A SET OF FIVE INDEPENDENT POSTULATES FOR BOOLEAN ALGEBRAS, WITH APPLICATION TO LOGICAL CONSTANTS*

BY

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Introduction.

Postulate-sets for determining the class of Boolean algebras† have been given by SCHRÖDER,‡ WHITEHEAD,§ and HUNTINGTON.‖ SCHRÖDER's set of ten postulates assumes—in addition to an undefined class K, common to all these postulate-sets—an undefined dyadic relation, $\nleqq$, and BOOLE's¶ undefined binary K-rules** of combination, $+$ and $\times$; WHITEHEAD's two sets, the first of thirteen, and the second of fifteen, postulates, and HUNTINGTON's first set, of ten postulates, assume the same undefined K-rules of combination, $+$ and $\times$, which HUNTINGTON writes respectively $\oplus$ and $\odot$; HUNTINGTON's second set, of nine postulates, assumes SCHRÖDER's undefined relation $\nleqq$,

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† We employ the term *Boolean algebras* in its plural form for the following reasons: (1) none of the equivalent postulate-sets here referred to is in terms of its undefined entities one-valued ("categorical")—that is, each determines not a single algebra but a class of algebras; one should not speak, therefore, of "*der* identische Kalkul" (SCHRÖDER), "*the* Algebra of Symbolic Logic" (WHITEHEAD), or "*the* algebra of logic" (HUNTINGTON); (2) PEANO's *Formulario Mathematico* and WHITEHEAD and RUSSELL's *Principia Mathematica*, each of which includes, as a part, the algebras under consideration, have a far stronger title to the name "algebra of logic"; (3) "The Algebra of Symbolic Logic, viewed as a distinct algebra, is due to Boole" (WHITEHEAD, *loc. cit.*, p. 115). "This algebra in all its essential particulars was invented and perfected by Boole" (*ib.*, p. 35, footnote).

‡ ERNST SCHRÖDER: *Vorlesungen über die Algebra der Logik (Exakte Logik)*, Erster Band, 1890. The postulates, under various names, are scattered throughout the volume; collected into one list by E. MÜLLER: *Abriss der Algebra der Logik*, Erster Teil (1909), pp. 20, 21.

§ A. N. WHITEHEAD: *A Treatise on Universal Algebra*, Vol. I, 1898, pp. 35–37.

‖ E. V. HUNTINGTON: *Sets of Independent Postulates for the Algebra of Logic.* These T r a n s a c t i o n s , vol. 5 (1904), pp. 288–309.

¶ G. BOOLE: *An Investigation of the Laws of Thought.* London, 1854.

** An *n-ary rule of combination* ϕ, is an agreement according to which any n (distinct or non-distinct) logical entities, a_1, a_2, a_3, $\cdots$, in a definite order, determine a unique logical entity $\phi (a_1, a_2, a_3, \cdots)$; in other words, a rule of combination is a *one-valued* logical function. If the entity $\phi (a_1, a_2, a_3, \cdots)$ is defined for all those and only those cases where all the n entities $a_1, a_2, a_3, \cdots$ are elements of some class K, then ϕ is a *K-rule of combination*; for a *binary* K-rule of combination, $\phi (a, b)$ is also conveniently written $a \circ b$. If, for a K-rule of combination ϕ, the entity $\phi (a_1, a_2, a_3, \cdots)$ is always a K-element, then ϕ is *K-closed*.

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which he writes $\oslash$; and his third set, of nine postulates, the undefined K-rule of combination $\oplus$. Each of these sets contains three existence-postulates, namely, those demanding the existence of (1) the special Boolean* element z; (2) the special Boolean* element u; and (3) for any K-element a, its corresponding Boolean element $\bar{a}$. The independence of all the postulates of each set is proved only for Huntington's sets; and Huntington was the first to show that any two of the concepts $\oplus$, $\odot$, and $\oslash$ are definable in terms of the third.

