Formalization of the Telegrapher's Equations using Higher-Order-Logic Theorem Proving

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Abstract

The telegrapher's equations constitute a set of linear partial differential equations that establish a mathematical correspondence between the electrical current and voltage within transmission lines, taking into account factors, such as distance and time. These equations find wide applications in the design and analysis of various systems, including integrated circuits and antennas. This paper proposes the utilization of higher-order-logic theorem proving for a formal analysis of the telegrapher's equations, also referred to as the transmission line equations. Specifically, we present a formal model of the telegrapher's equations in both time and phasor domains. Subsequently, we employ the HOL Light theorem prover to formally verify the solutions of the telegrapher's equations in the phasor domain. Furthermore, we established a connection between phasor and time-domain functions to formally verify the general solutions for the time-domain partial differential equations for the current and voltage in an electric transmission line. To demonstrate the practical effectiveness of our proposed formalization, we conduct a formal analysis of a terminated transmission line and its special cases, i.e., short- and open-circuited transmission lines commonly used in antenna design, by formally verifying the load impedance and the voltage reflection coefficient.

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1 Introduction

Transmission line theory provides a fundamental framework for understanding and analyzing the behavior of transmission lines in the context of their application in various domains, such as integrated circuits and antennas. It serves as a mathematical foundation, capturing an efficient power transfer, ensuring dependable communication, optimizing system design and achieving an electromagnetic compatibility. Therefore, electrical transmission lines play a pivotal role in the conveyance of signals and electrical energy, primarily for the transmission of power, from a source to a load. For instance, in real life, a transmission line acts as a conduit for distributing electricity to homes, businesses, industries, and hospitals, working in conjunction with power generation plants and substations. Sometimes a disruption in the power supply resulting from a transmission line breakage can lead to serious consequences. For instance, in a hospital environment, power failures directly jeopardize the safety of patients and medical personnel. Consequently, emphasizing the crucial importance of guaranteeing the dependability of the electrical elements in the transmission line is essential.

Transmission lines are comprised of a minimum of two conductors that facilitate an efficient and a reliable transmission of information and energy. A two-conductor transmission line supports a *transverse electromagnetic* (TEM) wave [1], where the electric and magnetic fields are perpendicular to each other and transverse to the direction of propagation of waves along the transmission line. TEM waves have a fundamental property of establishing a distinct relationship between the electric **E** and the magnetic **H** fields, which are specifically related to the voltage V and current I, respectively as the following Maxwell's equations:

$$V = -\int_L \mathbf{E}.d\mathbf{l}, \tag{1}$$

$$I \;=\; \oint_L \mathbf{H}.d\mathbf{l} \tag{2}$$

The analysis of transmission lines can be made simpler by only focusing on the circuit quantities, V and I, rather than directly solving the complex line integral based Maxwell's equations (Equations (1) and (2)) and boundary conditions involving electric and magnetic fields (**E** and **H**). In this regard, we employ an equivalent circuit in order to represent the transmission line's behavior. The purpose of developing an equivalent circuit model is to simplify the intricate electromagnetic interactions inherent to the transmission line, thereby reducing them to a set of lumped elements amenable to analysis through circuit theory.

Following the construction of the equivalent circuit, the telegrapher's, also referred to as the transmission line equations, can be derived using circuit analysis techniques. The behavior of transmission lines is elucidated through the utilization of the telegrapher's equations that are based on Partial Differential Equations (PDEs) and rigorously capture the complex electromagnetics and propagation dynamics occurring within these transmission systems. Next, by applying appropriate boundary conditions and simplification of assumptions, the telegrapher's equations provide a useful mathematical model for analyzing transmission lines. Furthermore, comprehending and analyzing their solutions is crucial to ensure the reliability and safety of our everyday electrical systems. For instance, solutions derived from the telegrapher's equations can be seamlessly integrated into the modeling of signal processing and communications systems, such as filters, matching networks, transmission lines, transformers, and small-signal models for transistors. Thus, our goal is to establish foundational concepts rooted in the telegrapher's equations, aiming to subsequently expand this basis for the analysis of more complex engineering applications.

There are numerous analytical and numerical approaches that have been used to solve the PDE based transmission line equations. For example, finite differences [2] and iterative [3] methods are a few numerical approaches that are applied to obtain the solution of these equations. These numerical techniques are highly efficient approaches as they use recursive algorithms to find out the solution of these PDEs. However, due to the finite precision of computer arithmetic and the involvement of round off approximations, these methods cannot guarantee the accuracy of the analysis. On the other hand, analytical solutions may provide a complementary point of view by deriving a closed-form exact solution. However, such analysis is usually done using paper-and-pencil proof methods and is hence prone to human error, especially, for larger systems. Therefore, these conventional methods cannot be trusted to provide accurate analysis, in particular for safety-critical applications.

Several approaches have been used to find the solutions of PDEs for the analysis of the telegrapher's equations. For instance, Konane et al. [4] proposed an exact solution of the telegrapher's equations for voltage monitoring of electrical transmission lines. Kühn [5] developed a general solution of the telegrapher's equations for electrically short transmission lines based on circuit theory. Similarly, Biazar et al. [3] proposed an iterative method to obtain an approximate solution of the telegrapher's equation. However, all these contributions are based on traditional analysis methods.

Formal methods, in particular interactive theorem proving, have also been used for analyzing other forms of PDEs. For example, Boldo et al. [6] formally verified the numerical solution of the wave equation [7] using the Coq theorem prover[1]. Similarly, Deniz et al. [8] formalized the one-dimensional heat equation [9] and verified the general solution of the equation and its convergence in the HOL Light theorem prover[2]. However, none of the aforementioned contributions focused on the telegrapher's equations. In this paper, we present a framework for formally analyzing the telegrapher's equations and their analytical solutions within higher-order-logic theorem proving. We first provide the formal definitions of the telegrapher's equations and their alternate representations, i.e., the wave equations both in the time and phasor domains by proving the relationship between these equations in the phasor domain. We also develop the reasoning steps for the verification of the analytical solutions of these equations, which, to the best of our knowledge, are not available in other theorem provers. In addition, we prove some important properties of special types of transmission line which are lossless and distortionless. In order to demonstrate the utilization of our work, we formally analyze the terminated, short- and open circuited transmission lines. We opted to use the HOL Light theorem prover for the proposed formalization of the telegrapher's equations due to the availability of rich libraries of the multivariate calculus. The HOL Light code developed in this paper is available at [10].

The rest of the paper is structured as follows: We present the proposed framework for the formalization of the telegrapher's equations in higher-order-logic in Section 2. Section 3 describes some preliminary details of the multivariate libraries of the HOL Light theorem prover that are necessary for understanding the rest of the paper. We present the formalization of the telegrapher's equations and a derived form of the wave equations in time and phasor domains alongside a verification of their relationship in the phasor domain in Section 4. In Section 5, we provide the formal verification of the analytical solutions of the telegrapher's equations. Section 6 provides the formal analysis of a terminated, short-circuited and open-circuited transmission lines that illustrate the practical effectiveness of our proposed formalizations. We discuss the difficulties encountered during our work and gained experience in Section 7. Finally, Section 8 concludes the paper.

2 Proposed Methodology

The proposed approach for formally analyzing the telegrapher's equations and their derived form (the wave equations) using higher-order-logic theorem proving is depicted in Figure 1.

[1] https://coq.inria.fr/
[2] https://www.cl.cam.ac.uk/jrh13/hol-light/

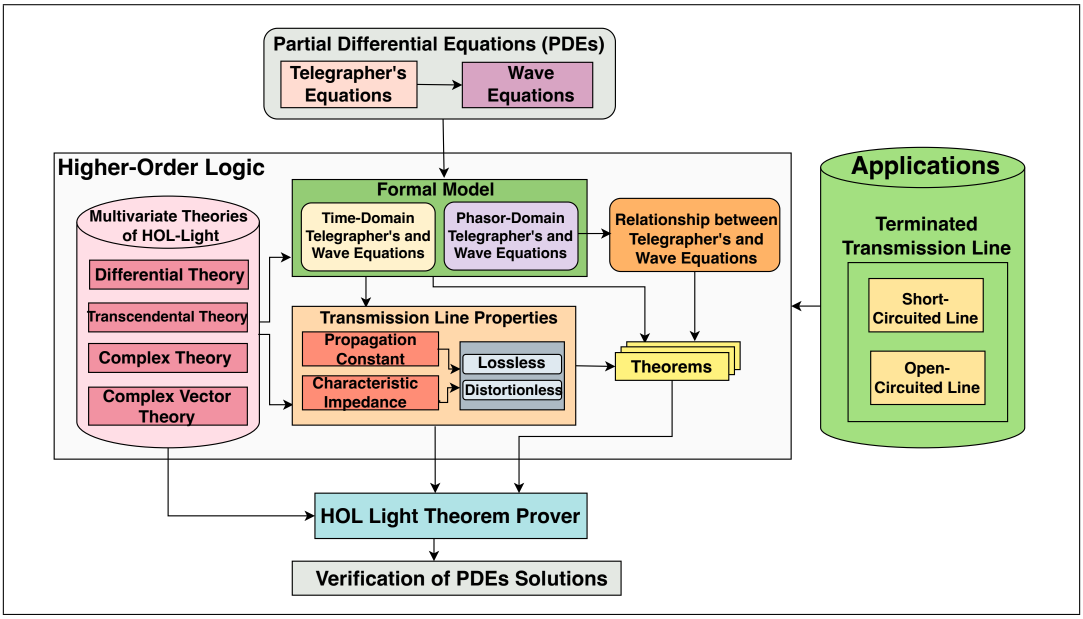

Figure 1: Proposed Methodology

The first step of our proposed approach is to formalize the telegrapher's and the wave equations in time and phasor domains. The formalization of these equations requires HOL Light's libraries of multivariate calculus, such as differential, transcendental and complex vectors. The next step is to establish theorems that enable the formal verification of solutions for these equations by leveraging the advantages of the phasor-domain representation of these equations, which simplifies the time-domain PDEs. Moreover, the relationship between the telegrapher's and the wave equations in the phasor domain is formally verified using these theorems. Subsequently, we use the solutions in the phasor domain to verify the PDEs by establishing a relationship between the corresponding functions in the phasor and time domains. All theorems of the proposed framework of the telegrapher's equations are verified in HOL Light in a generic way in order to obtain general (universally quantified) solutions of the related PDEs. The next step is to represent some important properties of transmission lines, such as the propagation constant and the characteristic impedance specifically focusing on the case of lossless and distortionless lines. Moreover, in order to demonstrate the practical effectiveness of the proposed formalization, we conduct a formal analysis of terminated transmission line and its special cases short-circuited and open-circuited tranmission lines, which are extensively used in electrical and telecommunication systems.

