Solutions to Exercise Set 1

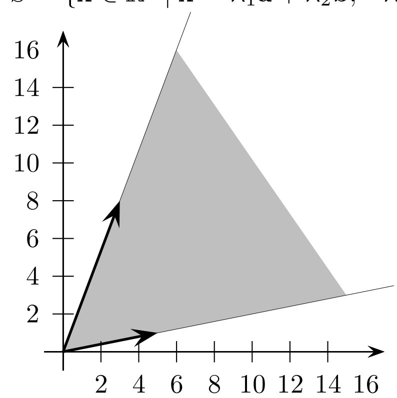
Problem 1

a) The set of linear combinations of two independent vectors in the plan is $\mathbb{R}^2$.

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}, \quad \lambda_1,\lambda_2 \in \mathbb{R}\}$$

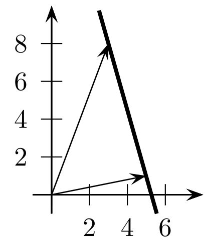
b) The set of conic combinations of two indepedent vectors in the plan is the cone generated by these two vectors.

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}, \quad \lambda_1,\lambda_2 \geq 0, \quad \lambda_1,\lambda_2 \in \mathbb{R}\}$$



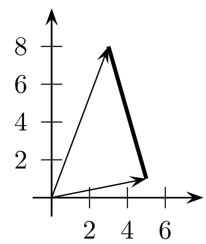
c) The set of affine combination of two independent vectors in the plane is the straight line running through the end points of these vectors.

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}, \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1,\lambda_2 \in \mathbb{R}\}$$



d) The set of convex combination of two independent vectors in the plane is the segment line running through the end points of these vectors.

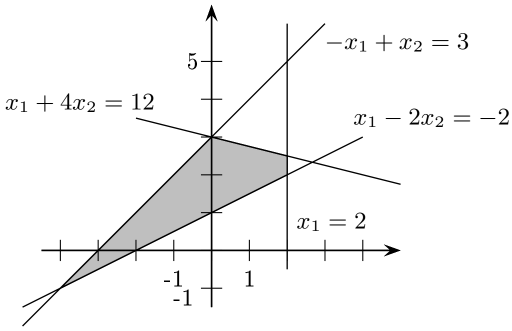
$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b}, \quad \lambda_1 + \lambda_2 = 1, \quad \lambda_1,\lambda_2 \geq 0, \quad \lambda_1,\lambda_2 \in \mathbb{R}\}$$



Problem 2

(i)

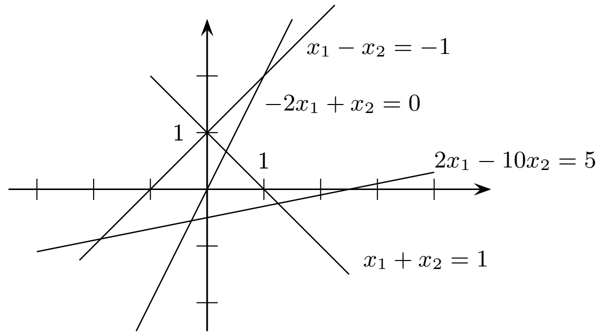
a)



b) This is a minimal set.

(ii)

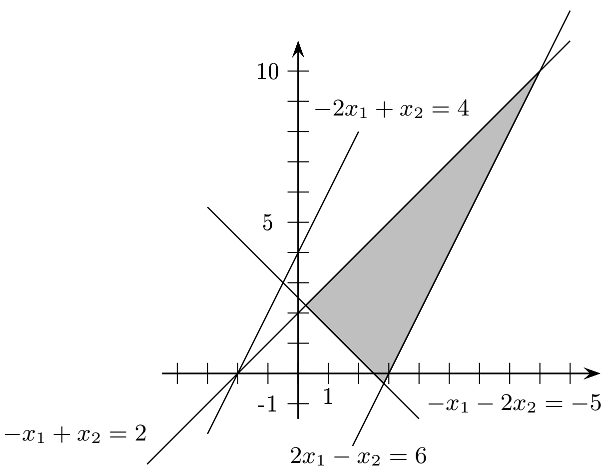
a) The set of solutions is empty ($\varnothing$)



b) As an example, here is a minimal set: $\left\{ \begin{array}{rcrcr} x_1 & + & x_2 & \leq & 1 \\ -x_1 & - & x_2 & \leq & -2 \end{array} \right.$

