Naïve Learning of Preference Orders

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1 The Model

Preferences. We assume a set $A = \{a_1, \dots, a_m\}$ of alternatives. A preference order σ over A is a linear order on A, and we write $\mathcal{L}_A$ for the set of all linear orders on A. If σ is a preference on A and $x \in A$ is an alternative, we write σ_{-x} for the restriction of σ on $A \setminus \{x\}$, i.e., the preference order that is just like σ except that it ignores x.

If σ and σ' are preferences on A, the *Kendall tau* distance $d(\sigma, \sigma')$ counts the number of pairs of alternatives on which σ and σ' differ.

If σ^* is a reference ranking over m alternatives and $k \in \{0, 1, \dots, \binom{m}{2}\}$, I_m^k is the number of rankings at distance k from σ^*.

Example 1. For $A = \{a, b, c\}$ and $\sigma^* = abc$, we have that $I_3^0 = 1$, corresponding to ranking abc, $I_3^1 = 2$, corresponding to bac and acb, $I_3^2 = 2$, corresponding to bca and cab, and $I_3 = 3$, corresponding to cba.

Noise models. The Mallows model assumes a reference order σ^*, and assigns a preference order σ the following probability:

$$\mathbb{P}[\sigma] = \frac{1}{Z} \varphi^{d(\sigma, \sigma^*)},$$

for $\varphi \in (0, 1)$ and $Z_m = (1 + \varphi)(1 + \varphi + \varphi^2) \dots (1 + \varphi + \dots + \varphi^{m-1})$. Note that:

$$Z_m = Z_{m-1} \cdot (1 + \varphi + \varphi^2 + \dots + \varphi^{m-1}).$$

Importantly, we also have that:

$$Z_m = I_m^0 \cdot \varphi^0 + I_m^1 \cdot \varphi^1 + \dots + I_m^{\binom{m}{2}} \cdot \varphi^{\binom{m}{2}}.$$

Agents. We assume a set $N = \{1, \dots, n\}$ of agents, each of which is a node in a directed graph $G = (N, E)$. The graph G represents the social network of the agents, with an edge $(i, j) \in E$ encoding the fact that i takes j into account when updating its preference. The neighborhood $N(i)$ of $i \in N$ is defined as

$N(i) = \{j \in N \mid (i,j) \in E\}$, i.e., as the agents in N that i pays attention to. Additionally, each edge $(i,j) \in E$ is associated with a weight $w_{ij} > 0$, such that:

$$\sum_{j \in N(i)} w_{ij} = 1,$$

i.e., the weights of the outgoing edges of i add up to 1. The graph G is strongly connected if there is a path from i to j, for every $i, j \in N$, and aperiodic if the greatest common divisor of the cycles in G is 1.

2 Learning

Without loss of generality we assume here that $\sigma^* = a_1 \dots a_m$.

We say that a network G_n is *wise* if, for any alternatives $x, y \in A$ such that $r^*(x) < r^*(y)$, we have that:

$$\lim_{n \to \infty} \mathbb{P}[r^+(x) - r^+(y) \geq \varepsilon] = 0,$$

for any $\varepsilon > 0$. Intuitively, if x is ranked above y in the reference order σ^*, this guarantees that society ends up ranking x above y in the consensus ranking. More concretely, the expression says that as the number of agents grows, the probability of a consensus in which x is ranked higher than y effectively goes to 0.

In the following we will focus on the top choice, which we will call a for ease of exposition, and show that society ends up putting a on top. Even more particular, we will show that a ends up before b, the second-ranked alternative:

$$\lim_{n \to \infty} \mathbb{P}[r^+(a) - r^+(b) \geq \varepsilon] = 0,$$

for any $\varepsilon > 0$.

We start by computing $\mathbb{E}[r(a)]$, the expected rank of alternative a in a ranking $\sigma \sim$ *Mallows*(σ^*, φ). Note, first, that if σ is a ranking in which a has rank k, it holds that:

$$d(\sigma^*, \sigma) = d(\sigma^*_{-a}, \sigma_{-a}) + (k-1).$$

Intuitively, $d(\sigma^*, \sigma)$ is obtained by computing the distance from the orders obtained by ignoring a, and then adding to this the amount by which a is shifted in σ with respect to σ^*: since a has rank 1 in σ^* and k in σ, the shift amounts to $k-1$.

