Formal Verification of Coupled Transmission Lines using Theorem Proving

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Abstract. Coupled transmission lines are essential components of modern electronic systems, which facilitate a reliable and an efficient transmission of high-frequency signals from source to destination and are widely used in various industries, including telecommunications, aerospace, and automotive. Moreover, their dynamics are generally represented by a set of differential equations involving voltages and currents, known as the telegrapher's equations. This paper proposes to use Higher-Order-Logic (HOL) theorem proving for formal modeling and verification of coupled transmission lines. In particular, we formalize the equations capturing the line voltages and currents, and their relationship in a system of coupled transmission lines. We then formally verify the equivalence between these equations and their matrix representations. Finally, we conduct a formal proof of the correctness of the general solutions of these generalized telegrapher's equations using the HOL Light theorem prover.

Keywords: Coupled Transmission Lines · Telegrapher's Equations · Higher-Order Logic · Theorem Proving · HOL Light

1 Introduction

The transmission of electrical signals and power is a pivotal achievement of engineering technology, significantly advancing modern civilization. These electrical systems transmit a wide range of communication signals, including data and control over distances reaching thousands of miles. Furthermore, electrical transmission engineering encompasses not only long transmission systems but also a vast array of shorter transmission line segments that perform numerous functions within the terminal units of the system [1]. Beyond their role in carrying information and energy, they can be also used as circuit elements for passive circuits such as impedance transformers [2], resonators [3] and baluns [4]. Coupled transmission lines (CTLs), in particular, play an important role in building the functionality of modern high speed communication systems.

Electromagnetic coupling occurs when two or more unshielded transmission lines are in close proximity due to the interaction of their electric and magnetic fields. This effect is particularly noticeable when the line axes are parallel, defining them as CTLs [5]. CTLs typically consist of two transmission lines but may include more than two. Furthermore, coupled line structures are applicable to all forms and types of transmission lines. For instance, microstriplines [6] and coplanar waveguides [7] are among the most popular planar forms [8]. When the coupled lines are identical (also known as symmetrical coupled lines), they can be analyzed in terms of *even* and *odd* modes to understand their behavior and characteristics. By applying even- and odd-mode excitations separately and then combining their solutions, engineers conveniently analyze the behavior of symmetric coupled transmission lines. This simplifies the problem by breaking it down into two more manageable parts, making it easier to understand and design transmission lines for specific applications.

Traditionally, the analysis of coupled transmission lines involves paper-and-pencil methods and simulation techniques. In the former approach, the lines are modeled using the telegrapher's equations [9], and the resulting system of coupled transmission line equations is expressed in matrix form. Although this analytical method provides closed-form mathematical solutions, conducting such analyses manually is prone to human error, especially when dealing with complex transmission line configurations. The latter method, which includes commonly used numerical techniques such as the finite-difference time-domain (FDTD) modeling of electromagnetic equations [10] and the transmission line modeling (TLM) method [11], has been shown to be quite time-consuming in many electromagnetic and transmission line problems, such as waveguide structures and high-frequency circuit designs. In addition to requiring a significant amount of memory and computational time, these techniques cannot provide perfectly accurate results because of the discretization of continuous parameters and the use of unverified numerical algorithms.

To address the inaccuracy problems mentioned earlier, formal methods-based techniques are capable of overcoming these issues. In the most pertinent related study on formally analyzing transmission systems using theorem proving [12], the authors formalized the telegrapher's equations for single Transmission Line (TL) and verified the analytical solutions of the equations. Moreover, they formally analyzed the terminated transmission line and its special cases, i.e., short- and open-circuited lines in the HOL Light theorem prover. However, it should be noted that single transmission lines may not offer the same level of versatility as CTLs, which allow for signal interaction and are therefore better suited for more complex applications such as power transmission from Power Grids to users [13].

The primary objective of this paper is to enhance the formal reasoning support within the domain of transmission lines. In this paper, we propose to use Higher-Order Logic (HOL) theorem proving to formally model and analyze CTLs. HOL Light was selected due to the availability of a library for single TL and its potential to connect this library with CTLs. Moreover, the HOL Light theorem prover offers users the flexibility to develop and apply customized automation methods.

The rest of the paper is organized as follows: In Section 2, we present some of the fundamental formal definitions of the multivariate calculus theories of HOL Light that are necessary for understanding the rest of the paper. Section 3 describes the mathematical modeling of CTLs. In Section 4, we provide the formal modeling of CTLs. In Section 5, we present the formal verification of the analytical solutions of the generalized telegrapher's equations, which are used to model CTLs. Finally, Section 6 concludes the paper.

2 Preliminaries

In this section, we present some HOL Light definitions that are used in our proposed formalization and are important to understand the rest of the paper.

2.1 Complex Vectors and Matrices

Here, we explain some of the commonly used HOL Light fuctions in the proposed formalization as follows:

Definition 1. *Vector*

```
⊢ ∀l. vector l = (lambda i. EL (i - 1) l)
```

The function `vector` takes an arbitrary list `l` : α `list` and returns a vector having each component of data-type α . It uses the function `EL i l`, which accepts an index `i` and a list `l`, and returns the i^{th} element of a list `l`. In HOL Light, the lambda operator is utilized to construct a vector from its individual components. A complex vector is defined as a vector having every elements as a complex number.

