The Simplex Algorithm: Phase II

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The Simplex Algortithm in a Tableau

Simplex Algorithm (phase II)

- Maximizing the objective function with pivoting
- Optimal and non-bounded tableaus
- Phase II
- Degeneracy
- Bland's rule
- Performance of the simplex algorithm
- Polynomial algorithms

How to Maximize z with Pivotings ?

Let's consider a tableau $\boldsymbol{T_B}$ associated with a feasible basis $\boldsymbol{B}$:

$$
\boldsymbol{T_B} \overset{P}{=}
\begin{array}{cccc}
\boldsymbol{x_N} & \boldsymbol{x_B} & z & \\
\hline
\multicolumn{1}{|c|}{\boldsymbol{B}^{-1}\boldsymbol{N}} & \multicolumn{1}{c|}{\boldsymbol{I}} & \multicolumn{1}{c||}{0} & \multicolumn{1}{c|}{\beta} \\
\hline
\multicolumn{1}{|c|}{-\gamma_{\boldsymbol{N}}} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c||}{1} & \multicolumn{1}{c|}{\zeta} \\
\hline
\end{array}
$$

The basic solution associated with $\boldsymbol{T_B}$ is

$$\boldsymbol{x_B} = \beta = \boldsymbol{B}^{-1}\boldsymbol{b},\ \boldsymbol{x_N} = \boldsymbol{0} \text{ and } z = \zeta = \boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{b} \text{ (since } \boldsymbol{x_N} = \boldsymbol{0})$$

The last row of the tableau contains the reduced costs $-\gamma_{\boldsymbol{N}}$:

$$-\gamma_{\boldsymbol{N}} = \boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}$$

How to Maximize z with Pivotings ? (Cont'd)

- In basis $\boldsymbol{B}$, the objective function can be written as:

$$z = \zeta + \boldsymbol{\gamma_N x_N} = \zeta + \sum_{k \in \mathcal{N}} \gamma_k x_k = \zeta - \sum_{k \in \mathcal{N}} (-\gamma_k) x_k,$$

  where $\mathcal{N}$ is **the set of non-basic variables**

- **Reduced costs** measure the sensitivity of the objective function to a unit change in non-basic variables (if we assume that current basis keeps optimal)

- if $-\gamma_k < 0$, a unit increase in variable x_k increases the objective function by $\gamma_k > 0$ (if we assume that current basis keeps optimal)

- So when we consider a **maximization** problem, negative reduced costs represent potential increases in terms of the objective function !

How to Maximize z with Pivotings ? (Cont'd)

All the feasible solutions can be obtained by choosing $\boldsymbol{x_N} = \boldsymbol{s} \geq \mathbf{0}$ and by computing:

$$\boldsymbol{x_B} = \beta - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{s},$$

with $\boldsymbol{x_B} \geq 0$ and $\boldsymbol{x_N} = \boldsymbol{s} \geq 0$

Case 1: If $-\gamma_r \geq 0 \;\; \forall\, r \in \mathcal{N}$, any increase of the non-basic variables causes a **decrease in the value of z**. The current basic solution is **optimal**:

$$\boldsymbol{x_B^*} = \beta, \qquad \boldsymbol{x_N^*} = \mathbf{0} \qquad \text{and} \qquad z^* = \zeta$$

How to Maximize z with Pivotings ? (Cont'd)

Case 2: If $\exists\ r \in \mathcal{N}$ such that $-\gamma_r < 0$, any increase of x_r causes an **increase in z**

If all the variables that are not in the basis, except x_r, are kept to 0, then the solution of the system of constraints is

$$\boldsymbol{x_B} = \boldsymbol{\beta} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{s} = \boldsymbol{\beta} - \boldsymbol{\alpha}^{(r)} x_r$$

where $\boldsymbol{s}^T = (0 \ldots x_r \ldots 0), \boldsymbol{\alpha}^{(r)} = \left(\boldsymbol{B}^{-1}\boldsymbol{N}\right)_{\cdot r}$ is the column of $\boldsymbol{T_B}$ associated to x_r

Case 2: $-\gamma_r < 0$

We are in this case :

x_1	$\dots$	x_k	$\dots$	x_{n+m}	z	
		α_{1r}			0	$\oplus$
		$\vdots$			$\vdots$	$\vdots$
		α_{mr}			0	$\oplus$
$*$	$\dots$	$-$	$\dots$	$*$	1	$*$

Case 2: $-\gamma_r < 0$

What is the impact of increasing x_r on the basis variables ?

