Linear Programming

Optimization Methods in Management Science
Master in Management
HEC Lausanne

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Canonical and Standard Forms for Linear Programming

- Canonical and standard forms
- Matrix notation
- Transformation rules
- Examples

General Form of a Linear Program

$$
\begin{array}{llrcll}
\text{Opt} & z = & \sum_{j=1}^{n} c_j x_j & & & \\
\text{s.t.} & & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & i \in I \subseteq \{1,\ldots,m\} \\
& & \sum_{j=1}^{n} a_{kj} x_j & \geq & b_k & k \in K \subseteq \{1,\ldots,m\} \\
& & \sum_{j=1}^{n} a_{rj} x_j & = & b_r & r \in R \subseteq \{1,\ldots,m\} \\
& & & l_j \leq x_j \leq u_j & & j = 1,\ldots,n
\end{array}
$$

- Opt = Max or Min,
- I, K et R are disjoint sets and $I \cup K \cup R = \{1,\ldots,m\}$,
- variables may be unbounded: $l_j = -\infty$ and/or $u_j = +\infty$

Canonical Form of a Linear Program

$$
\begin{array}{lrcll}
\text{Max} \quad z = & \displaystyle\sum_{j=1}^{n} c_j x_j & & & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & i = 1, \ldots, m \\
& x_j & \geq & 0 & j = 1, \ldots, n
\end{array}
$$

- Maximization problem
- All constraints are of type $\leq$
- All variables are non-negative

Standard Form of a Linear Program

$$
\begin{array}{llcll}
\text{Max} \quad z = & \displaystyle\sum_{j=1}^{n} c_j x_j & & & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j & = & b_i & i = 1, \ldots, m \\
 & x_j & \geq & 0 & j = 1, \ldots, n
\end{array}
$$

- Maximization problem
- All constraints are equalities
- All variables are non-negative

From Canonical to Standard Form

Addition of slack variables x_{n+i}

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad \rightarrow \qquad \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i$$

$$\begin{array}{llcccll}
\text{Max} & z = \sum_{j=1}^{n} c_j x_j & + & \sum_{i=1}^{m} 0 x_{n+i} & & & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij}x_j & + & x_{n+i} & = & b_i & i = 1, \dots, m \\
 & & & x_j & \geq & 0 & j = 1, \dots, n+m
\end{array}$$

Why Having Particular Forms?

- Check some prerequisites before applying algorithms
- Simplify the presentation of the algorithms

Important Remark

In this course, the reference form is the **canonical one** whose variables are called **decision variables** of the problem. They are denoted by $\boldsymbol{x_D}$. The standard form will always be obtained by adding some **slack variables** $\boldsymbol{x_E}$ to the canonical problem.

Matrix Notation

- A vector can be a row-vector or a column-vector. Examples:

$$\boldsymbol{c} = \left(c_1 \ \dots \ c_n\right), \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

  $\boldsymbol{c}$ is a row-vector and $\boldsymbol{x}$ is a column-vector

- The scalar product bewteen $\boldsymbol{c}$ and $\boldsymbol{x}$ is defined by $\sum_{i=1}^{n} c_i x_i$ and is simply denoted $\boldsymbol{cx}$ (matrix product)

- Note that, in general, the scalar product between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $\boldsymbol{x}^T\boldsymbol{y}$

Matrix Form of a Canonical PL

$$\begin{array}{llrcl} \text{Max} & z = & \boldsymbol{cx} & & \\ \text{s.t.} & & \boldsymbol{Ax} & \leq & \boldsymbol{b} \\ & & \boldsymbol{x} & \geq & 0 \end{array} \qquad \left[\begin{array}{llrcl} \text{Max} & z = & \boldsymbol{c_D x_D} & & \\ \text{s.t.} & & \boldsymbol{Ax_D} & \leq & \boldsymbol{b} \\ & & \boldsymbol{x_D} & \geq & 0 \end{array} \right]$$

where

$$\boldsymbol{c} = \boldsymbol{c_D} = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}, \qquad \boldsymbol{x} = \boldsymbol{x_D} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

