Linear Systems and Tableaus

Optimization Methods in Management Science
Master in Management
HEC Lausanne

Dr. Rodrigue Oeuvray

Linear Systems and Tableaus

- Reminder about linear systems
- Bases and basic solutions
- Pivoting and pivoting matrix
- Tableau
- Pivoting in a tableau
- Geometric interpretation

A Linear System

- We consider the system $\boldsymbol{Ax} = \boldsymbol{b}$ with

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- They are **4 variables** and **2 equations**
- What are the **solutions** of this system ?
- This is a classical problem in **linear algebra**

Reminders About Linear Systems

- Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a system of linear equation with $\boldsymbol{A}$ a $m \times n$ matrix with $m \leq n$ and $\text{rank}(\boldsymbol{A}) = m$
- To determine all the solutions of the system, we need to choose m **linearly independent columns** of $\boldsymbol{A}$ that form a basis of the column space of $\boldsymbol{A}$ (and consequently of $\mathbb{R}^m$) and to express the system in that base
- Let $\boldsymbol{B}$ be an invertible matrix comprised of m linearly independent columns of $\boldsymbol{A}$. After a permutation P of the columns of $\boldsymbol{A}$, then $\boldsymbol{A}$ is transformed into a matrix $(\boldsymbol{N} \mid \boldsymbol{B})$. This is denoted by

$$\boldsymbol{A} \overset{P}{=} (\boldsymbol{N} \mid \boldsymbol{B})$$

Reminders About Linear Systems (Cont'd)

With this permutation, the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ can be rewritten as:

$$(\boldsymbol{N} \mid \boldsymbol{B}) \begin{pmatrix} \boldsymbol{x_N} \\ \boldsymbol{x_B} \end{pmatrix} = \boldsymbol{b}$$

Let's multiply the preceding system by $\boldsymbol{B}^{-1}$

$$\boldsymbol{B}^{-1} (\boldsymbol{N} \mid \boldsymbol{B}) \begin{pmatrix} \boldsymbol{x_N} \\ \boldsymbol{x_B} \end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x_N} + \boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

To determine the solutions of the system, we arbitrarly choose some values of $\boldsymbol{x_N}$ (the free variables) and we compute the remaining values of $\boldsymbol{x_B}$

$$\begin{cases} \boldsymbol{x_N} & = & \boldsymbol{s} \in \mathbb{R}^{n-m} \\ \boldsymbol{x_B} & = & \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{s} \end{cases}$$

Reminders About Linear Systems (Cont'd)

From the following equations:

$$\begin{cases} \boldsymbol{x_N} & = & \boldsymbol{s} \in \mathbb{R}^{n-m} \\ \boldsymbol{x_B} & = & \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{s} \end{cases}$$

we conclude that the set of solutions S of the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is given by

$$S = \left\{ \begin{pmatrix} \boldsymbol{x_N} \\ \boldsymbol{x_B} \end{pmatrix} \mid \boldsymbol{x_N} \in \mathbb{R}^{n-m}, \boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x_N} \right\}$$

Example

We consider the previous example given by the system $\boldsymbol{Ax} = \boldsymbol{b}$ with

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We select columns 2 and 4 of $\boldsymbol{A}$ to form a basis of the column space $\mathcal{C}(\boldsymbol{A})$ of matrix $\boldsymbol{A}$. The matrix $\boldsymbol{B}$ formed by these 2 columns and its inverse are

$$\boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{B}^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

Example (Cont'd)

Let's pre-multiply the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$ by $\boldsymbol{B}^{-1}$

$$\boldsymbol{B}^{-1}(\boldsymbol{A} \mid \boldsymbol{b}) = \left(\begin{array}{cccc|c} -2 & 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

The system $\boldsymbol{Ax} = \boldsymbol{b}$ expressed in base $\boldsymbol{B}$ is:

$$\begin{cases} x_2 = 1 + 2x_1 - 2x_3 \\ x_4 = \quad - x_1 \end{cases}$$

Example (Cont'd)

Consequently, the set of solutions of $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\begin{cases} x_1 &= & & & s & & \\ {\color{red}x_2} &= & 1 & + & 2s & - & 2t \\ x_3 &= & & & & & t \\ {\color{red}x_4} &= & & - & s & & \end{cases}$$

with $s, t \in \mathbb{R}$. This can be expressed as:

$$S = \left\{ \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mid \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, s, t \in \mathbb{R} \right\}$$

Bases and Basic Variables

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a system of linear equations with $\boldsymbol{A} : m \times n$, $m \leq n$ and $\text{rank}(\boldsymbol{A}) = m$. Then:

- Every matrix $\boldsymbol{B}$ formed by m linearly independent column of $\boldsymbol{A}$ is an ordered basis of the system
- Columns forming $\boldsymbol{B}$ are called basic, the other non-basic
- Variables corresponding to the columns of $\boldsymbol{B}$ are called basic, the other non-basic
- By extension, we call a basis the **ordered** list of basic variables or their indices. It is denoted by $\mathcal{B}$

Basic Solutions

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a system of equations with $\boldsymbol{A} : m \times n$, $m \leq n$ and $\text{rank}(\boldsymbol{A}) = m$. Let $\boldsymbol{B}$ be a base of $\boldsymbol{A}$. Then:

- In base $\boldsymbol{B}$, the system can be written as

$$\boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x_N} + \boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

- The particular solution obtained by fixing non-basic variables $\boldsymbol{x_N}$ to $\boldsymbol{0}$ is called a basic solution corresponding to base $\boldsymbol{B}$. It is given by:

$$\boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} \qquad \text{and} \qquad \boldsymbol{x_N} = \boldsymbol{0}$$

Example

We consider the system $\boldsymbol{Ax} = \boldsymbol{b}$ with an augmented matrix $(\boldsymbol{A}\ \boldsymbol{b})$ given by:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For base $\boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ $(\mathcal{B} = \{2, 4\})$,

- the basic variables are $\boldsymbol{x_B} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$
- the non-basic variables are $\boldsymbol{x_N} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$

The basic solution corresponding to $\boldsymbol{B}$ is

$$\boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x_N} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Basic Solution of a Standard LP

- Let's consider a standard LP:

$$\begin{array}{ccccc} \boldsymbol{A}\boldsymbol{x_D} & + & \boldsymbol{I}\boldsymbol{x_E} & = & \boldsymbol{b} \\ \boldsymbol{x_D} & , & \boldsymbol{x_E} & \geq & \boldsymbol{0} \end{array}$$

- The matrix $(\boldsymbol{A} \mid \boldsymbol{I})$ has a size of $m \times (n + m)$ and is of full rank

- If we choose the initial base $\boldsymbol{B}_0 = \boldsymbol{I}$ formed by the slack variables, then the basic solution is given by

$$\boldsymbol{x_B} = \boldsymbol{x_E} = \boldsymbol{b} \qquad \text{and} \qquad \boldsymbol{x_N} = \boldsymbol{x_D} = \boldsymbol{0}$$

- They are m basic variables and n non-basic variables

Basic Solution of a Standard LP (Cont'd)

For any basis $\boldsymbol{B}$, its basic solution is obtained by fixing the n non-basic variables to 0

- If the ith slack variable (x_{n+i}) is fixed to 0, then the solutions lie in the following hyperlan:

$$a_{i1}x_1 + \ldots + a_{in}x_n = b_i$$

- If the jth decision variable is fixed to 0, the solutions lie in the following hyperplan:

$$x_j = 0$$

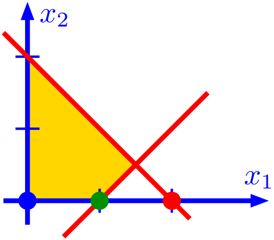
Consequently, the basic solution lies **at the intersection of the n hyerplans corresponding to the non-basic variables**

Basic Solution of a Standard LP (Cont'd)

- Consequence: for a problem with **two decision variables**, the basic solutions correspond to the intersections of the straight lines defined by the constraints, **including** non-negativity constraints
- If the basic solution lies on the edge of the feasible region, all the variables are non-negative and the solution is **feasible**
- If it is not the case, the solution is **not feasible**
- A base is **feasible** if and only if its basic solution is also **feasible**

Example

- The yellow zone is the feasible region
- The blue point is a **basic feasible** solution
- The green point is also a **basic feasible** solution
- The red point is a **basic solution** but it is not **feasible**



Pivoting: Reminder

A **pivoting** is a sequence of elementary operations on the rows of a matrix $(\boldsymbol{A} \mid \boldsymbol{b})$. More precisely, pivoting around a pivot $a_{ir} \neq 0$ corresponds to

(1) **Isolate variable x_r in the ith equation**. Concretely, divide the ith row of $(\boldsymbol{A} \mid \boldsymbol{b})$ by a_{ir}

(2) **Eliminate by substitution x_r in the other equations**. Concretely, substract to the other rows of $(\boldsymbol{A} \mid \boldsymbol{b})$ some multiple of the ith row in order to eliminate the variable x_r from the other equations

Example

We would like to find the basic solution corresponding to $\mathcal{B} = \{1, 2, 4\}$.