In this paper we offer, in § 1, a set of five independent postulates for Boolean algebras. This set, which like Huntington's third set assumes but one undefined K-rule of combination, differs from the previous sets (1) in the small number of postulates, and (2) in the fact that the set contains no existence-postulate for z, u, or $\bar{a}$.

In § 2 we apply our results to the problem of reducing the number of primitive logical constants.†

§ 1.   *Postulate-Set for Boolean Algebras.*

We assume:

I. A class K,

II. A binary K-rule of combination $|$,

III. The following properties of K and $|$:

1. There are at least two distinct K-elements.

2. Whenever a and b are K-elements, $a \mid b$ is a K-element.

$$Def. \quad a' = a \mid a.$$

3. Whenever a and the indicated combinations of a are K-elements,

$$(a')' = a.$$

4. Whenever a, b, and the indicated combinations of a and b are K-elements,

$$a \mid (b \mid b') = a'.$$

5. Whenever a, b, c, and the indicated combinations of a, b, and c are K-elements,

$$(a \mid (b \mid c))' = (b' \mid a) \mid (c' \mid a).$$

For convenience, $a \mid b$ may be read a *per* b.

Classification of Postulates 1–5.

Postulate 1 is an existence-postulate. Postulate 2, which demands that the K-rule of combination $|$ shall be K-closed,‡ is a *K-closing* postulate.

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* For z, BOOLE and SCHRÖDER write 0; for u, 1.   *Cf.* p. 486, footnote †.

† WHITEHEAD and RUSSELL: *Principia Mathematica*, vol. I, 1910, pp. 94–101.

‡ See p. 481, footnote **.

Postulate 3, which demands that a and $(a')'$ shall always be names for the same K-element—that the names a and $(a')'$ shall always be equivalent—is an *equivalence* postulate; so are 4 and 5.

Thus our set consists of an existence-postulate, a K-closing postulate, and three equivalence postulates. Moreover, if we do not wish to exclude systems which have but a single element, then 1 may be replaced by the weaker postulate

1'. There is at least one K-element.

Consistence of Postulates 1–5.

With the following interpretation of K and $|$, postulates 1–5 are satisfied: K has only two distinct elements, m and n; $m|m = n$, $m|n = n|m = n|n = m$.

Independence of Postulates 1–5.

With each of the interpretations of K and $|$ given in (1)–(5) below, all the postulates, except the one correspondingly numbered, are satisfied; that postulate is, therefore, independent of the remaining four.

(1) K has only one element m; $m \mid m = m$.

(2) K has any number, greater than one, of distinct elements; for any K-element m, $m \mid m = m$; for any two distinct K-elements, m and n, $m \mid n$ is not a K-element.

(3) K has only two distinct elements, m and n; $m \mid m = m \mid n = n \mid m = n \mid n = m$.

(4) K is the class of all rationals; for any K-elements, m and n, $m \mid n = -\frac{1}{2}(m+n)$. Postulate 4 holds only when $m = 0$.

(5) K has only three distinct elements, l, m, and n; $|$ is defined by the following table (for example: $m \mid l = n$).

$\mid$	l	m	n
l	l	m	n
m	n	n	l
n	m	l	m

Deductions from Postulates 1–5.

The proofs are given after theorem V.

A. Whenever a and b are K-elements, $a \mid b = b \mid a$.

B. Whenever a and b are K-elements, $a \mid a' = b \mid b'$.

Ia. Whenever a and b are K-elements, $(a \mid b)'$ is a K-element.

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Ib. Whenever a and b are K-elements, $a' \mid b'$ is a K-element.

IIa. There is a K-element z such that for any K-element a, $(a \mid z)' = a$.

IIb. There is a K-element u such that for any K-element a, $a' \mid u' = a$.

IIIa. Whenever $a, b, (a \mid b)'$, and $(b \mid a)'$ are K-elements, $(a \mid b)' = (b \mid a)'$.

IIIb. Whenever $a, b, a' \mid b'$, and $b' \mid a'$ are K-elements, $a' \mid b' = b' \mid a'$.