Matrix Form of a Standard PL

$$\begin{array}{lrcl} \text{Max} & z = \mathbf{cx} & & \\ \text{s.t.} & [\mathbf{A} \mid \mathbf{I}]\,\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array} \quad \left[\begin{array}{lrcrcl} \text{Max} & z = \mathbf{c_D x_D} & + & \mathbf{0 x_E} & & \\ \text{s.t.} & \mathbf{A x_D} & + & \mathbf{I x_E} & = & \mathbf{b} \\ & \mathbf{x_D} & , & \mathbf{x_E} & \geq & \mathbf{0} \end{array} \right]$$

where

$$\mathbf{c} = (\mathbf{c_D} \mid \mathbf{c_E}) = (\mathbf{c_D} \mid \mathbf{0}) = (c_1 \ \ldots \ c_n \mid 0 \ \ldots \ 0)$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{x_D} \\ - \\ \mathbf{x_E} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ - \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Matrix Form: Example

Starting from:

$$
\begin{array}{lrcrcrcr}
\text{Max} & z = & 250x_1 & + & 450x_2 & & \\
\text{s.t.} & & 2x_1 & + & 3x_2 & \leq & 42 \\
 & & -4x_1 & + & 6x_2 & \leq & 0 \\
 & & x_1 & & & \leq & 15 \\
 & & x_1 & , & x_2 & \geq & 0
\end{array}
$$

we get that:

$$
\boldsymbol{c} = \begin{pmatrix} 250 & 450 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
$$

$$
\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ -4 & 6 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 42 \\ 0 \\ 15 \end{pmatrix}.
$$

Some Transformation Rules

- **Minimization $\leftrightarrow$ maximization:**

$$\min f(\boldsymbol{x}) = -\max(-f(\boldsymbol{x}))$$

In order to min $z = \boldsymbol{cx}$, we just need to max $w = -\boldsymbol{cx} = (-\boldsymbol{c})\boldsymbol{x}$ and to multiply the optimal value of w by -1 to get the optimal value of z

- **Inequation "$\geq$" $\leftrightarrow$ inequation "$\leq$" :**

$$\boldsymbol{ax} \geq b \iff (-\boldsymbol{a})\boldsymbol{x} \leq -b$$

- **Equation $\rightarrow$ inequation "$\leq$" :**

$$\boldsymbol{ax} = b \iff \left\{ \begin{array}{ccc} \boldsymbol{ax} & \leq & b \\ \boldsymbol{ax} & \geq & b \end{array} \right. \iff \left\{ \begin{array}{rcr} \boldsymbol{ax} & \leq & b \\ (-\boldsymbol{a})\boldsymbol{x} & \leq & -b \end{array} \right.$$

Some Transformation Rules (Cont'd)

- **Inequation $\rightarrow$ equation** : we add a **slack variable**:

$$\begin{aligned} \boldsymbol{ax} \leq b \quad &\Longleftrightarrow \quad \boldsymbol{ax} + s = b, s \geq 0 \\ \boldsymbol{ax} \geq b \quad &\Longleftrightarrow \quad -\boldsymbol{ax} + s = -b, s \geq 0 \end{aligned}$$

- **Real variable $\rightarrow$ non-negative variable:** every real number can be written as the difference of two non-negative numbers

$$x \in \mathbb{R} \rightarrow \begin{cases} x = x^{+} - x^{-} \\ x^{+}, x^{-} \geq 0 \end{cases}$$

- **Variable with a lower bound:**

$$x \geq b \Longleftrightarrow \begin{cases} x' = x - b \\ x' \geq 0 \end{cases}$$

Transformation into Canonical Form

$$\begin{array}{llrcrcr}
\text{Min} & z = & -3x_1 & + & 4x_2 & & \\
\text{s.t.} & & x_1 & + & x_2 & = & 6 \\
& & x_1 & - & 2x_2 & \geq & 4 \\
& & x_1 & & & \in & \mathbb{R} \\
& & & & x_2 & \geq & 0
\end{array}$$

Initial LP

Transformations

$$\begin{array}{lcl}
\text{Min } z = -3x_1 + 4x_2 & \rightarrow & \text{Max } w = 3x_1 - 4x_2 \\
x_1 + x_2 = 6 & \rightarrow & \left\{ \begin{array}{rcr} x_1 + x_2 & \leq & 6 \\ -x_1 - x_2 & \leq & -6 \end{array} \right. \\
x_1 - 2x_2 \geq 4 & \rightarrow & -x_1 + 2x_2 \leq -4 \\
x_1 \in \mathbb{R} & \rightarrow & \left\{ \begin{array}{l} x_1 = x_1^+ - x_1^- \\ x_1^+, x_1^- \geq 0 \end{array} \right.
\end{array}$$

