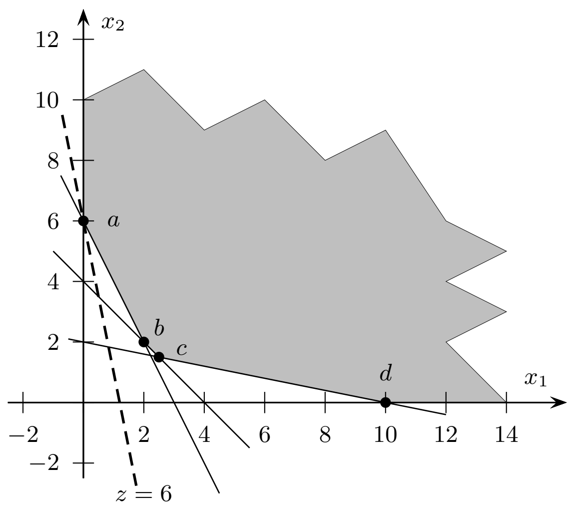
Review of Some Important Exercises

Problem 1

a) The grey zone corresponds to the feasible region. It is not bounded.



The optimal solution is located at $x_1 = 0$ and $x_2 = 6$ and has a value of 6.

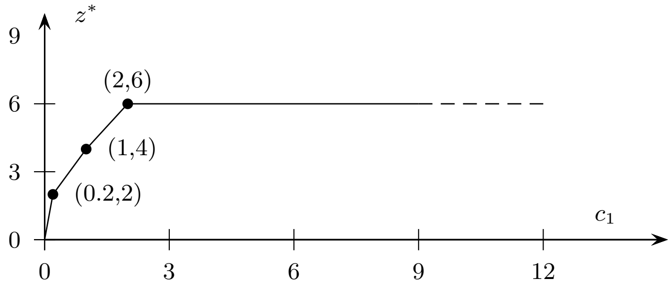
b) We assume that $z = c_1x_1 + x_2$. We want to determine the optimal solutions and the optimal values of the objective function depending on c_1. Depending on the slope s of the contour lines of the objective function, we get that the optimal solutions are given below. To get the value of the objective function $z = c_1x_1 + x_2$, we just have to replace x_1 and x_2 by their optimal values for each of the different cases.

$-\infty \leq s < -2$	The optimum is located at a and has a value of $z^* = 6$ with $x_1^* = 0$ and $x_2^* = 6$.
$-2 \leq s < -1$	The optimum is located at b and has value of $z^* = 2c_1 + 2$ with $x_1^* = 2$ and $x_2^* = 2$.
$-1 \leq s < -\frac{1}{5}$	The optimum is located at c and has a value of $z^* = \frac{5}{2}c_1 + +\frac{3}{2}$, with $x_1^* = 2.5$ and $x_2^* = 1.5$.
$-\frac{1}{5} \leq s \leq 0$	The optimum is located at d and has a value of $z^* = 10c_1$, with $x_1^* = 10$ and $x_2^* = 0$.
$s > 0$	The problem is unbounded.

As $z = c_1x_1 + x_2$, this can be rewritten as $x_2 = z - c_1x_1$. We conclude that the slope of the contour lines is given by $s = -c_1$. If we express the above condition in function of c_1 rather than s, we finally get that:

$c_1 < 0$	The problem is unbounded.
$0 \leq c_1 \leq \frac{1}{5}$	The optimum is located at d and has a value of $z^* = 10c_1$, with $x_1^* = 10$ and $x_2^* = 0$.
$\frac{1}{5} < c_1 \leq 1$	The optimum is located at c and has a value of $z^* = \frac{5}{2}c_1 + \frac{3}{2}$, with $x_1^* = 1.5$ and $x_2^* = 2.5$.
$1 < c_1 \leq 2$	The optimum is located at b and has value of $z^* = 2c_1 + 2$, with $x_1^* = 2$ and $x_2^* = 2$.
$2 < c_1 \leq \infty$	The optimum is located at a and has a value of $z^* = 6$, with $x_1^* = 0$ and $x_2^* = 6$.

If we plot z in function of c_1, then we get a concave piecewise linear function as illustrated below.



