Duality in Linear Programming

Optimization Methods in Management Science
Master in Management
HEC Lausanne

Dr. Rodrigue Oeuvray

Fall 2019 Semester

Duality in Linear Programming

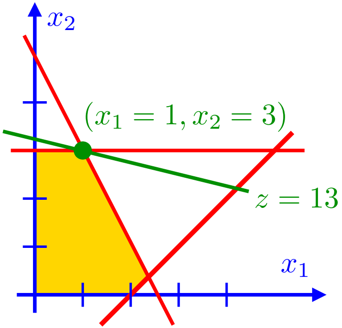
Duality:

- Motivation
- Dual problem of a canonical LP
- Dualization rules
- Weak and strong duality
- Complementary slack theorem
- Economic interpretation of the dual variabes at the optimum

Motivation

Let's consider the following LP:

$$\begin{array}{llrcrcl}
\text{Max} & z = & x_1 & + & 4x_2 & & \\
\text{s.t.} & & x_1 & - & x_2 & \leq & 2 \\
& & 2x_1 & + & x_2 & \leq & 5 \\
& & & & x_2 & \leq & 3 \\
& & x_1 & , & x_2 & \geq & 0
\end{array}$$



with a feasible solution: $x_1 = 1, x_2 = 3$ and $z = 13$

How to find a bound w to this LP such that $z \leq w$ for all the feasible solutions ?

Reminders About Inequalities

- Every conical combination (non-negative coefficients) of inequalities of the same type still provides a valid inequation
- Every linear combination of equalities still provides a valid equation
- It is possible to combine inequations with equations to obtain a valid inequation

Reminders About Inequalities: Examples

Example 1:

$$
\begin{array}{rcrcrcr}
2x_1 & + & x_2 & \leq & 3 & \times & 3 \\
5x_1 & - & 3x_2 & \leq & 5 & \times & 1 \\
\hline
11x_1 & & & \leq & 14 & &
\end{array}
$$

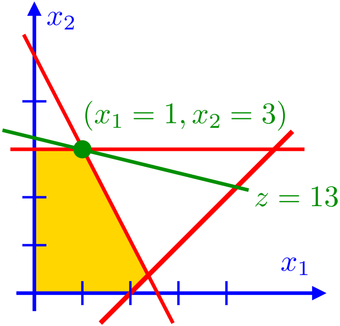
Example 2:

$$
\begin{array}{rcrcrcr}
x_1 & + & 2x_2 & \leq & 3 & \times & 4 \\
3x_1 & - & x_2 & = & 5 & \times & -1 \\
x_1 & + & x_2 & \geq & 2 & \times & -2 \\
\hline
-x_1 & + & 7x_2 & \leq & 3 & &
\end{array}
$$

How to Find a Bound to a LP ?

Same example has before :

$$
\begin{array}{llrcrcl}
\text{Max} & z = & x_1 & + & 4x_2 & & \\
\text{s.t.} & & x_1 & - & x_2 & \leq & 2 \\
 & & 2x_1 & + & x_2 & \leq & 5 \\
 & & & & x_2 & \leq & 3 \\
 & & x_1 & , & x_2 & \geq & 0
\end{array}
$$



with a feasible solution: $x_1 = 1, x_2 = 3$ and $z = 13$

How to Find a Bound to a LP (Cont'd) ?

Let's multiply by 4 the second constraint ($2x_1 + x_2 \leq 5$):

$$8x_1 + 4x_2 \leq 20$$

For every pair (x_1, x_2) with **non-negative** values:

$$x_1 + 4x_2 \leq 8x_1 + 4x_2$$

Consequently, for every feasible solution of the LP we have that:

$$z = x_1 + 4x_2 \leq 8x_1 + 4x_2 \leq 20$$

We conclude that the optimal solution is **bounded** by 20:

$$z^* \leq 20$$

Can We Do Better ?

