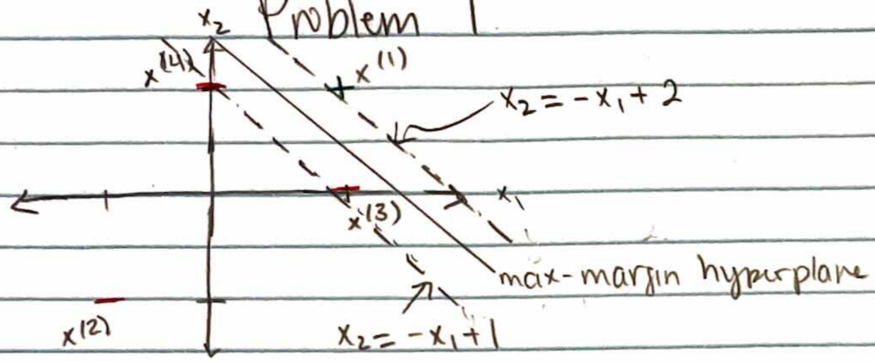
# Problem 1

1.




slope of max-margin hyperplane = slope of negative class support vector

$$w = \frac{1-0}{0-1} = -1.$$

y-intercept of max-margin hyperplane = halfway point between positive and negative class support vectors $\Rightarrow b = \frac{3}{2}$

$$\boxed{\text{max-margin hyperplane: } x_2 = -x_1 + \frac{3}{2}}$$

2. support vector(s) for positive class: $x^{(1)} = (1,1)^T$
support vector(s) for negative class: $x^{(3)} = (1,0)$, $x^{(4)} = (0,1)$

3. Original optimization problem: $\min_{\vec{w},b} \frac{1}{2}\|\vec{w}\|^2$

constraint: $(\vec{w}\cdot\vec{x}^{(j)} + b)y^{(j)} \geq 1 \ \forall j$

rewrite constraint: $(\vec{w}x^{(j)} + b)y^{(j)} - 1 \geq 0$.

$$\boxed{\text{Lagrangian: } L(\vec{w}, b, \vec{\alpha}) = \frac{1}{2}\|\vec{w}\|^2 - \sum_j \alpha_j \left[ (\vec{w}\vec{x}^{(j)} + b)y^{(j)} - 1 \right]}$$

4. $\min_{\vec{w},b} L(\vec{w}, b, \vec{\alpha})$ $\to$ $\nabla_{\vec{w}} L(\vec{w}, b, \vec{\alpha}) = 0$ (1)
$\to$ $\frac{\partial}{\partial b} L(\vec{w}, b, \vec{\alpha}) = 0$. (2).

Solving (1):

$$\nabla_w \left[ \frac{1}{2}\|\vec{w}\|^2 - \sum_j \alpha_j \left[ (\vec{w}\vec{x}^{(j)} + b)y^{(j)} - 1 \right] \right] = 0$$

$$\|\vec{w}\| - \sum_j \alpha_j y^{(j)} \vec{x}^{(j)} = 0$$

$$\boxed{\vec{w} = \sum_j \alpha_j y^{(j)} \vec{x}^{(j)}}$$

Solving (2):

$$\frac{\partial}{\partial b}\left[\frac{1}{2}\|\vec{w}\|^2 - \sum_j \alpha_j\left[(\vec{w}\vec{x}^{(j)}+b)y^{(j)}-1\right]\right]=0$$

$$0-\sum_j \alpha_j y^{(j)}=0$$

$$\boxed{\sum_j \alpha_j y^{(j)}=0}$$

5. $\max_{\vec{\alpha}}\left(\min_{\vec{w},b} \frac{1}{2}\|\vec{w}\|^2 - \sum_j \alpha_j\left[(\vec{w}x^{(j)}+b)y^{(j)}-1\right]\right)$

From part 4, $\vec{w}=\sum_j a_j y^{(j)}\vec{x}^{(j)}$. Substitute this into the equation above

$$\max_{\vec{\alpha}}\left[\min_{\vec{w},b}\left(\frac{1}{2}\|\vec{w}\|^2 - \sum_j a_j y^{(j)}\vec{w}\vec{x}^{(j)} + \underbrace{\cancel{a_j y^{(j)} b}}_{\to 0} - a_j\right)\right]$$

from question 4, $\sum_j a_j y^{(j)}=0$ so $\sum_j a_j y^{(j)} b = 0$ as well.

$$\Rightarrow \max_{\vec{\alpha}}\left[\frac{1}{2}\sum_j\sum_k a_j a_k y^{(j)} y^{(k)} (\vec{x}^{(j)}\cdot\vec{x}^{(k)}) - \sum_j\sum_k a_j a_k y^{(j)} y^{(k)} (\vec{x}^{(j)}\cdot\vec{x}^{(k)}) + \sum_j a_j\right]$$ rearrange.

$$\Rightarrow \boxed{\max_{\vec{\alpha}}\left[\sum_j a_j - \frac{1}{2}\sum_j\sum_k a_j a_k y^{(j)} y^{(k)} (\vec{x}^{(j)}\cdot\vec{x}^{(k)})\right]}$$ $\square$.

6. From the question statement, the constraints in Eq. (1) can be removed for non-support vectors $\hookrightarrow$ i.e. $(\vec{w}\cdot\vec{x}^{(j)}+b)y^{(j)}\geq 1$.

