Exercise Set 1

Problem 1

For the two planar vectors $\mathbf{a} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$, draw their a) linear, b) conic, c) affine, and d) convex combinations.

Reminders:

a) The set of linear combinations of two independent vectors is:

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b}, \quad \lambda_1, \lambda_2 \in \mathbb{R}\}$$

b) The set of conic combinations of two indepedent vectors is:

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b}, \quad \lambda_1, \lambda_2 \geq 0, \quad \lambda_1, \lambda_2 \in \mathbb{R}\}$$

c) The set of affine combination of two independent vectors is:

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b}, \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1, \lambda_2 \in \mathbb{R}\}$$

d) The set of convex combination of two independent vectors is:

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b}, \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1, \lambda_2 \geq 0, \quad \lambda_1, \lambda_2 \in \mathbb{R}\}$$

Problem 2

We consider the following systems of linear inequalities:

$$\text{(i)} \quad \left\{ \begin{array}{rcrcr} x_1 & + & 4x_2 & \leq & 12 \\ -x_1 & + & x_2 & \leq & 3 \\ x_1 & - & 2x_2 & \leq & -2 \\ x_1 & & & \leq & 2 \end{array} \right. \qquad \text{(ii)} \quad \left\{ \begin{array}{rcrcr} x_1 & + & x_2 & \leq & 1 \\ x_1 & - & x_2 & \leq & -1 \\ 2x_1 & - & 10x_2 & \leq & 5 \\ -2x_1 & + & x_2 & \leq & 0 \end{array} \right.$$

$$\text{(iii)} \quad \left\{ \begin{array}{rcrcr} -2x_1 & + & x_2 & \leq & 4 \\ -2x_1 & - & 2x_2 & \leq & -5 \\ 2x_1 & - & x_2 & \leq & 6 \\ -x_1 & + & x_2 & \leq & 2 \end{array} \right. \qquad \text{(iv)} \quad \left\{ \begin{array}{rcrcr} 3x_1 & - & 2x_2 & \leq & 6 \\ -3x_1 & + & 2x_2 & \leq & -6 \\ -3x_1 & + & x_2 & \leq & -3 \\ & & x_2 & \leq & 3 \end{array} \right.$$

For each system:

a) Determine graphically the set of solutions.

b) Is the number of inequations that describe the set of solutions minimal ? If not, then determine a minimal set of inequations. *Remark:* if this number is not minimal, then it means that we can remove at least one inequlity without changing the set of solutions.

Problem 3

We consider the following linear system:

$$\begin{array}{llrcrcr} \text{Min} & z = & 5x_1 & + & x_2 & & \\ \text{s.t.} & & 2x_1 & + & x_2 & \geq & 6 \\ & & x_1 & + & x_2 & \geq & 4 \\ & & 2x_1 & + & 10x_2 & \geq & 20 \\ & & x_1 & & & \geq & 0 \\ & & & & x_2 & \geq & 0 \end{array}$$

a) Use the graphical method to solve it.

b) Analyze the variation of z with respect to the coefficient of x_1.

Problem 4

A drugstore chain would like to sell its stock of vitamin A (3 tonnes) and of vitamin C (5 tonnes) in order to maximize its revenue. However, for the same amounts of vitamins, it should not be cheaper to buy fresh fruits.

	kg of vitamin/tonne		
Fruits	A	C	Price/tonne
Bananas	6	7	42000
Oranges	4	8	20000
Tomatos	6	2	12000

a) Formulate this problem as a linear program and define explicitly what are the decision variables, the objective function, and the constraints.

b) Use the graphical method to solve this problem and give the optimal prices of the vitamins.