Point Estimation

CSE 446: Machine Learning
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Maximum likelihood estimation for a binomial distribution

Your first consulting job

- A bored Seattle billionaire asks you a question:
  - He says: I have thumbtack, if I flip it, what's the probability it will fall with the nail up?
  - You say: Please flip it a few times:
  - You say: The probability is:
  - **He says: Why???**
  - You say: Because…

Thumbtack – Binomial distribution

- P(Heads) = θ, P(Tails) = 1-θ
- Flips are i.i.d.:
  - Independent events
  - Identically distributed according to a binomial distribution
- Sequence D of α_H heads (H) and α_T tails (T)
- $P(D \mid \theta)$ =

The learning task

- Want to learn a model of thumbtack flips from experience
- **Example 1:** Maximum likelihood estimation
  What value of θ maximizes the likelihood of having seen the observed sequence (according to my model)?
- What is a likelihood function?

Maximum likelihood estimation

- **Data:** Observed set D of α_H heads (H) and α_T tails (T)
- **Hypothesis:** Binomial distribution
- Learning θ is an optimization problem
  - What's the objective function?
- **MLE:** Choose θ that maximizes the likelihood of observed data

$$\hat{\theta} = \arg\max_{\theta} P(D \mid \theta)$$
$$= \arg\max_{\theta} \ln P(D \mid \theta)$$

Your first learning algorithm

$$\hat{\theta} = \arg\max_{\theta} \ln P(D \mid \theta)$$
$$= \arg\max_{\theta} \ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

- Set derivative to zero: $\frac{d}{d\theta} \ln P(D \mid \theta) = 0$

How many flips do I need?

$$\hat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$$

- Billionaire says: I flipped 3 heads and 2 tails.
- You say: θ = 3/5, I can prove it!
- He says: What if I flipped 30 heads and 20 tails?
- You say: Same answer, I can prove it!
- **He says: What's better?**
- You say: Humm… The more the merrier???
- He says: Is this why I am paying you the big bucks???

Simple bound (based on Hoeffding's Inequality)

- For $N = \alpha_H + \alpha_T$ and $\hat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$
- Let θ^* be the true parameter. For any $\varepsilon > 0$:

$$P(|\hat{\theta}_{MLE} - \theta^*| \geq \epsilon) \leq 2e^{-2N\epsilon^2}$$

PAC learning

- **PAC:** Probably Approximate Correct
- Billionaire says: I want to know the thumbtack parameter θ within $\varepsilon = 0.1$, with probability at least $1-\delta = 0.95$. How many flips do I need?

$$P(|\hat{\theta}_{MLE} - \theta^*| \geq \epsilon) \leq 2e^{-2N\epsilon^2}$$

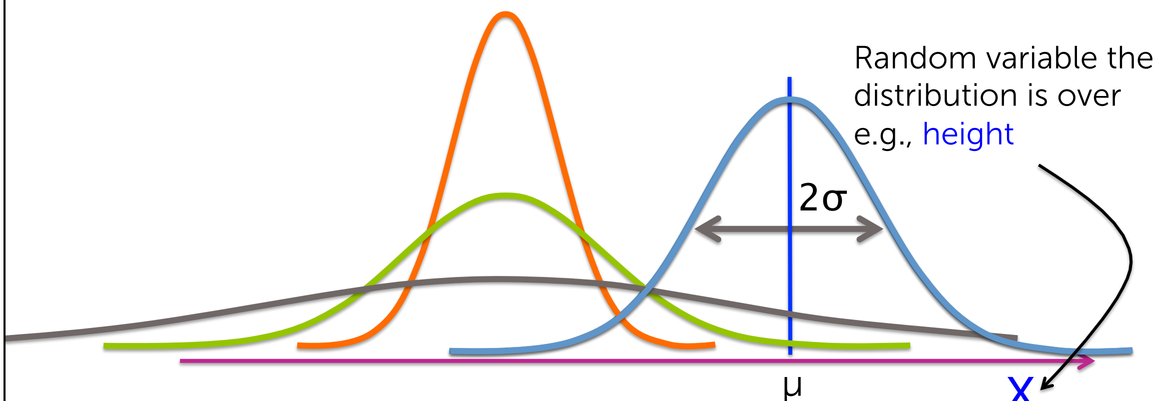
What about continuous-valued data?

What about continuous variables?

