Introduction to Linear Programming

Optimization Methods in Management Science
Master in Management
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Table of Contents

Definition of Linear Programmig

- A linear program is an optimization problem consisting in maximizing (or minimizing) a linear objective function of n **real** variables subject to a set of constraints expressed as linear equations or linear inequalities

- The term “Linear Programming” is due to G. B. Dantzig, who is considered as the father of the simplex algorithm

Formulation

A linear problem with m constraints is given by :

$$
\begin{array}{llcll}
\text{Max (Min)} \quad z = & \sum_{j=1}^{n} c_j x_i & & & \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & i = 1, \ldots, m_1 \\
 & \sum_{j=1}^{n} a_{kj} x_j & \geq & b_k & k = m_1 + 1, \ldots, m_2 \\
 & \sum_{j=1}^{n} a_{rj} x_j & = & b_r & r = m_2 + 1, \ldots, m
\end{array}
$$

The abreviation s.t. means subjet to.

Linear Programming: an Example

A company operates two canning plants A and B. The growers G_1, G_2, G_3 are willing to supply cereals in the following amounts:

- G_1: 200 tonnes at \$11 per tonne
- G_2: 310 tonnes at \$10 per tonne
- G_3: 420 tonnes at \$9 per tonne

Shipping costs in \$ per tonne are:

	To Plant A	To Plant B
G_1	3	3.5
G_2	2	2.5
G_3	6	4

Linear Programming: an Example

Plant capacities and labour costs are:

	Plant A	Plant B
Capacity	460 tonnes	560 tonnes
Labor cost	$26 per tonne	$21 per tonne

After processing, cereals are sold at $50 per tonne to the distributors. The company can sell at this price all they can produce.

Problem

The objective is to find the best mixture of the quantities supplied by the three growers to the two plants so that the company maximizes its profit.

Linear Programming: Problem Formulation

- **Variables**: the quantity to supply from each of the three growers to each of the two canning plants. Let x_{ij} be the number of tonnes supplied from grower i to plant j where $x_{ij} \geq 0, i = 1, 2, 3; j = 1, 2$.
- **Objective function**:

$$\max \quad \sum_{i,j} 50x_{ij} - 11(x_{11} + x_{12}) - 10(x_{21} + x_{22}) - 9(x_{31} + x_{32}) - 3x_{11}$$
$$- 2x_{21} - 6x_{31} - 3.5x_{12} - 2.5x_{22} - 4x_{32} - \sum_{i} 26x_{i1} - \sum_{i} 21x_{i2}$$

- **Grower supply constraints**:

$$\begin{aligned} x_{11} + x_{12} &\leq 200 \\ x_{21} + x_{22} &\leq 310 \\ x_{31} + x_{32} &\leq 420 \end{aligned}$$

- **Plant capacity constraints**:

$$\begin{aligned} x_{11} + x_{21} + x_{31} &\leq 460 \\ x_{12} + x_{22} + x_{32} &\leq 560 \end{aligned}$$

Terminology

- Variables $x_1, \ldots, x_n$ are called the decision variables of the problem
- The linear function to optimize is called the objective function
- Constraints can be linear **equations** or linear **inequalities**
- Constraints of type

$$l_j \leq x_j \leq u_j \qquad l_j, u_j \in \mathbb{R} \cup \{\pm\infty\}$$

are called constraint bounds. They are generally treated in a special way by the algorithms. In many cases, constraint bounds are just expressed as non-negativity constraints $x_j \geq 0$

The Fundamental Assumptions of Linear Programming

(1) **Linearity:** the impact of decision variables is linear in constraints and in objective function

(2) **Divisibility:** non-integer values of decision variables are acceptable

(3) **Certainty:** values of parameters are known and constant

Applications of Linear Programming

- Production management
- Logistics
- Inventory management
- Transportation
- ...

In some applications, the number of variables may be very high (several million) but there also exists very efficient linear programming packages able to solve them !

Definitions

- A solution is **feasible** if it satisfies all the constraints of the problem (including bound constraints)
- The value of the solution is the value of the objective function evaluated at that point
- The **feasible region** corresponds to the set of all the feasible solutions of the problem

Geometry of the Constraints

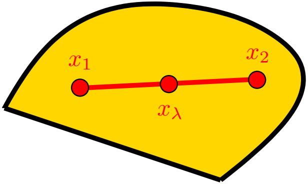
- The set of solutions of a linear inequality corresponds to a **half-space** in $\mathbb{R}^n$ (a half-plane in $\mathbb{R}^2$)
- The set of solution of a linear equation corresponds to an **hyperplan** in $\mathbb{R}^n$ (a straight line in $\mathbb{R}^2$)
- The set of solutions of a system of equations and inequalities (all linear) correspond to the intersection of half-spaces and hyperplans defined by each element of the system
- This intersection is the feasible region. It is a **convex** set and defines a **polyhedron** in $\mathbb{R}^n$

Reminders About Convex Sets

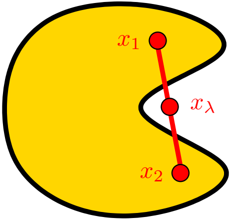
- A set $C \subseteq \mathbb{R}^n$ is **convex** if for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in C$

$$\boldsymbol{x}_\lambda = \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2 \in C$$

  $\forall \lambda \in [0, 1]$.
- Consequently, a set is convex if and only if every convex combination of its elements belongs to the set itself.