Problem 2

a) Gauss elimination: we start with the augmented matrix of the sytem:

$$\left(\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & 5 & 7 \\ 2 & -4 & 5 & 6 & 11 & 10 \\ 3 & -6 & 8 & 10 & 11 & 3 \\ 0 & 0 & 1 & 5 & -2 & -9 \end{array}\right) \begin{array}{c} \\ l_2 - 2l_1 \\ l_3 - 3l_1 \\ \\ \end{array} \rightarrow \left(\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & 5 & 7 \\ 0 & 0 & 1 & 6 & 1 & -4 \\ 0 & 0 & 2 & 10 & -4 & -18 \\ 0 & 0 & 1 & 5 & -2 & -9 \end{array}\right) \begin{array}{c} l_1 - 2l_2 \\ \\ l_3 - 2l_2 \\ l_4 - \frac{1}{2}l_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & -12 & 3 & 15 \\ 0 & 0 & 1 & 6 & 1 & -4 \\ 0 & 0 & 0 & -2 & -6 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} l_1 - 6l_3 \\ l_2 + 3l_3 \\ -\frac{1}{2}l_3 \\ \\ \end{array} \rightarrow \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 39 & 75 \\ 0 & 0 & 1 & 0 & -17 & -34 \\ 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The set of solutions of $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5 \;\middle|\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 75 \\ 0 \\ -34 \\ 5 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -39 \\ 0 \\ 17 \\ -3 \\ 1 \end{pmatrix}, \quad s,t \in \mathbb{R} \right\}.$$

b) We conclude that $\text{rank}(\boldsymbol{A}) = 3$.

c) The dimension of the column space is equal to the rank of $\boldsymbol{A}$. We just need to choose the columns of $\boldsymbol{A}$ based on the reduced row echelon form to get a basis $\mathcal{B}_c$ of the columns of $\boldsymbol{A}$:

$$\mathcal{B}_c = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 10 \\ 5 \end{pmatrix} \right\}$$

d) The dimension of the row space is equal to the rank of $\boldsymbol{A}$. We just need to choose the rows of $\boldsymbol{A}$ based on the reduced row echelon form to get a basis $\mathcal{B}_c$ of the rows of $\boldsymbol{A}$:

$$\mathcal{B}_l = \{(1 \quad -2 \quad 2 \quad 0 \quad 5), (2 \quad -4 \quad 5 \quad 6 \quad 11), (3 \quad -6 \quad 8 \quad 10 \quad 11)\}$$

Problem 3

$T_0 =$

x_1	x_2	x_3	x_4	z		ratio	
1	-1	1	0	0	1	1	←
-3	1	0	1	0	0	-	
-1	-4	0	0	1	0		
↑							

$T_1 =$

x_1	x_2	x_3	x_4	z	
1	-1	1	0	0	1
0	-2	3	1	0	3
0	-5	1	0	1	1

This problem has no finite optimum.

Problem 4

The initial tableau is given by:

$T_0 =$

x_1	x_2	x_3	x_4	x_5	z	
1	-1	1	0	0	0	2
-1	-2	0	1	0	0	-1
-2	3	0	0	1	0	-6
-2	-3	0	0	0	1	0

This tableau is not feasible, we apply phase I:

$T_0^{\text{aux}} =$

x_0	x_1	x_2	x_3	x_4	x_5	z	z'	
0	1	-1	1	0	0	0	0	2
-1	-1	-2	0	1	0	0	0	-1
-1	-2	3	0	0	1	0	0	-6
0	-2	-3	0	0	0	1	0	0
1	0	0	0	0	0	0	1	0

$T_1^{\text{aux}} =$

x_0	x_1	x_2	x_3	x_4	x_5	z	z'	
0	**1**	-1	1	0	0	0	0	2
0	1	-5	0	1	-1	0	0	5
1	2	-3	0	0	-1	0	0	6
0	-2	-3	0	0	0	1	0	0
0	-2	3	0	0	1	0	1	-6

$T_2^{\text{aux}} =$

x_0	x_1	x_2	x_3	x_4	x_5	z	z'	
0	1	-1	1	0	0	0	0	2
0	0	-4	-1	1	-1	0	0	3
1	0	-1	-2	0	-1	0	0	2
0	0	-5	2	0	0	1	0	4
0	0	1	2	0	1	0	1	-2

Phase I has completed but the optimal value is not null. The initial problem has no feasible solution.