In HOL Light, matrices are fundamentally formalized as vectors of vectors, where a M matrix is formally represented as of type $(\texttt{complex}^{\texttt{N}})^{\texttt{M}}$. For example, a 2×2 complex matrix can be formalized as follows:

Definition 2. 2×2 *Complex Matrix*

```
⊢ ∀a b c d. cmat2x2 a b c d = vector [vector [a; b]; vector [c; d]]
```

where `cmat2x2` accepts the complex numbers `a`:$\mathbb{C}$, `b`:$\mathbb{C}$, `c`:$\mathbb{C}$ and `d`:$\mathbb{C}$, and returns the corresponding 2×2 matrix.

2.2 Complex Analysis Library

This library includes fundamental concepts in complex analysis, including complex derivatives and transcendental functions.

Definition 3. *Cx and ii*

```
⊢ ∀a. Cx a = complex (a, &0)
⊢ ii = complex (&0, &1)
```

`Cx` is a type casting function with a data-type $\mathbb{R} \rightarrow \mathbb{C}$. It accepts a real number and returns its corresponding complex number with the imaginary part as zero. The `&` operator has data-type $\mathbb{N} \rightarrow \mathbb{R}$ and is used to map a natural number to a real number. Similarly, the function `ii` (iota) represents a complex number with a real part equal to 0 and the magnitude of the imaginary part equal to 1.

Definition 4. *Exponential Functions*
`⊢ ∀x. exp x = Re (cexp (Cx x))`

The HOL Light functions `exp` and `cexp` with data-types $\mathbb{R} \to \mathbb{R}$ and $\mathbb{C} \to \mathbb{C}$ represent the real-valued and complex-valued exponential functions, respectively.

Definition 5. *Complex Derivative*
```
⊢ ∀f x. complex_derivative f x =
              (@f'.(f has_complex_derivative f') (at x))
```

The function `complex_derivative` describes the complex derivative in functional form. It accepts a function `f`: $\mathbb{C} \to \mathbb{C}$ and a complex number `x`, which is the point at which `f` has to be differentiated, and returns a variable of data-type $\mathbb{C}$, providing the derivative of `f` at `x`. Here, the term `at` indicates a specific point at which the differentiation is being evaluated, namely, at the value of `x`.

Definition 6. *Complex Derivative for Vectors*
```
⊢ ∀f x. complex_derivative_vector Fn x =
              (lambda i.complex_derivative (λx. (Fn_i) x) x)
```

The function `complex_vector_derivative` takes a vector `Fn`, whose elements are complex functions of data type $\mathbb{C} \to \mathbb{C}$ and a complex number `x`, which is the point at which every element of `Fn` has to be differentiated, and returns a vector data-type `Fn`: $(\mathbb{C} \to \mathbb{C})^N$, where each element corresponds to the derivatives of the complex functions. It is important to note that throughout the paper, we use a combination of HOL Light code and mathematical notation to enhance readability.

3 Mathematical Modeling of Coupled Transmission Lines

In various transmission line applications, the proximity of neighboring lines often results in a level of coupling. This close proximity leads to modifications in the electromagnetic fields, consequently influencing the propagating voltage and current waves and in turn, altering the characteristic impedance of the transmission line. While this coupling may pose a drawback where it leads to undesired signals, commonly referred to as "cross-talk," it can also serve as a mean of intentionally transferring a set amount of signal to another circuit for various purposes such as monitoring, measurement, or signal processing [9]. There exist two forms of coupling, namely electric and magnetic. The electric coupling results from charges on one line inducing charges on another, often explained by mutual capacitance. The magnetic coupling, on the other hand, arises from the interaction of magnetic flux between the lines and is typically described by mutual inductance. Figure 1 shows a generic circuit model for the CTLs. Under the assumption of lossless conditions, we consider two isolated transmission lines characterized by distributed inductances and capacitances per unit length, represented as L_i and C_i for $i = 1, 2$. The respective propagation velocities and

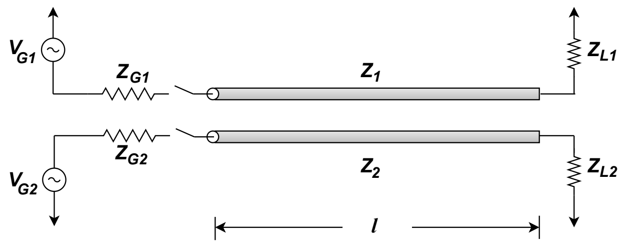

Fig. 1. Coupled Transmission Lines [14]

characteristic impedances are defined as $v_i = 1/\sqrt{L_i C_i}$ and $Z_i = \sqrt{L_i/C_i}$, respectively. To model an interaction between these lines, mutual inductance and capacitance per unit length, denoted as L_m and C_m, are introduced.
The dynamics of the CTLs can then be mathematically described as follows [9]:

$$\frac{\partial V_1}{\partial z} = -L_1 \frac{\partial I_1}{\partial t} - L_m \frac{\partial I_2}{\partial t} \tag{1}$$

$$\frac{\partial V_2}{\partial z} = -L_2 \frac{\partial I_2}{\partial t} - L_m \frac{\partial I_1}{\partial t} \tag{2}$$

$$\frac{\partial I_1}{\partial z} = -C_1 \frac{\partial V_1}{\partial t} + C_m \frac{\partial V_2}{\partial t} \tag{3}$$

$$\frac{\partial I_2}{\partial z} = -C_2 \frac{\partial V_2}{\partial t} + C_m \frac{\partial V_1}{\partial t} \tag{4}$$