$$\left(\begin{array}{rrrr|r} 2 & 0 & -2 & 4 & 6 \\ -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 4 & -4 & -1 \end{array}\right)$$

Pivot around a_{11}:
Divide the first row by a_{11}
Remove x_1 from the second row

$$\left(\begin{array}{rrrr|r} 1 & 0 & -1 & 2 & 3 \\ 0 & -1 & -2 & 5 & 6 \\ 0 & 1 & 4 & -4 & -1 \end{array}\right)$$

Pivot around a_{22}:
Change the sign of the 2nd row
Remove x_2 from the third row

Example (Cont'd)

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 2 & -5 & -6 \\ 0 & 0 & 2 & 1 & 5 \end{array}\right)$$

Pivot around a_{34}

$$\left(\begin{array}{cccc|c} 1 & 0 & -5 & 0 & -7 \\ 0 & 1 & 12 & 0 & 19 \\ 0 & 0 & 2 & 1 & 5 \end{array}\right)$$

Final output

The basic solution corresponding to $\mathcal{B} = \{1, 2, 4\}$ is

$$x_1 = -7, \quad x_2 = 19, \quad x_4 = 5 \qquad \text{and} \qquad x_3 = 0$$

Pivoting Matrix

Pivoting corresponds to **pre**multiplying the matrix $(\boldsymbol{A} \mid \boldsymbol{b})$ by a **pivoting matrix**. **It differs from the identity matrix by only 1 column**. For a pivot a_{ir}, the pivoting matrix is

$$
\begin{array}{cc}
\begin{array}{ccccc} & & r\text{th col} & & \end{array} & \\
\begin{pmatrix}
1 & & -a_{1r}/a_{ir} & & 0 \\
 & & \vdots & & \\
0 & \dots & 1/a_{ir} & \dots & 0 \\
 & & \vdots & & \\
0 & & -a_{mr}/a_{ir} & & 1
\end{pmatrix} & \begin{array}{c} \\ \\ i\text{th row} \\ \\ \\ \end{array}
\end{array}
$$

Explanation: the first row of this matrix corresponds to the operation consisting in substracting a_{1r}/a_{ir} times the ith row to the first row; the ith row to the division of ith row by a_{ir}

Example

The three pivoting matrices of the previous example are:

$$\boldsymbol{P}_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \boldsymbol{P}_3 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

One can check that:

$$\boldsymbol{P}_3\boldsymbol{P}_2\boldsymbol{P}_1 \left(\begin{array}{rrrr|r} 2 & 0 & -2 & 4 & 6 \\ -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 4 & -4 & -1 \end{array}\right) = \left(\begin{array}{rrrr|r} 1 & 0 & -5 & 0 & -7 \\ 0 & 1 & 12 & 0 & 19 \\ 0 & 0 & 2 & 1 & 5 \end{array}\right)$$

Definition of a Tableau

A **tableau** is an augmented matrix corresponding to a **standard** LP

Stand. LP

$$
\begin{array}{rcrcrcl}
\multicolumn{7}{c}{\text{Max} \quad (z, \text{ s.t. } \boldsymbol{x_D}, \boldsymbol{x_E} \geq \mathbf{0})} \\
\boldsymbol{A}\boldsymbol{x_D} & + & \boldsymbol{I}\boldsymbol{x_E} & & & = & \boldsymbol{b} \\
\hline
-\boldsymbol{c_D}\boldsymbol{x_D} & - & \mathbf{0}\boldsymbol{x_E} & + & z & = & 0
\end{array}
$$

Tableau

$$
\boldsymbol{T}_0 = \begin{array}{c}
\begin{array}{cccc} \quad\boldsymbol{x_D}\quad & \quad\boldsymbol{x_E}\quad & z & \end{array} \\
\begin{array}{|c|c|c||c|}
\hline
\boldsymbol{A} & \boldsymbol{I} & \mathbf{0} & \boldsymbol{b} \\
\hline
-\boldsymbol{c_D} & \mathbf{0} & 1 & 0 \\
\hline
\end{array}
\end{array}
$$

Initial Basis Associated With a Tableau

- A tableau is always associated with a basis of a system of equations
- For a standard LP obtained after adding some slack variables to a canonical form, the initial basis of the system is formed by the columns of the slack variables. We have that $\boldsymbol{B} = \boldsymbol{I}$ and $\boldsymbol{B}^{-1} = \boldsymbol{I}$
- A tableau has an additional row corresponding to the objective function of the problem. The basis associated with the initial tableau is formed by column of slack variables + variable z

$$\hat{\boldsymbol{B}} = \left(\begin{array}{c|c} \boldsymbol{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right) \qquad \text{and} \qquad \hat{\boldsymbol{B}}^{-1} = \hat{\boldsymbol{B}}$$

