Exercise Set 11

Problem 1

We consider the following problem:

$$\min_{(x,y)\in\mathbb{R}^2} f(x,y) = 3x^2 + 3y^2$$

and three different algorithms: 1) the steepest descent method (with the step obtained by exact minimization), 2) the Newton's method, 3) the conjuguate gradient method.

For each of these algorithms:

- Apply one iteration of the method starting at $(x_0,y_0) = (1,1)$.
- From a theoretical point of view, how many iterations are necessary to solve this problem?

Problem 2

We consider the following function:

$$\begin{aligned} f \ : \ & \mathbb{R}^2 \to \mathbb{R} \\ & (x,y) \mapsto f(x,y) = (x-2)^4 + (x-2)^2y^2 + (y+1)^2 \end{aligned}$$

a) Compute the gradient and the hessian of f for all $\mathbf{x} \in \mathbb{R}^2$.

b) The minimum of f is reached at $\mathbf{x}_* = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$ where $f(\mathbf{x}_*) = 0$. Apply Newton's method starting from $\mathbf{x}_0 = \begin{pmatrix} 1.0 & 1.0 \end{pmatrix}^T$ until $|f(\mathbf{x}_*) - f(\mathbf{x}_k)| < 10^{-2}$.

Problem 3

We consider the minimization problem where $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by:

$$f(x) = x_1^2 + 2x_1x_2 + 2x_2^2$$

Show that, independently from the starting point, the Newton's method converges in one iteration.

Problem 4

We consider the minimization problem where f is defined by:

$$f(x,y) = x^4 - 2x^2 + y^3 - 3y$$

and the points:

$$\{\ (2,2), (-1,1), (0,-1)\ \}$$

1. Are these points local minima?
2. Apply Newton's method to each of these points.
3. Check if each step satisfies Armijo rule with $\beta = 0.1$.

Problem 5

We consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by:

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{b}^T\mathbf{x}$$

$$\text{with} \quad \mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 25 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = (1\ 1\ 1)^T$$

(a) Find the unique minimum of f over $\mathbb{R}^3$.

(b) Compute the steepest descent direction at point $\mathbf{x} = (0\ 0\ 0)^T$

Problem 6

We consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x,y) = \frac{1}{2}(x^2 - y)^2 + \frac{1}{2}(1 - x)^2$$

(a) What is the minimum of f?

(b) Apply Newton's method to minimize f starting from $(x^0,y^0) = (2,2)$. Is it a good step?