The inequation that we get by adding the first constraint and 5 times the third one is:

$$
\begin{array}{rcrcrcc}
x_1 & - & x_2 & \leq & 2 & \times & 1 \\
 & & x_2 & \leq & 3 & \times & 5 \\
\hline
x_1 & + & 4x_2 & \leq & 17 & &
\end{array}
$$

For every feasible solution of the LP, we get that:

$$z = x_1 + 4x_2 \leq 17$$

This result also holds at the optimum:

$$z^* \leq 17$$

Can We Do Better ? (Cont'd)

- By adding $1/2$ times the second constraint and $7/2$ times the third one, we get:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & \leq & 5 & \times & 1/2 \\ & & x_2 & \leq & 3 & \times & 7/2 \\ \hline x_1 & + & 4x_2 & \leq & 13 & & \end{array}$$

- For every feasible solution, we get:

$$z = x_1 + 4x_2 \leq 13$$

- We conclude that the LP is bounded by 13
- As the feasible solution $x_1 = 1, x_2 = 3$ has a value of 13 corresponding to the value of this bound, then the inequality $z = x_1 + 4x_2 \leq 13$ gives us an **optimality certificate**
- Concretely, it is not possible to find another feasible solution for which the objective function value is strictly larger than 13

Generalization

- Every conical combination of constraints of a canonical LP is still valid, i.e. is satisfied by the system of constraints:

$$\begin{array}{cccccccl}
a_{11}x_1 & + & \ldots & + & a_{1n}x_n & \leq & b_1 & \times\ y_1 \geq 0 \\
\ldots & & \ldots & & \ldots & & \ldots & \\
a_{m1}x_1 & + & \ldots & + & a_{mn}x_n & \leq & b_m & \times\ y_m \geq 0 \\
\hline
\sum_{i=1}^{m}(y_i a_{i1})x_1 & + & \ldots & + & \sum_{i=1}^{m}(y_i a_{in})x_n & \leq & \sum_{i=1}^{m} y_i b_i &
\end{array}$$

- If $c_1 \leq \sum_{i=1}^{m} y_i a_{i1}, \ldots, c_n \leq \sum_{i=1}^{m} y_i a_{in}$, then we have that

$$z = c_1x_1 + \ldots + c_nx_n \leq \sum_{i=1}^{m}(y_i a_{i1})x_1 + \ldots + \sum_{i=1}^{m}(y_i a_{in})x_n \leq \sum_{i=1}^{m} y_i b_i$$

since $x_1 \geq 0, \ldots, x_n \geq 0$

Generalization (Cont'd)

To find the best upper bound, i.e. the smallest one, we need to solve the following LP:

$$
\begin{array}{llcl}
\text{Min} & w = \displaystyle\sum_{i=1}^{m} y_i b_i & & \\
\text{s.t.} & \displaystyle\sum_{i=1}^{m} y_i a_{i1} & \geq & c_1 \\
 & & \cdots & \\
 & \displaystyle\sum_{i=1}^{m} y_i a_{in} & \geq & c_n \\
 & y_1, \ldots, y_m & \geq & 0
\end{array}
$$

This problem is called **the dual linear program** of the initial **primal** canonical LP

Dual Program of a Canonical LP

To each canonical LP:

$$\begin{array}{llrcl} \text{Max} & z = & \boldsymbol{cx} & & \\ \text{s.t.} & & \boldsymbol{Ax} & \leq & \boldsymbol{b} \\ & & \boldsymbol{x} & \geq & 0 \end{array} \qquad \text{(PLP)}$$

corresponds a **dual program** given by:

$$\begin{array}{llrcl} \text{Min} & w = & \boldsymbol{yb} & & \\ \text{s.t.} & & \boldsymbol{A}^T\boldsymbol{y}^T & \geq & \boldsymbol{c}^T \\ & & \boldsymbol{y} & \geq & 0 \end{array} \qquad \text{(DLP)}$$

Important Remark

We consider here $\boldsymbol{y}$ and $\boldsymbol{c}$ as **row** vectors and $\boldsymbol{b}$ as a **column** vector. By doing so, $\boldsymbol{yb}$ is the scalar product between $\boldsymbol{y}$ and $\boldsymbol{b}$. The product $\boldsymbol{A}^T\boldsymbol{y}^T$ is a column vector as well as $\boldsymbol{c}^T$. Note that $\boldsymbol{A}^T\boldsymbol{y}^T \geq \boldsymbol{c}^T \iff \boldsymbol{yA} \geq \boldsymbol{c}$

Dual Program of a Canonical LP (Cont'd)