3 Preliminaries

In this section, we provide a brief overview of the HOL Light theorem prover, the HOL Light functions and symbols and some definitions from the theory of complex analysis of HOL Light that are necessary for understanding the rest of the paper.

3.1 HOL Light Theorem Prover

The HOL Light theorem prover [11] is a mechanized proof-assistant to construct mathematical proofs in higher-order-logic [12]. It is implemented in OCaml [13], which is a variant of the ML (Meta-Language) functional programming language [14]. HOL Light has a very small logical kernel, which includes some basic axioms and primitive inference rules.

HOL Light Symbols	Standard Symbols	Description
`@x.t(x)`	εx. t(x)	Some x such that t(x) is true
`&a`	$\mathbb{N} \rightarrow \mathbb{R}$	Type casting from Natural numbers to Reals
`&num`	$\{0, 1, 2..\}$	Positive Integers data type
`Cx(a)`	$\mathbb{R} \rightarrow \mathbb{C}$	Type casting from Reals to Complex
`real`	$\mathbb{R}$	Real data type
`complex`	$\mathbb{C}$	Complex data type
`csqrt x`	$\sqrt{x}$	Complex square root function

Table 1: HOL Light Symbols

Soundness is guaranteed by ensuring that every new theorem is verified by applying these basic axioms and inference rules or any other previously verified theorems/inference rules. In HOL Light, which is based on classical logic, a *theory* comprises types, constants, axioms, definitions, and theorems. HOL supports two interactive proof methods: forward and backward. In a forward proof, users begin with theorems that have already been proven and apply inference rules to arrive at the desired theorem. On the other hand, a backward or goal-directed proof method is the opposite of the forward approach. It relies on the concept of *tactics*, which are OCaml functions that reduce the goals into more manageable subgoals, which are verified to conclude with the proofs of theorems. Furthermore, HOL Light contains lemmas, which are proved as part of the more extensive proof process for theorems. The user can choose to either utilize established lemmas or prove new lemmas as they work towards their main objective of proving the theorems. One of the important features of HOL Light is the availability of many automatic proof procedures that help users in conducting proofs in an efficient manner. Table 1 summarizes some HOL functions and symbols and their meanings that are used in this paper.

3.2 Complex Analysis Library

We now present some of the common HOL Light functions that are used in the proposed analysis.

Definition 3.1. *Re and Im*
$\vdash_{def}$ `∀z. Re z = z$1`
$\vdash_{def}$ `∀z. Im z = z$2`

The functions `Re` and `Im` represent the real and imaginary parts of a complex number, respectively. Here, the notation `z$i` represents the i^{th} component of a vector `z`.

Definition 3.2. *Cx and ii*
$\vdash_{def}$ `∀a. Cx a = complex (a, &0)`
$\vdash_{def}$ `ii = complex (&0, &1)`

`Cx` is a type casting function with a data-type $\mathbb{R} \rightarrow \mathbb{C}$. It accepts a real number and returns its corresponding complex number with the imaginary part as zero. Also, the types $\mathbb{R}^2$ and $\mathbb{C}$ are synonymous. The `&` operator has data-type $\mathbb{N} \rightarrow \mathbb{R}$ and is used to map a natural number to a real number. Similarly, the function `ii` (iota) represents a complex number with a real part equal to 0 and the magnitude of the imaginary part equal to 1. In our formalization, the symbol `ii` is employed to represent j denoting the imaginary number.

Definition 3.3. *Exponential Functions*
$\vdash_{def}$ `∀x. exp x = Re (cexp (Cx x))`

The HOL Light functions `exp` and `cexp` with data-types $\mathbb{R} \rightarrow \mathbb{R}$ and $\mathbb{C} \rightarrow \mathbb{C}$ represent the real and complex exponential functions, respectively.

Definition 3.4. *Complex Derivative*
$\vdash_{def}$ `∀f x. complex_derivative f x =`
```
            (@f'. (f has_complex_derivative f') (at x))
```

The function `complex_derivative` describes the complex derivative in functional form. It accepts a function `f`: $\mathbb{C} \rightarrow \mathbb{C}$ and a complex number `x`, which is the point at which `f` has to be differentiated, and returns a variable of data-type $\mathbb{C}$, providing the derivative of `f` at `x`. Here, the term `at` indicates a specific point at which the differentiation is being evaluated, namely, at the value of `x`.

Definition 3.5. *Higher Complex Derivative*

```
⊢def ∀f x.
higher_complex_derivative 0 f x = f x ∧
(∀n. higher_complex_derivative (SUC n) f x
      = (complex_derivative (λx. higher_complex_derivative n f x) x))
```

The function `higher_complex_derivative` represents the n^{th}-order derivative of the function `f`. It accepts an order `n` of the derivative, a function `f`: $\mathbb{C} \to \mathbb{C}$ and a complex number `x`, and provides the n^{th} derivative of `f` at `x`.

To facilitate in the comprehension of the paper to a non-HOL user, we articulate the telegrapher's equations and the associated lemmas through a blend of Math/HOL Light notation, and some of the frequently used functions in our formalization, their meaning and the associated mathematical conventions are presented in Table 2.

HOL Light Functions	Mathematical Conventions	Description
`cexp x`	$\overrightarrow{\mathtt{e}}^{\mathtt{x}}$	Complex exponential function
`ctan`	$\overrightarrow{\mathtt{tan}}$	Tangent of a complex-valued function
`complex_derivative (λz. V(z)) z`	$\overrightarrow{\frac{\mathtt{dV(z)}}{\mathtt{dz}}}$	Derivative of a complex-valued function V w.r.t z
`higher_complex_derivative 2 (λz. V(z)) z`	$\overrightarrow{\frac{\mathtt{d^2V(z)}}{\mathtt{dz^2}}}$	Second-order derivative of a complex-valued function V w.r.t z
`complex_derivative (λz. V z t) z`	$\overrightarrow{\frac{\partial\mathtt{V(z,t)}}{\partial\mathtt{z}}}$	Partial derivative of a complex-valued function V w.r.t z
`higher_complex_derivative 2 (λz. V z t) z`	$\overrightarrow{\frac{\partial^2\mathtt{V(z,t)}}{\partial\mathtt{z}^2}}$	Second-order partial derivative of a complex-valued function V w.r.t z

Table 2: Conventions used for HOL Light Functions

4 Formalization of the Telegrapher's Equations

The telegrapher's equations are a pair of coupled linear PDEs that describe how the voltage and current change along a transmission line with respect to distance and time. Figure 2 depicts an equivalent circuit model of a two-conductor transmission line. Here, R represents the line parameter resistance, whereas the other line parameters are the inductance L, the capacitance C, and the conductance G, which are specified per unit length (Δz). Moreover, $V(z,t)$ and $V(z+\Delta z,t)$ are the input and output voltages, respectively. Similarly, $I(z,t)$ and $I(z+\Delta z,t)$ are the input and output currents, respectively. Moreover, both voltage and current are functions of space and time.

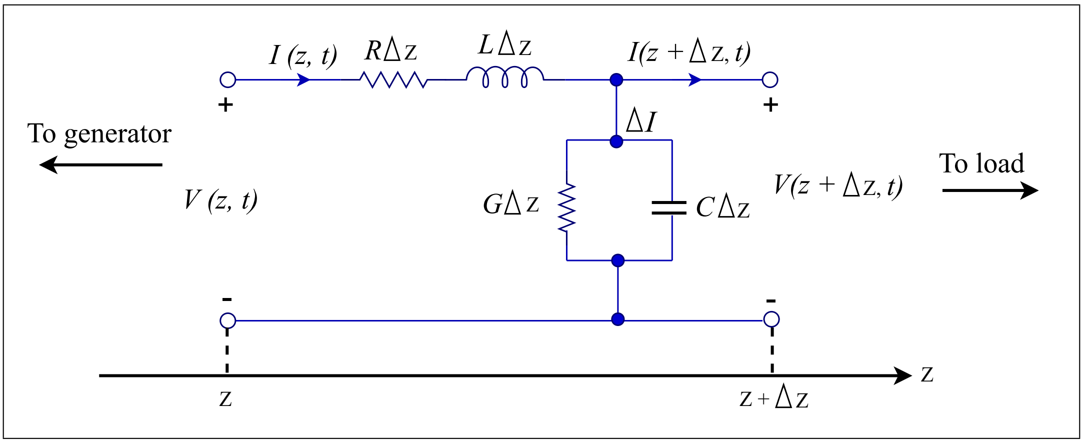

Figure 2: Equivalent Circuit of Two-Conductor Transmission Line [15]

4.1 Telegrapher's Equations in Time Domain

The Law of Conservation of Energy, attributed to Kirchhoff, asserts that there is no loss of voltage throughout a closed loop or circuit; instead, one returns to the initial point within the circuit and, consequently, to the same initial electric potential. Hence, any reductions in voltage within the circuit must balance out with the voltage sources encountered along the same route. By applying the Kirchhoff's voltage law to the circuit of two-conductor transmission line of Figure 2, we get the following equations [16]:

$$V(z+\Delta z,t) - V(z,t) = -R\Delta z I(z,t) - L\Delta z \frac{\partial I(z,t)}{\partial t} \tag{3}$$

Next, dividing Equation (3) by Δz and applying the limit $\Delta z \to 0$, we obtain:

$$\lim_{\Delta z \to 0} \frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = \frac{-R\Delta z I(z,t)}{\Delta z} - L\frac{\Delta z}{\Delta z}\frac{\partial I(z,t)}{\partial t}$$

Finally, by using the definition of the partial derivative, we get:

$$\frac{\partial V(z,t)}{\partial z} = -RI(z,t) - L\frac{\partial I(z,t)}{\partial t} \tag{4}$$

Similarly, by applying the Kirchhoff's current law to the circuit, we find [16]:

$$I(z+\Delta z,t) - I(z,t) = -G\Delta z V(z+\Delta z,t) - C\Delta z \frac{\partial V(z+\Delta z,t)}{\partial t} \tag{5}$$

Next, dividing Equation (5) by Δz and using the definition of the partial derivative, we get:

$$\frac{\partial I(z,t)}{\partial z} = -GV(z,t) - C\frac{\partial V(z,t)}{\partial t} \tag{6}$$

Equations (4) and (6) are known as the *telegrapher's equations* that provide a time-domain relationship between the voltage and current in any transmission line.

