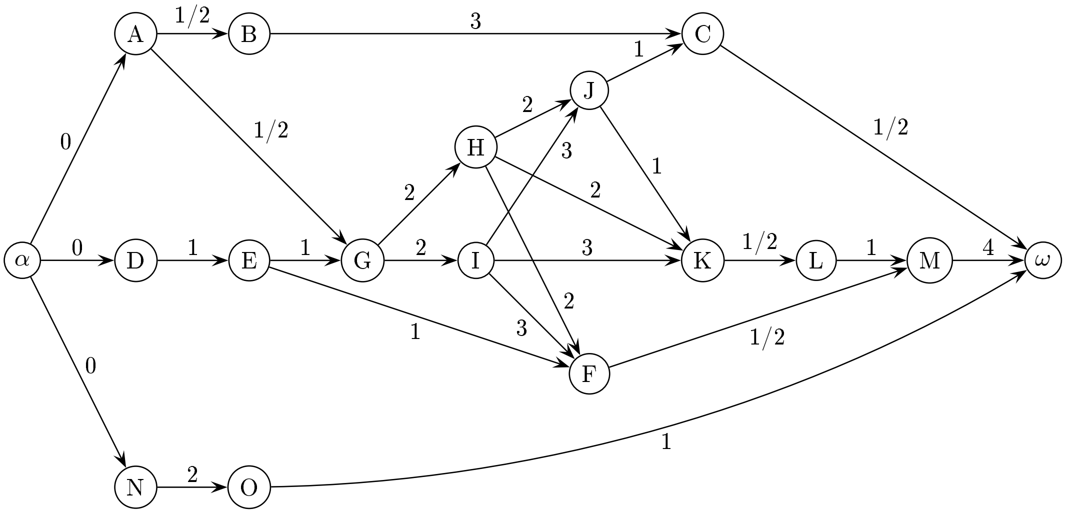
Solutions to Exercise Set 6

Problem 1

a) After adding fictive tasks α and ω, we get the following PERT network:



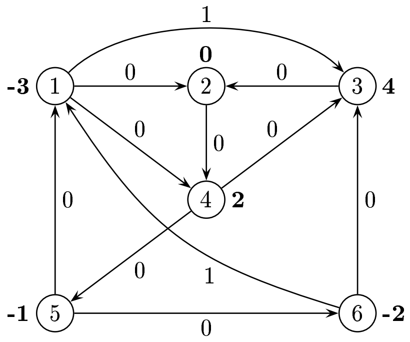
b) Critical Path Algorithm:

Number	Task	Duration	Pred.	Succ.	Start at the earliest δ_i	Start at the latest φ_i
1	α	0	-	A, D, N	0	0
2	A	1/2	α	B, G	0	1 1/2
3	D	1	α	E	0	0
4	N	2	α	O	0	10 1/2
5	B	3	A	C	1/2	10
6	E	1	D	G, F	1	1
7	O	1	N	ω	2	12 1/2
8	G	2	A, E	H, I	2	2
9	H	2	G	F, J, K	4	5
10	I	3	G	F, J, K	4	4
11	J	1	H, I	C, K	7	7
12	F	1/2	E, H, I	M	7	9
13	C	1/2	B, J	ω	8	13
14	K	1/2	H, I, J	L	8	8
15	L	1	K	M	8 1/2	8 1/2
16	M	4	L, F	ω	9 1/2	9 1/2
17	ω	0	C,M,O	-	13 1/2	13 1/2

