Solutions to Exercise Set 10

Problem 1

There are many ways to model this. Let's let x_3 be the amount of chemical we purchase. Here is one model:

$$\begin{aligned} max \quad & x_1(30-x_1)+x_2(50-2x_2)-3x_1-5x_2-10x_3 \\ \text{s.t.} \quad & x_1+x_2-x_3 \le 0 \\ & x_3 \le 17.25 \end{aligned}$$

The KKT conditions are the above feasibility constraints along with:

$$30-2x_1-3-\alpha_1=0 \tag{1}$$
$$50-4x_2-5-\alpha_1=0 \tag{2}$$
$$-10+\alpha_1-\alpha_2=0 \tag{3}$$
$$\alpha_1(x_1+x_2-x_3)=0 \tag{4}$$
$$\alpha_2(x_3-17.25)=0 \tag{5}$$
$$\alpha_1,\alpha_2 \ge 0 \tag{6}$$

There are four cases to check:

1. $\alpha_1=0,\alpha_2=0$. This gives us $-10=0$ in Equation (3), so there is no solution.
2. $\alpha_1=0,x_3=17.25$. This gives $\alpha_2=-10$ so there is no solution.
3. $x_1+x_2-x_3=0,\alpha_2=0$. This gives $\alpha_1=10,x_1=8.5,x_2=8.75,x_3=17.25$ and the solution is feasible.
4. $x_1+x_2-x_3=0,x_3=17.25$. This gives the same solution as before.

The objective function is concave (its hessian is negative semi-definite) and the constraints are linear. It is easy to show that the Slater's condition is satisfied. Any point that satisies the KKT conditions is a maximizer of this problem. So the unique maximum is given by $x_1=8.5,x_2=8.75$, and $x_3=17.25$.

Problem 2

Let us define the ground-set as $X=\{\boldsymbol{x}\in\mathbb{R}^n|x_j>0,j=1,\dots,n\}$, and let us dualize on the single equality constraint. The Lagrangian function takes on the form:

$$\begin{aligned} L(\boldsymbol{x},\alpha) &= 5x_1+7x_2-4x_3-\sum_{j=1}^{3} ln(x_j)+\alpha(x_1+3x_2+12x_3-37) \\ &= -37\alpha+(5+\alpha)x_1+(7+3\alpha)x_2+(-4+12\alpha)x_3-\sum_{j=1}^{3} ln(x_j) \end{aligned}$$

The dual function $L^*(\alpha)$ is constructed as $L^*(\alpha)=min_{\boldsymbol{x}\in X}L(\boldsymbol{x},\alpha)$. Now notice that the optimization problem above separates into three univariate optimization problems of a linear function minus a logarithm term for each of the three positive variables x_1,x_2, and x_3. Examining x_1, it holds that the minimization value will be $-\infty$ if $(5+\alpha)\le 0$, as we could set x_1 arbitrarily large. When $(5+\alpha)>0$, the problem of minimizing $(5+\alpha)x_1 - ln(x_1)$ is a convex optimization problem whose solution is given by setting the first derivative with respect x_1 equal to zero. This means solving:

$$(5+\alpha) - \frac{1}{x_1} = 0,$$

or in other words, setting

$$x_1 = \frac{1}{(5+\alpha)}.$$

Substituting this value of x_1, we obtain:

$$(5+\alpha)x_1 - ln(x_1) = 1 - ln(\frac{1}{5+\alpha}) = 1 + ln(5+\alpha).$$

Using parallel logic for the other two variables, we arrive at:

$$L^*(\alpha) = \begin{cases} -37\alpha + 3 + ln(5+\alpha) + ln(7+3\alpha) + ln(-4+12\alpha) & \text{if } \alpha > 1/3 \\ -\infty \quad \text{otherwise.} \end{cases}$$

Notice that $L^*(\alpha)$ is finite whenever $5+\alpha > 0$, $7+3\alpha > 0$, and $-4+12\alpha > 0$. These three inequalities in α are equivalent to the single inequality $\alpha > 1/3$. The dual problem is defined to be $\max_{\alpha \in \mathbb{R}} L^*(\alpha)$.