Basis Associated With a Tableau

For some basis $\boldsymbol{B}$ of a system of constraints of a standard LP, the basis matrix of the tableau and its inverse are:

$$\hat{\boldsymbol{B}} = \left(\begin{array}{c|c} \boldsymbol{B} & 0 \\ \hline -\boldsymbol{c_B} & 1 \end{array}\right) \qquad \text{and} \qquad \hat{\boldsymbol{B}}^{-1} = \left(\begin{array}{c|c} \boldsymbol{B}^{-1} & 0 \\ \hline \boldsymbol{c_B}\boldsymbol{B}^{-1} & 1 \end{array}\right).$$

The tableau corresponding to basis $\boldsymbol{B}$ (more precisely $\hat{\boldsymbol{B}}$) is

$$\boldsymbol{T}_{\hat{\boldsymbol{B}}} = \hat{\boldsymbol{B}}^{-1}\boldsymbol{T}_0 = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} \boldsymbol{x_D} & \boldsymbol{x_E} & z & \\ \hline \multicolumn{1}{|c|}{\boldsymbol{B}^{-1}\boldsymbol{A}} & \multicolumn{1}{c|}{\boldsymbol{B}^{-1}} & \multicolumn{1}{c||}{0} & \multicolumn{1}{c|}{\boldsymbol{B}^{-1}\boldsymbol{b}} \\ \hline \multicolumn{1}{|c|}{\boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{A} - \boldsymbol{c_D}} & \multicolumn{1}{c|}{\boldsymbol{c_B}\boldsymbol{B}^{-1}} & \multicolumn{1}{c||}{1} & \multicolumn{1}{c|}{\boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{b}} \\ \hline \end{array}$$

In a tableau $\boldsymbol{T}$, the variable z is always the last variable in the basis. We can associate to $\boldsymbol{T}$ either $\boldsymbol{B}$ or $\hat{\boldsymbol{B}}$

Example

Let's consider the following problem in standard form:

$$
\begin{array}{lrcrcrcrcrcrcr}
\text{Max} & \multicolumn{13}{l}{(z, \text{ s.t. } x_1, x_2, x_3, x_4, x_5 \geq 0)} \\
\text{with} & 2x_1 & + & 3x_2 & + & x_3 & & & & & & & = & 42 \\
& -4x_1 & + & 6x_2 & & & + & x_4 & & & & & = & 0 \\
& x_1 & & & & & & & + & x_5 & & & = & 15 \\
\hline
& -250x_1 & - & 450x_2 & & & & & & & + & z & = & 0
\end{array}
$$

$$
\text{and} \quad \boldsymbol{T}_0 = \begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & z \end{array} \\
\left[\begin{array}{cc|ccc|c||c}
2 & 3 & 1 & 0 & 0 & 0 & 42 \\
-4 & 6 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 15 \\
\hline
-250 & -450 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

Reading The Basic Solution

Let $\boldsymbol{B}$ be a basis of the system of constraints of a standard LP.

$$
\boldsymbol{T}_0 \stackrel{P}{=}
\begin{array}{c|c|c||c|}
\boldsymbol{x_N} & \boldsymbol{x_B} & z & \\
\hline
\boldsymbol{N} & \boldsymbol{B} & 0 & \boldsymbol{b} \\
\hline
-\boldsymbol{c_N} & -\boldsymbol{c_B} & 1 & 0 \\
\hline
\end{array}
$$

$$
\boldsymbol{T}_{\hat{\boldsymbol{B}}} \stackrel{P}{=}
\begin{array}{c|c|c||c|}
\boldsymbol{x_N} & \boldsymbol{x_B} & z & \\
\hline
\boldsymbol{B}^{-1}\boldsymbol{N} & \boldsymbol{I} & 0 & \boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
\boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N} & 0 & 1 & \boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{b} \\
\hline
\end{array}
$$

The values of the basic variables in $\boldsymbol{B}$ can be read from the **last** column of the tableau:

$$
\boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} \qquad (\boldsymbol{x_N} = \boldsymbol{0})
$$

Feasible Tableau

- A tableau is **feasible** only and only if its **basic solution** is **feasible**:

$$\boldsymbol{T_B} \quad \text{is feasible} \qquad \Longleftrightarrow \qquad \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

