Exercise Set 2

Problem 1

Formulate the following LP in canonical and standard forms.

a)
$$\begin{array}{llrcrcl}
\text{Max} & z = & x_1 & + & x_2 & & \\
\text{s.t.} & & 2x_1 & + & x_2 & \leq & 6 \\
& & x_1 & + & 2x_2 & \geq & 1 \\
& & x_1 & , & x_2 & \geq & 0
\end{array}$$

b)
$$\begin{array}{llrcrcr}
\text{Max} & z = & 2x_1 & - & x_2 & & \\
\text{s.t.} & & \frac{1}{3}x_1 & + & x_2 & = & 2 \\
& & -2x_1 & + & 5x_2 & \leq & 7 \\
& & x_1 & & & \geq & 0 \\
& & & & x_2 & \geq & -3
\end{array}$$

c)
$$\begin{array}{llrcrcrcl}
\text{Min} & z = & -x_1 & & & - & x_3 & & \\
\text{s.t.} & & x_1 & + & \frac{1}{2}x_2 & - & 3x_3 & \geq & 2 \\
& & & & 4x_2 & + & x_3 & = & 5 \\
& & x_1 & & , & & x_3 & \geq & 0 \\
& & & & x_2 & & & \leq & 0
\end{array}$$

Problem 2

Formulate as LPs the following optimization problems:

a)
$$\begin{array}{llrcrcrcl}
\text{Min} & z & = & \multicolumn{6}{l}{|2x_3 - x_1| + x_2} \\
\text{s.t.} & & & 4x_1 & - & x_2 & + & 2x_3 & = 6 \\
& & & & & 2x_2 & - & 4x_3 & \geq 4 \\
& & & x_1 & , & x_2 & , & x_3 & \geq 0
\end{array}$$

b)
$$\begin{array}{llrcrcrcl}
\text{Max} & z & = & \multicolumn{6}{l}{x_1 + \min\{2x_2 - 4, 4x_3 + x_1\}} \\
\text{s.t.} & & & x_1 & + & 3x_2 & + & 2x_3 & \leq 9 \\
& & & & & 5x_2 & - & x_3 & = 4 \\
& & & x_1 & , & x_2 & , & x_3 & \geq 0
\end{array}$$

c)
$$\begin{array}{llrcrcrcl}
\text{Max} & z & = & \multicolumn{6}{l}{x_1 + |x_1 - x_2|} \\
\text{s.t.} & & & x_1 & + & x_2 & & & \leq 5 \\
& & & & & x_2 & - & x_3 & \leq 7 \\
& & & 5x_1 & + & 2x_2 & - & 8x_3 & \leq 5 \\
& & & x_1 & , & x_2 & , & x_3 & \geq 0
\end{array}$$

Problem 3

Invert the matrix $\mathbf{A}$ by Gauss elimination and compute a factorization of $\mathbf{A}$ and $\mathbf{A}^{-1}$ in elementary matrices.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 2 \\ 3 & 0 & 4 \end{pmatrix}$$

Reminders:

- an elementary matrix $n \times n$ is denoted by
  a) $\mathbf{E}_i(c)$ if it is obtained by multiplying by c the row i of $\mathbf{I}_n$. Its inverse is $\mathbf{E}_i(1/c)$
  b) $\mathbf{E}_{ij}$ if it is obtained by permuting the row i and the row j of $\mathbf{I}_n$. Its inverse is $\mathbf{E}_{ij}$
  c) $\mathbf{E}_{ij}(c)$ if it is obtained by adding c times the row j to the row i of $\mathbf{I}_n$. Its inverse is $\mathbf{E}_{ij}(-c)$
- The inverse of the product of two invertible matrices $\mathbf{AB}$ is given by $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Problem 4

We consider the following vectors and matrix: $\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 & 0 & 5 \\ 2 & -4 & 5 & 6 & 11 \\ 3 & -6 & 8 & 10 & 11 \\ 0 & 0 & 1 & 5 & -2 \end{pmatrix}$,

$\mathbf{b} = \begin{pmatrix} 7 \\ 10 \\ 3 \\ -9 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$.

a) Determine the solution of the system $\mathbf{Ax} = \mathbf{b}$.

b) What is the rank of $\mathbf{A}$?

c) What is the dimension of the column space of $\mathbf{A}$? Give a basis of this space.

d) What is the dimension of the row space of $\mathbf{A}$? Give a basis of this space.

Reminders: let $\mathbf{A}$ be a $m \times n$ matrix. Then

- $rank(\mathbf{A})$ = dim of the column space of $\mathbf{A}$ = dim of the row space of $\mathbf{A}$
- $rank(\mathbf{A})$ = number of pivots in any echelon form of $\mathbf{A}$
- $rank(\mathbf{A})$ = the max number of linearly independent rows or columns of $\mathbf{A}$