

Linear Programming  
Optimization Methods in Management Science  
Master in Management  
HEC Lausanne

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# Canonical and Standard Forms for Linear Programming

- Canonical and standard forms
- Matrix notation
- Transformation rules
- Examples

# General Form of a Linear Program

$$\text{Opt } z = \sum_{j=1}^n c_j x_j$$

$$\text{s.t. } \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i \in I \subseteq \{1, \dots, m\}$$

$$\sum_{j=1}^n a_{kj} x_j \geq b_k \quad k \in K \subseteq \{1, \dots, m\}$$

$$\sum_{j=1}^n a_{rj} x_j = b_r \quad r \in R \subseteq \{1, \dots, m\}$$

$$l_j \leq x_j \leq u_j \quad j = 1, \dots, n$$

- Opt = Max or Min,
- $I, K$  et  $R$  are disjoint sets and  $I \cup K \cup R = \{1, \dots, m\}$ ,
- variables may be unbounded:  $l_j = -\infty$  and/or  $u_j = +\infty$

# Canonical Form of a Linear Program

$$\begin{array}{ll}\text{Max} & z = \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1, \dots, m \\ & x_j \geq 0 \quad j = 1, \dots, n\end{array}$$

- Maximisation problem
- All constraints are of type  $\leq$
- All variables are non-negative

# Standard Form of a Linear Program

$$\begin{array}{ll}\text{Max} & z = \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \sum_{j=1}^n a_{ij} x_j = b_i \quad i = 1, \dots, m \\ & x_j \geq 0 \quad j = 1, \dots, n\end{array}$$

- Maximization problem
- All constraints are equalities
- All variables are non-negative

# From Canonical to Standard Form

Addition of slack variables  $x_{n+i}$

$$\sum_{j=1}^n a_{ij}x_j \leq b_i \quad \rightarrow \quad \sum_{j=1}^n a_{ij}x_j + x_{n+i} = b_i$$

$$\begin{aligned} \text{Max} \quad z &= \sum_{j=1}^n c_j x_j + \sum_{i=1}^m 0 x_{n+i} \\ \text{s.t.} \quad &\sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i \quad i = 1, \dots, m \\ &x_j \geq 0 \quad j = 1, \dots, n + m \end{aligned}$$

# Why Having Particular Forms?

- Check some prerequisites before applying algorithms
- Simplify the presentation of the algorithms

## Important Remark

In this course, the reference form is the **canonical one** whose variables are called **decision variables** of the problem. They are denoted by  $x_D$ . The standard form will always be obtained by adding some **slack variables**  $x_E$  to the canonical problem.

# Matrix Notation

- A vector can be a row-vector or a column-vector. Examples:

$$\mathbf{c} = (c_1 \ \dots \ c_n), \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$\mathbf{c}$  is a row-vector and  $\mathbf{x}$  is a column-vector

- The scalar product between  $\mathbf{c}$  and  $\mathbf{x}$  is defined by  $\sum_{i=1}^n c_i x_i$  and is simply denoted  $\mathbf{c}\mathbf{x}$  (matrix product)
- Note that, in general, the scalar product between two vectors  $\mathbf{x}$  and  $\mathbf{y}$  is denoted by  $\mathbf{x}^T \mathbf{y}$



# Matrix Form of a Canonical PL

$$\begin{array}{ll} \text{Max} & z = \mathbf{c}\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \left[ \begin{array}{ll} \text{Max} & z = \mathbf{c}_D\mathbf{x}_D \\ \text{s.t.} & \mathbf{A}\mathbf{x}_D \leq \mathbf{b} \\ & \mathbf{x}_D \geq \mathbf{0} \end{array} \right]$$

where

$$\mathbf{c} = \mathbf{c}_D = (c_1 \ \dots \ c_n), \quad \mathbf{x} = \mathbf{x}_D = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

## Matrix Form of a Standard PL

$$\begin{array}{ll} \text{Max} & z = \mathbf{c}\mathbf{x} \\ \text{s.t.} & [\mathbf{A} \mid \mathbf{I}] \mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \left[ \begin{array}{ll} \text{Max} & z = \mathbf{c}_D \mathbf{x}_D + \mathbf{0} \mathbf{x}_E \\ \text{s.t.} & \mathbf{A} \mathbf{x}_D + \mathbf{I} \mathbf{x}_E = \mathbf{b} \\ & \mathbf{x}_D, \mathbf{x}_E \geq \mathbf{0} \end{array} \right]$$

where

$$\mathbf{c} = (\mathbf{c}_D \mid \mathbf{c}_E) = (\mathbf{c}_D \mid \mathbf{0}) = (c_1 \dots c_n \mid 0 \dots 0)$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_D \\ - \\ \mathbf{x}_E \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ - \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