- If $\alpha_{ir} = \boldsymbol{\alpha}_i^{(r)} \leq 0$, it causes an **increase** (or at least no decrease) of the ith basis variable $x_{\sigma(i)}$ since $\boldsymbol{x_B} = \boldsymbol{\beta} - \boldsymbol{\alpha}^{(r)} x_r$
- If $\alpha_{ir} = \boldsymbol{\alpha}_i^{(r)} > 0$, it causes a **decrease** of $x_{\sigma(i)}$ since $\boldsymbol{x_B} = \boldsymbol{\beta} - \boldsymbol{\alpha}^{(r)} x_r$

Case 2: -$\gamma_r < 0$

Case 2.a: $\boldsymbol{\alpha}^{(r)} \leq \mathbf{0}$ (the vector !), then the problem is **non-bounded** :

$$\boldsymbol{x_B} = \boldsymbol{\beta} - \boldsymbol{\alpha}^{(r)} x_r \geq 0 \qquad \forall\, x_r \geq 0$$

and as $z = \zeta + \boldsymbol{\gamma_N x_N} = \zeta + \sum_{k \in \mathcal{N}} \gamma_k x_k$:

$$\lim_{x_r \to \infty} z = \zeta + \lim_{x_r \to \infty} \gamma_r x_r = \infty$$

A tableau where $\exists\, r \in \mathcal{N}$ such that $-\gamma_r < 0$ and $\boldsymbol{\alpha}^{(r)} \leq \mathbf{0}$ is called **non-bounded** or **unbounded**

Case 2.b: -$\gamma_r < 0$ and $\alpha_{ir} > 0$

Case 2.b: it exists i such that $\alpha_{ir} > 0$. In this case:

$$x_{\sigma(i)} = \beta_i - \alpha_{ir} x_r = 0 \iff x_r = \beta_i / \alpha_{ir}, \qquad i \in \mathcal{B}$$

- By computing the ratios β_i / α_{ir} for each $i \in \mathcal{B}$ for which $\alpha_{ir} > 0$ and if we select the **smallest one**, then all the variables in $\boldsymbol{B}$ keep **non - negative**
- Let j be the index of the row having the smallest ratio. The solution obtained by fixing $x_r = \beta_j / \alpha_{jr}$ corresponds to the new basic solution. The variable x_r replaces $x_{\sigma(j)}$ in the new basis $\boldsymbol{B}'$

Case 2.b: -$\gamma_r < 0$ and $\alpha_{ir} > 0$ (Cont'd)

- We move from tableau $\boldsymbol{T_B}$ to tableau $\boldsymbol{T_{B'}}$ by pivoting around α_{jr}
- The new basis $\boldsymbol{B}'$ is feasible and the value of its basic solution is

$$\zeta' = \zeta + \gamma_r x_r = \zeta + \gamma_r \frac{\beta_j}{\alpha_{jr}} \geq \zeta$$

  since $\gamma_r > 0$, $\alpha_{jr} > 0$ and $\beta_j \geq 0$

- This just means that we have **improved** the current basic solution by performing a pivoting with these rules !
- These developments are the foundations of the simplex algorithm

Summary: Representations of $\boldsymbol{T_B}$

Let $\boldsymbol{B}$ be a feasible basis and $\boldsymbol{T_B}$ its associated tableau. There are two representations for $\boldsymbol{T_B}$:

$$\boldsymbol{T_B} = \begin{array}{c|c|c||c} \boldsymbol{x_D} & \boldsymbol{x_E} & z & \\ \hline \boldsymbol{B}^{-1}\boldsymbol{A} & \boldsymbol{B}^{-1} & \boldsymbol{0} & \beta \\ \hline -\gamma_{\boldsymbol{D}} & -\gamma_{\boldsymbol{E}} & 1 & \zeta \end{array} \qquad \beta = \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

$$\boldsymbol{T_B} \overset{P}{=} \begin{array}{c|c|c||c} \boldsymbol{x_N} & \boldsymbol{x_B} & z & \\ \hline \boldsymbol{B}^{-1}\boldsymbol{N} & \boldsymbol{I} & \boldsymbol{0} & \beta \\ \hline -\gamma_{\boldsymbol{N}} & \boldsymbol{0} & 1 & \zeta \end{array} \qquad \beta = \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

