

$$d_i(t) = \begin{cases} r_i^{\max} & l_i(t) > 0 \\ \min\left(r_i^{\max}, \bar{D}_i(t)\right) & \text{otherwise} \end{cases} \tag{5}$$

Fact

Ramp demand is now conserved by definition

Maximizing junction flux blocks onramp

- Using standard Piccoli model, 2×2 junctions permit a unique solution **without** priorities.

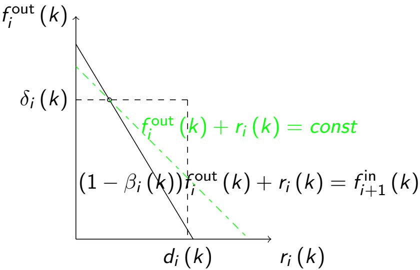

Figure : Flux maximization across junction blocks flux from onramp.

Maximizing junction flux blocks onramp

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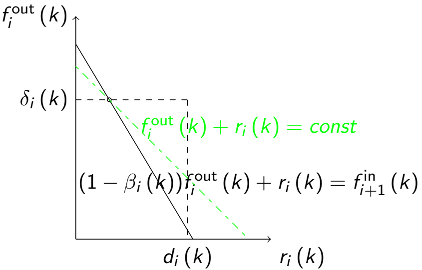
- Since turning ratio from onramp to offramp is 0, mainline flux will reach capacity before onramp flux allowed. See Figure 8



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- **Poor model of reality, where physical priorities still exist**

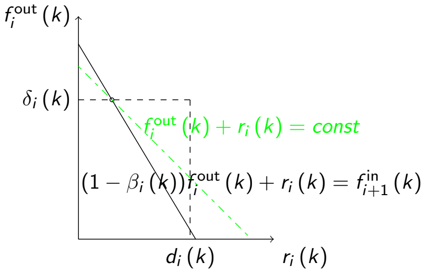


Figure : Flux maximization across junction blocks flux from onramp.

Solution: Maximize flux into downstream mainline

Instead of maximizing total flux across junction, $J = f_i + r_i$, we instead maximize the flux into just the downstream mainline,

$$J' = (1 - \beta) f_i + r_i \tag{6}$$

This makes the supply line and objective lines parallel, thus necessitating the reintroduction of a priority parameter, P.

Note: The same could be accomplished with a multi-objective cost function that maximizes total flux while penalizing deviations from priority.

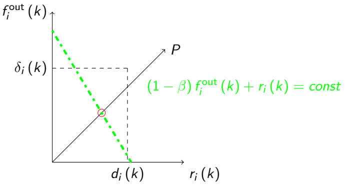


Figure : Reintroduction of priority vector to "unblock" ramp flux.

Plan

Consider a junction with one incoming mainline modeled by the real interval $]-\infty, 0]$ and an outgoing mainline modeled by the interval $[0, +\infty[$, one onramp and one offramp.

$$\begin{cases} \partial_t \rho_i + \partial_x f_i(\rho_i) = 0, & (t,x) \in \mathbb{R}^+ \times I_i, \\ \dfrac{dl(t)}{dt} = D_i(t) - r_i(t), & t \in \mathbb{R}^+, \\ \rho_i(0,x) = \rho_{i,0}(x), & \text{on } I_i \\ l(0) = l_0. & \end{cases} \tag{7}$$

Coupled with the following junction problem

$$\begin{aligned} d(t) &= \begin{cases} r_i^{\max} & \text{if } l(t) > 0, \\ \min\left(D_i(t), r_i^{\max}\right) & \text{if } l(t) = 0, \end{cases} \\ \delta(t) &= \min\left(f^{\max}, v\rho_i\right), \\ \sigma(t) &= \min\left(f^{\max}, w(\rho_{i+1} - \rho_{max})\right), \\ r_{i+1}(t) &= \beta f_i^{\text{out}}(\rho_i(t, 0-)), \\ f_{i+1}^{\text{in}}(\rho_{i+1}(t, 0+)) &= \min\left((1-\beta)\delta(t) + d(t), \sigma(t)\right). \end{aligned} \tag{8}$$

Definition

A collection of functions $(\rho_1, \rho_2, l) \in \prod_{i=1}^{2} \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap \mathrm{BV}(\mathbb{R})\right) \times \mathbf{W^{1,\infty}}(\mathbb{R}^+; \mathbb{R}^+)$ is a solution at J to (7) if

1. ρ_1,ρ_2 are weak solutions on I_1, I_2, i.e.,

$$\left(\int_{\mathbb{R}^+} \int_{I_i} \left(\rho_i \partial_t \varphi_i + f(\rho_i) \partial_x \varphi_i \right) dxdt \right) = 0 \quad i = 1, 2, \tag{9}$$

   for every $\varphi_i \in \mathcal{C}_c^1(\mathbb{R}^+ \times I_i)$.

2. $f_i^{\text{out}}(\rho_i(t, 0-)) + r_i(t) = f_{i+1}^{\text{in}}(\rho_{i+1}(t, 0+)) + r_{i+1}(t)$;
3. minimize $dist\,(\mathbf{f}, P)$ such that $f_{i+1}^{\text{in}}(\rho_{i+1}(t, 0+))$ is maximized;
4. l is a solution of the ODE for a.e. $t \in \mathbb{R}^+$.