The above telegrapher's equations for voltage and current (Equations (4) and (6)) can be formalized in HOL Light in the time domain as follows:

Definition 4.1. *Telegrapher's Equation for Voltage*

```
⊢def ∀V I L z t.
 telegraph_equation_voltage V I R L z t ⇔
   (complex_derivative (λz. V z t) z) =
     --(Cx L * complex_derivative (λt. I z t) t - Cx R * (I z t))
```

Definition 4.2. *Telegrapher's Equation for Current*

```
⊢def ∀V I C z t.
 telegraph_equation_current V I G C z t ⇔
   (complex_derivative (λz. I z t) z) =
     --(Cx C * complex_derivative (λt. V z t) t) - Cx G * (V z t)
```

where `telegraph_equation_voltage` and `telegraph_equation_current` mainly accept the functions `V` and `I` of type $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, representing the voltage and current, respectively, and return the corresponding telegrapher's equations. The variables `R`:$\mathbb{R}$, `L`:$\mathbb{R}$, `G`:$\mathbb{R}$ `C`:$\mathbb{R}$, `z`:$\mathbb{C}$, and `t`:$\mathbb{C}$ represent the resistance, inductance, conductance, capacitance, the spatial coordinate and the time variable, respectively.

It is important to note that we use `complex_derivative` to formalize the time-domain PDEs due to the involvement of the phasor-domain representations of the voltage and current functions in the analysis. Since a phasor-domain representation of a function is a vector in complex plane with some magnitude and angle, the variables `z` and `t` are considered as complex numbers for convenience and the corresponding voltages and currents equations equally hold under this choice.

Now, we can combine the telegrapher's equations (Equations (4) and (6)) to obtain their alternate representations that are commonly known as the *wave equations*, which are more practical to use and provide some additional physical insights and are mathematically expressed as follows:

$$\frac{\partial^2 V(z,t)}{\partial z^2} - LC\frac{\partial^2 V(z,t)}{\partial t^2} = (RC+GL)\frac{\partial V(z,t)}{\partial t} + RGV(z,t) \tag{7}$$

$$\frac{\partial^2 I}{\partial z^2} - LC\frac{\partial^2 I}{\partial t^2} = (RC+GL)\frac{\partial I(z,t)}{\partial t} + RGI(z,t) \tag{8}$$

where $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial t^2}$ capture the second-order partial derivative with respect to z and t, respectively.

To model the wave equations for voltage and current, we need the transmission line constants, such as R, L, G and C. Therefore, we use the type abbreviation in HOL Light providing a compact representation of these constants as follows:

Definition 4.3. *Transmission Line Constants*

```
new_type_abbrev ("R",‘:ℝ’)
new_type_abbrev ("L",‘:ℝ’)
new_type_abbrev ("G",‘:ℝ’)
new_type_abbrev ("C",‘:ℝ’)
new_type_abbrev ("trans_line_const",‘:R # L # G # C’)
```

Now, we formalize the wave equations for both voltage (Equation (7)) and current (Equation (8)) in the time domain as follows:

Definition 4.4. *Wave Equation for Voltage*

```
⊢def ∀V R L G C z t.
wave_voltage_equation V ((R,L,G,C):trans_line_const) z t ⇔
 higher_complex_derivative 2 (λz. V z t) z -
   Cx L * Cx C * (higher_complex_derivative 2 (λt. V z t) t =
     (Cx R * Cx C + Cx G * Cx L) * (complex_derivative (λt. V z t) t) +
                                                   Cx R * Cx G * (V z t))
```

Definition 4.5. *Wave Equation for Current*

```
⊢def ∀I R L G C z t.
 wave_current_equation I ((R,L,G,C):trans_line_const) z t ⇔
 higher_complex_derivative 2 (λz. I z t) z -
   Cx L * Cx C (higher_complex_derivative 2 (λt. I z t) t =
     (Cx R * Cx C + Cx G * Cx L) * (complex_derivative (λt. I z t) t) +
                                                   Cx R * Cx G * (I z t))
```

Next, we express the space-time voltage and current functions as *phasors* in order to reduce the PDEs to Ordinary Differential Equations (ODEs), which will greatly facilitate the derivation of the general solutions of these equations.

The relationship between the space-time voltage and current functions and their phasors can be mathematically expressed as follows [17]:

$$V(z,t) = \mathcal{R}e\{V(z)e^{j\omega t}\}$$

$$I(z,t) = \mathcal{R}e\{I(z)e^{j\omega t}\}$$

where $V(z)$ and $I(z)$ are the phasor components corresponding to $V(z,t)$ and $I(z,t)$, respectively.

4.2 Telegrapher's Equations in Phasor Domain

The principal advantage of the phasor representation of the telegrapher's equations over the time-domain versions is that we no longer need the derivatives with respect to time and are left with the derivatives with respect to distance only. This considerably simplifies the corresponding equations. For instance, the sinusoidally time-varying case, the telegrapher's equations (Equations (4) and (6)) can be rewritten in terms of phasor quantities by replacing $\frac{\partial}{\partial t}$ with $j\omega$. We can derive the telegrapher equation for voltage from Equation (4) as follows:

$$\frac{\partial V(z,t)}{\partial z} = -RI(z,t) - L\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial}{\partial z}[\underbrace{\mathcal{R}e\{V(z)e^{j\omega t}\}}_{V(z,t)}] = -R[\underbrace{\mathcal{R}e\{I(z)e^{j\omega t}\}}_{I(z,t)}] - L\frac{\partial}{\partial t}[\underbrace{\mathcal{R}e\{I(z)e^{j\omega t}\}}_{I(z,t)}]$$

$$\mathcal{R}e\{e^{j\omega t}\frac{dV(z)}{dz}\} = \mathcal{R}e\{-RI(z)e^{jwt} - L(j\omega)e^{j\omega t}I(z)\}$$

$$\frac{dV(z)}{dz} = (-R - j\omega L)I(z)$$

From the above, we can rewrite the telegrapher's equations for voltage as:

$$\frac{dV(z)}{dz} + (R + j\omega L)I(z) = 0 \tag{9}$$

We can also derive the following Equation (10) from Equation (6) in a similar manner

$$\frac{dI(z)}{dz} + (G + j\omega C)V(z) = 0 \tag{10}$$

Here, Equations (9) and (10) are ODEs due to the fact that V and I are functions of the single variable z. Equation (9) indicates that the rate of change of the phasor voltage along the transmission line, as a function of position z, is equal to the series impedance of the line per unit length multiplied by the phasor current. Similarly, Equation (10) states that the rate of change of phasor current along the transmission line, as a function of position z, is equal to the shunt admittance of the line per unit length multiplied by the phasor voltage. We formalize the telegrapher's equation in the phasor domain for voltage (Equation (9)) as:

Definition 4.6. *Telegrapher's Equation*

```
⊢def ∀V I R L w z.  telegraph_equation_phasor_voltage V I R L w z ⇔
                                    telegraph_voltage V I R L w z = Cx(&0)
```

where `telegraph_equation_phasor_voltage` accepts the complex functions `V`:$\mathbb{C} \to \mathbb{C}$ and `I`:$\mathbb{C} \to \mathbb{C}$, the line parameters `R`:$\mathbb{R}$ and `L`:$\mathbb{R}$, the angular frequency ω:$\mathbb{R}$, the spatial coordinate `z`:$\mathbb{C}$, and returns the corresponding telegrapher's equation. Here, the function `telegraph_voltage` models the left-hand side of Equation (9), and is formalized as follows:

Definition 4.7. *Left-Hand Side of Equation (11)*

```
⊢def ∀V I R L w z.
 telegraph_voltage V I R L w z =
   complex_derivative (λz. V(z)) z + (Cx R + ii * Cx w * Cx L) * I(z)
```

Similarly, we formalize Equation (10) in HOL Light as follows:

Definition 4.8. *Telegrapher's Equation*

```
⊢def ∀V I G C w z.  telegraph_equation_phasor_current V I G C w z ⇔
                                    telegraph_current V I G C w z = Cx(&0)
```

with

Definition 4.9. *Left-Hand Side of Equation (12)*

```
⊢def ∀V I G C w z.  telegraph_current V I G C w z =
        complex_derivative (λz. I(z)) z + (Cx G + ii * Cx w * Cx C) * V(z)
```

where `telegraph_current` models the left-hand side of Equation (10).

4.3 Relationship between Telegrapher's and Wave Equations in Phasor Domain

A limitation in using the above form of the telegrapher's equations (Equations (9) and (10)) is that we need to solve each of them for both voltage and current.

