Math 206

Linear Algebra and Matrix Theory

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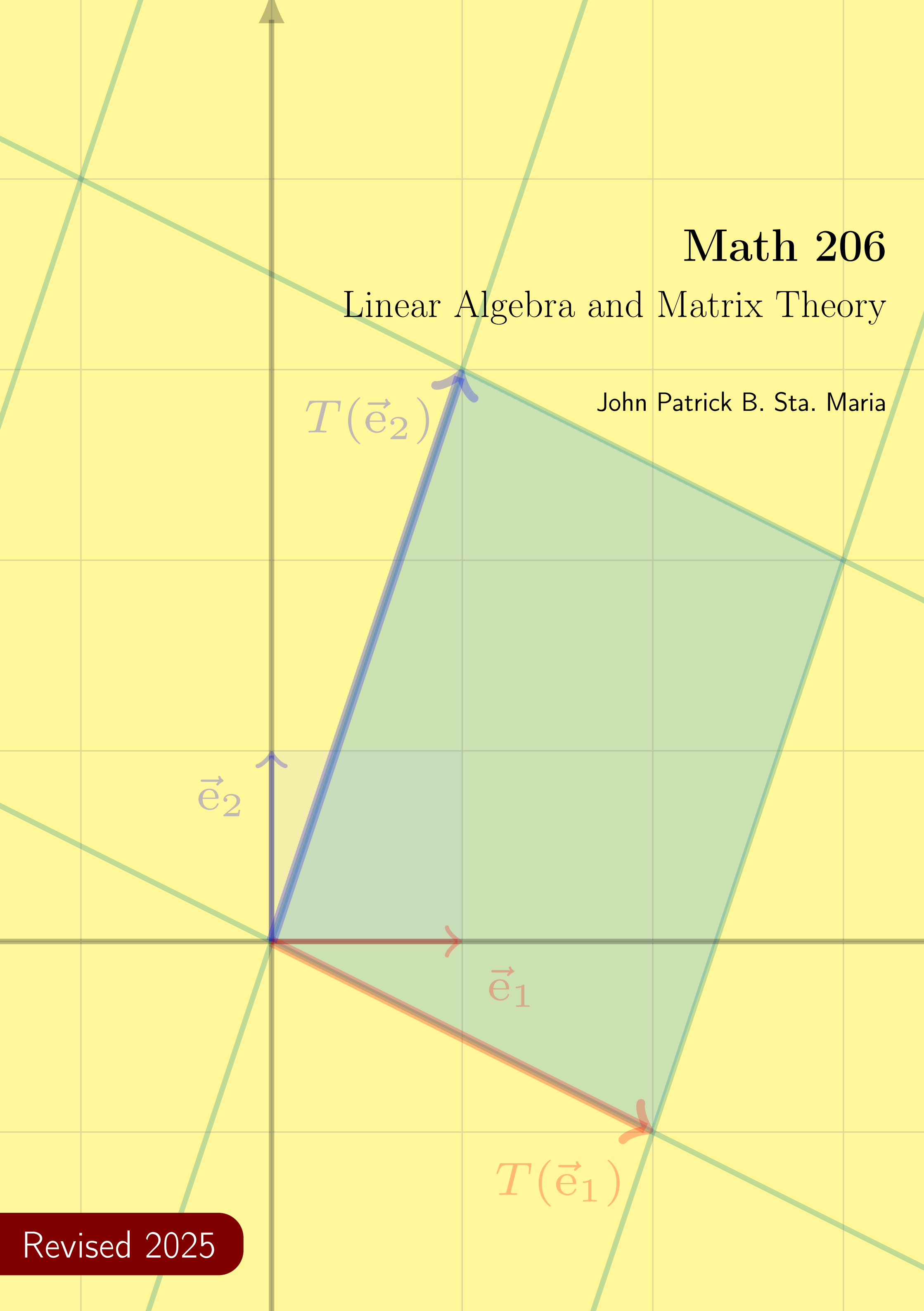

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Lesson 2.2: Nonsingular Matrices

Learning Outcomes. At the end of the lesson, the students are able to:

1. establish necessary and sufficient conditions for a square matrix to be nonsingular.

2. calculate inverses of nonsingular matrices.

3. use the properties of matrix inverse to solve linear systems.