In Lagrangian format, Eq. (1):

$$\mathcal{L}(\vec{w},b,\vec{a})=\frac{1}{2}\|\vec{w}\|^2-\sum_j a_j \underbrace{(\vec{w}\cdot\vec{x}^{(j)}+b)y^{(j)}-1}_{\text{this is the same constraint from Eq. 1.}}$$

If the constraint is disregarded, then $\sum_j a_j(\vec{w}\cdot\vec{x}^{(j)}+b)y^{(j)}-1=0$ for some $j^*$.

Since $j^*$ isn't a support vector, $(\vec{w}\cdot\vec{x}^{(j^*)}+b)y^{(j^*)}-1\neq 0$.

$$\boxed{\begin{array}{l}\text{Thus the only way the constraint is 0 is if } a_{j^*}=0.\\ a_{j^*}=0 \text{ when } j^*=2.\end{array}}$$ $\square$.

7. From question 4: $\sum_j a_j y^{(j)} = 0$.

$$\rightarrow a_1 y_1 + \underbrace{a_2 y_2}_{0 \text{ since } a_2 = 0 \text{ from question 6.}} + a_3 y_3 + a_4 y_4 = 0$$

$$\rightarrow a_1 y_1 + a_3 y_3 + a_4 y_4 = 0$$

Plug in values of $y_1 = 1$, $y_3 = -1$, $y_4 = -1$:

$$\rightarrow a_1 - a_3 - a_4 = 0$$

$$\boxed{a_1 = a_3 + a_4}$$

8. $\max\limits_{\vec{\alpha}} \left( \sum_j a_j - \frac{1}{2} \sum_j \sum_k a_j a_k y_j y_k (\vec{x}^{(j)} \cdot \vec{x}^{(k)}) \right)$

Expand the sum and simplify the terms by plugging in the values of $y_j$ and $x^{(j)}$

Note that the $a_2$ term was excluded because from part (6), we determined $a_2 = 0$

$$\max_{\vec{\alpha}} \left( a_1 + a_3 + a_4 - \frac{1}{2}(2a_1^2 + a_3^2 + a_4^2 - 2a_1 a_4 - 2a_1 a_3) \right)$$

$$= \max_{\vec{\alpha}} \left( a_1 + a_3 + a_4 - a_1^2 - \frac{1}{2}a_3^2 - \frac{1}{2}a_4^2 + a_1 a_4 + a_1 a_3 \right).$$

Substitute $a_1 = a_3 + a_4$ from part (7) and simplify:

$$= \max_{\vec{\alpha}} \left( -\frac{1}{2}(a_3^2 + a_4^2) + 2a_3 + 2a_4 \right)$$

$$\frac{\partial}{\partial a_3} \left( \max_{\vec{\alpha}} (\ldots) \right) \Rightarrow -a_3 + 2 = 0 \rightarrow a_3 = 2$$

$$\frac{\partial}{\partial a_4} \left( \max_{\vec{a}} (\ldots) \right) \Rightarrow -a_4 + 2 = 0 \rightarrow a_4 = 2.$$

$a_1 = a_3 + a_4 = 2 + 2$

$$\boxed{\begin{array}{l} a_1 = 4 \\ a_3 = 2 \\ a_4 = 2 \end{array}}$$

9. From (4), $\vec{w} = \sum_j a_j y^{(j)} \vec{x}^{(j)}$

$$= a_1 y^{(1)} \vec{x}^{(1)} + \cancel{a_2 y_2 \vec{x}^{(2)}} + a_3 y_3 \vec{x}^{(3)} + a_4 y_4 \vec{x}^{(4)}$$

$$= (4)(1)\begin{bmatrix}1\\1\end{bmatrix} + (2)(-1)\begin{bmatrix}1\\0\end{bmatrix} + (2)(-1)\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \begin{bmatrix}4-2-0\\4-0-2\end{bmatrix}$$

$$\boxed{\vec{w} = \begin{bmatrix}2\\2\end{bmatrix}}$$

10. Constraint: $(\vec{w}\cdot\vec{x}^{(j)} + b)y^{(j)} \geq 1 \quad \forall j$

For support vectors: $(\vec{w}\cdot\vec{x}^{(j)} + b)y^{(j)} = 1 \quad \forall j$

$$b = \frac{1}{y^{(j)}} - \vec{w}\cdot\vec{x}^{(j)} \quad \forall j.$$

Pick $j=1$: $\vec{x}^{(j)} = \begin{bmatrix}1\\1\end{bmatrix}$, $y^{(j)} = 1$, $\vec{w} = \begin{bmatrix}2\\2\end{bmatrix}$.

$$b = \frac{1}{1} - \begin{bmatrix}2\\2\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}$$

$$= 1-(2+2)$$

$$\boxed{b=-3}$$

11. The decision boundary takes the form $\vec{w}^T\vec{x} + b = 0$

Expanding: $\begin{bmatrix}w_1 & w_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + b = 0$.

$$w_1x_1 + w_2x_2 + b = 0$$

$$\rightarrow x_2 = \frac{-1}{w_2}(w_1x_1 + b)$$

$$\rightarrow x_2 = \frac{-1}{2}(2x_1 - 3)$$

$\rightarrow x_2 = -x_1 + 3/2$ which is the same hyperplane determined in question 1. $\square$