Advanced Machine Learning

Online Learning Basics

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Outline

- Prediction with expert advice
- Weighted Majority algorithm (WM)
- Randomized Majority algorithm (RWM)
- Online-to-batch conversion

Motivation

- PAC learning:
  - distribution fixed over time (training and test).
  - IID assumption.
- On-line learning:
  - no distributional assumption.
  - worst-case analysis (adversarial).
  - mixed training and test.
  - Performance measure: mistake model, regret.

General Online Setting

- For $t = 1$ to T do
  - receive instance $x_t \in X$.
  - predict $\widehat{y}_t \in Y$.
  - receive label $y_t \in Y$.
  - incur loss $L(\widehat{y}_t, y_t)$.
- Classification: $Y = \{0, 1\}, L(y, y') = |y' - y|$.
- Regression: $Y \subseteq \mathbb{R}, L(y, y') = (y' - y)^2$.
- Objective: minimize total loss $\sum_{t=1}^{T} L(\widehat{y}_t, y_t)$.

Prediction with Expert Advice

- For $t = 1$ to T do
  - receive instance $x_t \in X$ and advice $y_{t,i} \in Y, i \in [1, N]$.
  - predict $\widehat{y}_t \in Y$.
  - receive label $y_t \in Y$.
  - incur loss $L(\widehat{y}_t, y_t)$.
- Objective: minimize regret, i.e., difference of total loss incurred and that of the best expert,

$$\text{Regret}(T) = \sum_{t=1}^{T} L(\widehat{y}_t, y_t) - \min_{i=1}^{N} \sum_{t=1}^{T} L(\widehat{y}_{t,i}, y_t).$$

Halving Algorithm

[see (Mitchell, 1997)]

$\textsc{Halving}(H)$

1 $H_1 \leftarrow H$
2 **for** $t \leftarrow 1$ **to** T **do**
3 $\quad \textsc{Receive}(x_t)$
4 $\quad \widehat{y}_t \leftarrow \textsc{MajorityVote}(H_t, x_t)$
5 $\quad \textsc{Receive}(y_t)$
6 $\quad$ **if** $\widehat{y}_t \neq y_t$ **then**
7 $\quad\quad H_{t+1} \leftarrow \{c \in H_t : c(x_t) = y_t\}$
8 **return** H_{T+1}

Halving Algorithm - Bound

(Littlestone, 1988)

- Theorem: Let H be a finite hypothesis set, then the number of mistakes made by the Halving algorithm is bounded as follows:

$$M_{Halving(H)} \leq \log_2 |H|.$$

- Proof: At each mistake, the hypothesis set is reduced at least by half.

Weighted Majority Algorithm

(Littlestone and Warmuth, 1988)

WEIGHTED-MAJORITY(N experts) ▷ $y_t, y_{t,i} \in \{0, 1\}$. $\beta \in [0, 1)$.

for $i \leftarrow 1$ **to** N **do**
    $w_{1,i} \leftarrow 1$
for $t \leftarrow 1$ **to** T **do**
    RECEIVE(x_t)
    $\widehat{y}_t \leftarrow 1_{\sum_{y_{t,i}=1}^{N} w_t \geq \sum_{y_{t,i}=0}^{N} w_t}$ ▷ weighted majority vote
    RECEIVE(y_t)
    if $\widehat{y}_t \neq y_t$ **then**
        for $i \leftarrow 1$ **to** N **do**
            if $(y_{t,i} \neq y_t)$ **then**
                $w_{t+1,i} \leftarrow \beta w_{t,i}$
            else $w_{t+1,i} \leftarrow w_{t,i}$
return $\mathbf{w}_{T+1}$

Weighted Majority - Bound

- Theorem: Let m_t be the number of mistakes made by the WM algorithm till time t and m_t^* that of the best expert. Then, for all t,

$$m_t \leq \frac{\log N + m_t^* \log \frac{1}{\beta}}{\log \frac{2}{1+\beta}}.$$

  - Thus, $m_t \leq O(\log N) + \text{constant} \times \text{best expert}.$
  - Realizable case: $m_t \leq O(\log N)$.
  - Halving algorithm: $\beta = 0$.