To reduce such overhead, we can write the telegrapher's equations using one function (*V(z)* or *I(z)*) as equivalent wave equations. To do this, we first take the derivative of Equation (9) with respect to z:

$$\frac{d}{dz}\left\{\frac{dV(z)}{dz} = -(R + j\omega L)I(z)\right\}$$

which can be written as:

$$\frac{d^2V(z)}{dz^2} = -(R + j\omega L)\frac{dI(z)}{dz} \tag{11}$$

Next, we substitute Equation (10) in Equation (11), to obtain the following equation that involves only *V(z)*:

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \tag{12}$$

γ is the complex propagation constant and is mathematically expressed as:

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}. \tag{13}$$

where α is the attenuation coefficient and β is the phase coefficient and both are mathematically expressed as:

$$\alpha = \mathscr{R}e(\gamma) = \mathscr{R}e\{\sqrt{(R + j\omega L)(G + j\omega C)}\}$$

$$\beta = \mathscr{I}m(\gamma) = \mathscr{I}m\{\sqrt{(R + j\omega L)(G + j\omega C)}\}$$

In a similar manner, we derive the second wave equation by taking the derivative of Equation (10) and substituting Equation (9) in the resultant equation:

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z) \tag{14}$$

We can alternatively represent the wave equations (Equations (12) and (14)) as:

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \tag{15}$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \tag{16}$$

Now, to verify a relationship between the telegrapher's and wave equations for voltage and current in the phasor domain, we first formalize the propagation constant in HOL Light as follows:

Definition 4.10. *Propagation Constant*

```
⊢def ∀R L G C w.
 propagation_constant ((R,L,G,C):trans_line_const) w =
    csqrt ((Cx R + ii * Cx w * Cx L) * (Cx G + ii * Cx w * Cx C))
```

The function **propagation_constant** accepts the transmission line parameters R, L, G, C and angular frequency ω, and returns the corresponding function.
The wave equations (Equations (12) and (14)) in higher-order-logic are formalized as:

Definition 4.11. *Wave Equation for Voltage*

```
⊢def ∀V tlc w z.
 wave_equation_phasor_voltage V z tlc w ⇔
         wave_voltage V z tlc w = Cx(&0)
```

with

Definition 4.12. *Left-Hand Side of Equation (17)*

```
⊢def ∀V tlc w z.
  wave_voltage V z tlc w =
     higher_complex_derivative 2 (λz. V(z)) z -
          (propagation_constant tlc w) pow 2 * V(z)
```

Definition 4.13. *Wave Equation for Current*

```
⊢def ∀I tlc w z.
 wave_equation_phasor_current I z tlc w z ⇔
         wave_current I z tlc w = Cx(&0)
```

with

Definition 4.14. *Left-Hand Side of Equation (18)*

```
⊢def ∀I tlc w z.
  wave_current I z tlc w =
     higher_complex_derivative 2 (λz. I(z)) z -
          (propagation_constant tlc w) pow 2 * I(z)
```

Now, we formally verify the relationship between the telegrapher's and wave equations for voltage in the phasor domain as the following HOL Light theorem:

Theorem 4.1. *Relationship between Telegrapher's and Wave Equations for Voltage*

```
⊢thm ∀V I R L G C w z.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] (λz. complex_derivative (λz. V z) z) complex_differentiable at z ∧
 [A2] I complex_differentiable at z ∧
 [A3] telegraph_current V I G C w z = Cx(&0)
  ⇒ complex_derivative (λz. telegraph_voltage V I R L w z) z =
                                              wave_voltage V z tlc w
```

Assumption A1 ensures that the first-order derivative of function V is differentiable at z. Assumption A2 asserts the differentiability of the function I at z. Assumption A3 provides the telegrapher's equation for current, i.e., Equation (10). The proof of Theorem 4.1 is mainly based on the definitions of the telegrapher's and wave equations and some classical properties of the complex derivative along with some complex arithmetic reasoning. Similarly, we formally verify this relationship for current in the phasor domain.

Theorem 4.2. *Relationship between Telegrapher's and Wave Equations for Current*

```
⊢thm ∀V I R L G C w z.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] (λz. complex_derivative (λz. I z) z) complex_differentiable at z ∧
 [A2] V complex_differentiable at z ∧
 [A3] telegraph_voltage V I R L w z = Cx(&0)
  ⇒ complex_derivative (λz. telegraph_current V I G C w z) z =
                                              wave_current I z tlc w
```

The verification of Theorem 4.2 is very similar to that of Theorem 4.1. More details about their verification can be found at [10].

5 Formal Verification of Analytical Solutions of the Telegrapher's Equations

Analyzing transmission lines is mainly based on finding out solutions of these PDE based telegrapher's and wave equations that are further used to analyze various aspects of signal propagation, such as attenuation, distortion, reflection, and dispersion along the transmission line. One of the examples is to understand the behavior of high-frequency signals, where the distributed parameters of the transmission line significantly affect the signal integrity. In this section, we formally verify the correctness of the analytical solutions of the telegrapher's equations in the phasor domain pertaining to sinusoidal steady state and in the time domain that are concerned with arbitrary variations over time.

5.1 Verification of the Solutions in Phasor Domain

We can mathematically express the general solutions of the wave equations (and thus the telegrapher's equations) (Equations (15) and (16)) as folllows:

$$V(z) = V^+(z) + V^-(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \tag{17}$$

$$I(z) = I^+(z) + I^-(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \tag{18}$$

Here, V_0^+, V_0^-, I_0^+, I_0^- are the complex constants that can be determined by boundary conditions. Similarly, the transmission line voltage $V^+(z)$ and current $I^+(z)$ represent the forward-going waves (propagating in the $+z$ direction) and voltage $V^-(z)$ and current $I^-(z)$ are the backward-going waves (propagating in the $-z$ direction).
If we insert the solution for $V(z)$ in Equation (9), we get:

$$\frac{dV(z)}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} = -(R + j\omega L)I(z) \tag{19}$$

Next, we rearrange the above equation to obtain the current *I(z)*:

$$I(z) = \frac{\gamma}{R + j\omega L}(V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}) \tag{20}$$

Note that both expressions (Equations (18) and (20)) for the current are the same. The characteristic impedance, which is the ratio of the line voltage and current, is an important characteristic of transmission line and can be mathematically expressed as follows:

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \frac{R + J\omega L}{\gamma} = R_0 + jX_0 \tag{21}$$

where R_0 and X_0 are the real and imaginary parts of Z_0. The characteristic impedance Z_0 and the propagation constant γ are two important properties of the transmission line due to their direct dependence on the line parameters R, L, G, C and the phasor of the operation.

Next, we define the characteristic impedance in HOL Light as follows:

Definition 5.1. *Characteristic Impedance*

```
⊢def ∀R L G C w.
  characteristic_impedance (R,L,G,C) w =
    (let tlc = ((R,L,G,C):trans_line_const) in
      (Cx R + ii * Cx w * Cx L) / propagation_constant tlc w)
```

The next step is to formalize the general solutions (Equations (17) and (20)) in HOL Light:

Definition 5.2. *Wave Solution for Voltage*

```
⊢def ∀V1 V2 tlc w z.
    wave_solution_voltage_phasor V1 V2 tlc w z =
      V1 * cexp(--(propagation_constant tlc w) * z) +
       V2 * cexp((propagation_constant tlc w) * z)
```

where V1 and V2 in the formalization refer to the complex constants V_0^+ and V_0^- in Equation (17), respectively. The parameters w and z represent the angular frequency and the spatial coordinate, respectively.

Definition 5.3. *Wave Solution for Current*

```
⊢def ∀V1 V2 tlc w z.
    wave_solution_current_phasor V1 V2 tlc w z =
     Cx(&1) / characteristic_impedance tlc w *
       (V1 * cexp(--(propagation_constant tlc w) * z) -
        V2 * cexp((propagation_constant tlc w) * z))
```

Next, we formally verify the general solutions (Equations (17) and (20)) of the wave equations for voltage and current, (represented by Equations (15) and (16)), in the HOL Light theorem prover as follows:

Theorem 5.1. *Correctness of the Solution for Voltage*

```
⊢thm ∀V1 V2 V R L G C w z.
    let tlc = ((R,L,G,C):trans_line_const) in
      wave_equation_voltage_phasor
        (λz. wave_solution_voltage_phasor V1 V2 tlc w z) V tlc w
```

Theorem 5.2. *Correctness of the Solution for Current*

```
⊢thm ∀V1 V2 I R L G C w z.
    let tlc = ((R,L,G,C):trans_line_const) in
      wave_equation_current_phasor
        (λz. wave_solution_current_phasor V1 V2 tlc w z) I tlc w
```

The verification of Theorems 5.1 and 5.2 is mainly based on four lemmas about the complex differentiation of the solutions, given in Table 3.

Mathematical Form	Formalized Form
$$\frac{dV(z)}{dz} = -\gamma V1e^{-\gamma z} + \gamma V2e^{\gamma z}$$	**Lemma 1 (First-Order Derivative of General Solution for Voltage):** ∀V1 V2 R L G C w z. let tlc = ((R,L,G,C):trans_line_const) in complex_derivative (λz. wave_solution_voltage_phasor V1 V2 tlc w z) z = V1 * (--(propagation_constant tlc w)) * cexp (--(propagation_constant tlc w) * z) + V2 * (propagation_constant tlc w) * cexp ((propagation_constant tlc w) * z)
$$\frac{d^2V(z)}{dz^2} = \gamma^2 V1e^{-\gamma z} + \gamma^2 V2e^{\gamma z}$$	**Lemma 2 (Second-Order Derivative of General Solution for Voltage):** ∀V1 V2 R L G C w z. let tlc = ((R,L,G,C):trans_line_const) in higher_complex_derivative 2 (λz. wave_solution_voltage_phasor V1 V2 tlc w z) z = V1 * (propagation_constant tlc w) pow 2 * cexp (--(propagation_constant tlc w) * z) + V2 * (propagation_constant tlc w) pow 2 * cexp ((propagation_constant tlc w) * z
$$\frac{dI(z)}{dz} = \frac{1}{Z_0}(-\gamma V1e^{-\gamma z} - \gamma V2e^{\gamma z})$$	**Lemma 3 (First-Order Derivative of General Solution for Current):** ∀V1 V2 R L G C w z. let tlc = ((R,L,G,C):trans_line_const) in complex_derivative (λz. wave_solution_current_phasor V1 V2 tlc w z) z = Cx (&1) / characteristic_impedance tlc w * (V1 * (--propagation_constant tlc w) * cexp (--(propagation_constant tlc w) * z) - V2 * (propagation_constant tlc w) * cexp ((propagation_constant tlc w) * z))
$$\frac{d^2I(z)}{dz^2} = \frac{1}{Z_0}(\gamma^2 V1e^{-\gamma z} - \gamma^2 V2e^{\gamma z})$$	**Lemma 4 (Second-Order Derivative of General Solution for Current):** ∀V1 V2 R L G C w z. let tlc = ((R,L,G,C):trans_line_const) in higher_complex_derivative 2 (λz. wave_solution_current_phasor V1 V2 tlc w z) z = Cx (&1) / characteristic_impedance tlc w * (V1 * (propagation_constant tlc w) pow 2 * cexp (--(propagation_constant tlc w) * z) - V2 * (propagation_constant tlc w) pow 2 * cexp ((propagation_constant tlc w) * z))

Table 3: Lemmas of the Derivatives of General Solutions in Phasor Domain

Since, there exists a relationship between the telegrapher's and wave equations, as proven in Section 4, the solutions of the wave equations also satisfy the telegrapher's equations.