Summary: Optimal Tableau Signature

Lemma

If $-\gamma_r \geq 0 \ \forall\, r \in \mathcal{N}$*, then the tableau is* ***optimal***

Signature of an **optimal** tableau:

$$
\begin{array}{cccc|c||c}
x_1 & \dots & \dots & x_{n+m} & z & \\
\hline
 & & & & 0 & \oplus \\
 & & & & \vdots & \vdots \\
 & & & & 0 & \oplus \\
\hline
\oplus & \dots & \dots & \oplus & 1 & *
\end{array}
$$

Summary: Unbounded Tableau Signature

Lemma

If $\exists\, r \in \mathcal{N}$ so that $-\gamma_r < 0$ and $\boldsymbol{\alpha}^{(r)} \leq \mathbf{0}$, then the tableau is unbounded

Signature of an unbounded tableau:

$$
\begin{array}{ccccc|c||c}
x_1 & \dots & x_k & \dots & x_{n+m} & z & \\
\hline
 & & \ominus & & & 0 & \oplus \\
 & & \ominus & & & \vdots & \vdots \\
 & & \ominus & & & 0 & \oplus \\
\hline
* & \dots & - & \dots & * & 1 & * \\
\hline
\end{array}
$$

Summary: Choosing the Next Pivot

If the conditions from the two previous lemmas are not satisfied, then one can perform a new pivoting resulting in a new feasible tableau $\boldsymbol{T_{B'}}$ where the value of the **new basic solution** is **equal or larger** than the one associated to $\boldsymbol{T_B}$

The new pivot α_{jr} is chosen

1. in a column $r \in \mathcal{N}$ such that $-\gamma_r < 0$,
2. in a row j such that $\frac{\beta_j}{\alpha_{jr}} = \min\left\{\frac{\beta_i}{\alpha_{ir}} \mid \alpha_{ir} > 0\right\}$

The value of the new basic solution is:

$$\zeta' = \zeta + \gamma_r \frac{\beta_j}{\alpha_{jr}} \geq \zeta$$

Primal Simplex Algorithm - Phase II

Input: a **feasible** tableau **Output**: an **optimal** or an **unbounded** tableau

(1) **Entering** column: choose a non-basis column r with a negative reduced cost:

$$r \in \{k \in \mathcal{N} \mid -\gamma_k < 0\}.$$

It this column does not exist, then STOP: the current tableau is optimal

(2) **Exiting** column: choose a row j minimizing the ratio

$$j \in \left\{k \in \{1, \ldots, m\} \mid \frac{\beta_k}{\alpha_{kr}} = \min\left\{\frac{\beta_i}{\alpha_{ir}} \mid \alpha_{ir} > 0\right\}\right\}$$

If this row doesn't exist, then STOP: the current tableau is non-bounded

(3) Update the basis and the tableau: pivot around α_{jr} and go back to (1)

Example: Resolution with the Simplex Algorihm

We consider the following LP:

$$
\begin{array}{lrcrcrcr}
\text{Max} & z = & 800x_1 & + & 300x_2 & & \\
\text{s.t.} & & 2x_1 & + & x_2 & \leq & 400 \\
 & & x_1 & & & \leq & 150 \\
 & & & & x_2 & \leq & 200 \\
 & & x_1 & , & x_2 & \geq & 0
\end{array}
$$

Example - Tableau T_0

x_1	x_2	x_3	x_4	x_5	z		ratio	
2	1	1	0	0	0	400	200	
1	0	0	1	0	0	150	150	$\leftarrow$
0	1	0	0	1	0	200	-	
-800	-300	0	0	0	1	0		
$\uparrow$								

Example - Tableau T_1

x_1	x_2	x_3	x_4	x_5	z		ratio	
0	**1**	1	-2	0	0	100	100	$\leftarrow$
1	0	0	1	0	0	150	-	
0	1	0	0	1	0	200	200	
0	-300	0	800	0	1	120000		
	$\uparrow$							

Example - Tableau T_2

x_1	x_2	x_3	x_4	x_5	z	
0	1	1	-2	0	0	100
1	0	0	1	0	0	150
0	0	-1	2	1	0	100
0	0	300	200	0	1	150000

This tableau is **optimal**. The optimal solution is given by $x_1^* = 150$, $x_2^* = 100$. The optimal value is $z^* = 150000$

Does this Algorithm Work ?