Inverse of a Nonsingular Matrix

Let m and n be positive integers. The set of all m-by-n real matrices is denoted by $M_{m,n}(\mathbb{R})$. We also write $M_n(\mathbb{R}) := M_{n,n}(\mathbb{R})$. A matrix $A \in M_n(\mathbb{R})$ is called a **square matrix**.

Definition 85

A matrix $A \in M_n(\mathbb{R})$ is **nonsingular** if there exists $B \in M_n(\mathbb{R})$ such that

$$AB = I_n = BA.$$

Otherwise, we say that A is **singular**.

Let $A \in M_n(\mathbb{R})$ be nonsingular. Suppose that $B, \hat{B} \in M_n(\mathbb{R})$ such that

$$AB = I_n = BA \tag{2.6}$$

and

$$A\hat{B} = I_n = \hat{B}A. \tag{2.7}$$

Equations 2.6 and 2.7 imply that

$$B = BI_n = B(A\hat{B}) = (BA)\hat{B} = I_n\hat{B} = \hat{B}.$$

Consequently, the matrix B in Definition 85 is necessarily unique when A is nonsingular. The unique matrix B is called the **inverse** of the matrix A and is commonly denoted as

A^{-1}. Therefore, if A is nonsingular, then

$$AA^{-1} = I_n = A^{-1}A.$$

Example 86. The identity matrix I_n is nonsingular since $I_nI_n = I_n$. Furthermore, $I_n^{-1} = I_n$. On the other hand, the zero matrix 0_n is singular since $A0_n = 0_n \neq I_n$ for any $A \in M_n(\mathbb{R})$.

Example 87. Determine if the matrix $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ is nonsingular, and if so find A^{-1}.

Solution. Suppose that $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, and d are scalar. The relation $AA^{-1} = I_n$ means

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Performing the matrix multiplication on the left hand side and comparing columns yield the following systems of equations:

$$\begin{cases} a + 3c = 1 \\ a + 4c = 0 \end{cases} \quad \text{and} \quad \begin{cases} b + 3d = 0 \\ b + 4d = 1 \end{cases}$$

These linear systems may be solved by performing row-reduction to the following augmented matrices:

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & 4 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 4 & 1 \end{array}\right]$$

Since the coefficient matrices of the augmented matrices are identical, we can solve both the systems simultaneously by row-reducing the following matrix:

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

Performing a sequence of elementary row operation yield

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & 4 & -3 \\ 0 & 1 & -1 & 1 \end{array}\right].$$

It follows that

- RREF $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & 4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right] \implies \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$; and
- RREF $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 4 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \implies \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Therefore, A^{-1} exists and

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}.$$

One may verify this answer by verifying that $AA^{-1} = I_2$. ▨

We may generalize the method introduced in the previous example into matrices of arbitrary sizes.

Theorem 88: Method for Finding A^{-1}

Let $A \in M_n$. If $\mathrm{RREF}[A \mid I_n] = [I_n \mid B]$, then $B = A^{-1}$.

Example 89. For each of the following, find A^{-1} if it exists.

1. $A = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 3 & -3 \\ 3 & -10 & 9 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -4 & 3 \\ -1 & 1 & 0 \\ 0 & -3 & 3 \end{bmatrix}$

Solution.

1. One may verify that

$$\left.\begin{array}{ccc|ccc} 1 & -4 & 2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 0 & 1 & 0 \\ 3 & -10 & 9 & 0 & 0 & 1 \end{array}\right. \xrightarrow{\mathrm{RREF}} \left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -16 & -6 \\ 0 & 1 & 0 & 0 & -3 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right..$$

Therefore, A is nonsingular and $A^{-1} = \begin{bmatrix} 3 & -16 & -6 \\ 0 & -3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$.

2. Direct calculation shows that

$$\left.\begin{array}{ccc|ccc} 1 & -4 & 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & 3 & 0 & 0 & 1 \end{array}\right. \xrightarrow{\text{RREF}} \left.\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & -1/3 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right. .$$

   This means that the associated linear systems for finding A^{-1} is inconsistent. Thus, A^{-1} does not exist. Therefore, A is singular.