## Matrix Form: Example

Starting from:

$$\begin{array}{llllll} \text{Max} & z = & 250x_1 & + & 450x_2 & \\ \text{s.t.} & & 2x_1 & + & 3x_2 & \leq 42 \\ & & -4x_1 & + & 6x_2 & \leq 0 \\ & & x_1 & & & \leq 15 \\ & & x_1 & , & x_2 & \geq 0 \end{array}$$

we get that:

$$\mathbf{c} = ( 250 \quad 450 ), \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -4 & 6 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 42 \\ 0 \\ 15 \end{pmatrix}.$$

# Some Transformation Rules

- **Minimization**  $\leftrightarrow$  **maximization**:

$$\min f(\mathbf{x}) = -\max(-f(\mathbf{x}))$$

In order to  $\min z = \mathbf{c}\mathbf{x}$ , we just need to  $\max w = -\mathbf{c}\mathbf{x} = (-\mathbf{c})\mathbf{x}$  and to multiply the optimal value of  $w$  by  $-1$  to get the optimal value of  $z$

- **Inequation** " $\geq$ "  $\leftrightarrow$  **inequation** " $\leq$ " :

$$\mathbf{a}\mathbf{x} \geq b \iff (-\mathbf{a})\mathbf{x} \leq -b$$

- **Equation**  $\rightarrow$  **inequation** " $\leq$ " :

$$\mathbf{a}\mathbf{x} = b \iff \left\{ \begin{array}{l} \mathbf{a}\mathbf{x} \leq b \\ \mathbf{a}\mathbf{x} \geq b \end{array} \right. \iff \left\{ \begin{array}{l} \mathbf{a}\mathbf{x} \leq b \\ (-\mathbf{a})\mathbf{x} \leq -b \end{array} \right.$$

## Some Transformation Rules (Cont'd)

- **Inequation**  $\rightarrow$  **equation** : we add a **slack variable**:

$$\mathbf{ax} \leq b \iff \mathbf{ax} + s = b, s \geq 0$$

$$\mathbf{ax} \geq b \iff -\mathbf{ax} + s = -b, s \geq 0$$

- **Real variable**  $\rightarrow$  **non-negative variable**: every real number can be written as the difference of two non-negative numbers

$$x \in \mathbb{R} \rightarrow \begin{cases} x = x^+ - x^- \\ x^+, x^- \geq 0 \end{cases}$$

- **Variable with a lower bound**:

$$x \geq b \iff \begin{cases} x' = x - b \\ x' \geq 0 \end{cases}$$

# Transformation into Canonical Form

$$\begin{array}{ll} \text{Min} & z = -3x_1 + 4x_2 \\ \text{s.t.} & x_1 + x_2 = 6 \\ & x_1 - 2x_2 \geq 4 \\ & x_1 \in \mathbb{R} \\ & x_2 \geq 0 \end{array}$$

Initial LP

Transformations

$$\begin{array}{ll} \text{Min } z = -3x_1 + 4x_2 & \rightarrow \text{Max } w = 3x_1 - 4x_2 \\ x_1 + x_2 = 6 & \rightarrow \begin{cases} x_1 + x_2 \leq 6 \\ -x_1 - x_2 \leq -6 \end{cases} \\ x_1 - 2x_2 \geq 4 & \rightarrow -x_1 + 2x_2 \leq -4 \\ x_1 \in \mathbb{R} & \rightarrow \begin{cases} x_1 = x_1^+ - x_1^- \\ x_1^+, x_1^- \geq 0 \end{cases} \end{array}$$

# Transformation into Canonical Form

## Initial LP

$$\begin{array}{llllll} \text{Min} & z = & -3x_1 & + & 4x_2 & \\ \text{s.t.} & & x_1 & + & x_2 & = 6 \\ & & x_1 & - & 2x_2 & \geq 4 \\ & & x_1 & & & \in \mathbb{R} \\ & & & & x_2 & \geq 0 \end{array}$$