Problem 5

In order to produce 1000 items, the factory needs at least 1 ton of of M1, 0.6 ton of M2, and 0.3 ton of M3.

a) Let x_i be the quantity (in tons) of alloy i that the factory needs to purchase. The primal LP is given by:

$$(PLP)\left\{\begin{array}{llrcrcrcl}
\text{Min} & z = & 3x_1 & + & x_2 & + & 4x_3 & & \\
\text{s.t.} & & x_1 & + & 4x_2 & + & x_3 & \geq & 10 \\
& & 3x_1 & + & 6x_2 & + & 6x_3 & \geq & 6 \\
& & 6x_1 & & & + & 3x_3 & \geq & 3 \\
& & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}\right.$$

Note that each inequality has been multiplied by 10.

b) Dual problem:

$$(DLP)\left\{\begin{array}{llrcrcrcl}
\text{Max} & w = & 10y_1 & + & 6y_2 & + & 3y_3 & & \\
\text{s.t.} & & y_1 & + & 3y_2 & + & 6y_3 & \leq & 3 \\
& & 4y_1 & + & 6y_2 & & & \leq & 1 \\
& & y_1 & + & 6y_2 & + & 3y_3 & \leq & 4 \\
& & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}\right.$$

c) PLP in standard form:

$$\begin{array}{llrcrcrcrcrcrcl}
\text{Max} & z = & -3x_1 & - & x_2 & - & 4x_3 & & & & & & & & \\
\text{s.t.} & & -x_1 & - & 4x_2 & - & x_3 & + & x_4 & & & & & = & -10 \\
& & -3x_1 & - & 6x_2 & - & 6x_3 & & & + & x_5 & & & = & -6 \\
& & -6x_1 & & & - & 3x_3 & & & & & + & x_6 & = & -3 \\
& & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}$$

Slack variable x_{3+i} represents the surplus of metal M_i,$i = 1$,2,3. The initial tableau is dual-feasible but not primal-feasible. Let's apply the dual simplex algorithm (phase II):

$T_0 =$

	x_1	x_2	x_3	x_4	x_5	x_6	z	
	-1	**-4**	-1	1	0	0	0	-10
	-3	-6	-6	0	1	0	0	-6
	-6	0	-3	0	0	1	0	-3
	3	1	4	0	0	0	1	0
ratio	-3	$-1/4$	-4					

$T_1 =$

	x_1	x_2	x_3	x_4	x_5	x_6	z	
	1/4	1	1/4	-1/4	0	0	0	5/2
	-3/2	0	-9/2	-3/2	1	0	0	9
	-6	0	-3	0	0	1	0	-3
	11/4	0	15/4	1/4	0	0	1	-5/2
ratio	$-11/24$		$-5/4$					

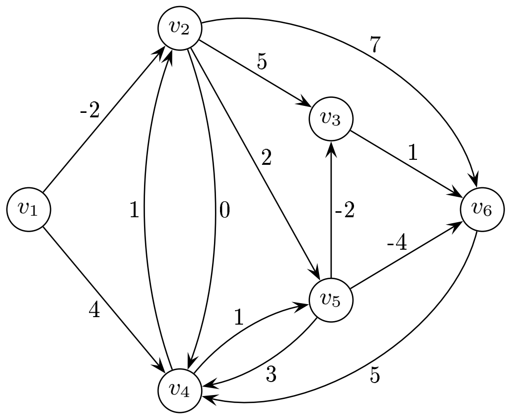
$T_2 =$

x_1	x_2	x_3	x_4	x_5	x_6	z	
0	1	1/8	-1/4	0	1/24	0	19/8
0	0	-15/4	-3/2	1	-1/4	0	39/4
1	0	1/2	0	0	-1/6	0	1/2
0	0	19/8	1/4	0	11/24	1	-31/8

Tableau T_2 is optimal. The factory needs to order $x_1 = 1/2$ ton of alloy 1, $x_2 = 19/8$ tons of alloy 2. There is no need of alloy 3. The surplus of metal M2 is $39/4$ tons. The minimal cost is $31/8$ kFrs.

Problem 6

We would like to determine the shortest path from v_1 to v_6:



As this network contains edges with negative weights, we must apply the generic algorithm:

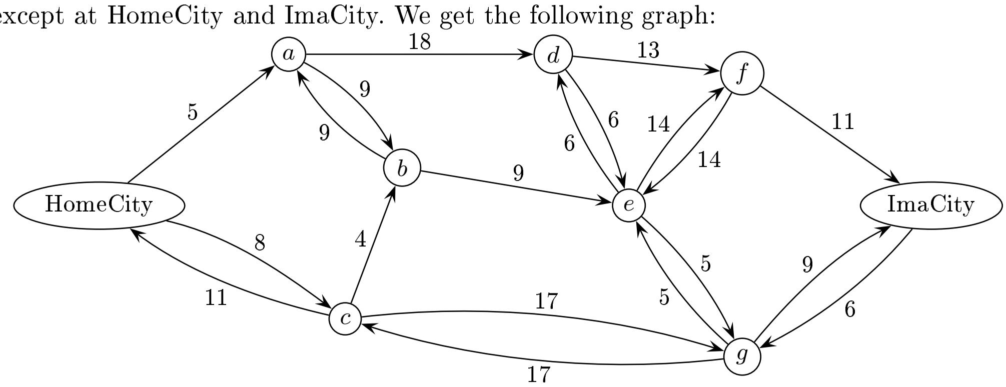
Iter.	v_i removed from L	Labels λ_i/ Predecessors $p(i)$						Cand. L
		v_1	v_2	v_3	v_4	v_5	v_6	
0		0/-	∞/-	∞/-	∞/-	∞/-	∞/-	$\{v_1\}$
1	v_1	0/-	-2/v_1	∞/-	4/v_1	∞/-	∞/-	$\{v_2,v_4\}$
2	v_2	0/-	-2/v_1	3/v_2	-2/v_2	0/v_2	5/v_2	$\{v_3,v_4,v_5,v_6\}$
3	v_3	0/-	-2/v_1	3/v_2	-2/v_2	0/v_2	4/v_3	$\{v_4,v_5,v_6\}$
4	v_4	0/-	-2/v_1	3/v_2	-2/v_2	-1/v_4	4/v_3	$\{v_5,v_6\}$
5	v_5	0/-	-2/v_1	-3/v_5	-2/v_2	-1/v_4	-5/v_5	$\{v_3,v_6\}$
6	v_3	0/-	-2/v_1	-3/v_5	-2/v_2	-1/v_4	-5/v_5	$\{v_6\}$
7	v_6	0/-	-2/v_1	-3/v_5	-2/v_2	-1/v_4	-5/v_5	$\varnothing$

The shortest path from v_1 to v_6 in R is unique and has a value of -5. It is given by:

$$v_1 \longrightarrow v_2 \longrightarrow v_4 \longrightarrow v_5 \longrightarrow v_6.$$

Problem 7

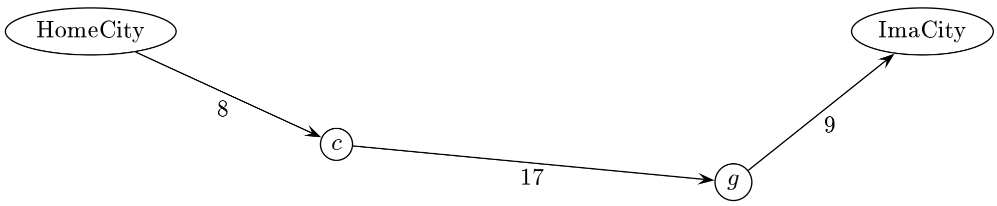
Let's first replace every edge by two arcs in opposite direction and let's add to each arc a duration of 3 minutes except at HomeCity and ImaCity. We get the following graph:



As the “weights” are non-negative, we can apply Dijkstra’s algorithm:

It	i_{min}	Label (predecessor) at the end of the iteration								
		HC	a	b	c	d	e	f	g	IC
0		0	∞	∞	∞	∞	∞	∞	∞	∞
1	HC	0	5(HC)	∞	8(HC)	∞	∞	∞	∞	∞
2	a		5(HC)	$14(a)$	8(HC)	$23(a)$	∞	∞	∞	∞
3	c			$12(c)$	8(HC)	$23(a)$	∞	∞	$25(c)$	∞
4	b			$12(c)$		$23(a)$	$21(b)$	∞	$25(c)$	∞
5	e					$23(a)$	$21(b)$	$35(e)$	$25(c)$	∞
6	d					$23(a)$		$35(e)$	$25(c)$	∞
7	g							$35(e)$	$25(c)$	$34(g)$
8	IC							$35(e)$		$34(g)$
9	f							$35(e)$		

The optimal path is:



Anne needs 34 minutes to go from HomeCity to ImaCity.

Problem 8

a) Shortest paths from α:

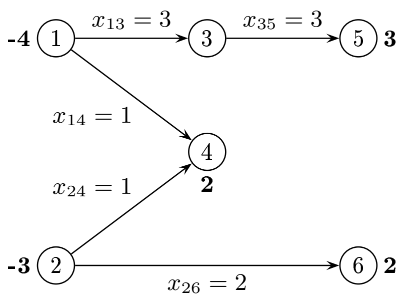
Vertex	k (top. sort)	$\lambda_k/p(k)$
α	1	$0/NULL$
A	2	$0/\alpha$
D	3	$0/\alpha$
N	4	$0/\alpha$
B	5	$0.5/A$
E	6	$1/D$
O	7	$2/N$
G	8	$0.5/A$
H	9	$2.5/G$
I	10	$2.5/G$
J	11	$4.5/H$
F	12	$2/E$
C	13	$3.5/B$
K	14	$4.5/H$
L	15	$5/K$
M	16	$2.5/F$
ω	17	$3/O$