These equations are generalizations of the telegrapher's equations incorporating the mutual inductance and capacitance, which were originally developed for a single transmission line.

To overcome the considerable challenges of solving time-domain PDEs [15], we utilize the *phasor* concept to transform them into a set of coupled Ordinary Differential Equations (ODEs) for the voltages and currents. For sinusoidal steady-state (*phasor*) excitation of the lines, we obtain by replacing $\partial/\partial t \Rightarrow j\omega$ [16]:

$$\frac{dV_1}{dz} = -j\omega L_1 I_1(z) - j\omega L_m I_2(z) \tag{5}$$

$$\frac{dV_2}{dz} = -j\omega L_m I_1(z) - j\omega L_2 I_2(z) \tag{6}$$

$$\frac{dI_1}{dz} = -j\omega C_1 V_1(z) + j\omega C_m V_2(z) \tag{7}$$

$$\frac{dI_2}{dz} = j\omega C_m V_1(z) - j\omega C_2 V_2(z) \tag{8}$$

Any system of linear equations can be represented in a compact form by a matrix-vector multiplication equation. For our case, we present Equations (5)-(8), in matrix form describing the relationship between the currents and voltages on the coupled transmission line as [9]:

$$\frac{d\mathbf{V}}{dz} = -j\omega \underbrace{\begin{bmatrix} L_1 & L_m \\ L_m & L_2 \end{bmatrix}}_{\mathbf{L}} \mathbf{I} \tag{9}$$

$$\frac{d\mathbf{I}}{dz} = -j\omega \underbrace{\begin{bmatrix} C_1 & -C_m \\ -C_m & C_2 \end{bmatrix}}_{\mathbf{C}} \mathbf{V} \tag{10}$$

where $\mathbf{V}$ and $\mathbf{I}$ are the column vectors. Moreover, the specific line inductance L and capacitance C in single transmission line have been replaced with 2×2 matrices denoted as $\mathbf{L}$ and $\mathbf{C}$. This modification provides a more detailed representation of the interaction between two coupled transmission lines, and hence a more comprehensive understanding of their dynamics.

4 Formal Modeling of Coupled Transmission Lines

In order to formalize the telegrapher's equations (Equations (5)-(8)) and their matrix-based representations (Equations (9) and (10)), we first model voltages and currents in HOL Light. Furthermore, we model the distributed and mutual inductance as well as the distributed and mutual capacitance using the feature of type abbreviation as follows:

```
new_type_abbrev (``vol'',`:(V1 × V2)')
new_type_abbrev (``cur'',`:(I1 × I2)')
new_type_abbrev (``vol_cur'',`:(V1 × V2) × (I1 × I2)')
new_type_abbrev (``ind_ctls'',`:(L1 × L2) × Lm')
new_type_abbrev (``cap_ctls'',`:(C1 × C2) × Cm')
```

Here, V_1, V_2 are of types voltage functions and I_1 and I_2 are of types current functions and they are modeled in HOL Light as:

```
new_type_abbrev (``vol_fun'',`:(ℂ → ℂ)')
new_type_abbrev (``cur_fun'',`:(ℂ → ℂ)'))
```

Here, the `vol_fun` type is employed to represent a voltage function $V_1(z)$, where z is a variable of complex type $\mathbb{C}$.

Now, we formalize Equations (5) and (6) capturing the voltages on CTLs in HOL Light as follows:

Definition 7. *First Equation for Voltage*

```
⊢ ∀V1 V2 I1 I2 L1 L2 Lm w z.
  coupled_vol_ode_fst ((V1,V2),(I1,I2))((L1,L2),Lm) w z ⇔
    complex_derivative (λz. V1(z)) z =
        --ii * Cx w * (Cx L1 * I1(z) + Cx Lm * I2(z))
```

Definition 8. *Second Equation for Voltage*

```
⊢ ∀V1 V2 I1 I2 L1 L2 Lm w z.
  coupled_vol_ode_snd ((V1,V2),(I1,I2))((L1,L2),Lm) w z  ⇔
     complex_derivative (λz. V2(z)) z =
         --ii * Cx w * (Cx Lm * I1(z) + Cx L2 * I2(z))
```

where coupled_vol_ode_fst and coupled_vol_ode_snd use the complex-derivative function in HOL Light to model the telegrapher's equations. The variables L1:ℝ and Lm:ℝ represent the distributed and mutual inductance per unit length, respectively. Here, the variables z:ℂ refers to the spatial coordinate, while w:ℝ denotes the angular frequency.

