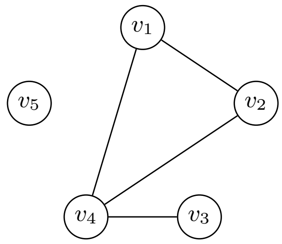
Solutions to Exercise Set 5

Problem 1

a) After numbering the rows and the columns of the matrix from v_1 to v_5, we get the following graph:



b)

incidence matrix

$$\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array}\\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}\end{array}$$

adjacency matrix

$$\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{array}\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array}\\ \begin{pmatrix} 0 & 1 & 1 & 0 & 1\\ 1 & 0 & 1 & 1 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}\end{array}$$

incidence function

$$\begin{aligned} V &= \{v_1,v_2,v_3,v_4,v_5\}\\ E &= \{e_1,e_2,e_3,e_4,e_5,e_6\} \end{aligned}$$

ψ	e_1	e_2	e_3	e_4	e_5	e_6
$u(e)$	v_1	v_1	v_2	v_2	v_1	v_4
$v(e)$	v_2	v_3	v_3	v_4	v_5	v_5

Problem 2

a)

i) incidence matrix

$$\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array}\\ \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & -1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}\end{array}$$

adjacency matrix

$$\begin{array}{c} \\ v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{array}\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array}\\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}\end{array}$$

ii) incidence matrix

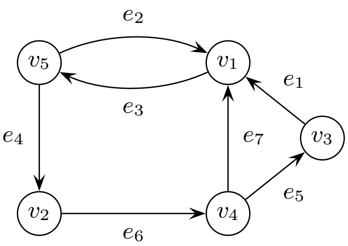
$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1
\end{pmatrix}
\end{array}
$$

adjacency matrix

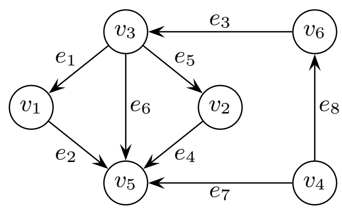
$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6
\end{array}
\begin{array}{c}
\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

b) After numbering the rows from v_1 to v_5 and the columns from e_1 to e_7 for the matrix given in i and the rows from v_1 to v_6 and the columns from e_1 to e_8 for the matrix given in ii, we get the following graphs:

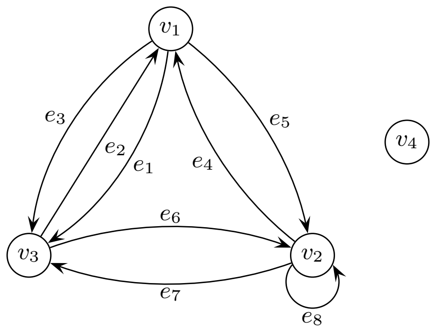
i)



ii)

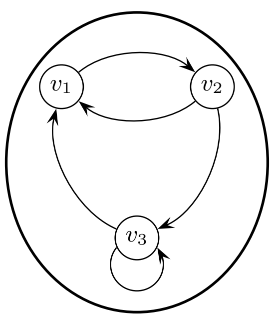


c)

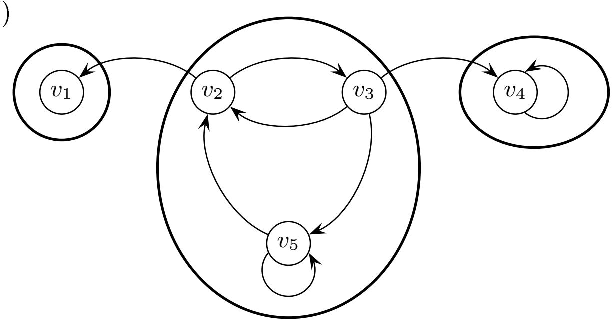


Problem 3

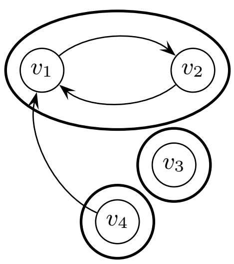
a)



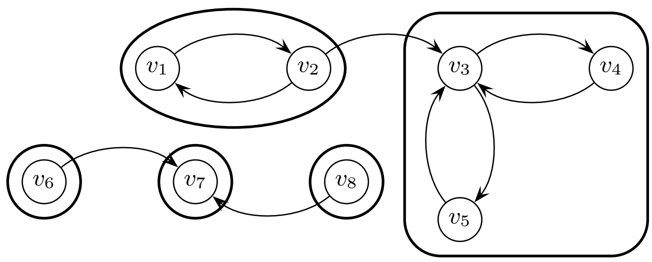
b)



c)

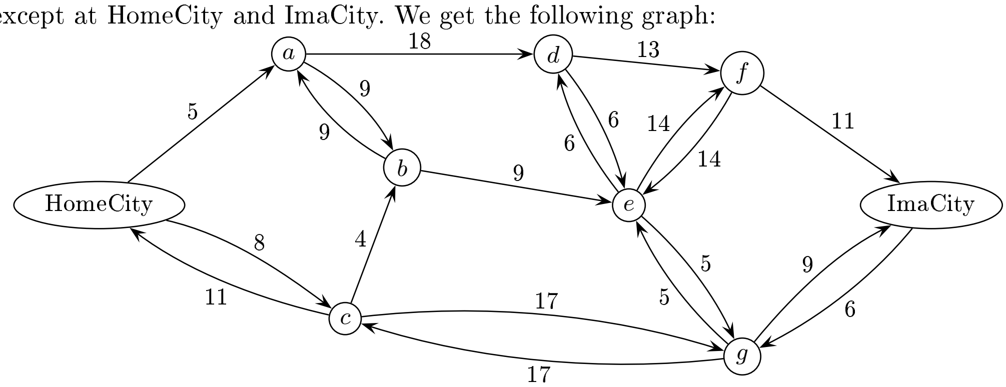


d)