Theorem 5.3. *General Solution of the Telegrapher's Equation for Voltage*

```
⊢thm ∀V1 V2 V I R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] Cx R + ii * Cx w * Cx L ≠ Cx(&0) ∧
 [A2] (∀z. V z = wave_solution_voltage_phasor V1 V2 tlc w z) ∧
 [A3] (∀z. I z = wave_solution_current_phasor V1 V2 tlc w z)

  ⇒ telegraph_equation_phasor_voltage V I R L w z
```

Assumption A1 ensures that expression $R + j\omega L$ is not equal to zero. Assumptions A2 and A3 provide solutions of the wave equations for the voltage and the current, respectively. The verification of the above theorem is based on the properties of the complex differentiation along with some complex arithmetic reasoning.

Theorem 5.4. *General Solution of the Telegrapher's Equation for Current*

```
⊢thm ∀V1 V2 V I R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] Cx R + ii * Cx w * Cx L ≠ Cx(&0) ∧
 [A2] (∀z. V z = wave_solution_voltage_phasor V1 V2 tlc w z) ∧
 [A3] (∀z. I z = wave_solution_current_phasor V1 V2 tlc w z)

  ⇒ telegraph_equation_phasor_current V I G C w z
```

5.2 Verification of Properties of Transmission Line

A transmission line is characterized by two essential properties, namely its propagation constant γ and characteristic impedance Z_0. These properties are specified by the angular frequency ω and the line parameters R, L, G and C. Understanding and optimizing the transmission line characteristics help engineers and designers to achieve efficient signal transmission, maintain signal integrity, and ensure the reliable operation of these systems. In this section, we formally verify these transmission line properties for the case of lossless and distortionless lines.

5.2.1 Lossless Line

The main purpose of a transmission line is to facilitate the transmission of information between distant locations with minimal signal degradation that can be achieved by reducing the signal loss. This is one of the crucial requirements in the construction of an efficient and a reliable transmission line. In the case of a lossless transmission line, the elements R (resistance) and G (conductance) can be considered as negligible or effectively zero:

$$R = G = 0$$

The characteristic impedance of a lossless transmission line can now be expressed in a simplified form by using the above values of R and G in Equation (21) as:

$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}}$$

Similarly, the attenuation and phase constants expressed in Equation (13) becomes:

$$\alpha = 0 \tag{22}$$

$$\beta = \sqrt{LC} \tag{23}$$

This implies that the transmission line has no signal attenuation, and as a result, the propagation constant can be represented by a purely imaginary number:

$$\gamma = j\beta = j\omega\sqrt{LC}$$

5.2.2 Distortionless Line

A distortionless line refers to a transmission medium characterized by an attenuation constant α that exhibits no variation with changes in frequency while the phase constant β is linearly dependent on frequency.
For a distortionless transmission line, the elements R and G are related as:

$$\frac{R}{L} = \frac{G}{C}$$

Now, the characteristic impedance of the transmission line is expressed as:

$$Z_0 = \sqrt{\frac{R(1 + j\omega L/R)}{R(1 + j\omega C/G)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$$

The propagation constant (Equation (13)) becomes:

$$\gamma = \sqrt{RG\left(1 + \frac{j\omega L}{R}\right)\left(1 + \frac{j\omega C}{G}\right)}$$

$$\gamma = \sqrt{RG}\left(1 + \frac{j\omega C}{G}\right) = \alpha + j\beta$$

or

$$\alpha = \sqrt{RG}, \qquad \beta = \omega\sqrt{LC} \tag{24}$$

We can see that the attenuation constant α is independent of the frequency, whereas β is a linear function of frequency.

The verified properties, i.e., propagation constant and characteristic impedance of the lossless and distortionless transmission lines are given in Table 4.

Case	**Propagation Constant**	**Characteristic Impedance**
Lossless	**Theorem 1** (***Attenuation Constant***) $\vdash_{thm}$ ∀R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 ∧ [A5] G = &0 ⇒ Re(propagation_constant tlc w) = &0 **Theorem 2** (***Phase Constant***) $\vdash_{thm}$ ∀R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 ∧ [A5] G = 0 ⇒ Im(propagation_constant tlc w) = w$\sqrt{LC}$	**Theorem 3** (***Characteristic Impedance***) $\vdash_{thm}$ ∀R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 ∧ [A5] G = &0 [A6] ii * Cx w ≠ Cx (&0) [A7] csqrt (Cx L * Cx C) ≠ Cx (&0) ⇒ characteristic_impedance tlc w = csqrt(Cx(L) * Cx(C)) / Cx(C)
Distortionless	**Theorem 4** (***Attenuation Constant***) $\vdash_{thm}$ ∀R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] L > &0 ∧ [A2] R > &0 ∧ [A3] G > &0 ∧ [A4] R / L = G / C ⇒ Re(propagation_constant tlc w) = $\sqrt{RG}$ **Theorem 5** (***Phase Constant***) $\vdash_{thm}$ ∀R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] L > &0 ∧ [A2] R > &0 ∧ [A3] G > &0 ∧ [A4] C > &0 ∧ [A5] R / L = G / C ⇒ Im(propagation_constant tlc w) = w$\sqrt{LC}$	**Theorem 6** (***Characteristic Impedance***) $\vdash_{thm}$ ∀R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] L > &0 ∧ [A2] C > &0 ∧ [A3] &0 < R ∧ [A4] G > &0 ∧ [A5] Cx G + ii * Cx w * Cx C ≠ Cx (&0) ∧ [A6] R / L = G / C ⇒ characteristic_impedance tlc w = csqrt(Cx(L) * Cx(C)) / Cx(C)

Table 4: Properties of Transmission Lines

In the following section, we verify the general solutions of the time-domain PDEs by considering a lossless line, where we assume both resistance R and conductance G to be zero.

5.3 Verification of the Solutions in Time Domain

It is useful to examine the complete time functions for understanding the nature of the voltage and current within a transmission line. We can find the corresponding time-domain expressions for voltage and current (solution in the time domain) on the line by multiplying the phasor of the voltage and current with the harmonic time variation term e^{jwt} and taking its real part as follows:

$$V(z,t) = \mathscr{R}e\{V(z)e^{j\omega t}\} \tag{25}$$

$$I(z,t) = \mathscr{R}e\{I(z)e^{j\omega t}\} \tag{26}$$

Next, we use Equation (17) in the time-domain solution (Equation (25)) and get:

$$V(z,t) = \mathscr{R}e\{(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z})e^{j\omega t}\}$$

$$V(z,t) = \mathscr{R}e\{V_0^+ e^{-\gamma z} e^{j\omega t} + V_0^- e^{\gamma z} e^{j\omega t}\}$$

By splitting the propagation constant in real and imaginary parts, i.e., $\gamma = \alpha + j\beta$, we can write the above equation for voltage as follows:

$$V(z,t) = \mathscr{R}e\{V_0^+ e^{-(\alpha+j\beta)z} e^{j\omega t} + V_0^- e^{(\alpha+j\beta)z} e^{j\omega t}\}$$

We know that α is equal to zero for a lossless transmission line. Thus, we get:

$$V(z,t) = \mathscr{R}e\{V_0^+ e^{j(\omega t-\beta z)} + V_0^- e^{j(\omega t+\beta z)}\} \tag{27}$$

After applying Euler's formula to the above equation and taking the real part of the solution, we have:

$$V(z,t) = V_0^+ cos(\omega t - \beta z) + V_0^- cos(\omega t + \beta z) \tag{28}$$

where we assume V_0^+ and V_0^- to be real.
Using Definition 5.2, we formalize the general solution (Equation (25)) in the time-domain for voltage as follows:

Definition 5.4. *General Solution for Voltage in Time Domain*

```
⊢def ∀V1 V2 tlc w z t.
 wave_solution_voltage_time V1 V2 tlc w z t =
  Re((wave_solution_voltage_phasor V1 V2 tlc w z) * cexp(ii * Cx w * t))
```

where the function wave_solution_voltage_time uses the phasor given by the voltage function wave_solution_voltage_phasor to construct the formal definition of Equation (25).

Next, we formally verify the general solution for voltage in the time domain in HOL Light as follows:

Theorem 5.5. *General Solution of Wave Equation for Voltage*

```
⊢thm ∀V1 V2 R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧
 [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧
 [A10](∀z t. V z t = Cx(wave_solution_voltage_time V1 V2 tlc w z t))
      ⇒ wave_voltage_equation V tlc z t
```

Assumptions A1-A3 ensure that the angular frequency ω, the line parameters L and C are positive real values. Assumptions A4-A5 assert that the line parameters R and G are equal to zero, which is an assumption for a lossless transmission line. Assumptions A6-A7 ensure that the imaginary parts of the variables z and t are equal to zero in the time domain. Assumptions A8-A9 guarantee that the coefficients V1 and V2 are real. Assumption A10 provides the solution of the wave equation for voltage, i.e., Equation (28). The proof of the above theorem is mainly based on the following Lemma 5.1 which gives the relationship between phasor and time-domain functions as well as four important lemmas about the complex differentiation of the time-domain solution with respect to the parameters z and t, which are given in Table 5.

Lemma 5.1. *Relationship between Phasor and Time-Domain Functions for Voltage*

```
⊢lem ∀V1 V2 R C L G w z t.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0
   ⇒ wave_solution_voltage_time V1 V2 tlc w z t =
     Re(V1) * (cos(w * Re t - (Im(propagation_constant tlc w)) * Re z)) +
      Re(V2) * (cos(w * Re t + Im((propagation_constant tlc w)) * Re z))
```

Assumptions A1-A5 are the same as those of Theorem 5.5. The verification of Lemma 5.1 is mainly based on Theorem 1 given in Table 4 and the properties of transcendental functions alongside some complex arithmetic reasoning.