Similarly, we can formalize Equations (7) and (8) capturing the currents on CTLs as:

Definition 9. *First Equation for Current*

```
⊢ ∀V1 V2 I1 I2 C1 C2 Cm w z.
  coupled_cur_ode_fst ((V1,V2),(I1,I2))((C1,C2),Cm) w z ⇔
     complex_derivative (λz. I1(z)) z =
         --ii * Cx w * (Cx (C1) * V1(z) - Cx (Cm) * V2(z))
```

Definition 10. *Second Equation for Current*

```
⊢ ∀V1 V2 I1 I2 C1 C2 Cm w z.
  coupled_cur_ode_snd ((V1,V2),(I1,I2))((C1,C2),Cm) w z  ⇔
     complex_derivative (λz. I2(z)) z =
         --ii * Cx w * (--Cx (Cm) * V1(z) + Cx (C2) * V2(z))
```

Next, we formalize the matrix representations of the linear system of equations for voltage and current (Equations (9) and (10)) as follows:

Definition 11. *Matrix Characterization of ODE System for Voltage*

```
⊢ ∀V1 V2 I1 I2 L1 L2 Lm w z.
  vol_ode_mat_rep ((V1,V2),(I1,I2))((L1,L2),Lm) w z ⇔
  (let ind = ((L1,L2),Lm):ind_ctls) in
      complex_derivative_vector (vector [V1;V2]) z =
         (--ii * Cx w) %% inductance_mat_const ind ** cur_vec (I1,I2) z)
```

where %% and ** model the scalar-matrix and matrix-vector multiplications, respectively.

Definition 12. *Matrix Characterization of ODE System for Current*

```
⊢ ∀V1 V2 I1 I2 C1 C2 Cm w z.
  cur_ode_mat_rep ((V1,V2),(I1,I2))((C1,C2),Cm) w z ⇔
  (let cap = ((C1,C2),Cm):cap_ctls) in
      complex_derivative_vector (vector [I1;I2]) z =
         (--ii * Cx w) %% capacitance_mat_const cap ** vol_vec (V1,V2) z)
```

Now, we formally verify the equivalence between the system of linear differential equations for the voltages (Equations (5) and (6)) and their matrix characterizations (Equation (9)) as the following HOL Light theorem:

Theorem 1. *Equivalence between ODE Systems and their Matrix Characterizations for Voltages*

```
⊢ ∀V1 V2 I1 I2 L1 L2 Lm w z.
    let vlcr = ((V1,V2),(I1,I2):vol_cur) and
        ind = ((L1,L2),Lm):ind_tls) in
        [A1] coupled_vol_ode_fst vlcr ind w z ∧
        [A2] coupled_vol_ode_snd vlcr ind w z ⇔
                 vol_ode_mat_rep vlcr ind w z
```

Assumptions A1 and A2 present the telegrapher's equations for the voltages, in phasor domain, i.e., Equations (5) and (6). The proof of Theorem 1 is based on properties of complex derivative, complex vectors and complex matrices alongside some complex arithmetic reasoning.

Next, we formally verify the equivalence of the telegrapher's equations for the current (Equations (7) and (8)) and their matrix representations (Equation (10)).

Theorem 2. *Equivalence between ODE Systems and their Matrix Characterizations for Currents*

```
⊢ ∀V1 V2 I1 I2 C1 C2 Cm w z.
    let vlcr = ((V1,V2),(I1,I2):vol_cur) and
        cap = ((C1,C2),Cm):cap_tls) in
        [A1] coupled_cur_ode_fst V1 vlcr cap z w ∧
        [A2] coupled_cur_ode_snd V2 vlcr cap z w ⇔
                 cur_ode_mat_rep vlcr cap w z
```

The verification of the above theorem is very similar to that of Theorem 1.

5 Formal Verification of Coupled Transmission Lines

To simplify the analysis of the telegrapher's equations, we consider the scenario of the identical transmission lines. In this case, we have $L_1 = L_2 \equiv L_0$ and $C_1 = C_2 \equiv C_0$, so that $\beta_1 = \beta_2 = \omega\sqrt{L_0 C_0} \equiv \beta$ and $Z_1 = Z_2 = \sqrt{L_0/C_0} \equiv Z_0$. Additionally, the wave propagation speed is defined as $v_0 = 1/\sqrt{L_0 C_0}$. If two lossless coupled lines have the same self-inductance parameters $L_1 = L_2 \equiv L_0$ and self-capacitance parameters $C_1 = C_2 \equiv C_0$, the coupled-line structure is considered symmetric. The final solution for symmetric coupled lines can be efficiently derived by combining two single-line scenarios. This is achieved by applying two specific types of excitations: *even* and *odd* mode excitations. In the *even* mode, currents in the conductors exhibit equal magnitudes and flow in parallel directions, while in the *odd* mode, currents in the conductors possess equal magnitudes but flow in opposite directions. It is important to emphasize that this paper primarily focuses on verifying the final solution of the telegrapher's equation rather than the derivation process of the solution.