(iii)

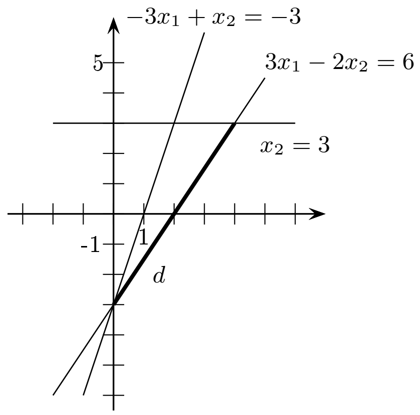
a)



b) We get a minimal set after removing the first inequation.

(iv)

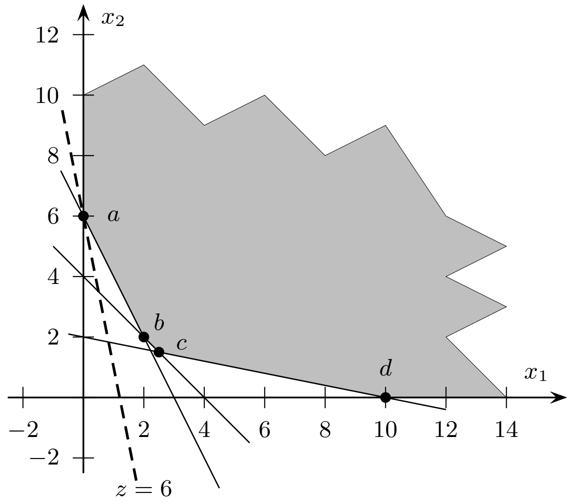
a) The set of solutions is the line segment d between points (0,-3) and (4,3).



b) This is a minimal set.

Problem 3

a) The grey zone corresponds to the feasible region. It is not bounded.



The optimal solution is located at $x_1 = 0$ and $x_2 = 6$ and has a value of 6.

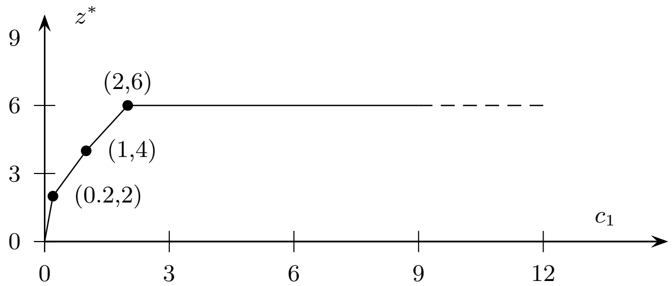
b) We assume that $z = c_1x_1 + x_2$. We want to determine the optimal solutions and the optimal values of the objective function depending on c_1. Depending on the slope s of the contour lines of the objective function, we get that the optimal solutions are given below. To get the value of the objective function $z = c_1x_1 + x_2$, we just have to replace x_1 and x_2 by their optimal values for each of the different cases.

$-\infty \leq s < -2$	The optimum is located at a and has a value of $z^* = 6$ with $x_1^* = 0$ and $x_2^* = 6$.
$-2 \leq s < -1$	The optimum is located at b and has value of $z^* = 2c_1 + 2$ with $x_1^* = 2$ and $x_2^* = 2$.
$-1 \leq s < -\frac{1}{5}$	The optimum is located at c and has a value of $z^* = \frac{5}{2}c_1 + \frac{3}{2}$, with $x_1^* = 2.5$ and $x_2^* = 1.5$.
$-\frac{1}{5} \leq s \leq 0$	The optimum is located at d and has a value of $z^* = 10c_1$, with $x_1^* = 10$ and $x_2^* = 0$.
$s > 0$	The problem is unbounded.

