Problem 3: SVD

1. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

$\downarrow$ $2R_1 - R_2 \rightarrow R_2$ to obtain leading 0 in second row

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ Matrix is now in row echelon form

Since A in row echelon form has 2 non-zero rows, $\text{rank}(A) = 2$. $\square$

2. $A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 3 \\ 8 & 13 & 5 \\ 3 & 5 & 2 \end{bmatrix}$

$A - \lambda \cdot I = \begin{bmatrix} 5 & 8 & 3 \\ 8 & 13 & 5 \\ 3 & 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0.$

$\begin{bmatrix} 5-\lambda & 8 & 3 \\ 8 & 13-\lambda & 5 \\ 3 & 5 & 2-\lambda \end{bmatrix} = 0.$ let this matrix be B'

$$\det(B') = (5-\lambda)[(13-\lambda)(2-\lambda) - 25] - 8[8(2-\lambda) - 15] + 3[40 - 3(13-\lambda)]$$
$$= (5-\lambda)(\lambda^2 - 15\lambda + 1) - 8(1 - 8\lambda) + 3(1 + 3\lambda)$$
$$= -\lambda^3 + 15\lambda^2 - \lambda + 5\lambda^2 - 75\lambda + 5 - 8 + 64\lambda + 3 + 9\lambda$$
$$= -\lambda^3 + 20\lambda^2 - 3\lambda$$
$$= -\lambda(\lambda^2 - 20\lambda + 3).$$

$\lambda_0 = 0$. since $\text{rank}(A) = 2$.

$$\lambda_1, \lambda_2 = \frac{20 \pm \sqrt{400 - 12}}{2} = 10 \pm \sqrt{97}.$$

$$\boxed{\sigma_0 = 0, \ \sigma_1 = \sqrt{10 + \sqrt{97}}, \ \sigma_2 = \sqrt{10 - \sqrt{97}}}$$

3. From (2), the singular values of A are $\sigma_1 = \sqrt{10+\sqrt{97}}$ and $\sigma_2 = \sqrt{10-\sqrt{97}}$.

To find V, calculate the orthonormal set of eigenvectors of A^TA
From (2), the eigenvalues are $\lambda_1 = 10+\sqrt{97}$, $\lambda_2 = 10-\sqrt{97}$, $\lambda_3 = 0$.
Since A^TA is symmetrical, the eigenvectors will be orthogonal.

For $\lambda_1 = 10+\sqrt{97}$: $A^TA - (10+\sqrt{97})I = \begin{bmatrix} 5-(10+\sqrt{97}) & 8 & 3 \\ 8 & 13-(10+\sqrt{97}) & 5 \\ 3 & 5 & 2-(10+\sqrt{97}) \end{bmatrix}$

an eigenvector $\vec{v}_1$ exists such that $[A^TA - (10+\sqrt{97})I]\vec{v}_1 = 0$

$$\begin{bmatrix} 5-(10+\sqrt{97}) & 8 & 3 \\ 8 & 13-(10+\sqrt{97}) & 5 \\ 3 & 5 & 2-(10+\sqrt{97}) \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{2,1} \\ v_{3,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

which produces a system of equations:

$[5-(10+\sqrt{97})]v_{1,1} + 8v_{2,1} + 3v_{3,1} = 0$

$8v_{1,1} + [13-(10+\sqrt{97})]v_{2,1} + 5v_{3,1} = 0$

$3v_{1,1} + 5v_{2,1} + [2-(10+\sqrt{97})]v_{3,1} = 0$

Let $v_{1,1} = 1$. Then solving the system of equations yields

$v_{2,1} = \dfrac{49+5\sqrt{97}}{31+3\sqrt{97}}$, $v_{3,1} = \dfrac{18+2\sqrt{97}}{31+3\sqrt{97}}$ so $\vec{v}_1 = \begin{bmatrix} 1 \\ 49+5\sqrt{97}/31+3\sqrt{97} \\ 18+2\sqrt{97}/31+3\sqrt{97} \end{bmatrix}$.

$\|v_1\| = \sqrt{v_{1,1}^2 + v_{2,1}^2 + v_{3,1}^2} \simeq 2.01.$

unit length vector $\vec{v}_1 = \begin{bmatrix} 0.499 \\ 0.809 \\ 0.311 \end{bmatrix}.$

Doing the same process for $\lambda_2 = 10-\sqrt{97}$ yields

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -49+5\sqrt{97}/-31+3\sqrt{97} \\ -18+2\sqrt{97}/-31+3\sqrt{97} \end{bmatrix} \xrightarrow[\text{vector}]{\text{unit length}} \vec{v}_2 = \begin{bmatrix} 0.646 \\ -0.109 \\ -0.755 \end{bmatrix}.$$

$\|v_2\| = 1.547.$

Doing the same process with $\lambda_3 = 0$ yields

$$\vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \xrightarrow[\text{vector}]{\text{unit length}} \vec{v_3} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\text{So } V = \begin{bmatrix} 0.499 & 0.646 & 0.577 \\ 0.809 & -0.109 & -0.577 \\ 0.311 & -0.755 & 0.577 \end{bmatrix} \quad \text{(orthonormal)}$$

To find U, we use the formula $u_i = \frac{1}{\sigma_i} A v_i$

$$u_1 = \frac{1}{\sqrt{10+\sqrt{97}}} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.499 \\ 0.809 \\ 0.311 \end{bmatrix} = \frac{1}{\sqrt{10+\sqrt{97}}} \begin{bmatrix} 0.499 + 1.618 + 0.311 \\ 0.998 + 2.427 + 0.311 \end{bmatrix}$$

$$= \begin{bmatrix} 0.545 \\ 0.839 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{10-\sqrt{97}}} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.646 \\ -0.109 \\ -0.755 \end{bmatrix} = \frac{1}{\sqrt{10-\sqrt{97}}} \begin{bmatrix} 0.646 - 0.218 - 0.755 \\ 1.292 - 0.327 - 0.755 \end{bmatrix}$$

$$= \begin{bmatrix} -0.838 \\ 0.545 \end{bmatrix}$$

$$U = \begin{bmatrix} 0.545 & -0.838 \\ 0.839 & 0.545 \end{bmatrix}$$

$$\boxed{V = \begin{bmatrix} 0.499 & 0.646 & 0.577 \\ 0.809 & -0.109 & -0.577 \\ 0.311 & -0.755 & 0.577 \end{bmatrix} \qquad U = \begin{bmatrix} 0.545 & -0.838 \\ 0.839 & 0.545 \end{bmatrix}}$$

Σ is a 2×3 matrix where the singular values are on the diagonal.

$$\Sigma = \begin{bmatrix} \sqrt{10+\sqrt{97}} & 0 & 0 \\ 0 & \sqrt{10-\sqrt{97}} & 0 \end{bmatrix}$$