We now mathematically express the final solutions of the telegrapher's equations for the CTLs in terms of even and odd modes for the voltages and currents as follows:

$$V_1(z) = \underbrace{\frac{e^{-j\beta_+ z} + \mathit{\Gamma}_{L+} e^{-2j\beta_+ l} e^{j\beta_+ z}}{1 - \mathit{\Gamma}_{G+}\mathit{\Gamma}_{L+} e^{-2j\beta_+ l}} V_+}_{\text{even}} + \underbrace{\frac{e^{-j\beta_- z} + \mathit{\Gamma}_{L-} e^{-2j\beta_- l} e^{j\beta_- z}}{1 - \mathit{\Gamma}_{G-}\mathit{\Gamma}_{L-} e^{-2j\beta_- l}} V_-}_{\text{odd}} \quad (11)$$

$$V_2(z) = \underbrace{\frac{e^{-j\beta_+ z} + \mathit{\Gamma}_{L+} e^{-2j\beta_+ l} e^{j\beta_+ z}}{1 - \mathit{\Gamma}_{G+}\mathit{\Gamma}_{L+} e^{-2j\beta_+ l}} V_+}_{\text{even}} - \underbrace{\frac{e^{-j\beta_- z} + \mathit{\Gamma}_{L-} e^{-2j\beta_- l} e^{j\beta_- z}}{1 - \mathit{\Gamma}_{G-}\mathit{\Gamma}_{L-} e^{-2j\beta_- l}} V_-}_{\text{odd}} \quad (12)$$

Similarly, the general solutions for the currents can be mathematically express as:

$$I_1(z) = \frac{1}{Z_+}\left[\underbrace{\frac{e^{-j\beta_+ z} - \mathit{\Gamma}_{L+} e^{-2j\beta_+ l} e^{j\beta_+ z}}{1 - \mathit{\Gamma}_{G+}\mathit{\Gamma}_{L+} e^{-2j\beta_+ l}} V_+}_{\text{even}} + \underbrace{\frac{e^{-j\beta_- z} - \mathit{\Gamma}_{L-} e^{-2j\beta_- l} e^{j\beta_- z}}{1 - \mathit{\Gamma}_{G-}\mathit{\Gamma}_{L-} e^{-2j\beta_- l}} V_-}_{\text{odd}}\right] \quad (13)$$

$$I_2(z) = \frac{1}{Z_-}\left[\underbrace{\frac{e^{-j\beta_+ z} - \mathit{\Gamma}_{L+} e^{-2j\beta_+ l} e^{j\beta_+ z}}{1 - \mathit{\Gamma}_{G+}\mathit{\Gamma}_{L+} e^{-2j\beta_+ l}} V_+}_{\text{even}} - \underbrace{\frac{e^{-j\beta_- z} - \mathit{\Gamma}_{L-} e^{-2j\beta_- l} e^{j\beta_- z}}{1 - \mathit{\Gamma}_{G-}\mathit{\Gamma}_{L-} e^{-2j\beta_- l}} V_-}_{\text{odd}}\right] \quad (14)$$

In this context, the parameters $\beta_\pm$ and $Z_\pm$ indicate the wave numbers and the impedances, respectively and they can be mathematically express as follows:

$$\beta_+ = \omega\sqrt{(L_0 + L_m)(C_0) - C_m} \quad (15) \qquad \beta_- = \omega\sqrt{(L_0 - L_m)(C_0) + C_m} \quad (16)$$

and

$$Z_+ = \sqrt{\frac{L_0 + L_m}{C_0 - C_m}} \quad (17) \qquad Z_- = \sqrt{\frac{L_0 - L_m}{C_0 + C_m}} \quad (18)$$

In order to formalize the general solutions of telegrapher's equations for the voltages and currents, we first define the types of the reflection coefficients, i.e., `g1`, `g2`, `g3`, `g4` denoted by $\mathit{\Gamma}_{L+}$, $\mathit{\Gamma}_{G+}$, $\mathit{\Gamma}_{L-}$ and $\mathit{\Gamma}_{G-}$ and the transmission line constants for identical lines as 4-tuples, and the complex constants associated with V_+ and V_- in HOL Light. Also, the types of the coefficients are given in Table 1.

Table 1. Data Types of Coefficients

Parameter Description	Standard Symbol	HOL Light Symbol: Type
Reflection coefficient at the load in even mode	Γ_{L+}	g1: $\mathbb{C}$
Reflection coefficient at the generator in even mode	Γ_{G+}	g2: $\mathbb{C}$
Reflection coefficient at the load in odd mode	Γ_{L-}	g3: $\mathbb{C}$
Reflection coefficient at the generator in odd mode	Γ_{G-}	g4: $\mathbb{C}$
Complex constant	V_+	Vm: $\mathbb{C}$
Complex constant	V_-	Vp: $\mathbb{C}$

```
new_type_abbrev (``ref_cons'',`:(g1 × g2 × g3 × g4)`)
new_type_abbrev (``ind_cap'',`:(L1 × L2 × C1 × C2)`)
new_type_abbrev (``vol_const'',`:(Vp × Vm)`))
```