Example 90. Consider a general 2-by-2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Observe that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)I_2.$$

If $ad - bc \neq 0$, it follows that

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

On the other hand, if $ad - bc = 0$, then

$$\left.\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right. \xrightarrow{E_2(b)} \left.\begin{array}{cc|cc} a & b & 1 & 0 \\ bc & bd & 0 & b \end{array}\right. \xrightarrow{E_{21}(-d)} \left.\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 0 & -d & b \end{array}\right.,$$

which implies that A^{-1} does not exist. This means that $ad - bc \neq 0$ is a necessary and sufficient condition for the matrix to be nonsingular.

We now prove some properties of nonsingular matrices.

Theorem 91

Let A and B be an nonsingular matrices.

1. The matrix A^{-1} is nonsingular and

$$(A^{-1})^{-1} = A.$$

2. If r is a nonzero scalar, then rA is nonsingular and

$$(rA)^{-1} = r^{-1}A^{-1}.$$

3. The product AB is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

4. If k is a positive integer, A^k is nonsingular and

$$(A^k)^{-1} = (A^{-1})^k.$$

Proof. We shall only prove statement 3. Assume A and B are nonsingular n-by-n matrices. Then A^{-1} and B^{-1} both exist. Consider the matrix $X = B^{-1}A^{-1}$. Then

$$ABX = ABB^{-1}A^{-1} = I_n,$$

and

$$XAB = B^{-1}A^{-1}AB = I_n.$$

Therefore, $(AB)^{-1} = X = B^{-1}A^{-1}$. ▨

We like to remark that statement (3) may be extended to hold by induction to an arbitrary number of matrices. If $A_1, A_2, \ldots, A_k$ are nonsingular matrices of the same size, then $A_1 A_2 \cdots A_k$ is nonsingular and

$$(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}.$$

Example 92. Consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

From Example 90, both A and B are nonsingular. However,

$$A + B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

is singular. This means that the sum of two nonsingular matrices is not necessarily nonsingular.

Nonsingularity and Linear Systems

Consider the linear system

$$A\vec{x} = \vec{b},$$

where $A = [a_{ij}]$, $\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^t$ and $\vec{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^t$. Let $A \in M_n$ be nonsingular. Suppose that A is nonsingular. Then

$$\begin{aligned} A^{-1}(A\vec{x}) = A^{-1}\vec{b} &\implies (A^{-1}A)\vec{x} = A^{-1}\vec{b} \\ &\implies I_n\vec{x} = A^{-1}\vec{b} \implies \vec{x} = A^{-1}\vec{b}. \end{aligned}$$

The following theorem follows.

Theorem 93

If A is a nonsingular matrix, then the linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$.

Example 94. To illustrate the theorem, consider the linear system

$$\begin{cases} x + 3y = 5 \\ x + 4y = 7 \end{cases}.$$

Equivalently, we may write the linear system as the matrix equation

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}.$$

Since $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ is nonsingular and $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$, the theorem above implies that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Therefore, $x = -1$ and $y = 2$.

As a consequence of Theorem 93, we have the following.

Corollary 95

If A is nonsingular, then the homogeneous linear system $A\vec{x} = \vec{0}$ only has the trivial solution, i.e., $N(A) = \{\vec{0}\}$.

Elementary Matrices

Definition 96: Elementary Matrix

An **elementary matrix** is a matrix obtained from the identity matrix by performing a single elementary row operation. We say that an elementary matrix is type I, (type II or type III) if it is obtained from the identity matrix by performing a type I (type II or type III, respectively) elementary row operation.

Example 97. The following are elementary matrices.

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E_1 is a type I, E_2 is a type II and E_3 is a type III elementary matrix.

A type I elementary matrix obtained from I_n by interchanging row i and row j, where $i \neq j$ is denoted by E_{ij}. A type II elementary matrix obtained from I_n by multiplying row i by a nonzero constant α is denoted by $E_i(\alpha)$. A type III elementary matrix obtained from I_n by replacing row i by the sum of row i and α times row j is denoted by $E_{ij}(\alpha)$. For example, in $M_3(\mathbb{R})$,

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_1(-3) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_{23}(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Consider the 3-by-4 matrix

$$A = \begin{bmatrix} a & b & c & d \\ p & q & r & s \\ t & u & v & w \end{bmatrix}.$$

Note that

$$E_{13}A = \begin{bmatrix} t & u & v & w \\ p & q & r & s \\ a & b & c & d \end{bmatrix}, E_1(-3)A = \begin{bmatrix} 3a & 3b & 3c & 3d \\ p & q & r & s \\ t & u & v & w \end{bmatrix},$$

and

$$E_{23}(4)A = \begin{bmatrix} a & b & c & d \\ p+4t & q+4u & r+4v & s+4w \\ t & u & v & w \end{bmatrix}.$$

Theorem 98

Let A be an n-by-p matrix and let an elementary row operation be performed on A obtaining a matrix B. If E is the elementary matrix obtained from I_n by performing the same elementary row operation, then $B = EA$.