- The variables $\boldsymbol{y}$ given by

$$\begin{array}{llrcl} \text{Min} & w = & \boldsymbol{yb} & & \\ \text{s.t.} & & \boldsymbol{A}^T\boldsymbol{y}^T & \geq & \boldsymbol{c}^T \\ & & \boldsymbol{y} & \geq & \boldsymbol{0} \end{array} \qquad \text{(DLP)}$$

  are called decision variables of the **dual** problem and are denoted by $\boldsymbol{y_D}$
- The **dual** problem can also be expressed in **standard** form by adding some **slack** variables $\boldsymbol{y_E}$
- In the canonical from, $\boldsymbol{A}^T\boldsymbol{y}^T \geq \boldsymbol{c}^T$ is expressed as $-\boldsymbol{A}^T\boldsymbol{y}^T \leq -\boldsymbol{c}^T$
- Slack variables $\boldsymbol{y_E}$ are defined as:

$$-\boldsymbol{A}^T\boldsymbol{y}_D^T + \boldsymbol{I}\boldsymbol{y}_E^T = -\boldsymbol{c}_D^T \iff -\boldsymbol{y_D}\boldsymbol{A} + \boldsymbol{y_E}\boldsymbol{I} = -\boldsymbol{c_D}$$

Weak Duality Theorem (1)

Theorem

Let $\boldsymbol{x}$ $(= \boldsymbol{x_D})$ be a feasible solution of a canonical LP and $\boldsymbol{y}$ $(= \boldsymbol{y_D})$ a feasible solution of its dual problem, then

$$\boldsymbol{cx} \leq \boldsymbol{yb}$$

Proof :

- $\boldsymbol{y}$ is a dual solution:

$$\boldsymbol{cx} \leq (\boldsymbol{yA})\boldsymbol{x} \qquad \text{since } \boldsymbol{x} \geq \boldsymbol{0} \text{ and } \boldsymbol{c} \leq \boldsymbol{yA}$$

- $\boldsymbol{x}$ is a primal solution:

$$\boldsymbol{y}(\boldsymbol{Ax}) \leq \boldsymbol{yb} \qquad \text{since } \boldsymbol{y} \geq \boldsymbol{0} \text{ and } \boldsymbol{Ax} \leq \boldsymbol{b}$$

- Then:

$$\boldsymbol{cx} \leq \boldsymbol{yAx} \leq \boldsymbol{yb}$$

Weak Duality Theorem (2)

Corollary

Let $\mathbf{x}$ $(= \mathbf{x_D})$ be a feasible solution of PLP of value z and $\mathbf{y}$ $(= \mathbf{y_D})$ a feasible solution of DLP of value w. If $z = w$, then the solutions $\mathbf{x}$ and $\mathbf{y}$ are optimal for their respective problem.

Important Remark

If PLP has no finite optimum, then its dual cannot have a feasible solution without contradicting the weak duality theorem!

Dual Problem of a General LP

- Similarly to the canonical case, we look for a combination of constraints to obtain a new valid constraint providing an upper bound on the optimal value
- Multipliers providing a new valid constraint of **type** $\leq$ when we combine them:

$$\begin{array}{ccccccccc}
a_{i1}x_1 & + & \ldots & + & a_{in}x_n & \leq & b_i & \times & y_i & (y_i \geq 0) \\
a_{k1}x_1 & + & \ldots & + & a_{kn}x_n & = & b_k & \times & y_k & (y_k \in \mathbb{R}) \\
a_{r1}x_1 & + & \ldots & + & a_{rn}x_n & \geq & b_r & \times & y_r & (y_r \leq 0)
\end{array}$$

Dual Problem of a General LP (Cont'd)

If we impose the following constraints:

- $\sum_{i=1}^{m} y_i a_{ij} \geq c_j \quad$ if $x_j \geq 0$
- $\sum_{i=1}^{m} y_i a_{ij} = c_j \quad$ if $x_j \in \mathbb{R}$
- $\sum_{i=1}^{m} y_i a_{ij} \leq c_j \quad$ if $x_j \leq 0$

then we get an upper bound for each of the terms of type $c_j x_j$ in the objective function. Indeed:

- $c_j x_j \leq \left(\sum_{i=1}^{m} y_i a_{ij}\right) x_j \quad$ if $\sum_{i=1}^{m} y_i a_{ij} \geq c_j$ and $x_j \geq 0$
- $c_j x_j = \left(\sum_{i=1}^{m} y_i a_{ij}\right) x_j \quad$ if $\sum_{i=1}^{m} y_i a_{ij} = c_j$ and $x_j \in \mathbb{R}$
- $c_j x_j \leq \left(\sum_{i=1}^{m} y_i a_{ij}\right) x_j \quad$ if $\sum_{i=1}^{m} y_i a_{ij} \leq c_j$ and $x_j \leq 0$