Main Theorem

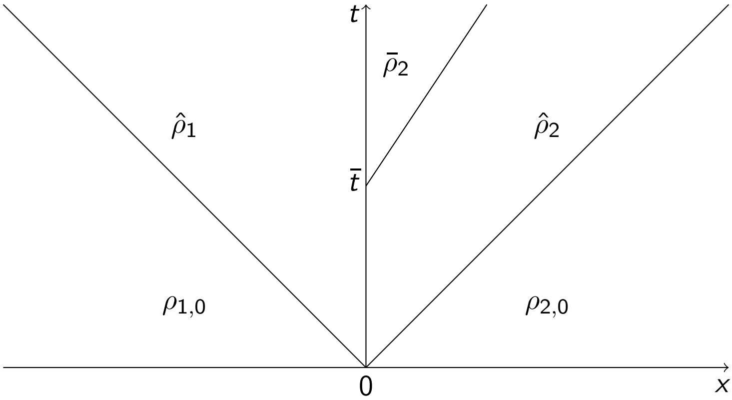
If we consider one junction with one incoming mainline modeled by the real interval $]-\infty, 0]$ and an outgoing mainline modeled by the interval $[0, +\infty[$, one onramp and one offramp it holds:

Theorem

Consider a junction J and fix a priority parameter $P \in]0, 1[$. *For every* $\rho_{1,0}, \rho_{2,0} \in [0, 1]$ *and* $l_0 \in [0, +\infty]$, *there exists a unique admissible solution* $(\rho_1(t, x), \rho_2(t, x), l(t))$ *in the sense of Definition 4 such that for a.e.* $t > 0$ *it holds*

$$(\rho_1(t, 0-), \rho_2(t, 0+)) = \mathcal{RS}_{l(t)}(\rho_1(t, 0-), \rho_2(t, 0+)).$$

Solution of $\mathcal{RS}$



Define $\gamma_{1,t} = f_i^{\text{out}}(\rho(t, 0-))$, $\gamma_{2,t} = f_{i+1}^{\text{in}}(\rho(t, 0+))$, $\gamma_{r,t} = r_i(t)$ and $\gamma_{1,t}^{\max} = \delta(t)$, $\gamma_{2,t}^{\max} = \sigma(t)$, $\gamma_{r,t}^{\max} = d(t)$.

Some Properties of the Riemann Solver:

- $\gamma_{i,t_0} = \gamma_{i,t_0}^{\max} \Rightarrow \gamma_{i,t_0+}^{\max} = \gamma_{i,t_0}^{\max}\ \forall i = 1, 2, r.$

  This holds because of the definitions of the functions $\tau(\rho)$, $\delta(t)$ and $\sigma(t)$ for the mainlines and the definition of $d(t)$ for the ramp.

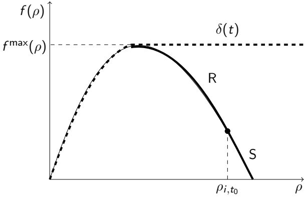


Figure : Incoming road.

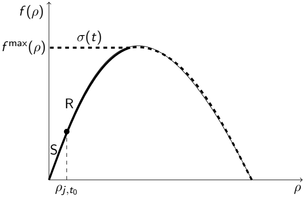


Figure : Outgoing road.

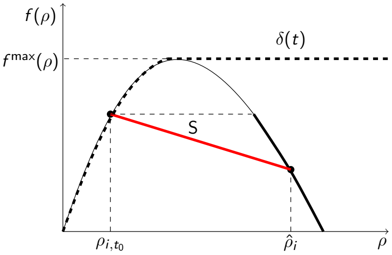
For the ramp:

$D > r^{\max} = \gamma_{r,t_0}^{\max} \Rightarrow \gamma_{r,t_0}^{\max} = \min\left(D, r^{\max}\right) = r^{\max} = \gamma_{r,t_0+}^{\max}.$

Properties of the $\mathcal{RP}$ II

- $\gamma_{i,t_0} < \gamma_{i,t_0}^{\max} \Rightarrow \gamma_{i,t_0}^{\max} \leq \gamma_{i,t_0+}^{\max} \; \forall i = 1, 2, r.$

  This holds because of the definitions of $\tau(\rho)$, $\delta(t)$, $\sigma(t)$ for the mainlines and the definition of $d(t)$ for the ramp.



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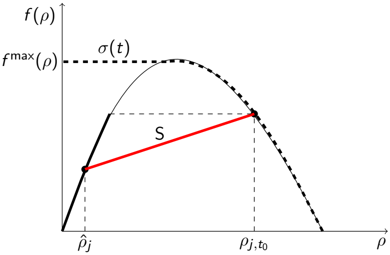


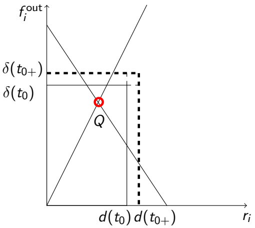
Figure : Outgoing road.

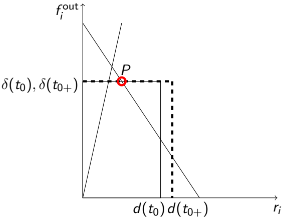
For the ramp: $\forall D, r^{\max} : r^{\max} \geq \min\left(D, r^{\max}\right).$

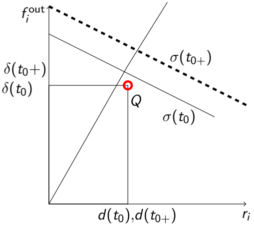
Properties of the $\mathcal{RP}$ III

- If the solution at time t_0 lies on a limiting side then the solution at time t_0+ will lie there too.
- The limiting side does not change.
  At time t_0+ the non limiting side can only increase while the limiting side's feasible set remains the same.

Sketch of the proof:







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Questions?

Thank you for your attention!