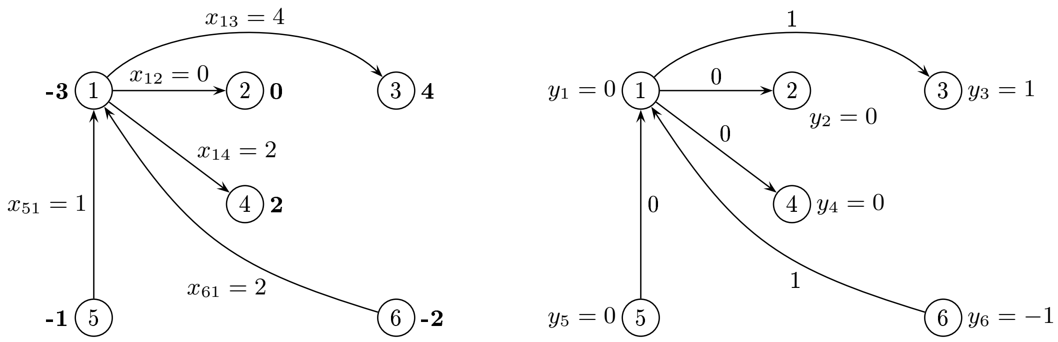
The minimal duration is 13.5 days. Critical tasks are D, E, G, I, J, K, L and M.

Problem 2

a) Definition of the auxiliary graph.
Auxiliary network $R'(V,E',b,c')$:



b) Initial tree-solution for $R'(V,E',b,c')$.
Initial tree-solution formed by arcs $\{(1,2),(1,3),(1,4),(5,1),(6,1)\}$.
The cost of this solution is $z_{aux} = \sum_{(i,j)\in T'} c_{ij}x_{ij} = 0+4+0+0+2 = 6$ (where T' is the set of basic arcs).



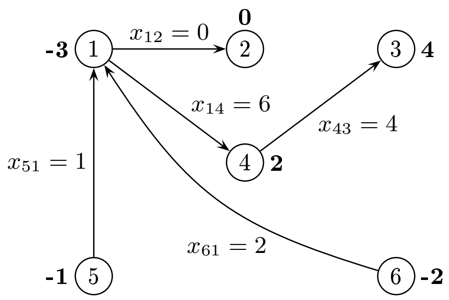
Remark. We have added arc (1,2) to form a tree even though it is not used ($x_{12} = 0$). We could also have added arcs (3,2) or (2,4). This is a degenerated solution.
To compute the dual basic solution, we set $y_1 = 0$ and we visit the tree-solution in the order: 2, 3, 4, 5, 6. We get $y_2 = 0$, $y_3 = 1$, $y_4 = 0$, $y_5 = 0$, and $y_6 = -1$.
The value of this solution is $w_{aux} = \sum_{i\in V} b_i y_i = 0+0+4+0+0+2 = 6$.

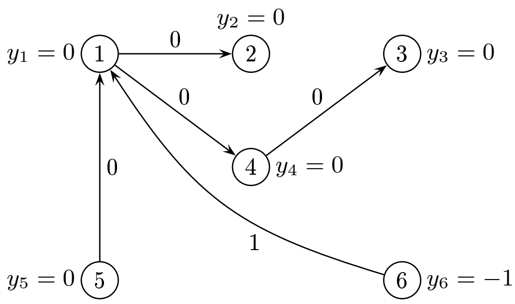
First iteration. We look for a violated dual constraint:

$$\begin{array}{lcl} (2,4) & : & y_4 - y_2 - c_{24} = 0-0-0 \le 0 \\ (3,2) & : & y_2 - y_3 - c_{32} = 0-1-0 \le 0 \\ (4,3) & : & y_3 - y_4 - c_{43} = 1-0-0 > 0 \end{array}$$

The arc (4,3) enters the basis. The cycle (4,3), (1,3), (1,4) has only one arc with an opposite orientation: (1,3). $\Delta = 4$ and the arc (1,3) exits the basis.
The new tree-solution is form by $\{(1,2),(1,4),(4,3),(5,1),(6,1\}$.
The new optimal solution is $x_{43} = 0+4 = 4$, $x_{13} = 4-4 = 0$, and $x_{14} = 2+4 = 6$, the other values are not modified.

The only dual variable that is modified is y_3. It decreases in value by $\varepsilon = y_3 - y_4 - c_{43} = 1 - 0 - 0 = 1$. The value of the new basic solutions is $z_{aux} = w_{aux} = 6 - 4 = 2$.

