The Simplex Algorithm: Phase I

Optimization Methods in Management Science
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Introduction

Let's consider the following tableau :

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & z & \end{array} \\
T_0 = & \left[\begin{array}{ccccc|c|c}
-1 & -1 & 1 & 0 & 0 & 0 & -3 \\
0 & -1 & 0 & 1 & 0 & 0 & -2 \\
1 & -1 & 0 & 0 & 1 & 0 & 1 \\
\hline
-2 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

- To apply **phase II** of the simplex algorithm, we need a **feasible** tableau
- But this is not the case

Phase I of the Simplex Algorithm

Question

What can we do when a primal tableau is not feasible ?

- Several approaches are possible. We focus in this presentation on the creation of an **auxiliary** LP whose solution provides a **feasible solution to the initial problem**
- Once a feasible solution is found, we can apply the phase II of the simplex algorithm

Construction of the Auxiliary Problem (1)

If the inital tableau associated with a standard LP is not feasible, then the system of constraints is of type:

$$\begin{array}{lclclcll} \boldsymbol{A}_1\boldsymbol{x_D} & + & \boldsymbol{I}\boldsymbol{x}_{\boldsymbol{E}}^{\boldsymbol{1}} & & & = & \boldsymbol{b}_1 & (\boldsymbol{b}_1 \geq \boldsymbol{0}) \\ \boldsymbol{A}_2\boldsymbol{x_D} & + & & & \boldsymbol{I}\boldsymbol{x}_{\boldsymbol{E}}^{\boldsymbol{2}} & = & \boldsymbol{b}_2 & (\boldsymbol{b}_2 < \boldsymbol{0}) \end{array}$$

By adding $x_0 \geq 0$ to the last column for constraints having $b_i < 0$, we get:

$$\begin{array}{lclclclcl} \boldsymbol{A}_1\boldsymbol{x_D} & + & \boldsymbol{I}\boldsymbol{x}_{\boldsymbol{E}}^{\boldsymbol{1}} & & & = & \boldsymbol{b}_1 & & \\ \boldsymbol{A}_2\boldsymbol{x_D} & + & & & \boldsymbol{I}\boldsymbol{x}_{\boldsymbol{E}}^{\boldsymbol{2}} & = & \boldsymbol{b}_2 & + & \boldsymbol{1}x_0 \end{array}$$

If x_0 takes a value δ which is sufficently large ($\geq \max\{|b_i| \mid b_i < 0\}$), the solution $\boldsymbol{x_D} = \boldsymbol{0}$, $\boldsymbol{x}_{\boldsymbol{E}}^{\boldsymbol{1}} = \boldsymbol{b}_1$, $\boldsymbol{x}_{\boldsymbol{E}}^{\boldsymbol{2}} = \boldsymbol{b}_2 + \boldsymbol{1}\delta$ is feasible

Construction of the Auxiliary Problem (2)

Let's consider a standard LP (P)

$$\text{Max} \quad (z, \text{ s.t. } \boldsymbol{x_D}, \boldsymbol{x_E} \geq \boldsymbol{0}) \qquad (P)$$

$$\begin{array}{lrcrcrcrcl}
\text{with} & \boldsymbol{A}_1\boldsymbol{x_D} & + & \boldsymbol{I}\boldsymbol{x_E^1} & & & & & = & \boldsymbol{b}_1 \qquad (\boldsymbol{b}_1 \geq \boldsymbol{0}) \\
& \boldsymbol{A}_2\boldsymbol{x_D} & + & & & \boldsymbol{I}\boldsymbol{x_E^2} & & & = & \boldsymbol{b}_2 \qquad (\boldsymbol{b}_2 < \boldsymbol{0}) \\
\hline
& -\boldsymbol{c_D}\boldsymbol{x_D} & - & \boldsymbol{0}\boldsymbol{x_E^1} & - & \boldsymbol{0}\boldsymbol{x_E^2} & + & z & = & 0
\end{array}$$

The **auxiliary** problem (P^{aux}) associated to (P) is:

$$\text{Max} \quad (z', \text{ s.t. } \boldsymbol{x_D}, \boldsymbol{x_E} \geq \boldsymbol{0}, x_0 \geq 0) \qquad (P^{aux})$$

$$\begin{array}{lrcrcrcrcrcrcl}
\text{with} & & & \boldsymbol{A}_1\boldsymbol{x_D} & + & \boldsymbol{I}\boldsymbol{x_E^1} & & & & & & & = & \boldsymbol{b}_1 \\
& -\boldsymbol{1}x_0 & + & \boldsymbol{A}_2\boldsymbol{x_D} & + & & & \boldsymbol{I}\boldsymbol{x_E^2} & & & & & = & \boldsymbol{b}_2 \\
\hline
& & - & \boldsymbol{c_D}\boldsymbol{x_D} & - & \boldsymbol{0}\boldsymbol{x_E^1} & - & \boldsymbol{0}\boldsymbol{x_E^2} & + & z & & & = & 0 \\
\hline
& x_0 & & & & & & & & & + & z' & = & 0
\end{array}$$

The objective function of (P^{aux}) is Max $z' = -x_0 \iff -\text{Min } z' = x_0$

