Introduction

Optimization Methods in Management Science
Master in Management
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Introduction to Optimization

- An optimization problem consists in finding the best solution from among the set of all **feasible** solutions
- It involves **maximizing** or **minimizing** a function $f(\boldsymbol{x})$ over a set of constraints that can be equalities or inequalities
- Depending on the nature of the problem, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \in \mathbb{Z}^n$, or $\boldsymbol{x} \in \mathbb{N}^n$

Reminders

- $\boldsymbol{x} \in \mathbb{R}^n \iff \boldsymbol{x} = (x_1, \ldots, x_n)$ and $x_i \in \mathbb{R}$
- $\mathbb{R}$: real numbers
- $\mathbb{Z}$: integers
- $\mathbb{N}$: non-negative integers

Important Remark

The assumption about the type of numbers we consider (real versus integer) is very important for the resolution of an optimization problem ! Different algorithms are used depending on the nature of these numbers

Simple Problem Transformation Rules

Here are a few **important** rules that are very useful from a modeling point of view:

- *max* **to** *min*:

$$max\ f(\boldsymbol{x}) \iff \text{-}min\ \text{-}f(\boldsymbol{x})$$

- $\leq$ **to** $\geq$:

$$g(\boldsymbol{x}) \leq 0 \iff \text{-}g(\boldsymbol{x}) \geq 0$$

- **equality to inequalities:**

$$g(\boldsymbol{x}) = 0 \iff g(\boldsymbol{x}) \leq 0 \text{ and } g(\boldsymbol{x}) \geq 0$$

Type of Optimization Problems

- **Continuous optimization** versus **discrete optimization**
- **Unconstrained optimization** versus **constrained optimization**
- **Deterministic optimization** versus **stochastic optimization**

Discrete Optimization

Some or all of the variables used in a **discrete** mathematical program are restricted to be discrete variables - that is, to assume only a discrete set of values, such as the integers.

Two notable branches of discrete optimization are:

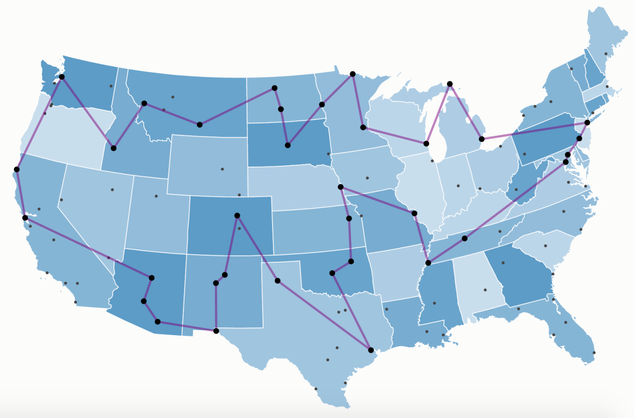
- **combinatorial optimization**, which refers to problems on graphs, matroids and other discrete structures
- **integer programming**

Combinatorial Optimization: An Example

Problem (TSP)

The travelling salesman problem (TSP) asks the following question: "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?" It is an **NP-hard** problem in combinatorial optimization, important in operations research and theoretical computer science. The TSP can be formulated as an integer linear program.

Combinatorial Optimization: TSP



Integer Linear Programming (ILP): Another Example

A company has a budget of \$ 5 million to invest in the coming year in its three plants (P_1, P_2, P_3). Expected profits depending on the investment (0,1,2,3,4, or 5 M) in the different plants are given below. What is the best investment policy that maximizes its profit ?

	0	1M	2M	3M	4M	5M
P_1	1	3	4	5	5.5	6
P_2	0.5	2.5	4	5	6	6.5
P_3	2	4	5.5	6.5	7	7.5

ILP Example: Problem Formulation

Variables: let x_{ij} represent the decision to invest ($x_{ij} = 1$) or not ($x_{ij} = 0$) j million in P_i.

$$\begin{aligned} \max_{i=1,\ldots,3,j=0,\ldots,5} & \quad \sum_{ij} EP_{ij} x_{ij} \\ \text{s.t.} & \quad \sum_i x_{i1} + 2x_{i2} + 3x_{i3} + 4x_{i4} + 5x_{i5} \leq 5, \end{aligned}$$

where EP_{ij} represents the expected profit from P_i if we invest j million in it

ILP: Another Example

Data: daily demand in drivers in a transport company

Mon	Tue	Wed	Thu	Fri	Sat	Sun
13	18	21	16	12	25	9

Drivers work 5 consecutive days and can start to work any day in a week.

