Review of Some Important Exercises

Problem 1

We consider the following linear system:

$$\begin{array}{lrrrrl} \text{Min} & z = & 5x_1 & + & x_2 & \\ \text{s.t.} & & 2x_1 & + & x_2 & \geq 6 \\ & & x_1 & + & x_2 & \geq 4 \\ & & 2x_1 & + & 10x_2 & \geq 20 \\ & & x_1 & & & \geq 0 \\ & & & & x_2 & \geq 0 \end{array}$$

a) Use the graphical method to solve it.
b) Analyze the variation of z with respect to the coefficient of x_1. Concretely determine the optimal solutions when we consider an objective function given by $z = c_1x_1 + x_2$.

Problem 2

We consider the following vectors and matrix: $\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 & 0 & 5 \\ 2 & -4 & 5 & 6 & 11 \\ 3 & -6 & 8 & 10 & 11 \\ 0 & 0 & 1 & 5 & -2 \end{pmatrix}$,

$\mathbf{b} = \begin{pmatrix} 7 \\ 10 \\ 3 \\ -9 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$.

a) Determine the solution of the system $\mathbf{Ax} = \mathbf{b}$.
b) What is the rank of $\mathbf{A}$?
c) What is the dimension of the column space of $\mathbf{A}$? Give a basis of this space.
d) What is the dimension of the row space of $\mathbf{A}$? Give a basis of this space.

Reminders: let $\mathbf{A}$ be a $m \times n$ matrix. Then
- $rank(\mathbf{A})$ = dim of the column space of $\mathbf{A}$ = dim of the row space of $\mathbf{A}$
- $rank(\mathbf{A})$ = number of pivots in any echelon form of $\mathbf{A}$
- $rank(\mathbf{A})$ = the max number of linearly independent rows or columns of $\mathbf{A}$

Problem 3

Solve the following LP with the simplex algorithm:

$$\begin{array}{rrrrrl} \text{Max } z & = & x_1 & + & 4x_2 & \\ s.t. & & x_1 & - & x_2 & \leq 1 \\ & & -3x_1 & + & x_2 & \leq 0 \\ & & x_1 & , & x_2 & \geq 0 \end{array}$$

Problem 4

Solve the following LP with the simplex algorithm:

$$\begin{array}{llrcrcl} \text{Max} & z = & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & & x_1 & - & x_2 & \leq & 2 \\ & & x_1 & + & 2x_2 & \geq & 1 \\ & & 2x_1 & - & 3x_2 & \geq & 6 \\ & & x_1 & , & x_2 & \geq & 0 \end{array}$$

Problem 5

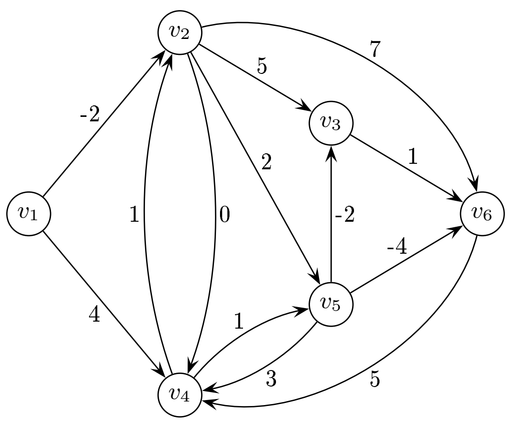
A steel factory would like to produce 1000 identical items. Each of them needs 1, 0.6, and 0.3 kg of metal M1, M2, and M3 respectively. These metals are present in different alloys that the factory purchases in the market. The price and the composition (in %) of the alloys are given in the table below:

	Alloy 1	Alloy 2	Alloy 3
M1	10%	40%	10%
M2	30%	60%	60%
M3	60%	0%	30%
Price per ton (kFrs)	3	1	4

a) The factory would like to minimize its costs. Formulate this problem as a LP (PLP).
b) Determine its dual problem (DLP).
c) Solve PLP with the dual simplex algorithm (phase II).
d) What is the optimal solution?