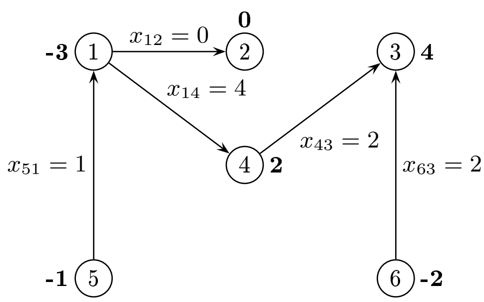


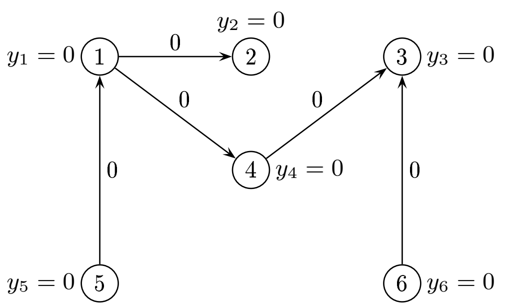


Second iteration. We look for a violated dual constraint:

$$
\begin{array}{lcl}
(1,3) & : & y_3 - y_1 - c_{13} = 0 - 0 - 1 \leq 0 \\
(2,4) & : & y_4 - y_2 - c_{24} = 0 - 0 - 0 \leq 0 \\
(3,2) & : & y_2 - y_3 - c_{32} = 0 - 0 - 0 \leq 0 \\
(4,5) & : & y_5 - y_4 - c_{45} = 0 - 0 - 0 \leq 0 \\
(5,6) & : & y_6 - y_5 - c_{56} = -1 - 0 - 0 \leq 0 \\
(6,3) & : & y_3 - y_6 - c_{63} = 0 + 1 - 0 > 0
\end{array}
$$

The arc (6,3) enters the basis. The cycle (6,3), (4,3), (1,4), (6,1) has three arcs with an opposite orientation: (4,3), (1,4) et (6,1). $\Delta = \min(x_{43}, x_{14}, x_{61}) = \min(4,6,2) = 2$ and the arc (6,1) exits the basis.
The new tree-solution is $\{(1,2),(1,4),(4,3),(5,1),(6,3)\}$.
The new primal solution is $x_{14} = 6 - 2 = 4$, $x_{43} = 4 - 2 = 2$ $x_{61} = 2 - 2 = 0$ and $x_{63} = 0 + 2 = 2$, the other values are not modified.
The only dual variable to be modified is y_6. She increases in value by $\varepsilon = y_3 - y_6 - c_{63} = 0 + 1 - 0 = 1$.
The value of the new basic solutions is $z_{aux} = w_{aux} = 2 - 2 = 0$.



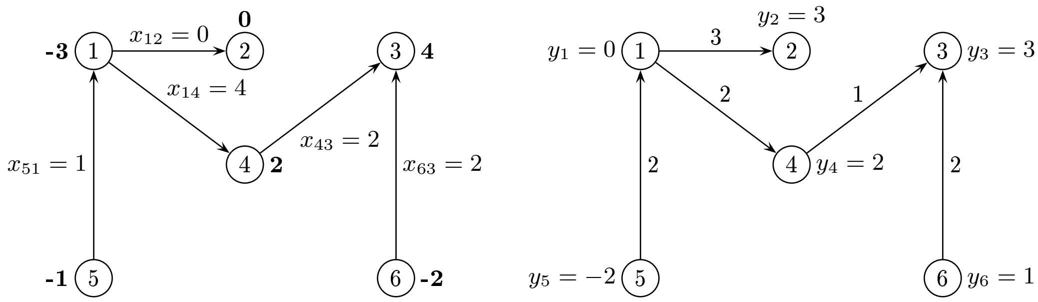


Third iteration. We look for a violated dual constraint:

$$
\begin{array}{lcl}
(2,4) & : & y_4 - y_2 - c_{24} = 0 - 0 - 0 \leq 0 \\
(3,2) & : & y_2 - y_3 - c_{32} = 0 - 0 - 0 \leq 0 \\
(4,5) & : & y_5 - y_4 - c_{45} = 0 - 0 - 0 \leq 0 \\
(5,6) & : & y_6 - y_5 - c_{56} = 0 - 0 - 0 \leq 0
\end{array}
$$

All the dual constraints are satisfied. The current basic solutions are optimal for the auxiliary network. Its value is 0 meaning that we have found a feasible solution for the initial problem.
Initial tree-solution for $R(V,E,b,c)$.