Problem 3

a) The lagrangian function is given by $L(x,y,z,\lambda) = xy + \lambda(3x^2 + y^2 - 6), \lambda \in \mathbb{R}$. KKT conditions are given by

$$\begin{aligned} y + 6\lambda x &= 0 \\ x + 2\lambda y &= 0 \\ 3x^2 + y^2 - 6 &= 0 \end{aligned}$$

This can be rewritten as

$$\begin{aligned} y &= -6\lambda x \quad (1) \\ x &= -2\lambda y \quad (2) \\ 3x^2 + y^2 - 6 = 0 & \quad\quad (3) \end{aligned}$$

Plugging the second equation into the first one gives

$$y = 12\lambda^2 y.$$

If y were 0, then x would be 0 too, which is impossible by (3). Thus we can divide by y to get that $12\lambda^2 = 1$. Then:

$$\begin{aligned} 6 &= 3x^2 + (6\lambda x)^2 \\ 6 &= 3x^2 + 3(12\lambda^2)x^2 \\ 6 &= 3x^2 + 3x^2. \end{aligned}$$

Thus $x \pm 1$ and $y = \pm\sqrt{3}$ by (3). They are four critical points: $\boldsymbol{a} = (1,\sqrt{3})$,$\boldsymbol{b} = (1,-\sqrt{3})$,$\boldsymbol{c} = (-1,\sqrt{3})$, and $\boldsymbol{d} = (-1,-\sqrt{3})$.

b) We have $f(\boldsymbol{a}) = f(\boldsymbol{d}) = \sqrt{3}$ and $f(\boldsymbol{b}) = f(\boldsymbol{c}) = -\sqrt{3}$. By Weierstrass extreme value theorem, this optimization problem have a maximum and a minimum. Thus $\boldsymbol{a}$, $\boldsymbol{d}$ are maxima and $\boldsymbol{b}$, $\boldsymbol{c}$ are minima.

Problem 4

Formulation:

$$\begin{array}{rl} min_{x,y\geq 0} & ax^2 + by^2 \\ \text{s.t.} & x + y = Q \end{array}$$

As a,$b > 0$ and as the cost of shipping is always non-negative for any quantity x,y, the latter problem is equivalent to:

$$\begin{array}{rl} min & ax^2 + by^2 \\ \text{s.t.} & x + y = Q \end{array}$$

Its associated lagrangian is

$$L(x,y,\lambda) = ax^2 + by^2 + \lambda(x + y - Q)$$

First-order conditions:

$$\begin{aligned} 2ax^* + \lambda^* &= 0 \\ 2by^* + \lambda^* &= 0 \\ x^* + y^* - Q &= 0 \end{aligned}$$

The candidate is given by:

$$x^* = \frac{bQ}{a+b},\ y^* = \frac{aQ}{a+b},\ \lambda^* = \frac{-2abQ}{a+b}$$

for a cost of $\frac{abQ^2}{a+b}$. The hessian matrix is

$$H(x,y) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

and is positive definite since $a > 0$ and $b > 0$. First-order conditions are sufficient when the objective function is convex with affine constraints. We conclude that (x^*,y^*) is the unique minimum of the problem.

Problem 5

The problem can be formulated as:

$$\begin{array}{rl} min & 0.04x^2 + 0.08y^2 + 0.02xy + 0.16z^2 + 0.04yz, \\ \text{s.t.} & 0.1x + 0.1y + 0.15z = 0.12 \\ & x + y + z = 1 \end{array}$$

Using the Lagrangian method, the following optimal solution is

$$x = 0.5,y = 0.1,z = 0.4,\lambda_1 = -1.8,\lambda_2 = 0.138,$$

where λ_1 is the Lagrange multiplier associated with the first constraint and λ_2 with the second constraint. The corresponding objective function value (i.e. the variance on the return) is 0.039. So the optimal portfolio consists in investing 50 % in fund 1, 10 % in fund 2, and 40 % in fund 3 for a standard deviation of 19.7 % ($\sqrt{0.039} \times 100$).