- A sufficient condition would be to demonstrate that the sequence of values associated to basic solutions is **strictly** increasing. Following a pivoting, the new value of the objective function is

$$\zeta' = \zeta + \gamma_r \frac{\beta_j}{\alpha_{jr}}$$

- $\gamma_r > 0$ ($\iff -\gamma_r < 0$) due to the choice of column r , $\alpha_{jr} > 0$ due to the choice of row j and $\beta_j \geq 0$ since the tableau is feasible
- If $\beta_j > 0$, we have that $\zeta' > \zeta$ and the **new** basis has never been **visited** yet
- If $\beta_j = 0$, the pivoting is **stationary** (meaning that $\zeta' = \zeta$) and after several stationary pivotings, **it is possible to visit a basis more than once**
- In the latter situation, the algorithm may **cycle** and never produce the optimal solution

Degeneracy

- A tableau and its associated basic solution are called **degenerated** if **at least** one of the components of vector β is null
- A standard LP is called **degenerated** if it has at least one **degenerated** basic solution
- Degeneracy occurs if, in the space of decision variables (i.e. $\mathbb{R}^n$), it exists more than n hyperplans defined by the the constraints (including bound constraints) **intersecting at the same point**
- It may create cycles in algorithms using pivotings

Bland's Rule

To avoid **cycling** and to be sure that the algorithm stops after a **finite** number of iterations, Bland has defined a very simple rule:

Rule

When several candidates may **enter** or **exit** a basis, **always** choose the variable x_r with the **smallest index**

We can apply this rule to **degenerated** tableaus only

Example: Bland's Rule

x_1	x_2	x_3	x_4	x_5	x_6	z	
1	1	1	-1	0	0	0	3
0	**1**	0	-1	1	0	0	1
0	1	-1	0	0	1	0	1
0	-1	-1	1	0	0	1	-3

- Two candidates for the **entering** variable: x_2 and x_3 (reduced costs are < 0)
- We take the one with the **smallest** index: x_2
- Then two candidates for the pivots (both have a ratio of 1)
  - If we choose α_{22}, then x_5 will leave the basis
  - If we choose α_{32}, then x_6 will leave the basis
- As x_5 has a smaller index that x_6, then we choose α_{22} as the **next pivot**

Theoretical Performance of the Simplex Algorithm

Complexity Theory

One of the key issue in **complexity theory** is to determine if, for a specific problem, there exists an algorithm whose **execution time** is bounded by a **polynomial** in the size of the instance of the problem we consider

If such an algorithm exists, it is said to be **polynomial** and the problem can generally be solved in an **effective** way even for instances of important size

Theoretical Performance of the Simplex Algorithm

Question

Is the simplex algorithm polynomial and, if it is not the case, does it exist polynomial algorithms for linear programming ?

Klee-Minty Cube (1972)

- In 1972, Klee and Minty showed that after perturbating the corner of an hypercube C:

$$C = \{\boldsymbol{x} \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, i = 1, \dots, n\},$$

  the simplex algorithm visited successively all the vertices of the feasible region before reaching the optimum

- The number of necessary pivoting is $2^n - 1$ in that case !

- The simplex algorithm is **exponential** time in the **worst** case !

Empirical Performance of the Simplex Algorithm

- However, in many situations, the simplex method is remarkably **efficient** in practice
- It is the reason why many numerical packages include the simplex algorithm

Polynomial Algorithms for Linear Programming

- The ellipsoid algorithm from Khachiyan. He was the first researcher to prove the **polynomial-time** solvability of linear programs. But this algorithm converges very slowly and is rarely used in practice

- Karmarkar's algorithm falls within the class of interior point methods: the current guess for the solution does not follow the **boundary of the feasible set** as in the simplex method, but it moves through the **interior of the feasible region**, improving the approximation of the optimal solution by a definite fraction with every iteration, and converging to an optimal solution

Karmarkar's Algorithm