To get the initial feasible solution, we need to remove the artificial edges and to restore the initial weighting.
The cost of this solution is $z = \sum_{(i,j)\in T} c_{ij}x_{ij} = 0 + 8 + 2 + 2 + 4 = 16$ (where T is the set of basic arcs).



In order to compute the dual basic solution, we first set $y_1 = 0$ and we visit the tree-solution in the order 2, 4, 3, 5, 6. We get $y_2 = 3$, $y_4 = 2$, $y_3 = 3$, $y_5 = -2$, and $y_6 = 1$.
The value of this solution is $w = \sum_{i\in V} b_i y_i = 0 + 0 + 12 + 4 + 2 - 2 = 16$

c) First iteration. We look for a violated dual constraint.

$$\begin{array}{lcl}
(2,4) & : & y_4 - y_2 - c_{24} = 2 - 3 - 4 \leq 0 \\
(3,2) & : & y_2 - y_3 - c_{32} = 3 - 3 - 1 \leq 0 \\
(4,5) & : & y_5 - y_4 - c_{45} = -2 + 2 - 1 \leq 0 \\
(5,6) & : & y_6 - y_5 - c_{56} = 1 + 2 - 5 \leq 0
\end{array}$$

All the dual constraints are satisfied. Current basic solutions are optimal.
The optimal transportation planing is given by the solution abobe and has a cost of 16.
Remark. The optimal solution is degenerated. We could replace the arc (1,2) by the arc (3,2) or (2,4) without affecting the optimal plan.

Problem 3

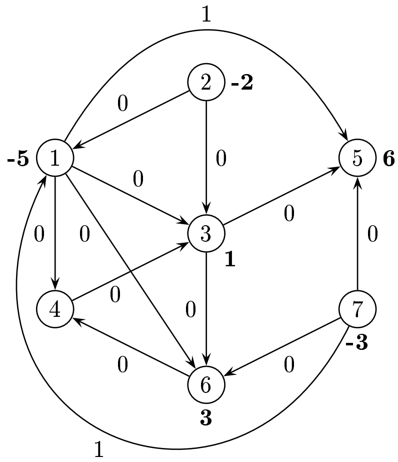
a) In order to get a transshipment problem from the shortest path problem from s to t, we just need to associate a supply of 1 with s and a demand of 1 with t and to consider the costs of the edges as their length.
b) In order to get a transshipment problem from the shortest path problem from s to any other vertices, we just need to associate a supply of $|V| - 1$ to s and a demand of 1 with the other vertices.

Problem 4

a) Definition of the auxiliary graph.
   - Let's consider vertex 1 as the main source.
   - We connect each source $s \neq 1$ of the graph to vertex 1 with an arc with a unit cost of 1. If such an arc already exists, we don't add it.
     So we add the arc (7,1).

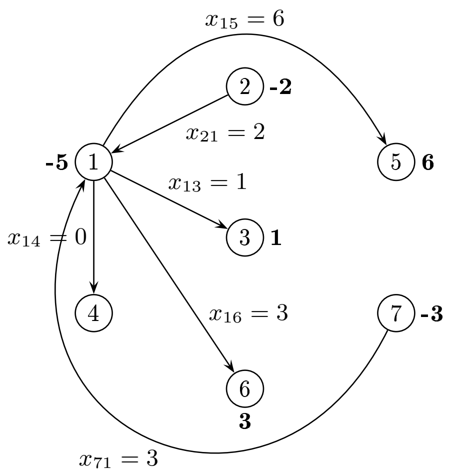
– We connect vertex 1 to each sink p with an artificial arc $(1,p)$ of cost 1 if such an arc doesn't exist. So we add the arc (1,5).
– We modify the unit cost of the arcs of the original graph. All of them are set to 0.

