Solutions to Exercise Set 4

Problem 1

Tableaus T_1, T_2, T_3 and T_4 have the following characteristics:

		T_1	T_2	T_3	T_4
i)	primal-feasible	☐	☐	☑	☐
ii)	dual-feasible	☑	☐	☑	☑
iii)	optimal	☐	☐	☑	☐
iv)	primal-unbounded	☐	☐	☐	☐
v)	dual-unbounded	☐	☐	☐	☑
vi)	primal-degenerated	☑	☐	☐	☐
vii)	dual-degenerated	☑	☑	☐	☐

Problem 2

a)

$$\begin{array}{llrcrcl}
\text{Min} & w = & -y_1 & + & 7y_2 & & \\
\text{s.t.} & & y_1 & - & 2y_2 & = & 0 \\
& & y_1 & + & y_2 & \leq & 7 \\
& & 2y_1 & + & y_2 & \geq & 2 \\
& & y_1 & & & \in & \mathbb{R} \\
& & & & y_2 & \geq & 0
\end{array}$$

b)

$$\begin{array}{llrcrcl}
\text{Max} & w = & -5y_1 & & & & \\
\text{s.t.} & & -y_1 & + & y_2 & \leq & 4 \\
& & 2y_1 & + & y_2 & = & 2 \\
& & y_1 & , & y_2 & \geq & 0
\end{array}$$

c)

$$\begin{array}{llrcrcr}
\text{Min} & w = & 4y_1 & + & 2y_2 & & \\
\text{s.t.} & & y_1 & & & = & 3 \\
& & -y_1 & + & 2y_2 & \geq & -5 \\
& & y_1 & + & y_2 & \geq & 0 \\
& & y_1 & & & \geq & 0 \\
& & & & y_2 & \in & \mathbb{R}
\end{array}$$

d)

$$\begin{array}{lllrcrcrcr}
\text{Max} & w & = & 5y_1 & + & 12y_2 & + & 6y_3 & & \\
\text{s.t.} & & & 2y_1 & & & + & y_3 & = & 3 \\
& & & -y_1 & + & 3y_2 & & & \leq & 0 \\
& & & 2y_1 & + & 4y_2 & & & \leq & -1 \\
& & & y_1 & & & & & \geq & 0 \\
& & & & & y_2 & & & \in & \mathbb{R} \\
& & & & & & & y_3 & \leq & 0
\end{array}$$

e)

$$\begin{array}{llrcrcrcl}
\text{Min} & w = & 8y_1 & + & 6y_2 & - & 3y_3 & & \\
\text{s.t.} & & -y_1 & + & 2y_2 & & & \geq & 0 \\
 & & 4y_1 & & & & & \geq & 5 \\
 & & 2y_1 & + & y_2 & + & y_3 & = & 1 \\
 & & y_1 & & & & & \geq & 0 \\
 & & & & y_2 & & & \in & \mathbb{R} \\
 & & & & & & y_3 & \leq & 0
\end{array}$$

Problem 3

In order to produce 1000 items, the factory needs at least 1 ton of of M1, 0.6 ton of M2, and 0.3 ton of M3.

a) Let x_i be the quantity (in tons) of alloy i that the factory needs to purchase. The primal LP is given by:

$$(PLP)\left\{\begin{array}{llrcrcrcl}
\text{Min} & z = & 3x_1 & + & x_2 & + & 4x_3 & & \\
\text{s.t.} & & x_1 & + & 4x_2 & + & x_3 & \geq & 10 \\
 & & 3x_1 & + & 6x_2 & + & 6x_3 & \geq & 6 \\
 & & 6x_1 & & & + & 3x_3 & \geq & 3 \\
 & & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}\right.$$

Note that each inequality has been multiplied by 10.

b) Dual problem:

$$(DLP)\left\{\begin{array}{llrcrcrcl}
\text{Max} & w = & 10y_1 & + & 6y_2 & + & 3y_3 & & \\
\text{s.t.} & & y_1 & + & 3y_2 & + & 6y_3 & \leq & 3 \\
 & & 4y_1 & + & 6y_2 & & & \leq & 1 \\
 & & y_1 & + & 6y_2 & + & 3y_3 & \leq & 4 \\
 & & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}\right.$$

c) PLP in standard form:

$$\begin{array}{llrcrcrcrcrcrcl}
\text{Max} & z = & -3x_1 & - & x_2 & - & 4x_3 & & & & & & & & \\
\text{s.t.} & & -x_1 & - & 4x_2 & - & x_3 & + & x_4 & & & & & = & -10 \\
 & & -3x_1 & - & 6x_2 & - & 6x_3 & & & + & x_5 & & & = & -6 \\
 & & -6x_1 & & & - & 3x_3 & & & & & + & x_6 & = & -3 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}$$

Slack variable x_{3+i} represents the surplus of metal M_i, $i = 1,2,3$. The initial tableau is dual-feasible but not primal-feasible. Let's apply the dual simplex algorithm (phase II):

$T_0 =$

	x_1	x_2	x_3	x_4	x_5	x_6	z	
	-1	**-4**	-1	1	0	0	0	-10
	-3	-6	-6	0	1	0	0	-6
	-6	0	-3	0	0	1	0	-3
	3	1	4	0	0	0	1	0
ratio	-3	$-1/4$	-4					

$T_1 =$

	x_1	x_2	x_3	x_4	x_5	x_6	z	
	1/4	1	1/4	-1/4	0	0	0	5/2
	-3/2	0	-9/2	-3/2	1	0	0	9
	-6	0	-3	0	0	1	0	-3
	11/4	0	15/4	1/4	0	0	1	-5/2
ratio	$-11/24$		$-5/4$					

$T_2 =$

x_1	x_2	x_3	x_4	x_5	x_6	z	
0	1	1/8	-1/4	0	1/24	0	19/8
0	0	-15/4	-3/2	1	-1/4	0	39/4
1	0	1/2	0	0	-1/6	0	1/2
0	0	19/8	1/4	0	11/24	1	-31/8

Tableau T_2 is optimal. The factory needs to order $x_1 = 1/2$ ton of alloy 1, $x_2 = 19/8$ tons of alloy 2. There is no need of alloy 3. The surplus of metal M2 is $39/4$ tons. The minimal cost is $31/8$.

Problem 4

a) Primal constraints can be written as:

$$\begin{array}{rcrl} x_1 & - & x_2 & \leq 2 \\ x_1 & - & x_2 & \geq 3 \end{array}$$

We conclude that there is no feasible solution.

b) Dual problem:

$$\begin{array}{llrcrcr} \text{Min} & z = & 2y_1 & - & 3y_2 & & \\ \text{s.t.} & & y_1 & - & y_2 & \geq & 3 \\ & & -y_1 & + & y_2 & \geq & -2 \\ & & y_1 & , & y_2 & \geq & 0 \end{array}$$

c) Dual constraints can be written as:

$$\begin{array}{rcrl} y_1 & - & y_2 & \geq 3 \\ y_1 & - & y_2 & \leq 2 \end{array}$$

The dual problem has no solution neither.