Weighted Majority - Proof

- Potential: $\Phi_t = \sum_{i=1}^{N} w_{t,i}$.
- Upper bound: after each error,

$$\Phi_{t+1} \leq \left[1/2 + 1/2\,\beta\right]\Phi_t = \left[\frac{1+\beta}{2}\right]\Phi_t.$$

Thus, $\Phi_t \leq \left[\frac{1+\beta}{2}\right]^{m_t} N.$

- Lower bound: for any expert i, $\Phi_t \geq w_{t,i} = \beta^{m_{t,i}}$.
- Comparison: $\beta^{m_t^*} \leq \left[\frac{1+\beta}{2}\right]^{m_t} N$

$$\Rightarrow m_t^* \log \beta \leq \log N + m_t \log\left[\frac{1+\beta}{2}\right]$$

$$\Rightarrow m_t \log\left[\frac{2}{1+\beta}\right] \leq \log N + m_t^* \log\frac{1}{\beta}.$$

Weighted Majority - Notes

- Advantage: remarkable bound requiring no assumption.
- Disadvantage: no deterministic algorithm can achieve a regret $R_T = o(T)$ with the binary loss.
  - better guarantee with randomized WM.
  - better guarantee for WM with convex losses.

Exponential Weighted Average

- Algorithm:
  - weight update: $w_{t+1,i} \leftarrow w_{t,i}\, e^{-\eta L(\widehat{y}_{t,i}, y_t)} = e^{-\eta L_{t,i}}$.
  - prediction: $\widehat{y}_t = \frac{\sum_{i=1}^N w_{t,i} y_{t,i}}{\sum_{i=1}^N w_{t,i}}$.

total loss incurred by expert *i* up to time *t*

- Theorem: assume that L is convex in its first argument and takes values in $[0, 1]$. Then, for any $\eta > 0$ and any sequence $y_1, \ldots, y_T \in Y$, the regret at time T satisfies

$$\text{Regret}(T) \leq \frac{\log N}{\eta} + \frac{\eta T}{8}.$$

For $\eta = \sqrt{8 \log N / T}$,

$$\boxed{\text{Regret}(T) \leq \sqrt{(T/2) \log N}.}$$

EW - Proof

- Potential: $\Phi_t = \log \sum_{i=1}^N w_{t,i}$.
- Upper bound:

$$\begin{aligned}
\Phi_t - \Phi_{t-1} &= \log \frac{\sum_{i=1}^N w_{t-1,i}\, e^{-\eta L(\widehat{y}_{t,i}, y_t)}}{\sum_{i=1}^N w_{t-1,i}} \\
&= \log \left(\mathop{\mathrm{E}}_{w_{t-1}} [e^{-\eta L(\widehat{y}_{t,i}, y_t)}] \right) \\
&= \log \left(\mathop{\mathrm{E}}_{w_{t-1}} \left[\exp \left(-\eta \left(L(\widehat{y}_{t,i}, y_t) - \mathop{\mathrm{E}}_{w_{t-1}} [L(\widehat{y}_{t,i}, y_t)] \right) - \eta \mathop{\mathrm{E}}_{w_{t-1}} [L(\widehat{y}_{t,i}, y_t)] \right) \right] \right) \\
&\leq -\eta \mathop{\mathrm{E}}_{w_{t-1}} [L(\widehat{y}_{t,i}, y_t)] + \frac{\eta^2}{8} \quad \text{(Hoeffding's ineq.)} \\
&\leq -\eta L(\mathop{\mathrm{E}}_{w_{t-1}} [\widehat{y}_{t,i}], y_t) + \frac{\eta^2}{8} \quad \text{(convexity of first arg. of } L\text{)} \\
&= -\eta L(\widehat{y}_t, y_t) + \frac{\eta^2}{8}.
\end{aligned}$$

EW - Proof

- Upper bound: summing up the inequalities yields

$$\Phi_T - \Phi_0 \leq -\eta \sum_{t=1}^{T} L(\widehat{y}_t, y_t) + \frac{\eta^2 T}{8}.$$

- Lower bound:

$$\Phi_T - \Phi_0 = \log \sum_{i=1}^{N} e^{-\eta L_{T,i}} - \log N \geq \log \max_{i=1}^{N} e^{-\eta L_{T,i}} - \log N$$
$$= -\eta \min_{i=1}^{N} L_{T,i} - \log N.$$

- Comparison:

$$-\eta \min_{i=1}^{N} L_{T,i} - \log N \leq -\eta \sum_{t=1}^{T} L(\widehat{y}_t, y_t) + \frac{\eta^2 T}{8}$$

$$\Rightarrow \sum_{t=1}^{T} L(\widehat{y}_t, y_t) - \min_{i=1}^{N} L_{T,i} \leq \frac{\log N}{\eta} + \frac{\eta T}{8}.$$

EW - Proof

- Advantage: bound on regret per bound is of the form

$$\frac{R_T}{T} = O\left(\sqrt{\frac{\log(N)}{T}}\right).$$

- Disadvantage: choice of η requires knowledge of horizon T.