Mathematical Form	Formalized Form
$\frac{\partial V(z,t)}{\partial z} = V1sin(\omega t - \beta z)\beta - V2sin(\omega t + \beta z)\beta$	**Lemma 1** (First-Order Partial Derivative of General Solution for Voltage with respect to distance): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ complex_derivative (λz. wave_solution_voltage_time V1 V2 tlc w z t) z = Cx (Re V1 * (--sin (w * Re t - (w * sqrt (L * C)) * Re z))· (--(w * sqrt (L * C)))) + Re V2 * (--sin (w * Re t + (w * sqrt (L * C)) * Re z))* ((w * sqrt (L * C)))
$\frac{\partial^2 V(z,t)}{\partial z^2} = -V1cos(\omega t - \beta z)\beta^2 - V2cos(\omega t + \beta z)\beta^2$	**Lemma 2** (Second-Order Partial Derivative of General Solution for Voltage with respect to distance): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ higher_complex_derivative 2 (λz. wave_solution_voltage_time V1 V2 tlc w z t) z = Cx (Re V1 * (--cos (w * Re t - (w * sqrt (L * C)) * Re z))· ((w * sqrt (L * C))) pow 2 + Re V2 * (--cos (w * Re t + (w * sqrt (L * C)) * Re z)) * ((w * sqrt (L * C))) pow 2)
$\frac{\partial V(z,t)}{\partial t} = -V1sin(\omega t - \beta z)\omega - V2sin(\omega t + \beta z)\omega$	**Lemma 3** (First-Order Partial Derivative of General Solution for Voltage with respect to time): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ complex_derivative (λt. wave_solution_voltage_time V1 V2 tlc w z t) t = Cx (Re V1 * (--sin (w * Re t - (w * sqrt (L * C)) * Re z)) * w + Re V2 * (--sin (w * Re t + (w * sqrt (L * C)) * Re z)) * w)
$\frac{\partial^2 V(z,t)}{\partial t^2} = -V1cos(\omega t - \beta z)\omega^2 - V2cos(\omega t + \beta z)\omega^2$	**Lemma 4** (Second-Order Partial Derivative of General Solution for Voltage with respect to time): ∀V1 V2 R C L G w. let tlc = ((R,L,G,C):trans_line_const) [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ higher_complex_derivative 2 (λt. wave_solution_voltage_time V1 V2 tlc w z t) t = Cx (Re V1 * (--cos (w * Re t - (w * sqrt (L * C)) * Re z)) * w pow 2 + Re V2 * (--cos (w * Re t + (w * sqrt (L * C)) * Re z)) * w pow 2)

Table 5: Lemmas of the Derivatives of General Solutions for Voltage in Time Domain

Similarly, we use Equation (20) in the time domain solution (Equation (26)) for current as follows:

$$I(z,t) = \mathscr{R}e\{\frac{\gamma}{R + j\omega L}(V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z})e^{j\omega t}\}$$

After rearranging the above equation, we have:

$$I(z,t) = \mathscr{R}e\{\frac{\gamma}{R + j\omega L}(V_0^+ e^{j(\omega t - \beta z)} - V_0^- e^{j(\omega t + \beta z)})\} \tag{29}$$

Next, by applying Euler's formula and taking the real part of the solution, we get:

$$I(z,t) = \frac{\gamma}{R + j\omega L}(V_0^+ cos(\omega t - \beta z) - V_0^- cos(\omega t + \beta z)) \tag{30}$$

Now, using Definition 5.3, we formalize the general solution (Equation (26)) in the time domain for current as follows:

Definition 5.5. *General Solution for Current in Time Domain*

```
⊢def ∀V1 V2 tlc w z t.
 wave_solution_current_time V1 V2 tlc w z t =
  Re((wave_solution_current_phasor V1 V2 tlc w z) * cexp(ii * Cx w * t))
```

where `wave_solution_current_time` accepts the phasor solution of the current `wave_solution _current_phasor` that is multiplied with the harmonic time variation term and returns its real part.

Theorem 5.6. *General Solution of Wave Equation for Current*

```
⊢thm ∀V1 V2 R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧
 [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧
 [A10] Im(Cx(&1)/characteristic_impedance tlc w) = &0 ∧
 [A11] (∀z t. I z t = Cx(wave_solution_current_time V1 V2 tlc w z t))
     ⇒ wave_current_equation I tlc z t
```

Assumptions `A1-A9` are the same as those of Theorem 5.5. Assumption `A10` ensures that the imaginary part of the inverse characteristic impedance is equal to zero. Assumption `A11` provides the solution of the wave equation for current, i.e., Equation (30). Similarly, the proof of Theorem 5.6 is primarily based on the formally verified lemmas about the relationship between phasor and time-domain functions, i.e., Lemma 5.2 and derivatives of the general solution for current as given in Table 6.

Mathematical Form	Formalized Form
$\frac{\partial I(z,t)}{\partial z} = \frac{1}{Z_0}(V1sin(\omega t - \beta z)\beta + V2sin(\omega t + \beta z)\beta)$	**Lemma 1** (First-Order Partial Derivative of General Solution for Current with respect to distance): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 ∧ [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ⇒ complex_derivative (λz. wave_solution_current_time V1 V2 tlc w z t) z = Cx (Re ((Cx (&1) / characteristic_impedance tlc w)) * (Re V1 * --sin (w * Re t - (w * sqrt (L * C)) * Re z) * --(w * sqrt (L * C)) + Re V2 * sin (w * Re t + (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C))))
$\frac{\partial^2 I(z,t)}{\partial z^2} = \frac{1}{Z_0}(-V1cos(\omega t - \beta z)\beta^2 + V2cos(\omega t + \beta z)\beta^2)$	**Lemma 2** (Second-Order Partial Derivative of General Solution for Current with respect to distance): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ higher_complex_derivative 2 (λz. wave_solution_current_time V1 V2 tlc w z t) z = Cx (Re (Cx (&1) / characteristic_impedance tlc w) * (Re V1 * --cos (w * Re t - (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C)) pow 2 + Re V2 * cos (w * Re t + (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C)) pow 2))
$\frac{\partial I(z,t)}{\partial t} = \frac{1}{Z_0}(-V1sin(\omega t - \beta z)\omega + V2sin(\omega t + \beta z)\omega)$	**Lemma 3** (First-Order Partial Derivative of General Solution for Current with respect to time): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ complex_derivative (λt. wave_solution_current_time V1 V2 tlc w z t) t = Cx (Re (Cx (&1) / characteristic_impedance tlc w) * Cx (Re V1 * (--sin (w * Re t - (w * sqrt (L * C)) * Re z)) * w + Re V2 * (sin (w * Re t + (w * sqrt (L * C)) * Re z)) * w)
$\frac{\partial^2 I(z,t)}{\partial t^2} = \frac{1}{Z_0}(-V1cos(\omega t - \beta z)\omega^2 + V2cos(\omega t + \beta z)\omega^2)$	**Lemma 4** (Second-Order Partial Derivative of General Solution for Current with respect to time): ∀V1 V2 R L G C w. let tlc = ((R,L,G,C):trans_line_const) in [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧ [A4] R = &0 [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧ [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧ ⇒ higher_complex_derivative 2 (λt. wave_solution_current_time V1 V2 tlc w z t) t = Cx (Re (Cx (&1) / characteristic_impedance tlc w) * (Re V1 * --cos (w * Re t - (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C)) pow 2 + Re V2 * cos (w * Re t + (w * sqrt (L * C)) * Re z) * (w * sqrt (L * C)) pow 2))

Table 6: Lemmas of the Derivatives of General Solutions for Current in Time Domain

Lemma 5.2. *Relationship between Phasor and Time-Domain Functions for Current*

```
⊢lem ∀V1 V2 R L G C w z t.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0
  ⇒ wave_solution_current_time V1 V2 tlc w z t =
     Re(Cx(&1) / characteristic_impedance tlc w) *
       (Re V1 * cos(w * Re t - Im(propagation_constant tlc w) * Re z) -
          Re V2 * cos(w * Re t + Im(propagation_constant tlc w) * Re z))
```

Assumptions A1-A5 are the same as those of Lemma 5.1. The verification of the above lemma is similar to that of Lemma 5.1.

Since the wave and telegrapher's equations are related to each other, the general solutions of the wave equations satisfy the telegrapher's equations in the time domain and are verified as the following HOL Light theorems:

Theorem 5.7. *General Solution of Telegrapher's Equation for Voltage*

```
⊢thm ∀V1 V2 R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0 ∧ [A7] (∀t. Im t = &0) ∧
 [A8] (∀z. Im z = &0) ∧ [A9] Im V1 = &0 ∧ [A10] Im V2 = &0 ∧
 [A11] (∀z t. V z t = Cx (wave_solution_voltage_time V1 V2 tlc w z t))
    ⇒ telegraph_equation_voltage V I R L z t
```

Theorem 5.8. *General Solution of Telegrapher's Equation for Current*

```
⊢thm ∀V1 V2 R L G C w.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] w > &0 ∧ [A2] L > &0 ∧ [A3] C > &0 ∧
 [A4] R = &0 ∧ [A5] G = &0 ∧ [A6] (∀t. Im t = &0) ∧
 [A7] (∀z. Im z = &0) ∧ [A8] Im V1 = &0 ∧ [A9] Im V2 = &0 ∧
 [A10] Im (Cx(&1)/characteristic_impedance tlc w) = &0 ∧
 [A11] (∀z t. I z t = Cx (wave_solution_current_time V1 V2 tlc w z t))
    ⇒ telegraph_equation_current V I G C z t
```

Assumptions of the above theorems are the same as those of Theorems 5.5 and 5.6. Similar to the verification of the wave equations in the time domain, we used Lemmas 5.1 and 5.2 as well as the verified lemmas of the derivatives for voltage and current in order to verify the correctness of the wave solutions for the telegrapher's equations. More details about the verification of the time-domain PDEs can be found in our proof script [10].