Problem (Covering Problem)

Determine the minimum number of drivers to meet all the requirements

Example: A Covering Problem

Let x_i be the number of drivers stating on day i:

x_1 : number of drivers in the team starting on Monday

x_2 : number of drivers in the team starting on Tuesday

...

x_7 : number of drivers in the team starting on Sunday

Example: A Covering Problem (Cont'd)

Objective function: we want to minimize the number of drivers

$$z = x_1 + \ldots + x_7$$

Constraints: we must satisfy the requirements

$$\begin{array}{ccccccccccccccl}
x_1 & & & & & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 13 \quad \text{(Mon)} \\
x_1 & + & x_2 & & & & & + & x_5 & + & x_6 & + & x_7 & \geq & 18 \quad \text{(Tue)} \\
 & & & & & & & & \cdots & & & & & & \\
 & & & & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 9 \quad \text{(Sun)}
\end{array}$$

Bound constraints: the number of drivers in each team should be a non-negative integer.

$$x_i \geq 0 \text{ and integer}, \ i = 1, \ldots, 7.$$

Example: A Covering Problem (Cont'd)

Formulation:

$$\begin{array}{llccccccccccccccc}
\text{Min} & z = & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & & \\
\text{s.t.} & & x_1 & & & & & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 13 \\
& & x_1 & + & x_2 & & & & & + & x_5 & + & x_6 & + & x_7 & \geq & 18 \\
& & x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & + & x_7 & \geq & 21 \\
& & x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & + & x_7 & \geq & 16 \\
& & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & & & & \geq & 12 \\
& & & & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & & & \geq & 25 \\
& & & & & & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 9 \\
& & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & , & x_7 & \geq & 0
\end{array}$$

and $x_i, \ldots, x_7$ should be **integers**. This is not a linear program but an integer linear program !

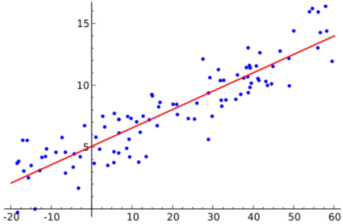
Continuous Optimization

- **Continuous optimization:** the variables in the model are allowed to take on any **real** value within a range
- **Constrained vs unconstrained:** an important distinction is between problems with no constraints on the variables and problems with constraints on the variables

Unconstrained Optimization: An Example

Linear least squares regression. Let's assume that we have a set of N points (x_i, y_i). We would like to determine β_0 and β_1 in order to minimize the sum of the squared residuals

$$\min_{\beta_0,\beta_1} \sum_{i=1}^{N} (y_i - \beta_1 x_i - \beta_0)^2$$



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Constrained Optimization: An Example

Portfolio optimization: let w_i represent the weight invested in asset $i = 1, \ldots, n$.

$$
\begin{aligned}
min_{\boldsymbol{w} \in \mathbb{R}^n} \quad & \boldsymbol{w}^T \Sigma \boldsymbol{w} \\
s.t. \quad & \boldsymbol{w}^T \mathbf{1} = 1, \\
& \boldsymbol{w}^T \boldsymbol{r} \geq m,
\end{aligned}
$$

where $\boldsymbol{r}$ is the vector of expected returns , $\boldsymbol{\Sigma}$ is the return covariance matrix, and m is the minimum acceptable portfolio return. This is an example of convex quadratic problem.

Reminder : the scalar product between $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^n$ is defined by

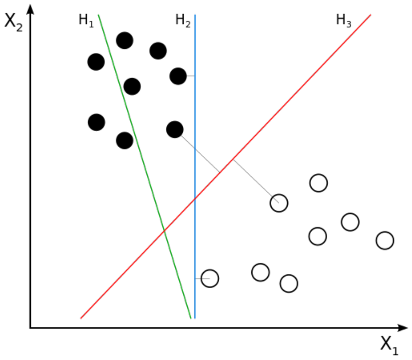
$$
\boldsymbol{x}^T \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i
$$

Constrained Optimization: Another Example

Linear SVM (Machine Learning)

- In machine learning, support vector machines are supervised learning models with associated learning algorithms that analyze data used for classification and regression analysis
- We would like to construct an hyperlane that separates the space in two sets of points

Constrained Optimization: Linear SVM



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Constrained Optimization: Linear SVM

(Primal Problem Formulation for Linear SVM)

Let $\{(\boldsymbol{x}_i, y_i), i = 1, \ldots, l\}$ be a set of labelled data with $\boldsymbol{x}_i \in \mathbb{R}^n$, and $y_i \in \{-1, 1\}$. A support vector machine is a linear classifier associated with the following decision function: $D(\boldsymbol{x}) = sign(\boldsymbol{w}^T\boldsymbol{x} + b)$ where $\boldsymbol{w} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. These parameters are the solution of the following problem:

$$
\begin{aligned}
min_{\boldsymbol{w},b} \quad & \frac{1}{2}\|\boldsymbol{w}\|^2 \\
s.t. \quad & y_i(\boldsymbol{w}^T\boldsymbol{x}_i + b) \geq 1 \quad i = 1, \ldots, l
\end{aligned}
$$

This is another example of **convex quadratic** problem

Stochastic Optimization

- Until now, we have only considered **deterministic** problems
- For **stochastic** problems, the **random** variables appear in the formulation of the optimization problem itself, which involve a random objective functions or random constraints
- In this course, we won't address this type of problems