Characteristics of a **feasible** tableau

$$\begin{array}{c|c|c||c|}
\multicolumn{1}{c}{\boldsymbol{x_D}} & \multicolumn{1}{c}{\boldsymbol{x_E}} & \multicolumn{1}{c}{z} & \multicolumn{1}{c}{} \\
\hline
 & & 0 & \oplus \\
 & & \vdots & \vdots \\
 & & 0 & \oplus \\
\hline
* \quad \ldots \quad * & * \quad \ldots \quad * & 1 & * \\
\hline
\end{array}$$

- The symbol $*$ means any real value and $\oplus$ any non-negative number (zero or a number larger than zero)

Example

The initial tableau below is feasible

$$\boldsymbol{T}_0 = \begin{array}{c} \\ \end{array} \begin{array}{cc|ccc|c||c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 2 & 3 & 1 & 0 & 0 & 0 & 42 \\ -4 & 6 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 15 \\ \hline -250 & -450 & 0 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

Its basic solution is:

$$\boldsymbol{x_B} = \boldsymbol{x_E} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 42 \\ 0 \\ 15 \end{pmatrix} \geq \mathbf{0} \quad \text{and} \quad \boldsymbol{x_N} = \boldsymbol{x_D} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Pivoting in a Tableau

- Pivoting around element (i, r) has the effect of replacing the ith variable in the basis by x_r
- In every tableau $\boldsymbol{T_B}$, the matrix $\hat{\boldsymbol{B}}^{-1}$ can be found below the slack variables and variable z

$$\boldsymbol{T_B} = \begin{array}{c|c|c||c} \boldsymbol{x_D} & \boldsymbol{x_E} & z & \\ \hline \boldsymbol{B}^{-1}\boldsymbol{A} & \boldsymbol{B}^{-1} & 0 & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \hline \boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{A} - \boldsymbol{c_D} & \boldsymbol{c_B}\boldsymbol{B}^{-1} & 1 & \boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{b} \end{array}$$

Pivoting and Adjacency

- Two bases are **adjacent** if they only differ by one variable
- This relationship is defined on **non-ordered** bases: $\mathcal{B}_1 = \{1, 7, 3, 5\}$ and $\mathcal{B}_2 = \{5, 1, 7, 2\}$ are adjacent
- Two basic solutions or two tableaus are **adjacent** if their bases are adjacent

Index of the Variables in the Current Basis

- To which variable corresponds the ith variable in the current basis ?
- We denote by σ the function returning the index of the ith basic variable
- For instance, we have $\sigma(1) = 2$, $\sigma(2) = 6$ and $\sigma(3) = 3$ for the basis $\mathcal{B} = \{2, 6, 3\}$

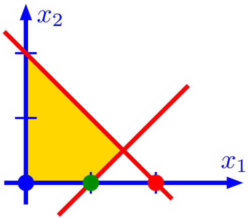
Geometric Interpretation of a Pivoting (1)

Let's consider the following canonical LP:

$$\begin{array}{lllllll} \text{Max} & z = & 2x_1 & + & x_2 & & \\ \text{s.t.} & & x_1 & + & x_2 & \leq & 2 \\ & & x_1 & - & x_2 & \leq & 1 \\ & & x_1 & , & x_2 & \geq & 0 \end{array}$$

$$\boldsymbol{T}_0 = \begin{array}{cccc|c|c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 1 & 1 & 1 & 0 & 0 & 2 \\ 1 & -1 & 0 & 1 & 0 & 1 \\ \hline -2 & -1 & 0 & 0 & 1 & 0 \end{array}$$

The **basic** solution corresponds to $x_3 = 2$, $x_4 = 1$, $x_1 = x_2 = 0$. We are at the origin of the axes (blue point). This solution is **feasible** and $\sigma(1) = 3$, $\sigma(2) = 4$



Geometric Interpretation of a Pivoting (2)

$$\begin{array}{lrcrcrcl} \text{Max} & z = & 2x_1 & + & x_2 & & \\ \text{s.t.} & & x_1 & + & x_2 & \leq & 2 \\ & & x_1 & - & x_2 & \leq & 1 \\ & & x_1 & , & x_2 & \geq & 0 \end{array}$$

$$T_0 = \begin{array}{cccc|c|c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 1 & 1 & 1 & 0 & 0 & 2 \\ 1 & -1 & 0 & 1 & 0 & 1 \\ \hline -2 & -1 & 0 & 0 & 1 & 0 \end{array}$$

If x_1 enters the basis, then it can replace x_3 or x_4. As x_2 keeps outside the basis, then we move along the x_1 axis

- If x_1 replaces x_3 (pivot α_{11}), then $x_1 = 2, x_2 = 0, x_3 = 0, x_4 = -1$ (red point). This solution is **not feasible**
- If x_1 replaces x_4 (pivot α_{21}), then $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ (green point). This solution is **feasible**