## Equivalent canonical LP

$$\begin{array}{llllllll} \text{Max} & w = & 3x_1^+ & - & 3x_1^- & - & 4x_2 & \\ \text{s.t.} & & x_1^+ & - & x_1^- & + & x_2 & \leq 6 \\ & & -x_1^+ & + & x_1^- & - & x_2 & \leq -6 \\ & & -x_1^+ & + & x_1^- & + & 2x_2 & \leq -4 \\ & & x_1^+ & , & x_1^- & , & x_2 & \geq 0 \end{array}$$

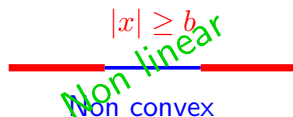
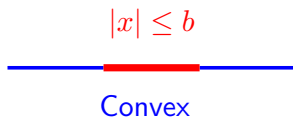
Don't forget that  $z_{opt} = -w_{opt}$  !

# Particular Transformations

- Min-max or max-min problem:

$$\begin{aligned} \text{Min } z = \max\{\mathbf{c}_1\mathbf{x}, \dots, \mathbf{c}_k\mathbf{x}\} &\iff \begin{aligned} \text{Min } z = & t \\ \text{s.t. } & t \geq \mathbf{c}_1\mathbf{x} \\ & \dots \\ & t \geq \mathbf{c}_k\mathbf{x} \\ & t \in \mathbb{R} \end{aligned} \end{aligned}$$

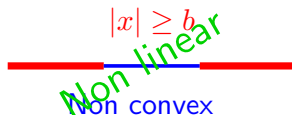
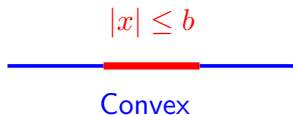
- Absolute value** (with  $b > 0$ ):  $|x| \leq b \iff \begin{cases} x \leq b \\ x \geq -b \end{cases}$





# What Can We Do With a Constraint of Type $|x| \geq b$

- This is not a linear constraint !
- We have to decompose the original problem into two sub-problems :
  - ▶ Replace  $|x| \geq b$  with  $x \leq -b$  in the first one
  - ▶ Replace  $|x| \geq b$  with  $x \geq b$  in the second one
- The final solution is given by the best solution of these two sub-problems



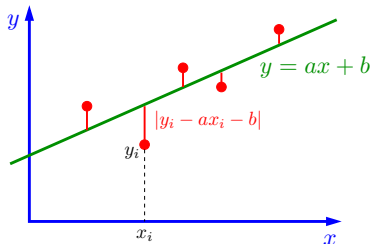
## Example: Chebychev Approximation

### Problem

Determine a linear approximation  $y = \mathbf{a}x + b$  minimizing the largest estimation error

**Data:**  $m$  measurements:

$$(\mathbf{x}_i, y_i) \in \mathbb{R}^{n+1}, i = 1, \dots, m$$



**Formulation:**

$$\text{Min } z = \max_{i=1, \dots, m} \{ |y_i - \mathbf{a}x_i - b| \}$$

Decision variables are  $\mathbf{a} \in \mathbb{R}^n$  and  $b \in \mathbb{R}$

## Chebyshev Approximation (Cont'd)

We can rewrite the problem as

$$\begin{array}{ll} \text{Min} & z = \max_{i=1,\dots,m} \{\Delta_i\} \\ \text{s.t.} & \Delta_i = |y_i - \mathbf{a}\mathbf{x}_i - b| \quad i = 1, \dots, m \end{array}$$

which is equivalent to:

$$\begin{array}{ll} \text{Min} & z = t \\ \text{s.t.} & t \geq |y_i - \mathbf{a}\mathbf{x}_i - b| \quad i = 1, \dots, m \end{array}$$

with  $t \geq 0$

## Chebyshev Approximation (Cont'd)

**Final formulation** (not in canonical nor in standard form) :

$$\begin{array}{ll}\text{Min} & z = t \\ \text{s.t.} & t \geq y_i - \mathbf{a}\mathbf{x}_i - b \quad i = 1, \dots, m \\ & t \geq -y_i + \mathbf{a}\mathbf{x}_i + b \quad i = 1, \dots, m\end{array}$$

with  $\mathbf{a} \in \mathbb{R}^n$ ,  $b \in \mathbb{R}$  and  $t \geq 0$