Convex



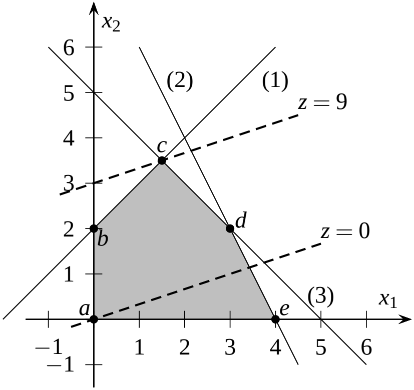
Non convex

Linear Program with Two Decision Variables

- We describe here how to solve a linear program with **two** decision variables, using the so-called **graphical method**

- To illustrate it, let's consider the following example:

$$\begin{array}{llrl} \text{Max} & z & = & -x_1 + 3x_2 \\ \text{s.t.} & & (1) & -x_1 + x_2 \leq 2 \\ & & (2) & 2x_1 + x_2 \leq 8 \\ & & (3) & x_1 + x_2 \leq 5 \\ & & & x_1, x_2 \geq 0 \end{array}$$



- The grey area corresponds to the feasible region

The Graphical Method (1)

- A coutour line is a curve in two dimensions on which the value of a function is a constant
- Let $z = f(x_1, x_2) = a_1 x_1 + a_2 x_2$, then its **gradient** is the vector given by
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$
- In the plane, the gradient is **orthogonal** to its contour line

The Graphical Method (2)

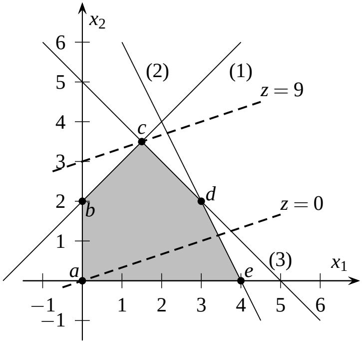
- Contour lines of a **linear** function are **parallel straight lines** in $\mathbb{R}^2$
- There exists a feasible solution of value z if its contour line intersects the feasible region D of the problem
- These points (at least one point) correspond to the **optimal** solution of the LP

The Graphical Method

To determine the **optimal** solution(s), you need to **move as far as possible** a contour line of the objective function in the direction of the gradient if it is a maximization problem (the opposite direction if it is a minimization problem) until it reaches the edge of the set D. This **intersection** corresponds to the **optimal** solution(s) of the problem.

Graphical Resolution in the Plane: Example Cont'd

$$\begin{array}{llrl} \text{Max} & z & = & -x_1 + 3x_2 \\ \text{s.t.} & & (1) & -x_1 + x_2 \leq 2 \\ & & (2) & 2x_1 + x_2 \leq 8 \\ & & (3) & x_1 + x_2 \leq 5 \\ & & & x_1, x_2 \geq 0 \end{array}$$

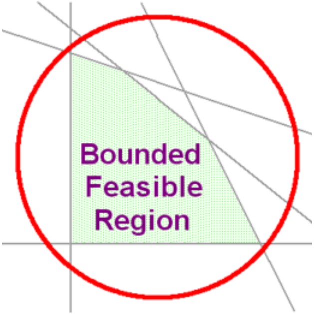


- The contour line through the origin is given by $z = -x_1 + 3x_2 = 0$
- The gradient of the objective function is the vector $(-1\ 3)^T$. This vector is perpendicular to the line given by $-x_1 + 3x_2 = 0$
- By moving this contour line into that direction, we get that the optimal solution is given by the intersection of (1) and (3)
- The optimal solution is $(1.5, 3.5)$ and its value is 9

Feasible Region in the Plane (1)

The feasible region of a LP can be (3 possibilities):

1. **Empty**: it means that the problem has no feasible solution and consequently no optimal solution
2. **Bounded**. A bounded feasible region may be enclosed in a circle. It will have both a maximum value and a minimum value for the objective function



Feasible Region in the Plane (2)

3. **Unbounded**. An unbounded feasible region cannot be enclosed in a circle, no matter how big the circle is. If the coefficients on the objective function are all positive, then an unbounded feasible region will have a minimum but no maximum. In the last case, we say that the LP has **no (finite) optimal solution and is unbounded**