Dualization Rules

Dualization Rules

Max problem	$\longleftrightarrow$	Min problem
Variable $x_j \geq 0$	$\longleftrightarrow$	jth constraint of type $\geq$
Variable $x_j \in \mathbb{R}$	$\longleftrightarrow$	jth constraint of type $=$
Variable $x_j \leq 0$	$\longleftrightarrow$	jth constraint of type $\leq$
ith constraint of type $\leq$	$\longleftrightarrow$	Variable $y_i \geq 0$
ith constraint of type $=$	$\longleftrightarrow$	Variable $y_i \in \mathbb{R}$
ith constraint of type $\geq$	$\longleftrightarrow$	Variable $y_i \leq 0$

Example

$$
(\text{PLP})\left\{\begin{array}{lrcrcrcl}
\text{Max } z = & 1x_1 & + & 3x_2 & & & \Rightarrow & \text{Min} \\
\text{s.t.} & 1x_1 & + & -1x_2 & \leq & 2 & \Rightarrow & y_1 \geq 0 \\
 & 2x_1 & + & 1x_2 & = & 5 & \Rightarrow & y_2 \in \mathbb{R} \\
 & & & 1x_2 & \leq & 3 & \Rightarrow & y_3 \geq 0 \\
 & x_1 & & & \in & \mathbb{R} & \Rightarrow & \text{1st const. of type } = \\
 & & & x_2 & \geq & 0 & \Rightarrow & \text{2nd const. of type } \geq
\end{array}\right.
$$

$$
(\text{DLP})\left\{\begin{array}{lrcrcrcr}
\text{Min } w = & 2y_1 & + & 5y_2 & + & 3y_3 & & \\
\text{s.t.} & 1y_1 & + & 2y_2 & & & = & 1 \\
 & -1y_1 & + & 1y_2 & + & 1y_3 & \geq & 3 \\
 & y_1 & & , & & y_3 & \geq & 0 \\
 & & & y_2 & & & \in & \mathbb{R}
\end{array}\right.
$$

Important Remarks

- If PLP is a max problem (resp. min), then DLP is a min problem (resp. max)
- If PLP has n variables and m constraints, then its dual problem has m variables and n constraints
- Each dual variable corresponds to a constraint of PLP and each dual constraint corresponds to a variable of PLP
- The dual problem of DLP is PLP

Dual Basic Solution in a Tableau (1)

Important Result

A basis in a tableau **univocally** defines a basis of the **dual** problem in its **standard** form. The row vector $\boldsymbol{y} = (-\gamma_{\boldsymbol{D}} \mid -\gamma_{\boldsymbol{E}})$ that can be found in the last row of a tableau is a **basic solution of the dual problem**. The m dual non-basic variables are the ones corresponding to the the m primal basic variables. Conversly, the n dual basic variables are the ones corresponding to the n non-basic primal variables. The **value** of this dual solution is ζ. Moreover, $\boldsymbol{y_D} = -\gamma_{\boldsymbol{E}}$ and $\boldsymbol{y_E} = -\gamma_{\boldsymbol{D}}$

Dual Basic Solution in a Tableau (2)

Concretely, the dual basic solution can be read in the last row of the tableau:

$$
T_B = \begin{array}{c|c|c||c|}
\mathbf{x_D} & \mathbf{x_E} & z & \\
\hline
\boldsymbol{B}^{-1}\boldsymbol{A} & \boldsymbol{B}^{-1} & \mathbf{0} & \beta \\
\hline
-\gamma_{\boldsymbol{D}} & -\gamma_{\boldsymbol{E}} & 1 & \zeta \\
\hline
\mathbf{y_E} = (y_{m+1} \dots y_{m+n}) & \mathbf{y_D} = (y_1 \dots y_m) & &
\end{array}
$$

A sketch of the demonstration of this important result is provided in the appendix at the end of this presentation

Strong Duality Theorem

Theorem (Strong Duality Theorem)

If a standard linear program has an optimal solution $\mathbf{x}^* = (\mathbf{x}_D^* \mid \mathbf{x}_E^*)$ of value $z^* = \mathbf{c}_D \mathbf{x}_D^*$ then its dual problem has also an optimal solution $\mathbf{y}^* = (\mathbf{y}_D^* \mid \mathbf{y}_E^*)$. Moreover, the value of this solution is $w^* = \mathbf{y}_D^* \mathbf{b} = z^*$