We now present the formalization of the general solutions of the telegrapher's equations (Equations (9) and (10)) for voltage and current. For brevity, we only provide the solutions for the first voltage and current, i.e., Equations (11) and (13). These solutions are formalized in HOL Light as follows:

Definition 13. *First Voltage Solution*

⊢ ∀Vm Vp L0 Lm C0 Cm g1 g2 g3 g4 z l w.
vol_sol_fst (Vm,Vp)((L0,Lm),(C0,Cm))(g1,g2,g3,g4) z l w =
(let tlc = ((L0,Lm),(C0,Cm)) in

$$\mathtt{Vm} * \frac{e^{-j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})z} + g1 * e^{-\mathrm{Cx}(\&2)j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})\mathrm{Cx}(l)} * e^{j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})z}}{\mathrm{Cx}(\&1) - g2 * g1 * e^{-\mathrm{Cx}(\&2)j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})\mathrm{Cx}(l)}} +$$
$$\mathtt{Vp} * \frac{e^{-j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})z} + g3 * e^{-\mathrm{Cx}(\&2)j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})\mathrm{Cx}(l)} * e^{j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})z}}{\mathrm{Cx}(\&1) - g4 * g3 * e^{-\mathrm{Cx}(\&2)j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})\mathrm{Cx}(l)}}$$

Definition 14. *First Current Solution*

⊢ ∀Vm Vp L0 Lm C0 Cm g1 g2 g3 g4 z l w.
cur_sol_fst (Vm,Vp)((L0,Lm),(C0,Cm))(g1,g2,g3,g4) z l w =
(let tlc = ((L0,Lm),(C0,Cm)) in $\mathrm{Cx}(\frac{\&1}{\mathtt{char\_imp\_fst\ tlc}}) *$

$$\mathtt{Vm} * \frac{e^{-j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})z} - g1 * e^{-\mathrm{Cx}(\&2)j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})\mathrm{Cx}(l)} * e^{j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})z}}{\mathrm{Cx}(\&1) - g2 * g1 * e^{-\mathrm{Cx}(\&2)j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})\mathrm{Cx}(l)}} -$$
$$\mathtt{Vp} * \frac{e^{-j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})z} - g3 * e^{-\mathrm{Cx}(\&2)j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})\mathrm{Cx}(l)} * e^{j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})z}}{\mathrm{Cx}(\&1) - g4 * g3 * e^{-\mathrm{Cx}(\&2)j\mathrm{Cx}(\mathtt{wn\_fst\ tlc\ w})\mathrm{Cx}(l)}}$$

where vol_sol_fst and cur_sol_fst accept the inductances L1:ℝ, L2:ℝ, the capacitances C1:ℝ, C2:ℝ, the complex constants Vm and Vp, the reflection coefficients g1, g2, g3, g4, the spatial coordinate z, the angular frequency ω:ℝ and the boundary condition l:ℝ and return the corresponding definitions. Moreover, wn_fst and wn_snd refer to the wave numbers in Equation (15) and (16), respectively. In addition, char_imp_fst and char_imp_snd correspond to the characteristic impedances in Equation (17) and (18), respectively. The second voltage and current solutions, i.e., Equations (12) and (14) are formalized in a similar manner. The details about these definitions can be found in the HOL Light proof script [17].

Next, utilizing Definitions 13 and 14, we formalize the general solutions for voltages and currents in vector form for more compact representation:

Definition 15. *Vector Forms of the General Solutions for the Voltages*

```
⊢ ∀Vm Vp V1 V2 L0 Lm C0 Cm I1 I2 g1 g2 g3 g4 z l w.
  vol_sol_vec ((V1,V2),(I1,I2))(Vm,Vp)((L0,Lm),(C0,Cm))(g1,g2,g3,g4) z l w ⇔
  (let vlcr = ((V1,V2),(I1,I2)) and
       tlc = ((L0,Lm),(C0,Cm)) and
        rc = (g1,g2,g3,g4) and
        vc = (Vm,Vp) in
        vector[V1 z; V2 z] = vector[vol_sol_fst vc tlc rc z l w;
                                   vol_sol_snd vc tlc rc z l w])
```

Here, `vol_sol_fst` and `vol_sol_snd` represent the general solutions for the voltages.