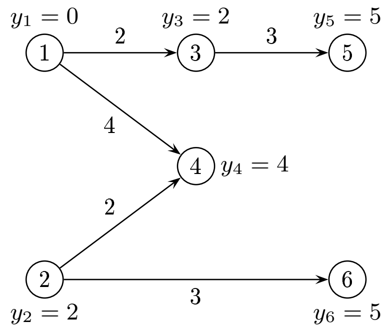
b) Longest paths from α:

Vertex	k (top. sort)	$\lambda_k/p(k)$
α	1	$0/NULL$
A	2	$0/\alpha$
D	3	$0/\alpha$
N	4	$0/\alpha$
B	5	$0.5/A$
E	6	$1/D$
O	7	$2/N$
G	8	$2/E$
H	9	$4/G$
I	10	$4/G$
J	11	$7/I$
F	12	$7/I$
C	13	$8/J$
K	14	$8/J$
L	15	$8.5/K$
M	16	$9.5/L$
ω	17	$13.5/M$

Problem 9

a) By removing successively vertices in the order 5, 3, 1, 4, 2, we get $x_{35} = 3$, $x_{13} = 3$, $x_{14} = 1$, $x_{24} = 1$, and $x_{26} = 2$.
The cost of this solution is $z = \sum_{(i,j)\in T} c_{ij}x_{ij} = 6 + 9 + 4 + 2 + 6 = 27$ (where T is the set of basic arcs).





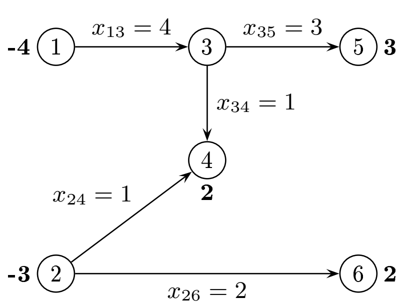
We set $y_1 = 0$. By visiting the vertices in the order 3, 5, 4, 2, 6, we get $y_3 = 2$, $y_5 = 5$, $y_4 = 4$, $y_2 = 2$, and $y_6 = 5$.
The value of this solution is $w = \sum_{i\in V} b_i y_i = 0 - 6 + 0 + 8 + 15 + 10 = 27$.

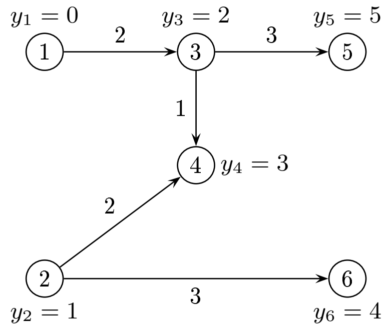
b) First iteration. We look for a violated dual constraint. We test the non-basic arcs in the lexicographical order:

$$(3,4) : y_4 - y_3 - c_{34} = 4 - 2 - 1 > 0$$

The arc (3,4) enters the basis. The cycle (3,4), (1,4), (1,3) has only one arc in the opposite orientation defined by (3,4): (1,4). $\Delta = x_{14} = 1$ and the arc (1,4) exits the basis.
The new tree-solution is formed by the arcs {(1,3),(2,4),(2,6),(3,4),(3,5)}. The new primal solution is $x_{13} = 3 + 1 = 4$, $x_{14} = 0$, $x_{34} = 1$, and no change for the other values.
The dual variables y_4, y_2 and y_6 are modified. They decrese in value by $\varepsilon = y_4 - y_3 - c_{34} = 4 - 2 - 1 = 1$.

The value of the new basic solutions is $z = w = 27 - 4 + 2 + 1 = 26$.