Proof. Let's consider the optimal tableau provided by the simplex algorithm:

$$\mathbf{T_B} = \begin{array}{c|c|c|c||c|}
 & \mathbf{x_D} & \mathbf{x_E} & z & \\
\hline
 & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} & \mathbf{0} & \beta \\
\hline
 & -\gamma_{\mathbf{D}} & -\gamma_{\mathbf{E}} & 1 & \zeta \\
\hline
 & \mathbf{y_E} & \mathbf{y_D} & &
\end{array}$$

Strong Duality Theorem

The tableau is optimal:

$$-\gamma_{\boldsymbol{D}} \geq \mathbf{0} \qquad \text{and} \qquad -\gamma_{\boldsymbol{E}} \geq \mathbf{0}$$

and its dual basic solution

$$\boldsymbol{y} = (\boldsymbol{y}_{\boldsymbol{D}} \mid \boldsymbol{y}_{\boldsymbol{E}}) = (-\gamma_{\boldsymbol{E}} \mid -\gamma_{\boldsymbol{D}})$$

is feasible. As this solution has the same value as the primal basic solution, then it is optimal !

Relationship between PLP and DLP

The table below summarizes the relationship between PLP and DLP:

	DLP		
	fo	onb	nfs
PLP: fo	**SD**	$\emptyset$	$\emptyset$
PLP: onb	$\emptyset$	$\emptyset$	**WD**
PLP: nfs	$\emptyset$	**WD**	**Pos**

fo = finite optimum, onb = optimum not bounded, nfs = no feasible solution, **SD** = **Strong Duality**, **WD** = **Weak Duality**, **Pos** = Possible, $\emptyset$ = empty set

The Slack Variables Theorem

Theorem (Slack Variables Theorem)

Let $\boldsymbol{x} = (\boldsymbol{x_D} \mid \boldsymbol{x_E})$ be a feasible solution to standard LP and $\boldsymbol{y} = (\boldsymbol{y_D} \mid \boldsymbol{y_E})$ a feasible solution of its dual problem. These solutions are optimal for their respective problem if and only if

$$\boldsymbol{y_E x_D} = 0 \qquad \text{and} \qquad \boldsymbol{y_D x_E} = 0$$

Proof. By definition, we have

$$\boldsymbol{x_E} = \boldsymbol{b} - \boldsymbol{A x_D} \qquad \text{and} \qquad \boldsymbol{y_E} = \boldsymbol{y_D A} - \boldsymbol{c_D}.$$

Moreover, as $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible:

$$z = \boldsymbol{c_D x_D} \leq (\boldsymbol{y_D A})\boldsymbol{x_D} = \boldsymbol{y_D}(\boldsymbol{A x_D}) \leq \boldsymbol{y_D b} = w$$

The Slack Variables Theorem (Cont'd)

1. Let's assume that the solutions are optimal. Then we have to show that the slack variabes conditions are satified. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal, then $z = w$ and the previous inequalities are indeed equalities. So:

$$\boldsymbol{c_D x_D} = (\boldsymbol{y_D A})\boldsymbol{x_D} \iff (\boldsymbol{y_D A} - \boldsymbol{c_D})\boldsymbol{x_D} = 0 \iff \boldsymbol{y_E x_D} = \boldsymbol{0}$$

$$\boldsymbol{y_D}(\boldsymbol{A x_D}) = \boldsymbol{y_D b} \iff \boldsymbol{y_D}(\boldsymbol{A x_D} - \boldsymbol{b}) = 0 \iff \boldsymbol{y_D x_E} = \boldsymbol{0}$$

The Slack Variables Theorem (Cont'd)

2. Conversely, if $\boldsymbol{y_E x_D} = 0$ and $\boldsymbol{y_D x_E} = 0$, then we have to show that these solutions are optimal

$$\boldsymbol{y_E x_D} = (\boldsymbol{y_D A} - \boldsymbol{c_D})\boldsymbol{x_D} = 0 \quad \Longleftrightarrow \quad (\boldsymbol{y_D A})\boldsymbol{x_D} = \boldsymbol{c_D x_D}$$

and

$$\boldsymbol{y_D x_E} = \boldsymbol{y_D}(\boldsymbol{A x_D} - \boldsymbol{b}) = 0 \quad \Longleftrightarrow \quad \boldsymbol{y_D}(\boldsymbol{A x_D}) = \boldsymbol{y_D b}$$

So

$$z = \boldsymbol{c_D x_D} = (\boldsymbol{y_D A})\boldsymbol{x_D} = \boldsymbol{y_D}(\boldsymbol{A x_D}) = \boldsymbol{y_D b} = w$$

and solutions $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal

The Slack Variables Theorem (Cont'd)