Characteristics the Auxiliary Problem

Characteristics of The Auxiliary Problem

- It always has a **feasible** solution
- It always has an **optimal** solution
- We can use the Phase II of the simplex algorithm to solve the **auxiliary** problem
- The **initial** problem has at least one **feasible** solution **if and only if** the **optimal** value of the **auxiliary** problem is **zero**
- If the **optimal** value of the **auxiliary** problem is **zero**, then it is easy to get a **feasible** tableau for the **initial** problem
- If we have determined a **feasible** solution to the **initial** problem after solving the auxiliary problem, then we can apply the Phase II of the simplex algorithm to solve the **initial** problem

Primal Simplex Algorithm: Phase I

Input: a **non-feasible** tableau
Output: a **feasible** tableau or a **certificate** that **no feasible solution exists**

(1) Construction of the auxiliary problem and of an initial feasible tableau:

- Introduce x_0 in all the constraints with $b_i < 0$
- Add the auxiliary objective function: Max $z' = -x_0$
- Enter x_0 into the basis by pivoting around α_{j1} where

$$j = \min\{i \mid b_i = \min\{b_k \mid b_k < 0\}\}$$

(2) Solve the auxiliary LP with phase II of the simplex algorithm with Bland's rule

- If $z' = 0$ at the optimum, remove the columns of x_0 and z' and the row of z'. The remaining tableau is feasible for the initial problem
- If $z' < 0$ at the optimum, the initial problem has no feasible solution

Example

Standard LP:

$$
\begin{array}{llrcrcrcr}
\text{Max} & z = & 2x_1 & - & x_2 & & & & \\
\text{s.t.} & & -x_1 & - & x_2 & + & x_3 & = & -3 \\
 & & & - & x_2 & + & x_4 & = & -2 \\
 & & x_1 & - & x_2 & + & x_5 & = & 1 \\
 & & x_i & \geq & 0 & & \multicolumn{3}{l}{i = 1, \ldots, 5}
\end{array}
$$

Initial tableau:

$$
T_0 = \begin{array}{ccccc|c|c}
x_1 & x_2 & x_3 & x_4 & x_5 & z & \\
\hline
-1 & -1 & 1 & 0 & 0 & 0 & -3 \\
0 & -1 & 0 & 1 & 0 & 0 & -2 \\
1 & -1 & 0 & 0 & 1 & 0 & 1 \\
\hline
-2 & 1 & 0 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
$$

The tableau is **not feasible**

Example (Cont'd)

We define the **auxiliary** problem:

$$\begin{array}{llrcrcrcrcr}
\text{Max} & z' = & -x_0 & & & & & & & & \\
\text{s.t.} & & -x_0 & - & x_1 & - & x_2 & + & x_3 & = & -3 \\
 & & -x_0 & & & - & x_2 & + & x_4 & = & -2 \\
 & & & & x_1 & - & x_2 & + & x_5 & = & 1 \\
 & & & - & 2x_1 & + & x_2 & + & z & = & 0 \\
 & & & & x_i & \geq & 0 & & \multicolumn{3}{l}{i = 0, \dots, 5}
\end{array}$$

Example (Cont'd)

Initial tableau of the **auxiliary** problem:

$$T_0^{aux} = \begin{array}{c} \begin{array}{ccccccccc} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & z & z' & \end{array} \\ \left[\begin{array}{c|ccccc|c|c|c} \mathbf{-1} & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -3 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The variable x_0 enters the basis and we pivot arount α_{11}

Example (Cont'd)

Then we apply the simplex algorithm:

$$T_1^{aux} = \begin{array}{c|cccccc|c|c|c|}
\multicolumn{1}{c}{x_0} & x_1 & x_2 & x_3 & x_4 & \multicolumn{1}{c}{x_5} & \multicolumn{1}{c}{z} & \multicolumn{1}{c}{z'} & \multicolumn{1}{c}{} \\ \hline
1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 3 \\
0 & \mathbf{1} & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline
0 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & -3 \\ \hline
\end{array}$$

Example (Cont'd)

$$T_2^{aux} = \begin{array}{c|ccccc|c|c|c|}
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & z & z' & \\
\hline
1 & 0 & \mathbf{1} & 0 & -1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & -2 & 2 & 0 & 1 & 0 & 2 \\
\hline
0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & -2 \\
\hline
\end{array}$$

Example (Cont'd)

$$T_3^{aux} = \begin{array}{c|c|c|c|c|c|c|c|c}
x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & z & z' & \\
\hline
1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\
\hline
-1 & 0 & 0 & -2 & 3 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\end{array}$$

- As soon as x_0 exits the basis, then $z' = 0$
- As $z' = 0$, this tableau is **optimal** for the objective funtion z'
- We get an **initial feasible tableau** for the **initial** problem by removing the columns x_0 and z' and the last row

Example (Cont'd)

- We get the following **feasible** tableau for the initial problem:

$$T_0 = \begin{array}{c} \\ \end{array}\begin{array}{ccccc|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \hline 0 & 1 & 0 & -1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ \hline 0 & 0 & -2 & 3 & 0 & 1 & 0 \end{array}$$

- Its basic solution is given by $x_1 = 1, x_2 = 2, x_5 = 2, x_3 = x_4 = 0$
- To determine the optimal solution, then we have to apply phase II of the simplex algorithm