Thus, the probability that a has rank k in such an order σ is:

$$\mathbb{P}[r(x) = k] = \frac{1}{Z_m} \sum_{i=0}^{\binom{m-1}{2}} I^i_{m-1} \cdot \varphi^{i+k-1}.$$

Intuitively, this probability is obtained by summing up over all rankings where a ends up in the k^{th} position. The probability of getting such a ranking, according to the Mallows model, is $\frac{1}{Z_m}\varphi^{i+k-1}$, i is the distance between σ^*_{-a} and σ_{-a}. Since ignoring a leaves us with $m-1$ alternatives, there are I^i_{m-1} rankings at distance i from σ^*_{-a}, and the maximum distance that can be obtained in this way is $\binom{m-1}{2}$. The expected rank of a comes out to:

$$\begin{aligned}
\mathbb{E}[r_i(a)] &= \frac{1}{Z_m}\left(1\cdot\sum_{0}^{\binom{m-1}{2}} I^i_{m-1}\cdot\varphi^i + 2\cdot\sum_{0}^{\binom{m-1}{2}} I^i_{m-1}\cdot\varphi^{i+1} + \cdots + m\cdot\sum_{0}^{\binom{m-1}{2}} I^i_{m-1}\cdot\varphi^{i+m-1}\right)\\
&= \frac{1}{Z_m}\Big(I^0_{m-1}(1\cdot\varphi^0 + \cdots + m\cdot\varphi^{m-1}) +\\
&\qquad I^1_{m-1}(1\cdot\varphi^1 + \cdots + m\cdot\varphi^{m}) +\\
&\qquad \cdots +\\
&\qquad\quad I^{\binom{m-1}{2}}_{m-1}(1\cdot\varphi^{\binom{m-1}{2}} + \cdots + m\cdot\varphi^{\binom{m-1}{2}+m-1})\Big)\\
&= \frac{1}{Z_m}\left(I^0_{m-1}\cdot\varphi^0 + \cdots + I^{\binom{m-1}{2}}_{m-1}\cdot\varphi^{\binom{m-1}{2}}\right)\left(1\cdot\varphi^0 + \cdots + m\cdot\varphi^{m-1}\right)\\
&= \frac{1}{Z_m}\cdot Z_{m-1}\left(1\cdot\varphi^0 + \cdots + m\cdot\varphi^{m-1}\right)\\
&= \frac{1\cdot\varphi^0 + 2\cdot\varphi^1 + \cdots + m\cdot\varphi^{m-1}}{1+\varphi+\varphi^2+\cdots+\varphi^{m-1}}.
\end{aligned}$$

As a sanity check, note that $\lim_{\varphi\to 0}\mathbb{E}[a] = 1$ and $\lim_{\varphi\to 1}\mathbb{E}[a] = \frac{m+1}{2}$, which is consistent with the fact that as φ approaches 0 the Mallows model assigns all probability to σ^*, and hence a is virtually guaranteed to end up in position 1; while for φ approaching 1 the probabilities approximate the uniform distribution, maaking a qually likely to end on on either position, and thus ending up in the middle of the ranking on average.

A similar calculation yields:

$$\mathbb{E}[r(b)] = \frac{1\cdot\varphi^1 + 2\cdot\varphi^0 + 3\cdot\varphi^1 + \cdots + m\cdot\varphi^{m-2}}{1+\varphi+\varphi^2+\cdots+\varphi^{m-1}}.$$

In general, the expected rank of an alternative x comes out to:

$$\mathbb{E}[r(x)] = \frac{1\cdot\varphi^{|1-r^*(x)|} + 2\cdot\varphi^{|2-r^*(x)|} + \cdots + m\cdot\varphi^{|m-r^*(x)|}}{1+\varphi+\varphi^2+\cdots+\varphi^{m-1}}$$

References