6 Application: Terminated Transmission Line

To illustrate the practical effectiveness of our proposed approach, we formally analyze the behavior of various transmission lines connected between a generator and a load. Particularly, we perform a formal analysis of a terminated transmission line by formally verifying the load impedance and the voltage reflection coefficient. Moreover, we formally analyze short-circuited and open-circuited transmission lines that are commonly used in the construction of resonant circuits and matching stubs. These lines correspond to the special cases of the load impedance: $Z_L = 0$ for a short-circuited line and $Z_L = \infty$ for an open-circuited line.

Terminated transmission lines in arbitrary complex load impedances are used in the majority of sinusoidal steady-state applications. They play a vital role in ensuring a smooth transfer of signals or power, especially in applications where signal quality and system performance are critical. We consider the essential behavior of line voltage, current, and impedance for a portion of a lossless transmission line terminated with a load Z_L, as shown in Figure 3. In this section, we formally analyze a terminated transmission line by formally verifying in HOL Light various important properties, such as load impedance and voltage reflection coefficient.

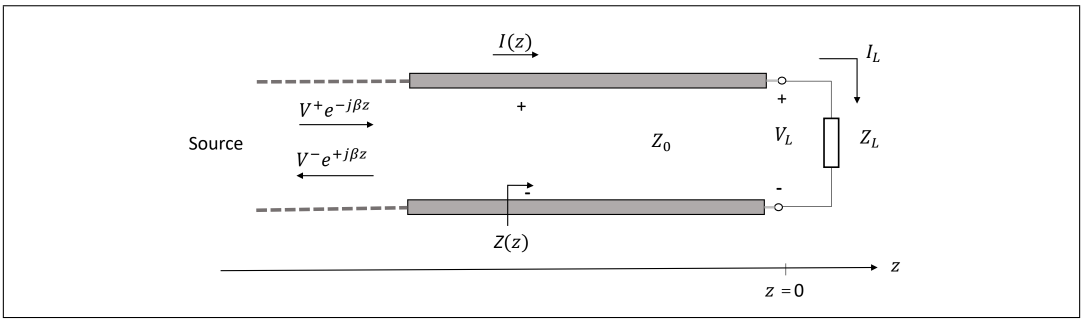

Figure 3: A Terminated Transmission Line [16]

Consider a line terminated by the load Z_L at $z = 0$ as depicted in Figure 3. The characteristic impedance is the ratio of the traveling voltage and current waves.

$$\frac{V_0^+}{I_0^+} = Z_0$$

Substituting the boundary condition $z = 0$, in Equations (17) and (20), we get

$$V(0) = V_0^+ + V_0^- \qquad (31) \qquad\qquad I(0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} \qquad (32)$$

We can define the line impedance $Z(z)$ at any position z on the line as seen in Figure 3:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}} \tag{33}$$

Here, the line impedance is not equal to Z_0 when the line is terminated, i.e., a leftward-traveling reflected wave exists. We can find the line impedance at the load position, i.e., Z_L, by dividing above two equations:

$$\frac{V(z)}{I(z)}|_{z=0} = \frac{V(0)}{I(0)} = Z_L = Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \tag{34}$$

Now, we define the line impedance in HOL Light as follows:

Definition 6.1. *Line Impedance*

```
⊢def ∀V1 V2 tlc w z.
  line_impedance V1 V2 tlc w z =
      wave_solution_voltage_phasor V1 V2 tlc w z /
         wave_solution_current_phasor V1 V2 tlc w z
```

where the HOL Light function `line_impedance` represents the ratio of the total voltage $V(z)$ to the total current $I(z)$ at any position z along the line.
Next, we formally verify that the voltage and current on the transmission line at point $z = 0$ have to abide to the boundary condition imposed by the load.

Theorem 6.1. *Line Impedance at the Load Position ($z = 0$)*

```
⊢thm ∀V1 V2 R L G C w z.
  let tlc = ((R,L,G,C):trans_line_const) in
  [A1] z = Cx(&0)
    ⇒ line_impedance V1 V2 tlc w z =
       characteristic_impedance tlc w * ((V1 + V2) / (V1 - V2))
```

The verification of Theorem 6.1 is based on the formalizations of line and characteristic impedances alongside some complex arithmetic reasoning.
We can rearrange Equation (34) as the ratio of the reflected voltage amplitude to the incident voltage amplitude

$$\frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{35}$$

This ratio of the phasors of the reverse and forward waves at the load position ($z = 0$) is defined as voltage reflection coefficient.

$$\Gamma_L = \frac{V_0^-(0)}{V_0^+(0)} = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{36}$$

Next, we define the voltage reflection coefficient in HOL Light as follows:

Definition 6.2. *Voltage Reflection Coefficient*

```
⊢def ∀V1 V2 tlc w z.
voltage_reflection_coefficient V1 V2 tlc w z =
(line_impedance V1 V2 tlc w z - characteristic_impedance tlc w) /
(line_impedance V1 V2 tlc w z + characteristic_impedance tlc w)
```

Now, we verify that the voltage reflection coefficient is equal to the ratio of reflected voltage to the incident voltage as the following HOL Light theorem:

Theorem 6.2. *Relating Forward-Going Voltage to Reflected Voltage*

```
⊢thm ∀V1 V2 R L G C w z.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] V1 ≠ V2 ∧ [A2] z = Cx(&0)
 [A3] characteristic_impedance tlc w ≠ Cx(&0)
   ⇒ voltage_reflection_coefficient V1 V2 tlc w z = V2 / V1
```

Assumption `A1` ensures that voltages are different from each other. Assumption `A2` represents the boundary condition `z` = 0. Assumption `A3` guarantees that the characteristic impedance is nonzero. The verification of the above theorem is mainly based on Theorem 6.1 along with some complex arithmetic reasoning.

We can also obtain the line impedance at the load ($z = 0$) from the reflection coefficient by rewriting the relationship in Equation (36):

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{37}$$

Here, the quantity Γ_L is known as the voltage reflection coefficient. Now, we verify the above relationship as the following HOL Light theorem.

Theorem 6.3. *Final Equation for Line Impedance at the Load Position*

```
⊢thm ∀V1 V2 R L G C w z.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] V1 ≠ V2 ∧ [A2] z = Cx(&0) ∧
 [A3] characteristic_impedance tlc w ≠ Cx(&0)
 ⇒ line_impedance V1 V2 tlc w z =
     characteristic_impedance tlc w *
      ((Cx(&1) + (voltage_reflection_coefficient V1 V2 tlc w z)) /
        (Cx(&1) - (voltage_reflection_coefficient V1 V2 tlc w z)))
```

Assumptions `A1-A3` are the same as those of Theorem 6.2. The verification of the above theorem is primarily based on Theorems 6.1 and 6.2 alongside some complex arithmetic reasoning.

In the following subsections, we formally analyze short-circuited and open-circuited transmission lines as special cases of a terminated transmission line.

6.1 Short-Circuited Line

When the load end of a transmission line is connected in such a way that it creates a short circuit, it is referred to as a short-circuited transmission line. These lines are extensively used in microwave engineering and Radio-Frequency (RF) systems to ensure a proper impedance matching, which is essential for an efficient power transmission and preserving the integrity of signals. Figure 4 depicts a transmission line of length l that is terminated by a short circuit ensuring a zero load impedance, i.e., $Z_L = 0$.

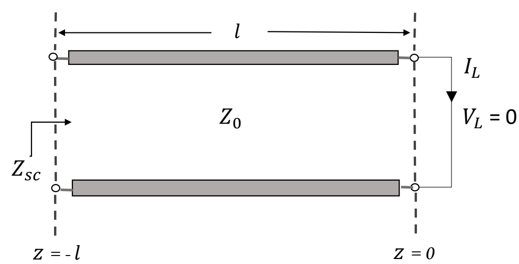

Figure 4: Short-Circuited Line [16]

Moreover, the short-circuited termination forces the load voltage V_L to zero. Therefore, from Equation (17), we have:

$$V_L = V(z)|_{z=0} = 0$$

$$V^+e^{-j\beta z} + V^-e^{j\beta z}|_{z=0} = 0$$

$$V^+ + V^- = 0$$

This implies

$$V^- = -V^+ \tag{38}$$

We employ Equation (20) to find the load current flowing through the short circuit by utilizing Equation (38) as:

$$
\begin{aligned}
I_L &= I(z)|_{z=0} \\
&= \frac{1}{Z_0}[V^+ - V^-|_{z=0} \\
&= \frac{2V^+}{Z_0} \qquad (39)
\end{aligned}
$$

Everywhere else on the transmission line, the voltage and current are mathematically expressed as [16]:

$$V(z) = V^+(e^{-j\beta z} - e^{j\beta z}) = -2V^+ jsin(\beta z)$$

$$I(z) = \frac{V^+}{Z_0}(e^{-j\beta z} + e^{j\beta z}) = \frac{2V^+}{Z_0}cos(\beta z)$$

The line impedance observed when looking towards the far end (short-circuited location) on the transmission line is:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0\frac{-2V^+ jsin(\beta z)}{2V^+ cos(\beta z)} = -jZ_0 tan(\beta z)$$

Next, we formally verify the short-circuited line in HOL Light as follows:

Theorem 6.4. *Short-Circuited Line*

```
⊢thm ∀V1 V2 R L G C w z.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] (V2 = --V1) ∧ [A2] w > &0 ∧ [A3] L > &0 ∧
 [A4] C > &0 ∧ [A5] R = &0 ∧ [A6] G = &0 ∧ [A7] V1 ≠ Cx (&0)
   ⇒ line_impedance V1 V2 tlc w z =
          --ii * characteristic_impedance tlc w *
             ctan (Cx (Im(propagation_constant tlc w)) * z)
```

Assumptions A1 provides the condition for the short-circuited line. Assumptions A2-A4 guarantee that the angular frequency ω and the parameters L and C cannot be negative or zero, respectively. Assumptions A5-A6 assert that the line parameters R and G are equal to zero, which is an assumptions for a lossless transmission line. Assumption A7 provides that the coefficient V1 is different than zero. The verification of Theorem 6.4 is primarily based on the following lemma:

Lemma 6.1. *Lemma for Short-Circuited Line*

```
⊢lem ∀V1 V2 R L G C w z.
 let tlc = ((R,L,G,C):trans_line_const) in
 [A1] (V2 = --V1) ∧ [A2] w > &0 ∧ [A3] L > &0 ∧
 [A4] C > &0 ∧ [A5] R = &0 ∧ [A6] G = &0 ∧ [A7] V1 ≠ Cx (&0)
  ⇒ line_impedance V1 V2 tlc w z = characteristic_impedance tlc w *
      ((-Cx(&2) * ii * V1 * csin(Cx (Im(propagation_constant tlc w)) * z)) /
           (Cx(&2) * V1 * ccos (Cx(Im(propagation_constant tlc w)) * z)))
```

Every assumption in the above lemma is the same as that of Theorem 6.4. The proof of Lemma 6.1 is mainly based on Theorems 1 and 2 provided in Table 4 and properties of the trancendental functions along with some complex arithmetic reasoning.