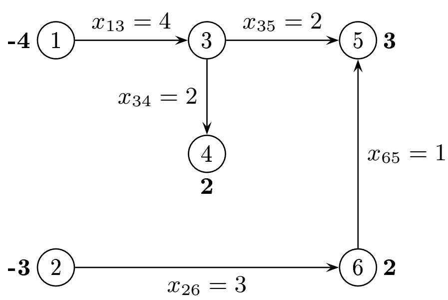


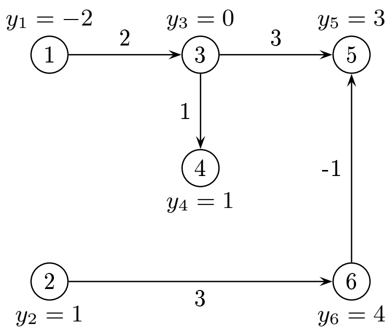


Second iteration. We look for a violated dual constraint:

$$
\begin{aligned}
(1,4) &: y_4 - y_1 - c_{14} = 3 - 0 - 4 \leq 0 \\
(4,5) &: y_5 - y_4 - c_{45} = 5 - 3 - 2 \leq 0 \\
(4,6) &: y_6 - y_4 - c_{46} = 4 - 3 - 4 \leq 0 \\
(6,5) &: y_5 - y_6 - c_{65} = 5 - 4 + 1 > 0
\end{aligned}
$$

The arc (6,5) enters the basis. The cycle (6,5), (3,5), (3,4), (2,4), (2,6) has two arcs in the opposite direction : (3,5) and (2,4). $\Delta = \min(x_{35}, x_{24}) = \min(3,1) = 1$ and the arc (2,4) exits the basis.
The new tree-solution is formed by the arcs $\{(1,3),(2,6),(3,4),(3,5),(6,5)\}$. The new primal solution is $x_{24} = 0$, $x_{34} = 1 + 1 = 2$, $x_{35} = 3 - 1 = 2$, $x_{26} = 2 + 1 = 3$ and $x_{65} = 1$, the other values are not modified.
All the dual variables are modified, except y_2 and y_6. They decrease in value by $\varepsilon = y_5 - y_6 - c_{65} = 5 - 4 + 1 = 2$.
The value of the new basic solution is $z = w = 26 - 1 - 3 + 1 - 2 + 3 = 24$.





Third iteration. We look for a violated dual constraint:

$$
\begin{aligned}
(1,4) &: y_4 - y_1 - c_{14} = 1 + 2 - 4 \leq 0 \\
(2,4) &: y_4 - y_2 - c_{24} = 1 - 1 - 2 \leq 0 \\
(4,5) &: y_5 - y_4 - c_{45} = 3 - 1 - 2 \leq 0 \\
(4,6) &: y_6 - y_4 - c_{46} = 4 - 1 - 4 \leq 0
\end{aligned}
$$

All the dual constraints are satisfied. The current basic solutions are optimal.

c) The current basis is optimal as long as

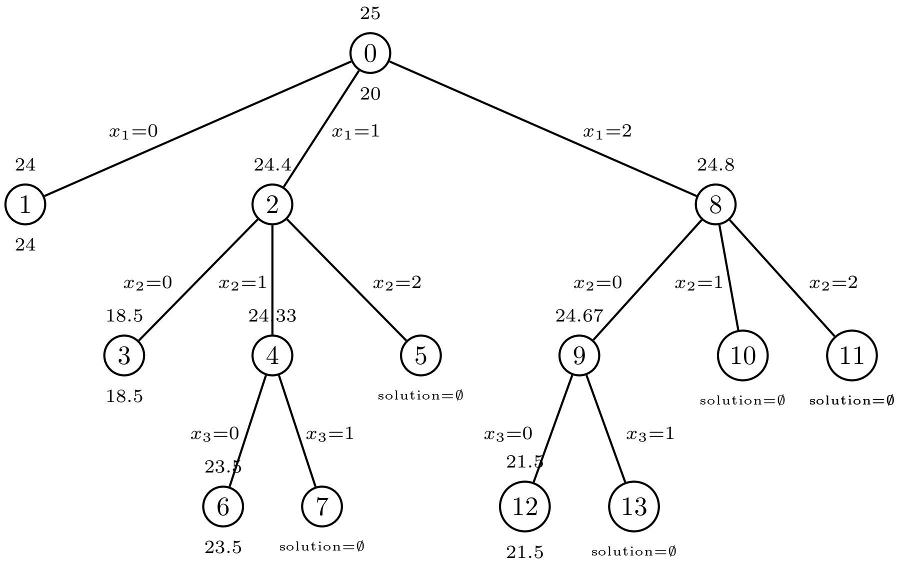
$$y_6 - y_4 - c_{46} \leq 0$$

i.e.:

$$c_{46} \geq y_6 - y_4 = 4 - 1 = 3 \quad \Longleftrightarrow \quad c_{46} \in [3,\infty).$$

Problem 10

It is quite obvious that x_1,$x_2 \in \{0,1,2\}$. By solving the relaxed LP for each node, we get the following enumeration tree:



The optimal value is 24 and corresponds to node 1. The optimal solution is given by $x_2 = 2$ and $x_1 = x_3 = x_4 = 0$.

