Exercise Set 9

Problem 1

Compute the eigenvalues and the eigenvectors of the following matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Reminders about eigenvalues and eigenvectors

We consider a $n \times n$ square matrix $\mathbf{A}$. A scalar λ is called an eigenvalue of $\mathbf{A}$ if there exists a non-zero vector $\mathbf{v}$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Any vector satisfying this relation is called an eigenvector of $\mathbf{A}$ belonging to the eigenvalue λ. To determine the eigenvectors, we first need to find all the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $\mathbf{A}$. These values are the zeros of the characteristic polynomial of $\mathbf{A}$ which is defined by $det(\mathbf{A} - \lambda\mathbf{I}) = 0$ where $\mathbf{I}$ is the $n \times n$ identity matrix. Then, for each of the eigenvalues λ_i, we weed to solve the linear system given $\mathbf{A}\mathbf{v} = \lambda_i\mathbf{v}$ to determine its eigenvectors.

Problem 2

Among the following functions, which ones are convex? Which ones are concave? Justify your answer. We remind that a function f is concave if $-f$ is convex, i.e. for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ and for any $\lambda \in [0,1]$:

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}).$$

a) $f : \mathbb{R} \mapsto \mathbb{R} : f(x) = 1 - x^2$

b) $f : \mathbb{R} \mapsto \mathbb{R} : f(x) = x^2 - 1$

c) $f : \mathbb{R}^2 \mapsto \mathbb{R} : f(x,y) = \sqrt{x^2 + y^2}$. For this case, no computations are necessary. Just note that f corresponds to the euclidean distance $\|\cdot\|_2$ and use the following results $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$ and $\|\lambda\mathbf{a}\| = |\lambda| \, \|\mathbf{a}\|$ for any $\mathbf{a},\mathbf{b} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$

d) $f : \mathbb{R} \mapsto \mathbb{R} : f(x) = x^3$

Problem 3

Show that the real symmetric matrix

$$M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite.

Problem 4

We consider the two following functions:

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad\qquad g : \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \mapsto f(x,y) = x^2 + y^2 \qquad\qquad (x,y) \mapsto g(x,y) = \frac{1}{3}x^3 + y^3 - x - y$$

a) Compute the gradient of f and g for all $\mathbf{x} \in \mathbb{R}^2$.
b) Compute the hessian of f and g for all $\mathbf{x} \in \mathbb{R}^2$. For which values of $\mathbf{x} \in \mathbb{R}^2$ are these matrices positive definite? What are your conclusions?
c) How many critical points have these fonctions? For each of them, determine if it is a local maximum, a local minimum or a saddle point.

Hint: a critical point whose hessian is indefinite (not positive semi-definite, nor negative semi-definite) is a saddle point.

Problem 5

We consider the following function:

$$f(x,y) = 5x^2 + 5y^2 - xy - 11x + 11y + 11$$

(a) Rewrite this function as follows:
$$f(\mathbf{z}) = \frac{1}{2}\mathbf{z}^T\mathbf{Q}\mathbf{z} + \mathbf{b}^T\mathbf{z} + c$$
where $\mathbf{z} = (x\ y)^T$, $\mathbf{Q} \in \mathbb{R}^{2\times 2}$, $\mathbf{b} \in \mathbb{R}^2$ et $c \in \mathbb{R}$.

(b) Find the unique minimum of f over $\mathbb{R}^2$.

Hint: a symmetric strictly diagonally dominant matrix $\mathbf{A}$ with real non-negative diagonal entries is positive definite. A square matrix $\mathbf{A}$ is said strictly diagonally dominant if $|A_{ii}| > \sum_{j\neq i} |A_{ij}| \quad \forall i$.

Problem 6

We consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x,y) = xe^y + ye^x$$

Does this function has a local minimum over $\mathbb{R}^2$?

Hint: if a square symmetric matrix has a negative determinant, then it cannot be positive semi-definite (its determinant is the product of its eigenvalues)