6.2 Open-Circuited Line

When a transmission line is open at the load end, it is referred to as an open-circuited transmission line. Since the terminal is characterized by an open circuit configuration, the signal or current is unable to propagate beyond the open-circuited point. Open-circuited transmission lines are employed in antenna design to model the behavior of open-ended radiating devices. Figure 5 depicts an open-circuited transmission line with an infinite load impedance, i.e., $Z_L = \infty$.

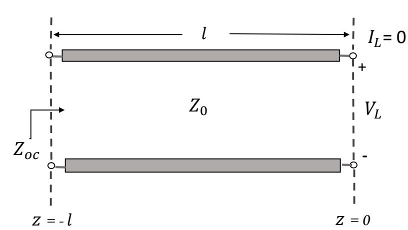

Figure 5: Open-Circuited Line [16]

An open-circuited transmission line forces the load current I_L to be zero. Therefore, by using Equation (20) we have:

$$I_L = I(z)|_{z=0} = 0$$

$$\frac{V^+}{Z_0}e^{-j\beta z} - \frac{V^-}{Z_0}e^{j\beta z}|_{z=0} = 0$$

$$\frac{V^+ + V^-}{Z_0} = 0$$

Thus,

$$V^- = V^+ \tag{40}$$

Note that the load voltage V_L appearing across the open circuit can be found from Equation (17) using Equation (40):

$$V_L = V(z)|_{z=0}$$

$$= V^+ e^{-j\beta z} + V^- e^{j\beta z}|_{z=0}$$

$$= V^+ + V^- = 2V^+ \tag{41}$$

Everywhere else on the transmission line, the voltage and current are mathematically expressed as [16]:

$$V(z) = V^+(e^{-j\beta z} + e^{j\beta z}) = 2V^+ cos(\beta z)$$

$$I(z) = \frac{V^+}{Z_0}(e^{-j\beta z} - e^{j\beta z}) = -\frac{2V^+}{Z_0} j sin(\beta z) = \frac{2V^+}{Z_0} e^{-j\pi/2} sin(\beta z)$$

Next, we formally verify the open-circuited line in HOL Light as follows:

Theorem 6.5. *Open-Circuited Line*

```
⊢thm ∀V1 V2 R L G C w z.
  let tlc = ((R,L,G,C):trans_line_const) in
  [A1] (V2 = V1) ∧ [A2] w > &0 ∧ [A3] L > &0 ∧
  [A4] C > &0 ∧ [A5] R = &0 ∧ [A6] G = &0 ∧ [A7] V1 ≠ Cx (&0)
    ⇒ line_impedance V1 V2 tlc w z =
          ii * characteristic_impedance tlc w *
             ccot (Cx (Im (propagation_constant tlc w)) * z)
```

Assumptions `A1` ensures the condition for the open-circuited line. The rest of the assumptions are the same as that of Theorem 6.4. Similar to Theorem 6.4, the proof of the above theorem is mainly based on the following lemma:

Lemma 6.2. *Lemma for Open-Circuited Line*

```
⊢lem ∀V1 V2 R L G C w z.
  let tlc = ((R,L,G,C):trans_line_const) in
  [A1] (V2 = V1) ∧ [A2] w > &0 ∧ [A3] L > &0 ∧
  [A4] C > &0 ∧ [A5] R = &0 ∧ [A6] G = &0 ∧ [A7] V1 ≠ Cx (&0) ∧
   ⇒ line_impedance V1 V2 tlc w z = characteristic_impedance tlc w *
     ((Cx(&2) * V1 * ccos(Cx (Im (propagation_constant tlc w)) * z)) /
      (--Cx(&2) * ii * V1 * csin (Cx (Im (propagation_constant tlc w)) * z)))
```

The proof of the above lemma is mainly based on the formally verified lemmas about the exponential functions alongwith some complex arithmetic reasoning. This completes the formal analysis of the terminated, short-circuited and open-circuited transmission lines. The details about the analysis can be found in the proof script [10].

7 Discussion

The main purpose of this work is the formal development of transmission line theory within the sound core of a higher-order-logic theorem prover to analyze transmission systems. For our constructive formalization, we first formally analyzed the variations of the line voltage and current utilizing the phasor representations of the telegrapher's equations because the phasor approach reduces the time-domain PDEs to ODEs. In the verification of the ODEs, we proved lemmas about the derivatives of the general solutions. One of the main challenges of the presented work was to formally verify the general solutions for the time-domain PDEs. The process began by translating solutions from the phasor domain, where they are articulated as complex-valued functions of frequency, into the time-domain as real-valued functions to establish solutions for PDEs. In the HOL Light proof process, we subsequently faced the requirement to transform the time-domain functions back into complex-valued forms. This was essential because the time-domain PDEs are defined using complex derivatives, and the challenge lays in adeptly employing these complex derivatives during the proof procedure. We also proved the necessary lemmas about the complex differentions of the general solutions with respect to the parameters z and t. In addition, we provided proofs of the attenuation and phase constants for the lossless line and some other theorems regarding exponential functions and complex numbers in order to verify the correctness of the wave solutions for the time-domain PDEs. Once we proved the required theorems and lemmas, the verification of the correctness of the equations just took several lines of proof steps.

For example, the proofs of the general solutions of the wave equations for voltage and current just took 19 and 22 lines, which clearly illustrates the benefit of the

Formalized Theorems	Proof Lines	Page Numbers	Woman Hours
Theorem 4.1	26	212	1
Theorem 4.2	25	212	1
Theorem 5.1	13	214	0.5
Theorem 5.2	25	214	0.5
Table 3. Lemma 1	4	215	0.5
Table 3. Lemma 2	33	215	1
Table 3. Lemma 3	4	215	0.5
Table 3. Lemma 4	57	215	1
Theorem 5.3	28	216	1
Theorem 5.4	49	216	1.5
Table 4. Theorem 1	34	218	3
Table 4. Theorem 2	37	218	4
Table 4. Theorem 3	81	218	3
Table 4. Theorem 4	85	218	7
Table 4. Theorem 5	146	218	2
Table 4. Theorem 6	220	218	3
Theorem 5.5	19	220	0.5
Lemma 5.1	61	220	20
Table 5. Lemma 1	23	221	3
Table 5. Lemma 2	46	221	5
Table 5. Lemma 3	25	221	4
Table 5. Lemma 4	33	221	7
Theorem 5.6	22	222	0.5
Lemma 5.2	82	224	10
Table 6. Lemma 1	54	223	4
Table 6. Lemma 2	32	223	6
Table 6. Lemma 3	59	223	5
Table 6. Lemma 4	38	223	9
Theorem 5.7	71	224	2
Theorem 5.8	107	224	3
Theorem 6.1	18	226	0.5
Theorem 6.2	70	227	3
Theorem 6.3	22	227	0.5
Theorem 6.4	33	229	0.5
Lemma 6.1	63	230	4
Theorem 6.5	48	231	0.5
Lemma 6.2	57	232	5

Table 7: Verification Details for Proven Theorems and Lemmas

formally verified lemmas and theorems. The amount of effort required for verifying each individual theorem in terms of proof lines and the corresponding woman-hours is presented in Table 7. It is noteworthy that the woman-hours needed to complete proofs are dependent on both the number of lines of code and the complexity of the proof. Consequently, there is no direct relation between the number of lines in the proof script and the amount of time required in woman-hours. For example, the verification process for the characteristic impedance of a distortionless line involves a greater number of proof lines compared to the verification of the attenuation constant. However, the woman-hours required for the former are actually less than those needed for the latter. Another difficulty encountered in this formalization pertains to the considerable level of user intervention. However, we developed several tactics that automate certain parts of our proofs resulting in a reduction of the length of proof scripts in many instances (e.g., reducing part of the code by around 240 lines) and make the proofs simpler and more compact. Examples of such tactics are `SHORT_TAC` and `EQ_DIFF_SIMP`, which allowed us to simplify complex arithmetics involved in the proof of the time-domain solutions. For instance, `EQ_DIFF_SIMP` is constructed to efficiently deal with the repetitive patterns in our proof procedure by consolidating them into a single tactic. This proves to be efficient in refining and optimizing our overall approach. The main advantage of the conducted formal proofs of the telegrapher's equations is that all the underlying assumptions can be explicitly written contrary to the case of paper-and-pencil proofs and proof-steps that are mechanically verified using a theorem prover. In addition, the formalization of the transmission line theory provides mathematicians and engineers with the ability to modify and reuse the formal library in HOL Light, in contrast to conventional manual mathematical analysis.

8 Conclusion

This paper advocates the usage of higher-order-logic theorem proving for the formalization of the telegrapher's equations and the verification of its general solutions. In particular, we formalized the telegrapher's equations and their alternate representations, i.e., wave equations in time and phasor domains using HOL Light. Furthermore, we verified the relationship between the telegrapher's equations and the wave equations in the phasor domain. Moreover, we constructed the formal proof for the general solutions of the telegrapher's equations in the phasor domain. Subsequently, we proved the relation between the phasor and the time-domain functions in order to formally verify the general solutions for the time-domain PDEs. Finally, in order to demonstrate the usefulness of our formalization work, we formally analyzed several practical applications, including terminated, short-circuited and open-circuited transmission lines. One of our future plans is to extend this formalization by formalizing the deviations of real circuits from the idealized model, with an aim of applying it to practical applications in real-world scenarios. Another potential area for future investigation involves analyzing the behavior of harmonics in transmission lines using the Fourier transform [18].

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