Problem 11

We have

$$\mathbf{z}^{\mathrm{T}}\mathbf{M}\mathbf{z} = (\mathbf{z}^{\mathrm{T}}\mathbf{M})\mathbf{z} = \begin{bmatrix}(2a-b) & (-a+2b-c) & (-b+2c)\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$
$$= 2a^2 - 2ab + 2b^2 - 2bc + 2c^2$$
$$= a^2 + (a-b)^2 + (b-c)^2 + c^2$$

This result is a sum of squares, and therefore non-negative. It is egal to zero only if a = b = c = 0, that is, when $\mathbf{z}$ is zero.

Problem 12

- Function f:
  Its gradient and its hessian are:

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad \forall \boldsymbol{x} \in \mathbb{R}^2$$
$$\nabla^2 f(\boldsymbol{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \forall \boldsymbol{x} \in \mathbb{R}^2$$

  The hessian is positive definite for all $\boldsymbol{x} \in \mathbb{R}^2$. There is a single critical point which is also the unique global minimum of f: $\nabla f(\boldsymbol{x}^*) = 0 \Leftrightarrow \boldsymbol{x}^* = (0,0)$.
- Function g:
  Its gradient and its hessian are:

$$\nabla g(\boldsymbol{x}) = \begin{pmatrix} x^2 - 1 \\ 3y^2 - 1 \end{pmatrix} \qquad \forall \boldsymbol{x} \in \mathbb{R}^2$$
$$\nabla^2 g(\boldsymbol{x}) = \begin{pmatrix} 2x & 0 \\ 0 & 6y \end{pmatrix} \qquad \forall \boldsymbol{x} \in \mathbb{R}^2$$

There are 4 critical points: $\boldsymbol{x} = (1,\sqrt{1/3})$, $\boldsymbol{x} = (1,-\sqrt{1/3})$, $\boldsymbol{x} = (-1,\sqrt{1/3})$ and $\boldsymbol{x} = (-1,-\sqrt{1/3})$. The first one is a local minimum (the hessian matrix is positive definite), the last one is a local maximum (the hessian is negative definite) and the two other are saddle points **since their hessians are indefinite (neither positive semi-definite nor negative semi-definite)**.

Problem 13

a) The lagrangian function is given by $L(x,y,z,\lambda) = xy + \lambda(3x^2 + y^2 - 6), \lambda \in \mathbb{R}$. KKT conditions are given by

$$\begin{aligned} y + 6\lambda x &= 0 \\ x + 2\lambda y &= 0 \\ 3x^2 + y^2 - 6 &= 0 \end{aligned}$$

This can be rewritten as

$$\begin{aligned} y &= -6\lambda x \quad (1) \\ x &= -2\lambda y \quad (2) \\ 3x^2 + y^2 - 6 &= 0 \quad (3) \end{aligned}$$

Plugging the second equation into the first one gives

$$y = 12\lambda^2 y.$$

If y were 0, then x would be 0 too, which is impossible by (3). Thus we can divide by y to get that $12\lambda^2 = 1$. Then:

$$\begin{aligned} 6 &= 3x^2 + (6\lambda x)^2 \\ 6 &= 3x^2 + 3(12\lambda^2)x^2 \\ 6 &= 3x^2 + 3x^2. \end{aligned}$$

Thus $x \pm 1$ and $y = \pm\sqrt{3}$ by (3). They are four critical points: $\boldsymbol{a} = (1,\sqrt{3})$,$\boldsymbol{b} = (1,-\sqrt{3})$,$\boldsymbol{c} = (-1,\sqrt{3})$, and $\boldsymbol{d} = (-1,-\sqrt{3})$.

b) We have $f(\boldsymbol{a}) = f(\boldsymbol{d}) = \sqrt{3}$ and $f(\boldsymbol{b}) = f(\boldsymbol{c}) = -\sqrt{3}$. By Weierstrass extreme value theorem, this optimization problem have a maximum and a minimum. Thus $\boldsymbol{a}$, $\boldsymbol{d}$ are maxima and $\boldsymbol{b}$, $\boldsymbol{c}$ are minima.