As $z = c_1x_1 + x_2$, this can be rewritten as $x_2 = z - c_1x_1$. We conclude that the slope of the contour lines is given by $s = -c_1$. If we express the above condition in function of c_1 rather than s, we finally get that:

$c_1 < 0$	The problem is unbounded.
$0 \leq c_1 \leq \frac{1}{5}$	The optimum is located at d and has a value of $z^* = 10c_1$, with $x_1^* = 10$ and $x_2^* = 0$.
$\frac{1}{5} < c_1 \leq 1$	The optimum is located at c and has a value of $z^* = \frac{5}{2}c_1 + \frac{3}{2}$, with $x_1^* = 2.5$ and $x_2^* = 1.5$.
$1 < c_1 \leq 2$	The optimum is located at b and has value of $z^* = 2c_1 + 2$, with $x_1^* = 2$ and $x_2^* = 2$.
$2 < c_1 \leq \infty$	The optimum is located at a and has a value of $z^* = 6$, with $x_1^* = 0$ and $x_2^* = 6$.

If we plot z in function of c_1, then we get a concave piecewise linear function as illustrated below.



Problem 4

a) Decision variables are:

p_i : price of vitamin i per kg $\qquad i = \text{A, C}$

Input data:

$$\mathbf{d} = \begin{pmatrix} 3000 \\ 5000 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 6 & 4 & 6 \\ 7 & 8 & 2 \end{pmatrix} \quad \text{et} \quad \mathbf{c} = \begin{pmatrix} 42000 \\ 20000 \\ 12000 \end{pmatrix}$$

where :

d_j = kg of vitamin i to sell $\qquad i = \text{A, C}$

q_{ij} = kg of vitamin i in a tonne of fruit j $\qquad j$ = banana, orange, tomato

c_j = price of a tonne of fruit j

Objective function:

$$z = d_{\text{A}}p_{\text{A}} + d_{\text{C}}p_{\text{C}}$$

Price constraints are:

$$p_{\text{A}}q_{\text{A}j} + p_{\text{C}}q_{\text{C}j} \leq c_j \qquad \forall j$$

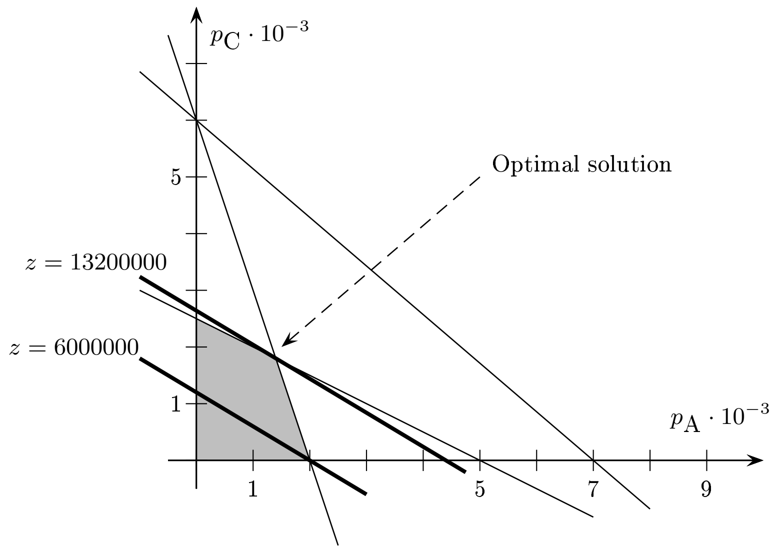
Non-negativity of the prices:

$$p_i \geq 0 \qquad \forall i$$

Finally, we get the following LP:

$$\begin{array}{lrcrcr} \text{Max } z = & 3000p_{\text{A}} & + & 5000p_C & & \\ s.t. & 6p_{\text{A}} & + & 7p_{\text{C}} & \leq & 42000 \\ & 4p_{\text{A}} & + & 8p_{\text{C}} & \leq & 20000 \\ & 6p_{\text{A}} & + & 2p_{\text{C}} & \leq & 12000 \\ & p_{\text{A}} & , & p_{\text{C}} & \geq & 0 \end{array}$$

b) Feasible region and contour lines of z are given by:



The optimal solution is given by:

$$\begin{aligned} z &= 13200000 \\ p_A &= 1400 \\ p_C &= 1800 \end{aligned}$$

Vitamin A can be sold at 1400 francs per kg and Vitamin C at 1800 francs per kg for a total revenue of 13'200'000 francs.