Problem 6

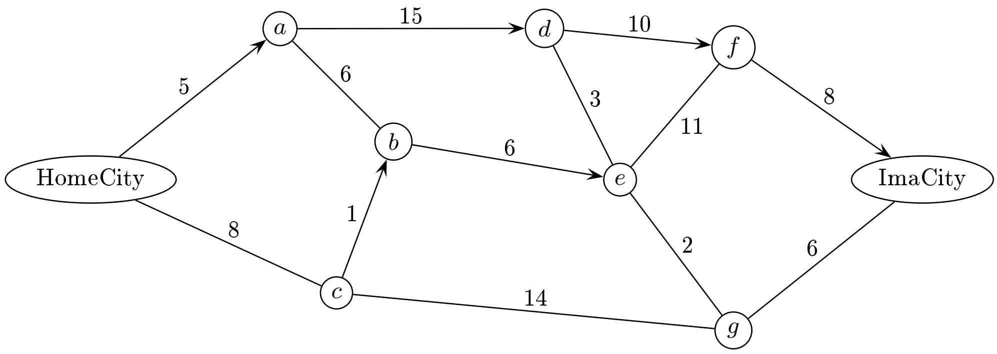
We consider the following network $R = (V,E,c)$:



Determine a shortest path from vertex v_1 to v_6.

Problem 7

Anne lives in HomeCity and works at ImaCity. She would like to determine what is the fastest itinerary between these two locations.

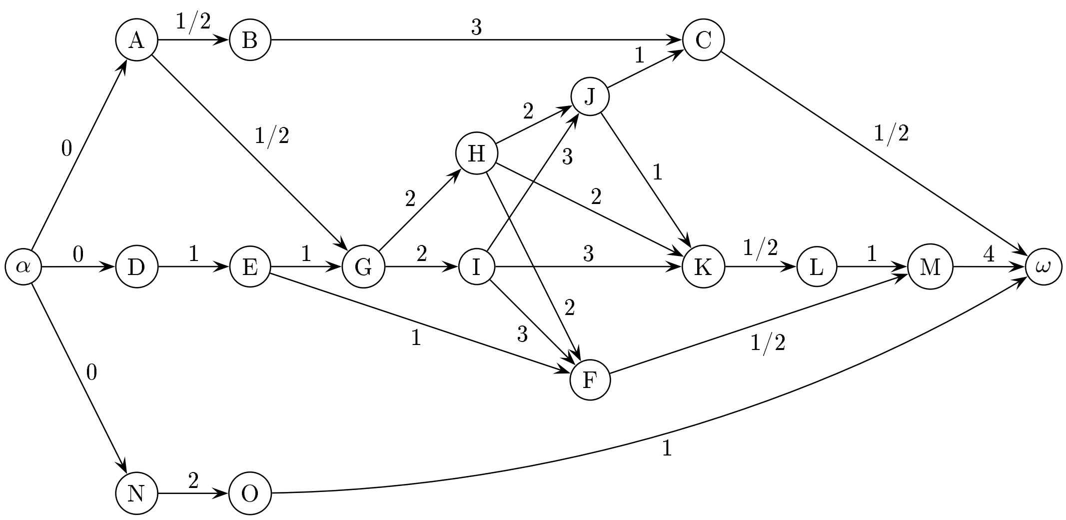


Vertices represent crossroads, the arcs are one-way roads, and the egdes are two-way roads. Values beside arcs and edges are amount of time in minutes from a crossroads to the next one. There is also a 3 minutes waiting time at each crossroads except at Homecity and at ImaCity.

Model this problem as a shortest path problem in a network. Determine the optimal path from HomeCity to ImaCity. How long is the travel?