Problem 14

Let us define the ground-set as $X = \{\boldsymbol{x} \in \mathbb{R}^n | x_j > 0, j = 1,\dots,n\}$, and let us dualize on the single equality constraint. The Lagrangian function takes on the form:

$$\begin{aligned} L(\boldsymbol{x},\alpha) &= 5x_1 + 7x_2 - 4x_3 - \sum_{j=1}^{3} ln(x_j) + \alpha(x_1 + 3x_2 + 12x_3 - 37) \\ &= -37\alpha + (5+\alpha)x_1 + (7+3\alpha)x_2 + (-4+12\alpha)x_3 - \sum_{j=1}^{3} ln(x_j) \end{aligned}$$

The dual function $L^*(\alpha)$ is constructed as $L^*(\alpha) = min_{\boldsymbol{x}\in X} L(\boldsymbol{x},\alpha)$. Now notice that the optimization problem above separates into three univariate optimization problems of a linear function minus a logarithm term for each of the three positive variables x_1,x_2, and x_3. Examining x_1, it holds that the minimization value will be $-\infty$ if $(5+\alpha) \leq 0$, as we could set x_1 arbitrarily large. When $(5+\alpha) > 0$, the problem of minimizing $(5+\alpha)x_1 - ln(x_1)$ is a convex optimization problem whose solution is given by setting the first derivative with respect x_1 equal to zero. This means solving:

$$(5+\alpha) - \frac{1}{x_1} = 0,$$

or in other words, setting

$$x_1 = \frac{1}{(5+\alpha)}.$$

Substituting this value of x_1, we obtain:

$$(5+\alpha)x_1 - ln(x_1) = 1 - ln(\frac{1}{5+\alpha}) = 1 + ln(5+\alpha).$$

Using parallel logic for the other two variables, we arrive at:

$$L^*(\alpha) = \begin{cases} -37\alpha + 3 + ln(5+\alpha) + ln(7+3\alpha) + ln(-4+12\alpha) & \text{if } \alpha > 1/3 \\ -\infty \quad \text{otherwise.} \end{cases}$$

Notice that $L^*(\alpha)$ is finite whenever $5+\alpha > 0$, $7+3\alpha > 0$, and $-4+12\alpha > 0$. These three inequalities in α are equivalent to the single inequality $\alpha > 1/3$. The dual problem is defined to be $\max_{\alpha\in\mathbb{R}} L^*(\alpha)$.

Problem 15

Steepest descent method with a step obtained by exact minimization

(a) The steepest descent direction of f in (x_0,y_0) is given by:

$$\mathbf{d} = -\begin{pmatrix} 6x_0 \\ 6y_0 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \end{pmatrix}$$

Computation of the step α_{min}:

$$\alpha_{min} = argmin_{\alpha\geq 0} g(\alpha) = argmin_{\alpha\geq 0} f((x_0\ y_0)^T + \alpha\mathbf{d})$$

We get that $g(\alpha) = f((1-6\alpha\ 1-6\alpha)^T) = 6(1-6\alpha)^2$. Moreover, as the function is strictly convex, the step is obtained by setting $g^{'}(\alpha) = 0$:

$$1 - 6\alpha = 0 \Rightarrow \alpha_{min} = \frac{1}{6}$$

The new iterate is:

$$(x_1\ y_1)^T = (x_0\ y_0)^T + \alpha_{min}\mathbf{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We find the minimum of f in one iteration.

(b) From a theoretical point of view, there is no result that gives the number of iterations necessary to converge in the general case for this method.

Newton's Method

(a) The Newton's direction is given by:

$$\mathbf{d} = -\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

The new iterate is:

$$(x_1\ y_1)^T = (x_0\ y_0)^T + \mathbf{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We find the minimum in one iteration.

(b) Newton's method converges in one iteration for strictly convex quadratic problems. Newton's direction is obtained by minimizing a quadratic function explaining why it converges in one iteration.

Conjuguate gradient method

(a) $\mathbf{Q}$ is a symmetric positive definite matrix given by:

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix},$$

$\mathbf{b} = (0\ 0)^T$ and $c = 0$. Let's set $\mathbf{x} = (x_0\ y_0)^T$.

The direction is given by:

$$\mathbf{d} = -\mathbf{Q}\mathbf{x} - \mathbf{b} = \begin{pmatrix} -6 \\ -6 \end{pmatrix}$$

The step is:

$$\alpha = -\frac{\mathbf{d}^T(\mathbf{Q}\mathbf{x} + \mathbf{b})}{\mathbf{d}^T\mathbf{Q}\mathbf{d}} = \frac{1}{6}$$

The new iterate:

$$(x_1\ y_1)^T = \mathbf{x} + \alpha\mathbf{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We find the minimum in one iteration.

(b) The maximal number of iterations for this method is given by the dimension of the problem, i.e. 2 in this example.