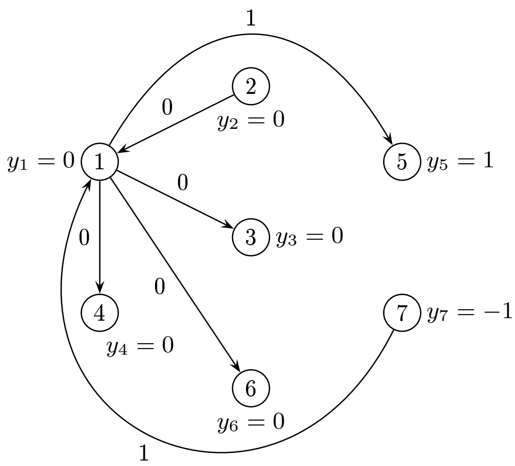
We get the following auxiliary network $R'(V,E',b,c')$:



b) Initial tree-solution for $R'(V,E',b,c')$.

The initial tree-solution is formed by the arcs {(1,3),(1,4),(1,6),(2,1),(1,5),(7,1)}.
The cost of this solution is $z_{aux} = \sum_{(i,j)\in T'} c_{ij}x_{ij} = 6+0+0+0+3 = 9$ (where T' is the set of basic arcs).





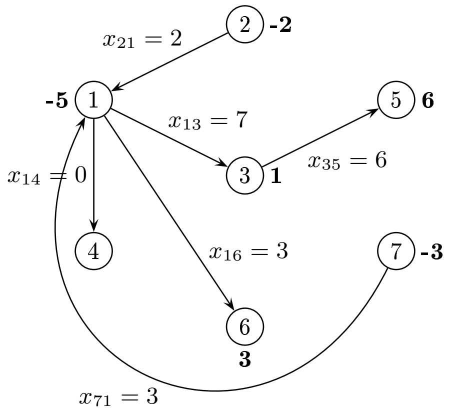
Remark. Even though it is not used ($x_{14} = 0$), we add the arc (1,4) to get a tree. This solution is degenerated.
To compute the dual basic solution, we set $y_1 = 0$ and we visit the vertices in the following order: 2, 3, 4, 5, 6, 7. We get $y_2 = 0$, $y_3 = 0$, $y_4 = 0$, $y_5 = 1$, $y_6 = 0$ et $y_7 = -1$.
The value of this solution is $w_{aux} = \sum_{i\in V} b_i y_i = 0+0+0+0+6+0+3 = 9$ (obviously $z_{aux} = w_{aux}$).

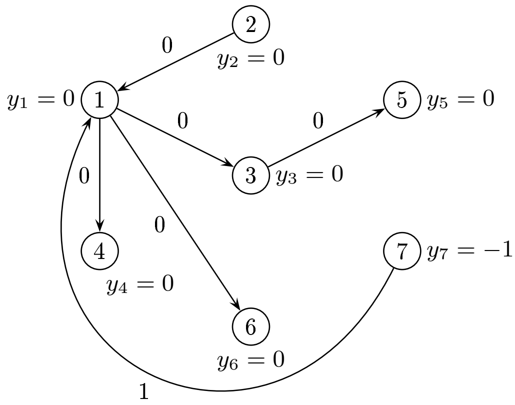
First iteration. We look for a violated dual constraint:

$$\begin{aligned}
(2,3) &\quad : \quad y_3 - y_2 - c_{23} = 0 - 0 - 0 \leq 0 \\
(3,5) &\quad : \quad y_5 - y_3 - c_{35} = 1 - 0 - 0 > 0
\end{aligned}$$

The arc (3,5) enters the basis. The circuit (3,5), (1,5), (1,3) has only one arc with an opposite orientation: (1,5). $\Delta = x_{15} = 6$ and the (1,5) exits the basis.