Remark. As solutions $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible, they are non-negative. Consequently:

$$\boldsymbol{y_E x_D} = 0 \quad \Longleftrightarrow \quad y_{m+j}x_j = 0 \quad \forall\, j = 1, \ldots, n$$

and

$$\boldsymbol{y_D x_E} = 0 \quad \Longleftrightarrow \quad y_i x_{n+i} = 0 \quad \forall\, i = 1, \ldots, m$$

Dual Variables at the Optimum

- If a standard LP has a finite optimum, this is also the case for its dual problem et we have that:

$$z^* = \sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} y_i^* b_i = w^*$$

- If the primal solution is non-degenerated, then the optimal value of the dual variable y_i represents the **marginal price** of resource i at the optimum:

$$\frac{\partial}{\partial b_i} z^* = y_i^*$$

Dual Variables at the Optimum (Cont'd)

- In other words, y_i^* is the potential increase of the optimal value of the problem if the resource i, limited to b_i, is increased by one unit (we assume that the current basis keeps optimal)
- Note that this interpretation is not valid if the primal solution is degenerated (the basis changes but the optimal value keeps the same)

Appendix. Dual Basic Solution in a Tableau

A basis in a tableau univocally defines a basis of the dual problem in its standard form. Moreover, the basic solution associated with this dual basis can be read in the last row of the tableau.

$$
\boldsymbol{T_B} = \begin{array}{c|c|c|c||c|}
 & \boldsymbol{x_D} & \boldsymbol{x_E} & z & \\
\hline
 & \boldsymbol{B}^{-1}\boldsymbol{A} & \boldsymbol{B}^{-1} & \mathbf{0} & \beta \\
\hline
 & -\gamma_{\boldsymbol{D}} & -\gamma_{\boldsymbol{E}} & 1 & \zeta \\
\hline
 & \boldsymbol{y_E} = (y_{m+1} \dots y_{m+n}) & \boldsymbol{y_D} = (y_1 \dots y_m) & &
\end{array}
$$

In order to show that the vector $\boldsymbol{y} = (\boldsymbol{y_D} \mid \boldsymbol{y_E}) = (-\gamma_{\boldsymbol{E}} \mid -\gamma_{\boldsymbol{D}})$ is a solution of the system of constraints of DLP, let's remind that

$$
-\gamma_{\boldsymbol{D}} = \boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{A} - \boldsymbol{c_D} \qquad \text{and} \qquad -\gamma_{\boldsymbol{E}} = \boldsymbol{c_B}\boldsymbol{B}^{-1}
$$

Appendix. Dual Basic Solution in a Tableau (Cont'd)

On the other hand, dual constraints $\boldsymbol{y_D A} \geq \boldsymbol{c_D}$ can be written in standard form as

$$-\boldsymbol{y_D A} + \boldsymbol{y_E I} = -\boldsymbol{c_D}$$

If we replace $\boldsymbol{y_D}$ and $\boldsymbol{y_E}$ by the expressions of $-\gamma_{\boldsymbol{E}}$ and $-\gamma_{\boldsymbol{D}}$, we get

$$-\boldsymbol{y_D A} + \boldsymbol{y_E I} = \gamma_{\boldsymbol{E}}\boldsymbol{A} - \gamma_{\boldsymbol{D}} = -\boldsymbol{c_B B}^{-1}\boldsymbol{A} + \boldsymbol{c_B B}^{-1}\boldsymbol{A} - \boldsymbol{c_D} = -\boldsymbol{c_D}$$

This shows that $\boldsymbol{y} = (\boldsymbol{y_D} \mid \boldsymbol{y_E}) = (-\gamma_{\boldsymbol{E}} \mid -\gamma_{\boldsymbol{D}})$ is a solution of the standard dual problem.

Appendix. Dual Basic Solution in a Tableau (Cont'd)

We still need to check that this dual solution is basic. As the reduced costs of the m basic primal variables are null (a basic primal variable has by construction a zero in its last row), then $\boldsymbol{y}$ has at least m null components.

Indeed, the m dual non-basic variables are the ones corresponding to the the m primal basic variables. Conversly, the n dual basic variables are the ones corresponding to the n non-basic primal variables.

As a final remark, the value of the basis dual solution is given by

$$w = \boldsymbol{y_D b} = \boldsymbol{c_B B}^{-1}\boldsymbol{b} = \zeta$$

and has the same value as the one corresponding to the primal basic solution.