IVa. Whenever a, b, c, $(a \mid b)'$, $(a \mid c)'$, $b' \mid c'$, $[a \mid (b' \mid c')]'$, and $[(a|b)']'|[(a|c)']'$ are K-elements, $[a|(b'|c')]' = [(a|b)']'|[(a|c)']'$.

IVb. Whenever a, b, c, $a' \mid b'$, $a' \mid c'$, $(b \mid c)'$, $a' \mid [(b \mid c)']'$, and $[(a'|b')|(a'|c')]'$ are K-elements, $a'|[(b|c)']'=[(a'|b')|(a'|c')]'$.

V*. If z and u of IIa and IIb are unique K-elements, then for any K-element a there is a K-element $\bar{a}$ such that $(a \mid \bar{a})' = u$ and $a' \mid (\bar{a})' = z$.

Proofs of the Preceding Theorems.

In the following proofs the use of postulate 2 is not always explicitly mentioned.

Proof of A.

$$a \mid b = [(a \mid b)']' \qquad \text{[by 3]}$$

$$= [(a \mid \{ b' \}')']' \quad \text{[by 3}$$

$$= [(\{ b' \}' \mid a)']' \quad \text{[by 5, } b \text{ replaced by } b' \text{ and } c \text{ by } b']$$

$$= [(b \mid a)']' \qquad \text{[by 3]}$$

$$= b \mid a \qquad\qquad \text{[by 3]}.$$

Proof of B.

$$a \mid a' = [(a \mid a')']' \qquad\qquad \text{[by 3]}$$

$$= [(a \mid a') \mid (b \mid b')]' \quad \text{[by 4, } a \text{ replaced by } a \mid a']$$

$$= [(b \mid b') \mid (a \mid a')]' \quad \text{[by } A]$$

$$= [(b \mid b')']' \qquad\qquad \text{[by 4, } a \text{ replaced by } b \mid b' \text{ and } b \text{ by } a]$$

$$= b \mid b' \qquad\qquad\qquad \text{[by 3]}.$$

Proof of Ia. Use 2 twice.

Proof of Ib. Use 2 thrice.

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* Other theorems, the proofs of which we omit, are:

C. $(a|b)|(a|b') = a$.

D. $[a'|(a|b)]' = a|(a'|b')$.

E. $[(a|b')|(a'|b)]' = (a|b)|(a'|b')$.

F. $a|[a|(b|c)] = b|[b|(c|a)] = c|[c|(a|b)]$
    $= a|(b|c)' \qquad = b|(c|a)' \qquad = c|(a|b)'$.

G. $(a'|a)|[(b'|a)|(c'|a)] = a(b|c)$.

Proof of IIa.

  There is a K-element—say, x [by 1].

  There is a K-element, $x \mid x'$, which we call z [by 2, used twice].

$$(a \mid z)' = (a')' \qquad\qquad\qquad \text{[by 4, } b \text{ replaced by } x\,]$$
$$= a . \qquad\qquad\qquad\qquad \text{[by 3].}$$

Proof of IIb.

  There is a K-element, z', which we call u [by IIa and 2].

$$a' \mid u' = a' \mid (z')'$$
$$= a' \mid z \quad \text{[by 3]}$$
$$= (a')' \quad \text{[by 4, } a \text{ replaced by } a' \text{ and } b \text{ by } x\,]$$
$$= a \qquad \text{[by 3].}$$

Proof of IIIa. $a \mid b = b \mid a$ [by A].

               Hence $(a \mid b)' = (b \mid a)'$.

Proof of IIIb. $a' \mid b' = b' \mid a'$

                     [by A, a replaced by a' and b by b'].