Doubling Trick

- Idea: divide time into periods $[2^k, 2^{k+1}-1]$ of length 2^k with $k = 0, \ldots, n$, $T \geq 2^n - 1$, and choose $\eta_k = \sqrt{\frac{8 \log N}{2^k}}$ in each period.
- Theorem: with the same assumptions as before, for any T, the following holds:

$$\text{Regret}(T) \leq \frac{\sqrt{2}}{\sqrt{2}-1}\sqrt{(T/2)\log N} + \sqrt{\log N/2}.$$

Doubling Trick - Proof

- By the previous theorem, for any $I_k = [2^k, 2^{k+1} - 1]$,

$$L_{I_k} - \min_{i=1}^{N} L_{I_k,i} \leq \sqrt{2^k/2 \log N}.$$

Thus,
$$L_T = \sum_{k=0}^{n} L_{I_k} \leq \sum_{k=0}^{n} \min_{i=1}^{N} L_{I_k,i} + \sum_{k=0}^{n} \sqrt{2^k (\log N)/2}$$
$$\leq \min_{i=1}^{N} L_{T,i} + \sum_{k=0}^{n} 2^{\frac{k}{2}} \sqrt{(\log N)/2}.$$

with
$$\sum_{k=0}^{n} 2^{\frac{k}{2}} = \frac{\sqrt{2}^{n+1} - 1}{\sqrt{2} - 1} = \frac{2^{(n+1)/2} - 1}{\sqrt{2} - 1} \leq \frac{\sqrt{2}\sqrt{T+1} - 1}{\sqrt{2} - 1} \leq \frac{\sqrt{2}(\sqrt{T} + 1) - 1}{\sqrt{2} - 1} \leq \frac{\sqrt{2}\sqrt{T}}{\sqrt{2} - 1} + 1.$$

Notes

- Doubling trick used in a variety of other contexts and proofs.
- More general method, learning parameter function of time: $\eta_t = \sqrt{(8 \log N)/t}$. Constant factor improvement:

$$\text{Regret}(T) \leq 2\sqrt{(T/2)\log N} + \sqrt{(1/8)\log N}.$$

General Setting

- Adversarial model with action set $X = \{1, \ldots, N\}$.
- For $t = 1$ to T do
  - player selects distribution p_t over X.
  - adversary selects loss $\mathbf{l}_t = (l_{t,1}, \ldots, l_{t,N})$.
  - player receives $\mathbf{l}_t$.
  - player incurs loss $\sum_{i=1}^{N} p_{t,i}\, l_{t,i}$.
- Objective: minimize (external) regret

$$R_T = \sum_{t=1}^{T} \mathop{\mathrm{E}}_{i \sim \mathsf{p}_t} [l_{t,i}] - \min_{i=1}^{N} \sum_{t=1}^{T} l_{t,i}.$$

Different Setups

- Deterministic vs. randomized.
- Full information vs. partial information (e.g. bandit setting).
- More general competitor class (e.g., swap regret).
- Oblivious vs. non-oblivious (or adaptive).
- Bounded memory.

Randomized Weighted Majority

(Littlestone and Warmuth, 1988)

RANDOMIZED-WEIGHTED-MAJORITY (N)

1 **for** $i \leftarrow 1$ **to** N **do**
2 $\quad w_{1,i} \leftarrow 1$
3 $\quad p_{1,i} \leftarrow 1/N$
4 **for** $t \leftarrow 1$ **to** T **do**
5 $\quad$ **for** $i \leftarrow 1$ **to** N **do**
6 $\quad\quad$ **if** $(l_{t,i} = 1)$ **then**
7 $\quad\quad\quad w_{t+1,i} \leftarrow \beta w_{t,i}$
8 $\quad\quad$ **else** $w_{t+1,i} \leftarrow w_{t,i}$
9 $\quad W_{t+1} \leftarrow \sum_{i=1}^{N} w_{t+1,i}$
10 $\quad$ **for** $i \leftarrow 1$ **to** N **do**
11 $\quad\quad p_{t+1,i} \leftarrow w_{t+1,i}/W_{t+1}$
12 **return** $\mathbf{w}_{T+1}$