- Billionaire says: If I am measuring a continuous variable, what can you do for me?
- **You say: Let me tell you about Gaussians…**

1D Gaussians

Fully specified by **mean** μ and **variance** σ^2
(or **standard deviation** σ)

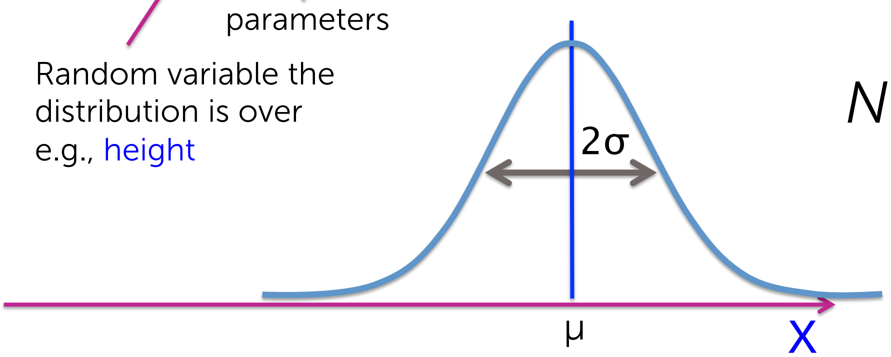


1D Gaussian probability density function

$$p(x \mid \underbrace{\mu, \sigma}_{\text{parameters}}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Random variable the
distribution is over
e.g., height

$N(\mu, \sigma^2)$



Some properties of Gaussians

- Affine transformation (multiplying by scalar and adding a constant)
  - $X \sim N(\mu,\sigma^2)$
  - $Y = aX + b \rightarrow Y \sim N(a\mu+b,a^2\sigma^2)$
- Sum of Gaussians
  - $X \sim N(\mu_X,\sigma^2_X)$
  - $Y \sim N(\mu_Y,\sigma^2_Y)$
  - $Z = X+Y \rightarrow Z \sim N(\mu_X+\mu_Y, \sigma^2_X+\sigma^2_Y)$

Learning a Gaussian

- Collect a bunch of data
  - Hopefully, i.i.d. samples
  - e.g., heights of students in class
- Learn parameters
  - Mean
  - Variance

$$p(x \mid \mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

MLE for Gaussian

- Prob. of i.i.d. samples D={x_1,…,x_N}:

$$p(D \mid \mu, \sigma) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \prod_{i=1}^N e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

- Log-likelihood of data:

$$\ln p(D \mid \mu, \sigma) = \ln\left[\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \prod_{i=1}^N e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}\right]$$

$$= -N \ln \sigma\sqrt{2\pi} - \sum_{i=1}^N \frac{(x_i-\mu)^2}{2\sigma^2}$$

Your second learning algorithm: MLE for mean of a Gaussian

- What's MLE for the mean?

$$\frac{d}{d\mu} \ln p(D \mid \mu, \sigma) = \frac{d}{d\mu}\left[-N \ln \sigma\sqrt{2\pi} - \sum_{i=1}^N \frac{(x_i-\mu)^2}{2\sigma^2}\right] = 0$$

MLE for variance

- Again, set derivative to zero:

$$\frac{d}{d\sigma}\ln p(D \mid \mu, \sigma) = \frac{d}{d\sigma}\left[-N \ln \sigma\sqrt{2\pi} - \sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

$$= \frac{d}{d\sigma}\left[-N \ln \sigma\sqrt{2\pi}\right] - \sum_{i=1}^{N}\frac{d}{d\sigma}\left[\frac{(x_i-\mu)^2}{2\sigma^2}\right] = 0$$

Learning Gaussian parameters

- MLE: $\hat{\mu}_{MLE} = \frac{1}{N}\sum_{i=1}^{N} x_i$

$$\hat{\sigma}^2_{MLE} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu}_{MLE})^2$$

- FYI, MLE for the variance of a Gaussian is **biased**
  - Expected value of estimator is **not** true parameter!
  - Unbiased variance estimator:

$$\hat{\sigma}^2_{unbiased} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu}_{MLE})^2$$

Recap of concepts

What you need to know…

- Learning is…
  - Collect some data
    - E.g., thumbtack flips
  - Choose a hypothesis class or model
    - E.g., binomial
  - Choose a loss function
    - E.g., data likelihood
  - Choose an optimization procedure
    - E.g., set derivative to zero to obtain MLE
  - Collect the big bucks

- Like everything in life, there is a lot more to learn…
  - Many more facets… Many more nuances…
  - The fun will continue…