Solutions to Exercise Set 9

Problem 1

The characteristic polynomial is $(\lambda-3)(\lambda-1)$. The eigenvalues are given by $\lambda_1 = 3$ and $\lambda_2 = 1$. For each of them, we need to solve $\boldsymbol{A}\boldsymbol{v} = \lambda_i \boldsymbol{v}$.

Case 1: $\lambda_1 = 3$. We need to solve

$$\boldsymbol{A}\boldsymbol{v} = 3\boldsymbol{v} \iff (\boldsymbol{A} - 3\boldsymbol{I})\boldsymbol{v} = 0,$$

where $\boldsymbol{I}$ is the 2×2 identity matrix. The eigenvectors belonging to $\lambda_1 = 3$ are given by the vectors $\boldsymbol{v}^T = (v_1\ v_2)$ for which $v_1 = v_2$.

Case 2: $\lambda_1 = 1$. We need to solve

$$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{v} \iff (\boldsymbol{A} - \boldsymbol{I})\boldsymbol{v} = 0,$$

where $\boldsymbol{I}$ is the 2×2 identity matrix. The eigenvectors belonging to $\lambda_1 = 1$ are given by the vectors $\boldsymbol{v}^T = (v_1\ v_2)$ for which $v_1 = $ -v_2.

Problem 2

a) $f : \mathbb{R} \mapsto \mathbb{R} : f(x) = 1 - x^2$ is concave:

$$\lambda f(x_1) + (1-\lambda)f(x_2) = 1 - \lambda x_1^2 - (1-\lambda)x_2^2$$

and

$$f(\lambda x_1 + (1-\lambda)x_2) = 1 - \lambda^2 x_1^2 - (1-\lambda)^2 x_2^2 - 2\lambda(1-\lambda)x_1 x_2$$

Then:

$$f(\lambda x_1 + (1-\lambda)x_2) - \lambda f(x_1) - (1-\lambda)f(x_2) = \lambda(1-\lambda)(x_1 - x_2)^2 \geq 0$$

b) $f : \mathbb{R} \mapsto \mathbb{R} : f(x) = x^2 - 1$ is convex:

$$\lambda f(x_1) + (1-\lambda)f(x_2) = \lambda x_1^2 + (1-\lambda)x_2^2 - 1$$

and

$$f(\lambda x_1 + (1-\lambda)x_2) = \lambda^2 x_1^2 + (1-\lambda)^2 x_2^2 + 2\lambda(1-\lambda)x_1 x_2 - 1$$

Then:

$$f(\lambda x_1 + (1-\lambda)x_2) - \lambda f(x_1) - (1-\lambda)f(x_2) = -\lambda(1-\lambda)(x_1 - x_2)^2 \leq 0$$

c) The function $f : \mathbb{R}^2 \mapsto \mathbb{R} : f(x,y) = \sqrt{x^2 + y^2}$ is convex.
Indeed :

$$\|\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\| \leq \|\lambda\mathbf{x}\| + \|(1-\lambda)\mathbf{y}\|$$

which can be rewritten as :

$$\|\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\| \leq \lambda\,\|\mathbf{x}\| + (1-\lambda)\,\|\mathbf{y}\|\,,$$

for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ and any $\lambda \in [0,1]$

d) The function $f : \mathbb{R} \mapsto \mathbb{R} : f(x) = x^3$ isn't convex, nor concave. Indeed, if we choose $x_1 = 1$ and $x_2 = -1$, and if consider the cases where $\lambda = 0.25$ and $\lambda = 0.75$, then we get :

- For $\lambda = \frac{1}{4}$, $f(\lambda x_1 + (1-\lambda)x_2) = f(\text{-}0.5) = -\frac{1}{8}$ and $\lambda f(x_1) + (1-\lambda)f(x_2) = -\frac{1}{2}$, then

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$$

- For $\lambda = \frac{3}{4}$, $f(\lambda x_1 + (1-\lambda)x_2) = f(0.5) = \frac{1}{8}$ and $\lambda f(x_1) + (1-\lambda)f(x_2) = \frac{1}{2}$, then

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

We conclude that f cannot be convex, nor concave.