Problem 8

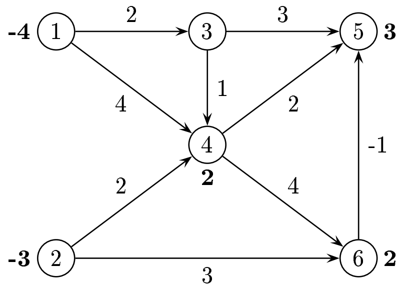
We consider the acyclic graph below.



a) Determine the shortest paths from α to the other vertices.
b) Determine the longest paths from α to the other vertices.

Problem 9

We consider the transshipment problem given by the following network where the numbers beside the edges are the unit usage costs, and the numbers at the vertices represent the supply (negative) and the demand (positive).



a) Determine the primal and dual basic solutions associated with the spanning tree defined by the arcs $\{(1,3),(1,4),(2,4),(2,6),(3,5)\}$.
b) Determine an optimal solution with the transshipment simplex algorithm.
c) In which range can vary the unit usage cost of arc (4,6) without affecting the optimal basis?

Problem 10

Using the B&B algorithm, solve the following knapsack problem where we can select several times items 1 and 2.

$$\begin{array}{lrcl} \max & 10x_1 + 12x_2 + 7x_3 + \frac{3}{2}x_4 & & \\ & & & \\ \text{s.t.} & 4x_1 + 5x_2 + 3x_3 + x_4 & \leq & 10 \\ & x_1,x_2 & \in & \mathbb{Z}_+ \\ & x_3,x_4 & \in & \{0,1\} \end{array}$$

Hint: It is quite obvious that $x_1,x_2 \in \{0,1,2\}$. When branching on x_i, then consider the cases $x_i = 0$, $x_i = 1$, $x_i = 2$ for $i = 1,2$.

Problem 11

Show that the real symmetric matrix

$$M = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite.

Problem 12

We consider the two following functions:

$$\begin{array}{rcl} f & : & \mathbb{R}^2 \to \mathbb{R} \\ & & (x,y) \mapsto f(x,y) = x^2 + y^2 \end{array} \qquad \begin{array}{rcl} g & : & \mathbb{R}^2 \to \mathbb{R} \\ & & (x,y) \mapsto g(x,y) = \frac{1}{3}x^3 + y^3 - x - y \end{array}$$

a) Compute the gradient of f and g for all $\mathbf{x} \in \mathbb{R}^2$.
b) Compute the hessian of f and g for all $\mathbf{x} \in \mathbb{R}^2$. For which values of $\mathbf{x} \in \mathbb{R}^2$ are these matrices positive definite? What are your conclusions?
c) How many critical points have these fonctions? For each of them, determine if it is a local maximum, a local minimum or a saddle point.

Hint: a critical point whose hessian is indefinite (not positive semi-definite, nor negative semi-definite) is a saddle point.

Problem 13

We consider the following function $f(x,y) = xy$ with the constraint that $3x^2 + y^2 = 6$.

a) Compute the points satisfying the KKT conditions.

b) For each of these points, determine if it is a maximum, a minimum, or none of these.

Problem 14

Let's consider the following logarithmic barrier problem:

$$\begin{array}{ll} min & 5x_1 + 7x_2 - 4x_3 - \sum_{j=1}^{3} ln(x_j) \\ \text{s.t.} & x_1 + 3x_2 + 12x_3 = 37 \\ & x_1 > 0, x_2 > 0, x_3 > 0 \end{array}$$

Give its dual problem.

Problem 15

We consider the following problem:

$$\min_{(x,y)\in\mathbb{R}^2} f(x,y) = 3x^2 + 3y^2$$

and three different algorithms: 1) the steepest descent method (with the step obtained by exact minimization), 2) the Newton's method, 3) the conjuguate gradient method.

For each of these algorithms:

– Apply one iteration of the method starting at $(x_0,y_0) = (1,1)$.

– From a theoretical point of view, how many iterations are necessary to solve this problem?