RWM - Bound

- Theorem: fix $\beta \in [\frac{1}{2}, 1)$. Then, for any $T \geq 1$, the expected cumulative loss of RWM can be bounded as follows:

$$\mathcal{L}_T \leq \frac{\log N}{1-\beta} + (2-\beta)\mathcal{L}_T^{\min}.$$

For $\beta = \max\left\{\frac{1}{2}, 1 - \sqrt{\frac{\log N}{T}}\right\}$,

$$\boxed{\mathcal{L}_T \leq \mathcal{L}_T^{\min} + 2\sqrt{T \log N}.}$$

RWM - Proof

- Potential: $W_t = \sum_{i=1}^{N} w_{t,i}$.
- Upper bound:

$$\begin{aligned} W_{t+1} = \sum_{i:\, l_{t,i}=0} w_{t,i} + \beta \sum_{i:\, l_{t.i}=1} w_{t,i} &= W_t + (\beta - 1) \sum_{i:\, l_{t,i}=1} w_{t,i} \\ &= W_t + (\beta - 1) W_t \sum_{i:\, l_{t,i}=1} p_{t,i} \\ &= W_t + (\beta - 1) W_t L_t \\ &= W_t (1 - (1 - \beta) L_t). \end{aligned}$$

Thus, $W_{T+1} = N \prod_{t=1}^{T} (1 - (1 - \beta) L_t)$.

- Lower bound: $W_{T+1} \geq \max_{i \in [1,N]} w_{T+1,i} = \beta^{\mathcal{L}_T^{\min}}$.

RWM - Proof

- Comparison:

$$\beta^{\mathcal{L}_T^{\min}} \leq N \prod_{t=1}^{T} (1 - (1-\beta) L_t) \implies \mathcal{L}_T^{\min} \log \beta \leq \log N + \sum_{t=1}^{T} \log(1 - (1-\beta) L_t)$$

$$(\forall x < 1, \log(1-x) \leq -x) \implies \mathcal{L}_T^{\min} \log \beta \leq \log N - (1-\beta) \sum_{t=1}^{T} L_t$$

$$\implies \mathcal{L}_T^{\min} \log \beta \leq \log N - (1-\beta) \mathcal{L}_T$$

$$\implies \mathcal{L}_T \leq \frac{\log N}{1-\beta} - \frac{\log \beta}{1-\beta} \mathcal{L}_T^{\min}$$

$$\implies \mathcal{L}_T \leq \frac{\log N}{1-\beta} - \frac{\log(1-(1-\beta))}{1-\beta} \mathcal{L}_T^{\min}$$

$$(\forall x \in [0, 1/2], -\log(1-x) \leq x + x^2) \implies \mathcal{L}_T \leq \frac{\log N}{1-\beta} + (2-\beta) \mathcal{L}_T^{\min}.$$

- For the second statement, use

$$\beta^{\mathcal{L}_T^{\min}} \leq \frac{\log N}{1-\beta} + (2-\beta) \mathcal{L}_T^{\min} \leq \frac{\log N}{1-\beta} + (1-\beta) T + \mathcal{L}_T^{\min}.$$

Lower Bound

- Theorem: let $N = 2$. There is a stochastic sequence of losses for which the expected regret of any algorithm verifies $\mathrm{E}[R_T] \geq \sqrt{T/8}$.

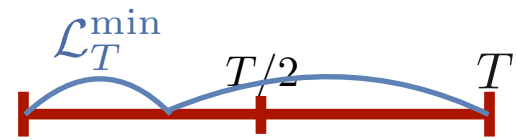
Lower Bound - Proof

- let $\mathbf{l}_t$ take values $\mathbf{l}_{01} = \binom{0}{1}$ or $\mathbf{l}_{10} = \binom{1}{0}$ with equal probability. Then,

$$\mathrm{E}[\mathcal{L}_T] = \mathrm{E}\left[\sum_{t=1}^{T} \mathsf{p}_t \cdot \mathbf{l}_t\right] = \sum_{t=1}^{T} \mathsf{p}_t \cdot \mathrm{E}[\mathbf{l}_t] = \sum_{t=1}^{T} \frac{1}{2} p_{t,1} + \frac{1}{2} p_{t,2} = T/2.$$

