# Exercise Set - Week 1

These exercises are reminders of some concepts that you should master. If you do not feel comfortable with these exercises, please review your bachelor courses in mathematics. Here are some references :

- Linear Algebra, S. Lipschutz, M. Lipson, Schaum's outlines.
- Calculus, F. Ayres, E. Mendelson, Schaum's outlines.

## Problem 1

What are the solutions of the following systems of equations/inequations :

**a.**

$$x + y = 0$$

**b.**

$$\begin{cases} x + y &= 0 \\ x - y &= 2 \end{cases}$$

**c.**

$$\begin{cases} x + y &= 0 \\ x + y &= 1 \end{cases}$$

**d.**

$$x + y + z = 0$$

**e.**

$$\begin{cases} x + y &= 0 \\ x, y &\geq 0 \end{cases}$$

**f.**

$$\begin{cases} x - y &= 0 \\ x, y &\geq 0 \end{cases}$$

## Problem 2

We consider the following problem :

$$\max_{x,y} f(x, y) = x + y \tag{1}$$

$$2x + y \leq 4 \tag{2}$$

$$x, y \geq 0 \tag{3}$$

**a.** Draw the region defined by (2) and (3).

**b.** Draw the three straight lines defined by :

$$x + y = 0 \tag{4}$$

$$x + y = 2 \tag{5}$$

$$x + y = 4 \tag{6}$$

**c.** Use point **b.** to determine the optimal solution and its value.

## Problem 3

We consider the following function $f(x) = x^3 - 3x^2 + 2x$.

**a.** Give the zeros of this function.

**b.** Compute the derivative of $f$.

**c.** Compute the second derivative of $f$.

**d.** Show that $x_1 = 1 - \frac{\sqrt{12}}{6}$ is a local maximum and $x_2 = 1 + \frac{\sqrt{12}}{6}$ a local minimum.

## Problem 4

We consider a $3 \times 3$ matrix $\mathbf{A}$, a $3 \times 3$ matrix $\mathbf{B}$, and two vectors $\mathbf{a}$ and $\mathbf{b}$ with three elements. They are given by :

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Compute the following expressions :

**a.** $\mathbf{Ab}$.

**b.** $\mathbf{AB}$.

**c.** $\mathbf{a}^T\mathbf{A}$.

**d.** $\mathbf{a}^T\mathbf{Ab}$.

## Problem 5

A variant of Gaussian elimination called Gauss-Jordan elimination can be used for finding the inverse of a matrix, if it exists. If $\mathbf{A}$ is an $n \times n$ square matrix, then one can use row reduction to compute its inverse matrix, if it exists. First, the $n \times n$ identity matrix is augmented to the right of $\mathbf{A}$, forming an $n \times 2n$ block matrix $[\mathbf{A}|\mathbf{I}]$. Now through application of elementary row operations, find the reduced echelon form of this $n \times 2n$ matrix. The matrix $\mathbf{A}$ is invertible if and only if the left block can be reduced to the identity matrix $\mathbf{I}$; in this case the right block of the final matrix is $\mathbf{A}^{-1}$. If the algorithm is unable to reduce the left block to $\mathbf{I}$, then $\mathbf{A}$ is not invertible.

For example, consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

To find the inverse of this matrix, one takes the following matrix augmented by the identity and row-reduces it as a $3 \times 6$ matrix:

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right].$$

By performing row operations, one can check that the reduced row echelon form of this augmented matrix is

$$[\mathbf{I}|\mathbf{B}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right].$$

One can think of each row operation as the left product by an elementary matrix. Denoting by $\mathbf{B}$ the product of these elementary matrices, we showed, on the left, that $\mathbf{BA} = \mathbf{I}$, and therefore, $\mathbf{B} = \mathbf{A}^{-1}$. On the right, we kept a record of $\mathbf{BI} = \mathbf{B}$, which we know is the inverse desired. This procedure for finding the inverse works for square matrices of any size.

**Application** : compute the inverse of the matrix given below with the Gauss-Jordan elimination.

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$$

# Problem 6

We assume that $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices. Then one can show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

**Application**. We consider the following matrices :

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Compute :

**a.** $\mathbf{A}^{-1}$

**b.** $\mathbf{B}^{-1}$

**c.** $\mathbf{B}^{-1}\mathbf{A}^{-1}$

**d.** $\mathbf{C} = \mathbf{AB}$

**e.** $\mathbf{C}^{-1}$

# Problem 7

We consider the following linear system:

$$\begin{array}{llrcrcr} \text{Min} & z = & 5x_1 & + & x_2 & & \\ \text{s.t.} & & 2x_1 & + & x_2 & \geq & 6 \\ & & x_1 & + & x_2 & \geq & 4 \\ & & 2x_1 & + & 10x_2 & \geq & 20 \\ & & x_1 & & & \geq & 0 \\ & & & & x_2 & \geq & 0 \end{array}$$

Use the graphical method to solve it.

# Problem 8

A drugstore chain would like to sell its stock of vitamin A (3 tonnes) and of vitamin C (5 tonnes) in order to maximize its revenue. However, for the same amounts of vitamins, it should not be cheaper to buy fresh fruits.

| | kg of vitamin/tonne | | |
|---|---|---|---|
| Fruits | A | C | Price/tonne |
| Bananas | 6 | 7 | 42000 |
| Oranges | 4 | 8 | 20000 |
| Tomatos | 6 | 2 | 12000 |

a) Formulate this problem as a linear program and define explicitly what are the decision variables, the objective function, and the constraints.

b) Use the graphical method to solve this problem and give the optimal prices of the vitamins.