Corollary 99

If a matrix B is row equivalent to A, then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $B = E_k E_{k-1} \cdots E_1 A$.

The following can be observed:

1. $E_{ij}E_{ij} = I_n$;
2. $E_i(\alpha)E_i(\alpha^{-1}) = I_n$; and
3. $E_{ij}(\alpha)E_{ij}(-\alpha) = I_n$

Hence, an elementary matrix is nonsingular and that the inverse of an elementary matrix is also an elementary matrix of the same type.

Corollary 100

If B is row equivalent to $A \in M_{n,p}$, then there exists a nonsingular matrix $P \in M_n$ such that $B = PA$.

We now state and prove the main theorem of the section.

Theorem 101: A Unifying Theorem

Let $A \in M_n$. The following are equivalent statements:

1. A is nonsingular.
2. The linear system $Ax = b$ has a unique solution.
3. The homogeneous linear system $Ax = 0$ only has the trivial solution.
4. $\mathrm{RREF}(A) = I_n$.
5. A is a product of elementary matrices.

Proof. The implications (1) $\implies$ (2) $\implies$ (3) are already established in Theorem 93 and Corollary 95. We now establish that (3) $\implies$ (4). Assume that $Ax = 0$ only has the trivial solution. Suppose that $\mathrm{RREF}(A) \neq I_n$. Then $\mathrm{RREF}(A)$ has a zero row and fewer than n pivots. Hence, by Theorem 3, the linear system $Ax = 0$ has a nontrivial solution. This contradicts (3). Therefore, $\mathrm{RREF}(A) = I_n$.
We next prove that (4) $\implies$ (5). Assume that $\mathrm{RREF}(A) = I_n$. Corollary (2) guarantees that there are elementary matrices $E_1, \ldots, E_k$ such that $E_k E_{k-1} \cdots E_1 A = I_n$. Therefore,

$$A = (E_k E_{k-1} \cdots E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

is a product of elementary matrices proving statement (5).
Finally, assume that (5) is true. Then A is a product of elementary matrices. Since elementary matrices are nonsingular, it follows that A is nonsingular, proving statement (1). This completes the proof. ▨

Note that statement (5) tells us that the elementary matrices are the basic building blocks of nonsingular matrices, just like living organisms are made up of cells, or the fact that a natural number greater than 1 is a product of prime numbers.

Example 102. Show that $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ is nonsingular and write A as a product of elementary matrices.

Solution. We first apply row-reduction via Gauss-Jordan as follows:

$$\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \xrightarrow{E_{12}} \begin{bmatrix} 1 & -1 \\ 3 & -4 \end{bmatrix} \xrightarrow{E_{21}(-3)} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \xrightarrow{E_{2}(-1)} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \xrightarrow{E_{12}(1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows that $\text{RREF}(A) = I_2$, so A is nonsingular. To write A as a product of elementary matrices, we take note of the elementary row operations applied to A in order to come up with I_2 and consider the elementary matrices corresponding to each elementary row operations.

$$E_{12}(1) \cdot E_2(-1) \cdot E_{21}(-3) \cdot E_{12} \cdot A = I_2.$$

It follows that A is the inverse of $E_{12}(1) \cdot E_2(-1) \cdot E_{21}(-3) \cdot E_{12}$ and by using the shoes and socks property,

$$\begin{aligned} A &= [E_{12}(1) \cdot E_2(-1) \cdot E_{21}(-3) \cdot E_{12}]^{-1} \\ &= E_{12}^{-1} \cdot E_{21}(-3)^{-1} \cdot E_2(-1)^{-1} \cdot E_{12}(1)^{-1} \\ &= E_{12} \cdot E_{21}(3) \cdot E_2(-1) \cdot E_{12}(-1). \end{aligned}$$

In other words,

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