Proof of IVa. $[\,a \mid (b' \mid c')\,]' = [\,(b')' \mid a\,] \mid [\,(c')' \mid a\,]$

                     [by 5, b replaced by b' and c by c']

$$= (b \mid a) \mid (c \mid a) \qquad\qquad \text{[by 3, used twice]}$$
$$= (a \mid b) \mid (a \mid c) \qquad\qquad \text{[by } A \text{, used twice]}$$
$$= [\,(a \mid b)'\,]' \mid [\,(a \mid c)'\,]' \qquad \text{[by 3, used twice].}$$

Proof of IVb.

$$a' \mid [\,(b \mid c)'\,]' = a' \mid (b \mid c) \qquad\qquad \text{[by 3]}$$
$$= [\,\{\, a' \mid (b \mid c)\,\}'\,]' \qquad \text{[by 3]}$$
$$= [\,(b' \mid a') \mid (c' \mid a')\,]' \quad \text{[by 5, } a \text{ replaced by } a'\,]$$
$$= [\,(a' \mid b') \mid (a' \mid c')\,]' \quad \text{[by } A \text{, used twice].}$$

Proof of V.

  Take $\bar{a} = a'$.   Then

$$(a \mid a')' = (x \mid x')' \quad \text{[by } B \text{, } b \text{ replaced by } x\,]$$
$$= z' \qquad \text{[by IIa]}$$
$$= u \qquad \text{[by IIb].}$$

$$a' \mid (a')' = a' \mid a \qquad \text{[by 3]}$$
$$= a \mid a' \qquad \text{[by } A\,]$$
$$= z \qquad\quad \text{[by } B \text{, } b \text{ replaced by } x\,].$$

<center>*Postulates* 1–5 *and Boolean Algebras.*</center>

The following is Huntington's first postulate-set* for Boolean algebras:

" [For this postulate-set] we take as the fundamental concepts a class K with two [binary K-] rules of combination $\oplus$ and $\odot$; and as the fundamental propositions the following ten postulates:

Ia. $a \oplus b$ is in the class whenever a and b are in the class.

Ib. $a \odot b$ is in the class whenever a and b are in the class.

IIa.   There is an element† z such that $a \oplus z = a$ for every element a.

IIb.   There is an element† u such that $a \odot u = a$ for every element a.

IIIa. $a \oplus b = b \oplus a$ whenever a, b, $a \oplus b$, and $b \oplus a$ are in the class.

IIIb. $a \odot b = b \odot a$ whenever a, b, $a \odot b$, and $b \odot a$ are in the class.

IVa. $a \oplus (b \odot c) = (a \oplus b) \odot (a \oplus c)$ whenever a, b, c, $a \oplus b$, $a \oplus c$,
       $b \odot c$, $a \oplus (b \odot c)$, and $(a \oplus b) \odot (a \oplus c)$ are in the class.

IVb. $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ whenever a, b, c, $a \odot b$, $a \odot c$,
       $b \oplus c$, $a \odot (b \oplus c)$, and $(a \odot b) \oplus (a \odot c)$ are in the class.

V.   If the elements z and u in postulates IIa and IIb exist and are unique,
      then for every element a there is an element $\bar{a}$ such that $a \oplus \bar{a} = u$
      and $a \odot \bar{a} = z$.

VI.   There are at least two elements, x and y, in the class such that $x \neq y$."‡

That set 1–5 is a postulate-set for Boolean algebras we shall prove by showing that this set and Huntington's first postulate-set are equivalent.

Proof.—If for any elements, a and b, of our class K we write

$$\bar{a} \text{ for } a', \quad a \oplus b \text{ for } (a \mid b)', \quad \text{and} \quad a \odot b \text{ for } a' \mid b',$$

theorems Ia-V and postulate 1 are precisely Huntington's first postulate-set; hence set 1–5 implies Huntington's set.

If for any elements, a and b, of Huntington's class we write

$$a \mid b \text{ for } \bar{a} \odot \bar{b},$$

Huntington's set implies set 1–5.§

<center>§ 2.   *Application to Primitive Logical Constants.*</center>

Since not only in special deductive systems but even in the foundations of logic not all propositions can be proved and not all non-propositional entities

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* HUNTINGTON, *loc. cit.*, pp. 292–3.

† For z and u respectively HUNTINGTON uses the symbols $\wedge$ and $\vee$, which he takes from PEANO's *Formulaire de Mathématiques*. These are, however, symbols for *logical constants*, just as 0 and 1 are symbols for *numerical constants*. We have replaced, therefore, BOOLE's and SCHRÖDER's 0 and 1, and HUNTINGTON's $\wedge$ and $\vee$, by z and u.