The new tree-solution is formed by the arcs {(1,3),(1,4),(1,6),(3,5),(7,1)}. The new primal solution $x_{35} = 6$, $x_{13} = 1 + 6 = 7$, and $x_{15} = 0$, the other values aren't modified.
The dual variable y_5 is modofied. It decreases in value by $\varepsilon = y_5 - y_3 - c_{35} = 1 - 0 - 0 = 1$.
The value of the new basic solutions is $z_{aux} = w_{aux} = 9 - 6 + 0 + 0 = 3$.

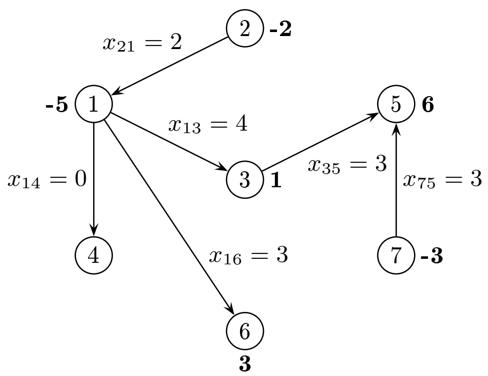


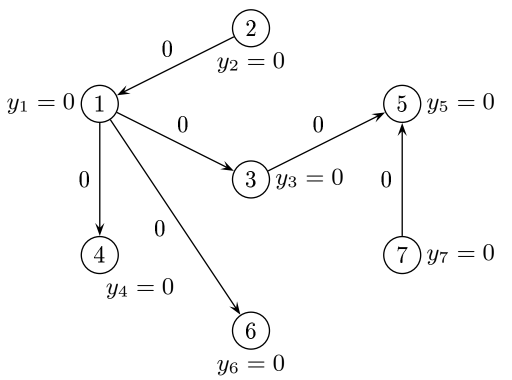


Second iteration. We look for a violated dual constraint:

$$
\begin{array}{lcl}
(1,5) & : & y_5 - y_1 - c_{15} = 0 - 0 - 0 \leq 0 \\
(2,3) & : & y_3 - y_2 - c_{23} = 0 - 0 - 0 \leq 0 \\
(3,6) & : & y_6 - y_3 - c_{36} = 0 - 0 - 0 \leq 0 \\
(4,3) & : & y_3 - y_4 - c_{43} = 0 - 0 - 0 \leq 0 \\
(6,4) & : & y_4 - y_6 - c_{64} = 0 - 0 - 0 \leq 0 \\
(7,5) & : & y_5 - y_7 - c_{75} = 0 + 1 - 0 > 0
\end{array}
$$

The arc (7,5) enters the base. The circuit (7,5), (3,5), (1,3), (7,1) has three arcs with an opposite direction: (3,5), (1,3) et (7,1). $\Delta = \min(x_{35}, x_{13}, x_{71}) = x_{71} = 3$ and the arc (7,1) exits the basis.
The new tree-solution is formed by the arcs {(1,3),(1,4),(1,6),(3,5),(7,5)}. The new primal solution is $x_{75} = 3$, $x_{35} = 6 - 3 = 3$, $x_{13} = 7 - 3 = 4$, and $x_{71} = 0$, the other values arn't modified.
The dual variable y_7 is modified. It increases in value by $\varepsilon = y_5 - y_7 - c_{64} = 0 + 1 - 0 = 1$.
The value of the new basic solutions is $z_{aux} = w_{aux} = 3 + 0 - 0 - 0 - 3 = 0$.





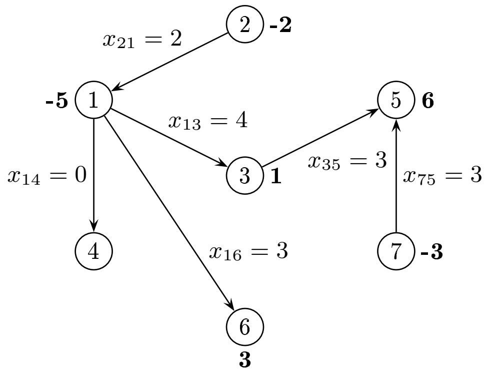
Third iteration. We look for a violated dual constraint:

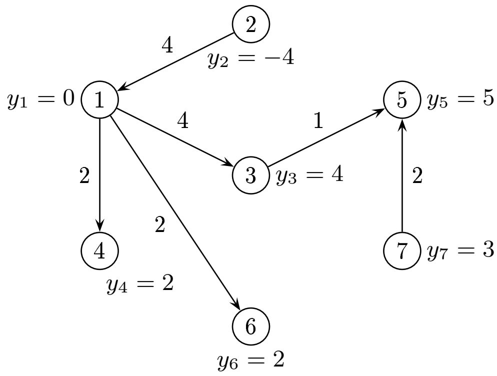
$$\begin{array}{rcl}
(1,5) & : & y_5 - y_1 - c_{15} = 0 - 0 - 0 \leq 0 \\
(2,3) & : & y_3 - y_2 - c_{23} = 0 - 0 - 0 \leq 0 \\
(3,6) & : & y_6 - y_3 - c_{36} = 0 - 0 - 0 \leq 0 \\
(4,3) & : & y_3 - y_4 - c_{43} = 0 - 0 - 0 \leq 0 \\
(6,4) & : & y_4 - y_6 - c_{64} = 0 - 0 - 0 \leq 0 \\
(7,1) & : & y_5 - y_7 - c_{75} = 0 - 0 - 0 \leq 0 \\
(7,6) & : & y_5 - y_7 - c_{75} = 0 - 0 - 0 \leq 0
\end{array}$$

All the dual constraints are satisfied. The current basic solutions are optimal for the auxiliary network. As the total cost is 0, then this solution is also feasible for the initial network $R(V,E,b,c)$.

Initial tree-solution for $R(V,E,b,c)$.
To get the initial feasible solution, we need to remove the artificial arcs and to relace the costs by the ones of the initial problem.

The cost of this solution is $z = \sum_{(i,j)\in T} c_{ij}x_{ij} = 16 + 0 + 6 + 8 + 3 + 6 = 39$ (where T is the set of basic arcs).





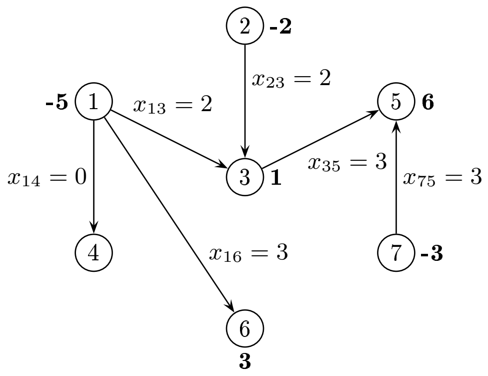
For the computation of the dual basic solution, we set $y_1 = 0$ and we visit the vertices in the lexicographical order: 2, 3, 5, 7, 4, 6. We get $y_2 = -4$, $y_3 = 4$, $y_5 = 5$, $y_7 = 3$, $y_4 = 2$, and $y_6 = 2$.
The value of this solution is $w = \sum_{i\in V} b_i y_i = 0 + 8 + 4 + 0 + 30 + 6 - 9 = 39$ (obviously $z = w$).

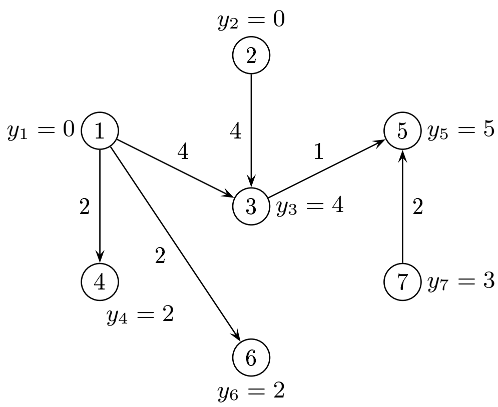
c) First iteration. We look for a violated dual constraint:

$$(2,3) \quad : \quad y_3 - y_2 - c_{23} = 4 + 4 - 4 > 0$$

The arc (2,3) enters the basis. The circuit (2,3), (1,3), (2,1) has two arcs with an opposite orientation: (1,3) and (2,1). $\Delta = \min(x_{13},x_{21}) = x_{21} = 2$ and the arc (2,1) exits the basis.
The new tree-solution is formed by the arcs $\{(1,3),(1,4),(1,6),(2,3),(3,5),(7,5)\}$. The new primal solution is $x_{23} = 2$, $x_{13} = 4 - 2 = 2$, and $x_{21} = 0$, the other values aren't modified.
The dual variable y_2 is modified. It decreases in value by $\varepsilon = y_3 - y_2 - c_{23} = 4 + 4 - 4 = 4$.

The value of the new basic solutions is: $z = w = 39 - 8 + 8 - 8 = 31$.

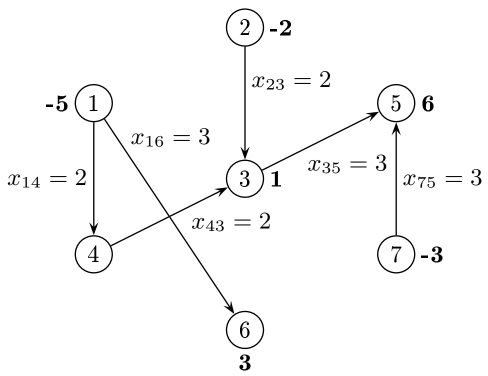


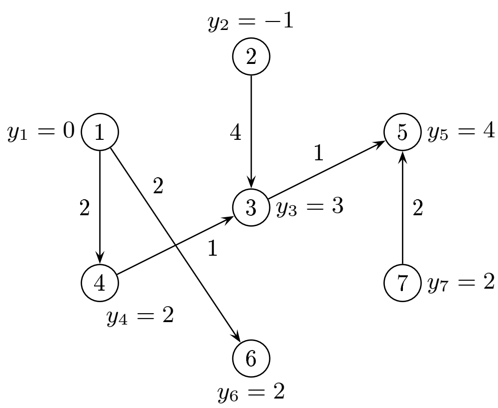


Second iteration. We look for a violated dual constraint:

$$\begin{array}{lcl} (2,1) & : & y_1 - y_2 - c_{21} = 0 - 0 - 4 \leq 0 \\ (3,6) & : & y_6 - y_3 - c_{36} = 2 - 4 - 1 \leq 0 \\ (4,3) & : & y_3 - y_4 - c_{43} = 4 - 2 - 1 > 0 \end{array}$$

The arc (4,3) enters the basis. The circuit (4,3), (1,3), (1,4) has only one arc with an opposite orientation (1,3). $\Delta = x_{13} = 2$ and the arc (1,3) exits the basis.
The new tree-solution is formed by the arcs {(1,4),(1,6),(2,3),(3,5),(4,3),(7,5)}. The new primal solution is $x_{43} = 2$, $x_{14} = 0 + 2 = 2$, and $x_{13} = 0$, the other values aren't modified.
The dual variables y_2, y_3, y_5 et y_7 are modified. They decrease in value by $\varepsilon = y_3 - y_4 - c_{43} = 4 - 2 - 1 = 1$.
The value of the new basic solutions is: $z = w = 31 - 8 + 4 + 2 = 29$.





Third iteration. We look for a violated dual constraint:

$$\begin{array}{lcl} (1,3) & : & y_3 - y_1 - c_{13} = 3 - 0 - 4 \leq 0 \\ (2,1) & : & y_1 - y_2 - c_{21} = 0 + 1 - 4 \leq 0 \\ (3,6) & : & y_6 - y_3 - c_{36} = 2 - 3 - 1 \leq 0 \\ (6,4) & : & y_4 - y_6 - c_{64} = 2 - 2 - 2 \leq 0 \\ (7,6) & : & y_6 - y_7 - c_{76} = 2 - 2 - 2 \leq 0 \end{array}$$

All the dual constraints are satisfied. The current basic solutions are optimal.
The optimal transportation planing is given by the basic solution above and has a cost of 29.