Transformation into Canonical Form

Initial LP

$$
\begin{array}{llrcrcr}
\text{Min} & z = & -3x_1 & + & 4x_2 & & \\
\text{s.t.} & & x_1 & + & x_2 & = & 6 \\
 & & x_1 & - & 2x_2 & \geq & 4 \\
 & & x_1 & & & \in & \mathbb{R} \\
 & & & & x_2 & \geq & 0
\end{array}
$$

Equivalent canonical LP

$$
\begin{array}{llrcrcrcr}
\text{Max} & w = & 3x_1^+ & - & 3x_1^- & - & 4x_2 & & \\
\text{s.t.} & & x_1^+ & - & x_1^- & + & x_2 & \leq & 6 \\
 & & -x_1^+ & + & x_1^- & - & x_2 & \leq & -6 \\
 & & -x_1^+ & + & x_1^- & + & 2x_2 & \leq & -4 \\
 & & x_1^+ & , & x_1^- & , & x_2 & \geq & 0
\end{array}
$$

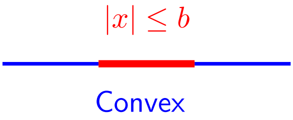
Don't forget that $z_{opt} = -w_{opt}$!

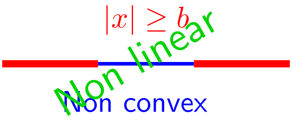
Particular Transformations

- **Min-max or max-min problem:**

$$\text{Min} \quad z = \max\{\boldsymbol{c_1 x}, \dots, \boldsymbol{c_k x}\} \quad \Longleftrightarrow \quad \begin{array}{llll} \text{Min} & z = & t & \\ \text{s.t.} & & t & \geq \boldsymbol{c_1 x} \\ & & & \dots \\ & & t & \geq \boldsymbol{c_k x} \\ & & t & \in \mathbb{R} \end{array}$$

- **Absolute value** (with b > 0): $|x| \leq b \Longleftrightarrow \begin{cases} x \leq b \\ x \geq -b \end{cases}$





What Can We Do With a Constraint of Type $|x| \geq b$

- This is not a linear constraint !
- We have to decompose the orginal problem into two sub-problems :
  - Replace $|x| \geq b$ with $x \leq -b$ in the first one
  - Replace $|x| \geq b$ with $x \geq b$ in the second one
- The final solution is given by the best solution of these two sub-problems

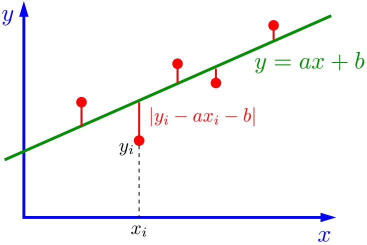
Example: Chebychev Approximation

Problem

Determine a linear approximation $y = \boldsymbol{ax} + b$ minimizing the largest estimation error

Data: m measurements:

$$(\boldsymbol{x}_i, y_i) \in \mathbb{R}^{n+1}, i = 1, \ldots, m$$



Formulation:

$$\text{Min } z = \max_{i=1,\ldots,m} \left\{ \, |y_i - \boldsymbol{ax}_i - b| \, \right\}$$

Decision variables are $\boldsymbol{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$

Chebychev Approximation (Cont'd)

We can rewrite the problem as

$$\begin{array}{llcl} \text{Min} & z = & \displaystyle\max_{i=1,\dots,m} \{\Delta_i\} & \\ \text{s.t.} & & \Delta_i \;=\; |y_i - \boldsymbol{ax}_i - b| & i = 1, \dots, m \end{array}$$

which is equivalent to:

$$\begin{array}{llcl} \text{Min} & z = & t & \\ \text{s.t.} & & t \;\geq\; |y_i - \boldsymbol{ax}_i - b| & i = 1, \dots, m \end{array}$$

with $t \geq 0$

Chebychev Approximation (Cont'd)

Final formulation (not in canonical nor in standard form) :

$$\begin{array}{llrcll}
\text{Min} & z = & t & & & \\
\text{s.t.} & & t & \geq & y_i - \boldsymbol{a}\boldsymbol{x}_i - b & i = 1, \ldots, m \\
 & & t & \geq & -y_i + \boldsymbol{a}\boldsymbol{x}_i + b & i = 1, \ldots, m
\end{array}$$

with $\boldsymbol{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$ and $t \geq 0$