Definition 16. *Vector Forms of the General Solutions for the Currents*

```
⊢ ∀Vm Vp V1 V2 L0 Lm C0 Cm I1 I2 g1 g2 g3 g4 z l w.
  vol_sol_vec ((V1,V2),(I1,I2))(Vm,Vp)((L0,Lm),(C0,Cm))(g1,g2,g3,g4) z l w ⇔
  (let vlcr = ((V1,V2),(I1,I2)) and
       tlc = ((L0,Lm),(C0,Cm)) and
        rc = (g1,g2,g3,g4) and
        vc = (Vm,Vp) in
        vector[I1 z; I2 z] = vector[cur_sol_fst vc tlc rc z l w;
                                   cur_sol_snd vc tlc rc z l w])
```

Similarly, `cur_sol_fst` and `cur_sol_snd` denote the general solutions for the currents. The final step is to formally verify the correctness of the solutions of the generalized telegrapher's equations as the following HOL Light theorem:

Theorem 3. *Verification of the General Solutions of the Telegrapher's Equation*

```
⊢ ∀V1 V2 I1 I2 C1 C2 L1 L2 Vm Vp L0 Lm C0 Cm g1 g2 g3 g4 l w.
   let tlc = ((L0,Lm),(C0,Cm)) and ind = ((L1,L2),Lm)
    and cap = (C1,C2),Cm) and vlcr = ((V1,V2),(I1,I2))
    and rc = (g1,g2,g3,g4) and vc = (Vm,Vp) in
    [A1] &0 < L1) ∧ [A2] &0 < L2 ∧ [A3] &0 < C1 ∧ [A4] &0 < C2
    [A5] Cm < C0) ∧ [A6] Lm < L0 ∧ [A7] &0 < Cm ∧ [A8] &0 < Lm
    [A9] L1 = L0 ∧ [A10] L2 = L0 ∧ [A11] C1 = C0 ∧ [A12] C2 = C0
    [A13] (∀z. vol_sol_vec vlcr tlc rc z l w) ∧
    [A14] (∀z. cur_sol_vec vlcr tlc rc z l w)
    ⇒ vol_ode_mat_rep vlcr ind w z ∧ cur_ode_mat_rep vlcr cap w z
```

Assumptions `A1-A4` ensure that the inductances and capacitances are positive quantities. Assumptions `A5-A6` indicate that the distributed capacitance and inductance are greater than the mutual inductance and capacitance, respectively. Assumptions A7-A8 guarantee that the mutual capacitance and inductance are greater than zero. Assumptions A9-A12 model the conditions pertaining identical transmission lines. Assumptions A13 and A14 provide the general solutions of the telegrapher's equations for the voltages and the currents in vector form. Finally, the conclusion of the theorem presents the generalized telegrapher's equations, i.e., Equations (9) and (10). The verification of Theorem 3 is mainly based on the following four important formally verified lemmas about the complex derivatives of the general solutions.

Lemma 1. *Verification of the First Voltage Solution*

```
⊢ ∀I1 I2 V1 Vm Vp g1 g2 g3 g4 L0 L1 Lm C0 Cm z l w.
    let vlcr = ((V1,V2),(I1,I2)) and tlc = ((L0,Lm),(C0,Cm))
      and ind = ((L1,L2),Lm) and rc = (g1,g2,g3,g4) and vc = (Vm,Vp) in
      [A1] L1 = L0 ∧ [A2] Cm < C0 ∧ [A3] Lm < L0 ∧ [A4] &0 < Cm ∧
      [A5] &0 < Lm ∧ [A6](∀z. V1 z = vol_sol_fst vc tlc rc z l w) ∧
      [A7](∀z. I1 z = cur_sol_fst vc tlc rc z l w) ∧
      [A8](∀z. I2 z = cur_sol_snd vc tlc rc z l w)
          ⇒ coupled_vol_ode_fst vlcr ind z w
```

Assumption A1 is the condition for the identical lines. Assumptions A2-A5 are same as those of Assumptions A5-A8 of Theorem 3. Assumption A6 provides the first voltage solution (Equation (11)) of the telegrapher's equation. Assumptions A7 and A8 provide the general solutions of the telegrapher's equations for the currents (Equations (13) and (14)). The conclusion of the lemma provides the telegrapher's equation for the first voltage (Equation (5)). The proof of Lemma 1 is mainly based on the properties of transcendental functions [18], complex derivatives [19] along with some complex arithmetic reasoning.

Lemma 2. *Verification of the Second Voltage Solution*

```
⊢ ∀I1 I2 V1 Vm Vp g1 g2 g3 g4 L0 L2 Lm C0 Cm z l w.
    let vlcr = ((V1,V2),(I1,I2)) and tlc = ((L0,Lm),(C0,Cm))
      and ind = ((L1,L2),Lm) and rc = (g1,g2,g3,g4) and vc = (Vm,Vp) in
      [A1] L2 = L0 ∧ [A2] Cm < C0 ∧ [A3] Lm < L0 ∧ [A4] &0 < Cm ∧
      [A5] &0 < Lm ∧ [A6](∀z. V2 z = vol_sol_snd vc tlc rc z l w) ∧
      [A7](∀z. I1 z = cur_sol_fst vc tlc rc z l w) ∧
      [A8](∀z. I2 z = cur_sol_snd vc tlc rc z l w)
          ⇒ coupled_vol_ode_snd vlcr ind z w
```

Assumption A1 is the condition for the identical lines. A2-A5 are the same as those of Lemma 1. Assumption A6 provides the second voltage solution (Equation (12)) of the telegrapher's equation. Assumptions A7-A8 are also the same as those of Lemma 1. The lemma concludes by providing the telegrapher's equation for the second voltage, as shown in Equation (6). The proof of the above lemma is similar to that of Lemma 1.