- Since $\mathcal{L}_{T,1} + \mathcal{L}_{T,2} = T$,

$$\mathcal{L}_T^{\min} = T/2 - |\mathcal{L}_{T,1} - T/2|.$$



- Thus, by the Khintchine-Kahane ineq.,

$$\mathrm{E}[R_T] = \mathrm{E}[\mathcal{L}_T] - \mathrm{E}[\mathcal{L}_T^{\min}] = \mathrm{E}[|\mathcal{L}_{T,1} - T/2|]$$
$$= \mathrm{E}\left[\left|\sum_{t=1}^{T} \frac{1+\sigma_t}{2} - T/2\right|\right] = \mathrm{E}\left[\left|\sum_{t=1}^{T} \frac{\sigma_t}{2}\right|\right] \geq \sqrt{T/8}.$$

Online-to-Batch Conversion

- Problem:
  - sample $((x_1, y_1), \ldots, (x_T, y_T)) \in (X \times Y)^T$ drawn i.i.d. according to D.
  - loss function L bounded by $M > 0$.
  - how do we combine the sequence $h_1, \ldots, h_{T+1}$ of hypotheses generated by a regret minimization algorithm to achieve a small generalization error?

Average Generalization

- Lemma: for any $\delta > 0$, with probability at least $1 - \delta$, the following holds

$$\frac{1}{T}\sum_{t=1}^{T} R(h_t) \leq \frac{1}{T}\sum_{t=1}^{T} L(h_t(x_t), y_t) + M\sqrt{\frac{2\log\frac{1}{\delta}}{T}}.$$

- Proof: for any $t \in [1, T]$, let $V_t = R(h_t) - L(h_t(x_t), y_t)$.
  - Then,

$$\mathrm{E}[V_t | x_{1:t-1}] = R(h_t) - \mathrm{E}[L(h_t(x_t), y_t) | x_{1:t-1}] = R(h_t) - R(h_t) = 0.$$

  - By Azuma's inequality,

$$\Pr[\frac{1}{T}\sum_{i=1}^{T} V_i \geq \epsilon] \leq \exp(-2T\epsilon^2/(2M)^2)).$$

Online-to-Batch Guarantee

- Theorem: for any $\delta > 0$, with probability at least $1 - \delta$, the following holds for a convex loss:

$$R\left(\frac{1}{T}\sum_{i=1}^{T} h_i\right) \leq \frac{1}{T}\sum_{i=1}^{T} L(h_i(x_i), y_i) + M\sqrt{\frac{2\log\frac{1}{\delta}}{T}}$$

$$R\left(\frac{1}{T}\sum_{i=1}^{T} h_i\right) \leq \inf_{h\in H} R(h) + \frac{R_T}{T} + 2M\sqrt{\frac{2\log\frac{2}{\delta}}{T}}.$$

Proof

- Convexity: $L\left(\frac{1}{T}\sum_{i=1}^{T} h_i(x), y\right) \leq \frac{1}{T}\sum_{i=1}^{T} L(h_i(x), y)$. Thus,

$$R\left(\frac{1}{T}\sum_{i=1}^{T} h_i\right) \leq \frac{1}{T}\sum_{i=1}^{T} R(h_i).$$

- By definition of the regret, with probability at least $1 - \delta/2$,

$$R\left(\frac{1}{T}\sum_{i=1}^{T} h_i\right) \leq \frac{1}{T}\sum_{i=1}^{T} L(h_i(x_i), y_i) + M\sqrt{\frac{2\log\frac{2}{\delta}}{T}}$$

$$\leq \inf_{h\in H} \frac{1}{T}\sum_{i=1}^{T} L(h(x_i), y_i) + \frac{R_T}{T} + M\sqrt{\frac{2\log\frac{2}{\delta}}{T}}.$$

Proof

- Assume that the infimum is reached at h^*. By Hoeffding's inequality, with probability at least $1 - \delta/2$,

$$\frac{1}{T}\sum_{i=1}^{T} L(h^*(x_i), y_i) \leq R(h^*) + M\sqrt{\frac{2\log\frac{2}{\delta}}{T}}.$$

- Thus, with probability at least $1 - \delta$,

$$\begin{aligned} R\left(\frac{1}{T}\sum_{i=1}^{T} h_i\right) &\leq \frac{1}{T}\sum_{i=1}^{T} L(h^*(x_i), y_i) + \frac{R_T}{T} + M\sqrt{\frac{2\log\frac{2}{\delta}}{T}} \\ &\leq R(h^*) + M\sqrt{\frac{2\log\frac{2}{\delta}}{T}} + \frac{R_T}{T} + M\sqrt{\frac{2\log\frac{2}{\delta}}{T}} \\ &= R(h^*) + \frac{R_T}{T} + 2M\sqrt{\frac{2\log\frac{2}{\delta}}{T}}. \end{aligned}$$

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