Problem 4

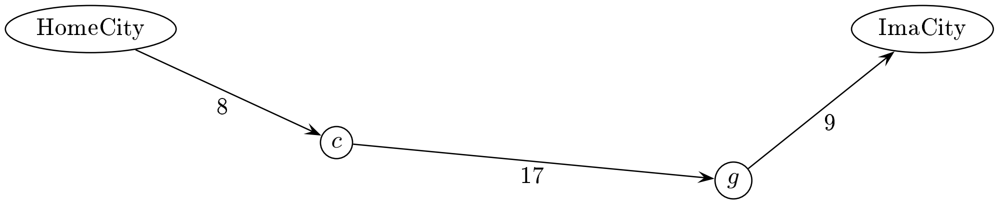
Let's first replace every edge by two arcs in opposite direction and let's add to each arc a duration of 3 minutes except at HomeCity and ImaCity. We get the following graph:



As the "weights" are non-negative, we can apply Dijkstra's algorithm:

It	i_{min}	Label (predecessor) at the end of the iteration								
		HC	a	b	c	d	e	f	g	IC
0		0	∞	∞	∞	∞	∞	∞	∞	∞
1	HC	0	5(HC)	∞	8(HC)	∞	∞	∞	∞	∞
2	a		5(HC)	14(a)	8(HC)	23(a)	∞	∞	∞	∞
3	c			12(c)	8(HC)	23(a)	∞	∞	25(c)	∞
4	b			12(c)		23(a)	21(b)	∞	25(c)	∞
5	e					23(a)	21(b)	35(e)	25(c)	∞
6	d					23(a)		35(e)	25(c)	∞
7	g							35(e)	25(c)	34(g)
8	IC							35(e)		34(g)
9	f							35(e)		

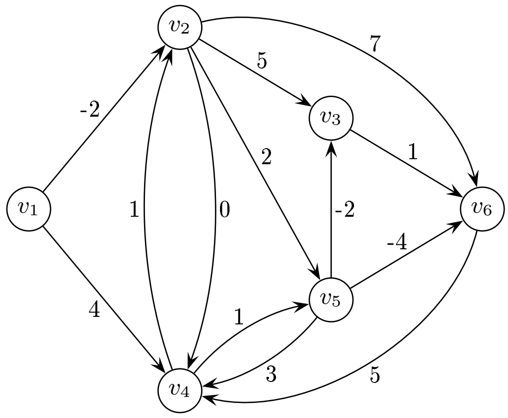
The optimal path is:



Anne needs 34 minutes to go from HomeCity to ImaCity.

Problem 5

We would like to determine the shortest path from v_1 to v_6:



As this network contains edges with negative weights, we must apply the generic algorithm:

Iter.	v_i removed from L	Labels λ_i/ Predecessors $p(i)$						Cand. L
		v_1	v_2	v_3	v_4	v_5	v_6	
0		0/-	∞/-	∞/-	∞/-	∞/-	∞/-	$\{v_1\}$
1	v_1	0/-	-2/v_1	∞/-	4/v_1	∞/-	∞/-	$\{v_2,v_4\}$
2	v_2	0/-	-2/v_1	3/v_2	-2/v_2	0/v_2	5/v_2	$\{v_3,v_4,v_5,v_6\}$
3	v_3	0/-	-2/v_1	3/v_2	-2/v_2	0/v_2	4/v_3	$\{v_4,v_5,v_6\}$
4	v_4	0/-	-2/v_1	3/v_2	-2/v_2	-1/v_4	4/v_3	$\{v_5,v_6\}$
5	v_5	0/-	-2/v_1	-3/v_5	-2/v_2	-1/v_4	-5/v_5	$\{v_3,v_6\}$
6	v_3	0/-	-2/v_1	-3/v_5	-2/v_2	-1/v_4	-5/v_5	$\{v_6\}$
7	v_6	0/-	-2/v_1	-3/v_5	-2/v_2	-1/v_4	-5/v_5	$\varnothing$

The shortest path from v_1 to v_6 in R is unique and has a value of -5. It is given by:

$$v_1 \longrightarrow v_2 \longrightarrow v_4 \longrightarrow v_5 \longrightarrow v_6.$$

Problem 6

a) Shortest paths from α:

Vertex	k (top. sort)	$\lambda_k/p(k)$
α	1	$0/NULL$
A	2	$0/\alpha$
D	3	$0/\alpha$
N	4	$0/\alpha$
B	5	$0.5/A$
E	6	$1/D$
O	7	$2/N$
G	8	$0.5/A$
H	9	$2.5/G$
I	10	$2.5/G$
J	11	$4.5/H$
F	12	$2/E$
C	13	$3.5/B$
K	14	$4.5/H$
L	15	$5/K$
M	16	$2.5/F$
ω	17	$3/O$

b) Longest paths from α:

Vertex	k (top. sort)	$\lambda_k/p(k)$
α	1	$0/NULL$
A	2	$0/\alpha$
D	3	$0/\alpha$
N	4	$0/\alpha$
B	5	$0.5/A$
E	6	$1/D$
O	7	$2/N$
G	8	$2/E$
H	9	$4/G$
I	10	$4/G$
J	11	$7/I$
F	12	$7/I$
C	13	$8/J$
K	14	$8/J$
L	15	$8.5/K$
M	16	$9.5/L$
ω	17	$13.5/M$