In the next two HOL Light lemmas, we formally verify the derivatives of the general solutions for currents.

Lemma 3. *Verification of the First Current Solution*

```
⊢ ∀I1 I2 V1 V2 Vm Vp g1 g2 g3 g4 L0 C0 C1 Lm Cm z l w.
    let vlcr = ((V1,V2),(I1,I2)) and tlc = ((L0,Lm),(C0,Cm))
      and cap = ((C1,C2),Cm) and rc = (g1,g2,g3,g4) and vc = (Vm,Vp) in
      [A1] C1 = C0 ∧ [A2] Cm < C0 ∧ [A3] Lm < L0 ∧ [A4] &0 < Cm ∧
      [A5] &0 < Lm ∧ [A6] (∀z. I1 z = cur_sol_fst vc tlc rc z l w) ∧
      [A7] (∀z. V1 z = vol_sol_fst vc tlc rc z l w) ∧
      [A8] (∀z. V2 z = vol_sol_snd vc tlc rc z l w )
          ⇒ coupled_cur_ode_fst vlcr cap z w
```

Assumption A1 is the condition for the identical lines. Assumptions A2-A5 are the same as those of the above lemmas. Assumption A6 provides the first current solution (Equation (13)) of the telegrapher's equation. Assumptions A7-A8 provide the general solutions for the voltages (Equations (11) and (12)). The conclusion of Lemma 3 provides the telegrapher's equation for the first current (Equation (7)). The verification of the above lemma is very similar to those of Lemmas 1 and 2.

Lemma 4. *Verification of the Second Current Solution*

```
⊢ ∀I1 I2 L1 V1 V2 Vm Vp g1 g2 g3 g4 L0 Lm C0 C2 Cm z l w.
    let vlcr = ((V1,V2),(I1,I2)) and tlc = ((L0,Lm),(C0,Cm))
      and cap = ((C1,C2),Cm) and rc = (g1,g2,g3,g4) and vc = (Vm,Vp) in
      [A1] C2 = C0 ∧ [A2] Cm < C0 ∧ [A3] Lm < L0 ∧ [A4] &0 < Cm ∧
      [A5] &0 < Lm ∧ [A6] (∀z. I2 z = cur_sol_snd vc tlc rc z l w) ∧
      [A7] (∀z. V1 z = vol_sol_fst vc tlc rc z l w) ∧
      [A8] (∀z. V2 z = vol_sol_snd vc tlc rc z l w )
          ⇒ coupled_cur_ode_snd vlcr cap z w
```

Assumption A1 is the condition for the identical lines. Assumptions A2-A5 are the same as those of the above lemmas. Assumption A6 provide the second current solution (Equation (14)) of the telegrapher's equation. Assumptions A7-A8 provide the general solutions for the voltages (Equations (11) and (12)). The conclusion of the lemma provides the telegrapher's equation for the second current (8)). The verification of the above lemma and the other lemmas and theorems can be found in our proof script [17].

Discussion

In this paper, we proposed to use the HOL Light proof assistant for the formal verification of coupled transmission lines. An important aspect of our work is the utilization of theorem proving into a domain that has been traditionally dominated by numerical techniques. The analysis of coupled transmission lines requires to understand various fundamental aspects, ranging from electromagnetic theory to microwave engineering. In particular, for those of us who are not experts in electromagnetics, it has been challenging to comprehend the formal definitions used to model transmission systems and phenomena. Another challenge encountered during this formalization was the mathematical proof itself.

We relied on snippets of proofs gathered from the literature including textbooks, articles and courses. However, we frequently found these traditional pen-and-paper proofs to be somewhat incomplete or lack rigorous details. Due to the nature of the analysis, we had to develop our own proof with all necessary details for the verification process. The primary benefit of this work includes the accuracy of verified results and the revelation of hidden assumptions, which are often omitted in textbooks and engineering literature. Furthermore, every verified theorem and lemma is made general, allowing for further extensions. We believe our work to be useful in the design and analysis of systems involving transmission lines from various engineering and physical science disciplines such as communication systems, electromagnetics, RF and microwave engineering.

6 Conclusion

Coupled transmission lines are traditionally described by a system of differential equations. In this paper, we first formalized the dynamics of the CTLs using the telegrapher's equations in phasor domain. Since the behavior of the line can be fully characterized using circuit theory parameters, such as matrices representing inductances, capacitances, resistances, and conductances per unit length, we modeled these equations in matrix forms for a more compact representation and ease of the formal analysis. We then formally verified the analytical solutions of the telegrapher's equations for the CTLs. It is important to note that our analysis is conducted under the assumption of lossless lines, where resistances and conductances are assumed to be zero. Our research revealed numerous promising directions for future work. Our first goal is to extend the phasor domain solutions into the time domain and verify their correctness for the time domain partial differential equations. Second, we intend to explore the possibility of formally analyzing the results to determine crosstalk in communication circuits. Finally, we aim to formally analyze cable coupling, which is significant in industrial automation systems where precise control and monitoring of machinery and processes are crucial.

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