Problem 3

We have

$$\begin{aligned}
\mathbf{z}^{\mathrm{T}}\mathbf{M}\mathbf{z} = (\mathbf{z}^{\mathrm{T}}\mathbf{M})\mathbf{z} &= \begin{bmatrix}(2a-b) & (-a+2b-c) & (-b+2c)\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}\\
&= 2a^2 - 2ab + 2b^2 - 2bc + 2c^2\\
&= a^2 + (a-b)^2 + (b-c)^2 + c^2
\end{aligned}$$

This result is a sum of squares, and therefore non-negative. It is egal to zero only if a = b = c = 0, that is, when $\mathbf{z}$ is zero.

Problem 4

– Function f:
Its gradient and its hessian are:

$$\begin{aligned}
\nabla f(\boldsymbol{x}) &= \begin{pmatrix} 2x \\ 2y \end{pmatrix} & \forall\,\boldsymbol{x} \in \mathbb{R}^2\\
\nabla^2 f(\boldsymbol{x}) &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \forall\,\boldsymbol{x} \in \mathbb{R}^2
\end{aligned}$$

The hessian is positive definite for all $\boldsymbol{x} \in \mathbb{R}^2$. There is a single critical point which is also the unique global minimum of f: $\nabla f(\boldsymbol{x}^*) = 0 \Leftrightarrow \boldsymbol{x}^* = (0,0)$.

– Function g:
Its gradient and its hessian are:

$$\begin{aligned}
\nabla g(\boldsymbol{x}) &= \begin{pmatrix} x^2 - 1 \\ 3y^2 - 1 \end{pmatrix} & \forall\,\boldsymbol{x} \in \mathbb{R}^2\\
\nabla^2 g(\boldsymbol{x}) &= \begin{pmatrix} 2x & 0 \\ 0 & 6y \end{pmatrix} & \forall\,\boldsymbol{x} \in \mathbb{R}^2
\end{aligned}$$

There are 4 critical points: $\boldsymbol{x} = (1,\sqrt{1/3})$, $\boldsymbol{x} = (1,-\sqrt{1/3})$, $\boldsymbol{x} = (-1,\sqrt{1/3})$ and $\boldsymbol{x} = (-1,-\sqrt{1/3})$. The first one is a local minimum (the hessian matrix is positive definite), the last one is a local maximum (the hessian is negative definite) and the two other are saddle points **since their hessians are indefinite (neither positive semi-definite nor negative semi-definite)**.

Problem 5

(a)

$$\nabla^2 f(x,y) = \mathbf{Q} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} -11 \\ 11 \end{pmatrix} \quad \text{and} \quad c = 11$$

(b) $\mathbf{Q}$ is a positive definite since it is a symmetric strictly diagonally dominant matrix.

The unique minimizer of f over $\mathbb{R}^2$ is given by $\mathbf{Qz} = -\mathbf{b}$

$$\begin{cases} 10x - y = 11 \\ -x + 10y = -11 \end{cases}$$

Then

$$\begin{cases} x^* = 1 \\ y^* = -1 \end{cases}$$

The solution is:

$$\mathbf{z}^* = (x^* \; y^*)^T = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Problem 6

The gradient of f is given by:

$$\nabla f(x,y) = \begin{pmatrix} e^y + ye^x \\ xe^y + e^x \end{pmatrix} \; \forall (x,y) \in \mathbb{R}^2$$

Let's determine the critical points:

$$\begin{cases} e^y + ye^x = 0 \\ xe^y + e^x = 0 \end{cases}$$

Then:

$$\begin{cases} x = -e^x/e^y \\ y = -e^y/e^x \end{cases}$$

Let's show that $x = y$. Indeed, if $x < y$, then $e^y/e^x < e^x/e^y$, i.e. $e^{2y} < e^{2x}$ which is a contradiction since the exponential is strictly increasing. A similar argument is used to show that assuming $x > y$ is not possible. So we conclude that $x = y$. Consequently the only candidate is $(x,y) = (-1,-1)$.
Let's have a look at the hessian at $(-1,-1)$:

$$\nabla^2 f(-1,-1) = e^{-1} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

A **necessary** condition to have a local minimum is that the hessian must be positive semi-definite. As the determinant of this matrix is negative $(-3e^{-1} < 0)$, the hessian cannot be positive semi-definite. We conclude that fhe function f has no local minimum over $\mathbb{R}^2$.