‡ That is, such that x and y are distinct.

§ By the " principle of duality " the results of § 1 hold also when $a|b$ is interpreted throughout as $\bar{a} \oplus \bar{b}$.

can be defined, some logical constants* must be primitive,† that is, either
unproved or undefined. A list of primitive logical constants—primitive
ideas and primitive propositions—in terms of which, presumably,‡ all other
logical constants can be either defined or proved, is given by Whitehead and
Russell in their *Principia Mathematica*.§ This list contains, among other
logical constants, the primitive ideas‖ *negation* (symbolized $\backsim$) and *dis-
junction* (symbolized v). Negation and disjunction are partly explained—
but, of course, not at all defined—by the statement that for any elementary
proposition p (*elementary proposition* being itself one of the *Principia's* primitive
ideas), $\backsim p$ means the elementary proposition *not-p*; and for any two elemen-
tary propositions, p and q, $p \vee q$ means the elementary proposition *either p
or q* (*or both*).

On these two primitive ideas, in view of the following interpretation of K
and $|$, our set 1–5 has an important bearing. For, if K is the class of all
propositions of a given logical type,¶ then whenever p and q are two propo-
sitions of this type, $p \mid q$ may be interpreted as the proposition *neither p nor q*;
in other words, $|$ has the properties of the logical constant *neither-nor*. This
logical constant we may symbolize by $\wedge$, and for obvious reasons we may
name *rejection*.**

THEOREM 1. *If in any list of primitive ideas for logic both negation and dis-
junction are primitive, they may be replaced by the single primitive idea rejection.*

Proof.—In terms of negation and disjunction, rejection is defined by the
Def.—For any two elementary propositions, p and q,

$$p \wedge q = \backsim (p \vee q).$$

In terms of rejection, negation is defined †† by the
Def.—For any elementary proposition p, $\backsim p = p \wedge p$.
In terms of rejection, disjunction is defined by the
Def.—For any two elementary propositions, p and q,

$$p \vee q = (p \wedge q) \wedge (p \wedge q).$$

By the following theorem, a similar reduction is possible for primitive
propositions.

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* WHITEHEAD and RUSSELL, *loc. cit.*, pp. 94–101.

† *Ib.*, p. 95.

‡ ". . . there must always be some element of doubt, since it is hard to be sure that one
never uses some principle unconsciously" (*ib.*, p. 94).

§ WHITEHEAD and RUSSELL, *loc. cit.*, partial list, pp. 95–101; the other primitives are scat-
tered throughout the rest of the book.

‖ *Ib.*, p. 97.

¶ *Ib.*, pp. 39–68.

** By analogy with *subject* and *object*, we may call $p \wedge q$ the *reject* of p and q .

†† Negation may thus be considered as a special case of rejection.

Two primitive propositions of the *Principia** are:

* 1.7.   If p is an elementary proposition, $\backsim p$ is an elementary proposition.

* 1.71.   If p and q are elementary propositions, $p \vee q$ is an elementary proposition.

THEOREM 2.   *If in any list of primitive propositions for logic both* * 1.7 *and* * 1.71 *are primitive, they may be replaced by the single primitive proposition.*

* 1.7′.   *If p and q are elementary propositions, $p \wedge q$ is an elementary proposition.*

Proof.—If in * 1.7 we replace p by $p \vee q$, * 1.7 and * 1.71 imply * 1.7′. If in * 1.7′ we replace q by p, * 1.7′ implies * 1.7.

If in * 1.7′ we replace p by $p \wedge q$ and q by $p \wedge q$, * 1.7′, used twice, implies * 1.71.†

Thus we have made it possible to reduce, by one each, the number of primitive ideas and of primitive propositions used in the *Principia* for the foundation of logic.

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* WHITEHEAD and RUSSELL, *loc. cit.*, p. 101.

† By the "principle of duality" the results of § 2 hold also when $p \wedge q$ is interpreted throughout as the logical constant *either not-p or not-q.*