Solutions to Exercise Set 2

Problem 1

a) Canonical form:

$$\begin{array}{llrcrcr} \text{Max} & z = & x_1 & + & x_2 & & \\ \text{s.t.} & & 2x_1 & + & x_2 & \leq & 6 \\ & & -x_1 & - & 2x_2 & \leq & -1 \\ & & x_1 & , & x_2 & \geq & 0 \end{array}$$

Standard form:

$$\begin{array}{ll} \text{Max} & z = x_1 + x_2 \\ \text{s.t.} & \end{array}$$

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & 6 \\ -x_1 & - & 2x_2 & + & x_4 & = & -1 \end{array}$$

with $x_1,x_2,x_3,x_4 \geq 0$.

b) i. First solution: we consider that $x_2 \in \mathbb{R}$ and that $x_2 \geq -3$ is a constraint.

Canonical form:

$$\begin{array}{llrcrcrcr} \text{Max} & z = & 2x_1 & - & x_2^+ & + & x_2^- & & \\ \text{s.t.} & & \frac{1}{3}x_1 & + & x_2^+ & - & x_2^- & \leq & 2 \\ & & -\frac{1}{3}x_1 & - & x_2^+ & + & x_2^- & \leq & -2 \\ & & -2x_1 & + & 5x_2^+ & - & 5x_2^- & \leq & 7 \\ & & & - & x_2^+ & + & x_2^- & \leq & 3 \\ & & x_1 & , & x_2^+ & , & x_2^- & \geq & 0 \end{array}$$

Standard form:

$$\begin{array}{llcrcrcrcr} \text{Max} & \multicolumn{9}{l}{z = 2x_1 - x_2^+ + x_2^-} \\ \text{s.t.} & \frac{1}{3}x_1 & + & x_2^+ & - & x_2^- & + & x_3 & = & 2 \\ & -\frac{1}{3}x_1 & - & x_2^+ & + & x_2^- & + & x_4 & = & -2 \\ & -2x_1 & + & 5x_2^+ & - & 5x_2^- & + & x_5 & = & 7 \\ & & & -x_2^+ & + & x_2^- & + & x_6 & = & 3 \end{array}$$

with $x_1,x_2^+,x_2^-,x_3,x_4,x_5,x_6 \geq 0$

ii. Second solution: change of variable $x_2' = x_2 + 3$.

Canonical form:

$$\begin{array}{llrcrcr} \text{Max} & z = & 2x_1 & - & x_2' & + & 3 \\ \text{s.t.} & & \frac{1}{3}x_1 & + & x_2' & \leq & 5 \\ & & -\frac{1}{3}x_1 & - & x_2' & \leq & -5 \\ & & -2x_1 & + & 5x_2' & \leq & 22 \\ & & x_1 & , & x_2' & \geq & 0 \end{array}$$

Standard form:

$$
\begin{array}{ll}
\text{Max} & z = 2x_1 - x_2' + 3 \\
\text{s.t.} & \\
& \begin{array}{rcrcrcr}
\frac{1}{3}x_1 & + & x_2' & + & x_3 & = & 5 \\
-\frac{1}{3}x_1 & - & x_2' & + & x_4 & = & -5 \\
-2x_1 & + & 5x_2' & + & x_5 & = & 22
\end{array}
\end{array}
$$

with $x_1, x_2', x_3, x_4, x_5 \geq 0$

c) Canonical form:

$$
\begin{array}{llrcrcrcr}
\text{Max} & \bar{z} = & x_1 & & & + & x_3 & & \\
\text{s.t.} & & -x_1 & + & \frac{1}{2}x_2^- & + & 3x_3 & \leq & -2 \\
& & & & -4x_2^- & + & x_3 & \leq & 5 \\
& & & & 4x_2^- & - & x_3 & \leq & -5 \\
& & x_1 & , & x_2^- & , & x_3 & \geq & 0
\end{array}
$$

with $z^* = -\bar{z}^*$.

Standard form:

$$
\begin{array}{ll}
\text{Max} & \bar{z} = x_1 + x_3 \\
\text{s.t.} & \begin{array}{rcrcrcrcr}
-x_1 & + & \frac{1}{2}x_2^- & + & 3x_3 & + & x_4 & = & -2 \\
& & -4x_2 & + & x_3 & + & x_5 & = & 5 \\
& & 4x_2 & - & x_3 & + & x_6 & = & -5
\end{array}
\end{array}
$$

with $x_1, x_2^-, x_3, x_4, x_5, x_6 \geq 0$ and $z^* = -\bar{z}^*$.

Problem 2

a) Let's define $x_4 \geq 0$ such that

$$x_4 \geq |2x_3 - x_1| \iff x_4 \geq (2x_3 - x_1) \text{ and } x_4 \geq (x_1 - 2x_3)$$

Equivalent LP :

$$
\begin{array}{llrcrcrcrcr}
\text{Min} & z = & & & x_2 & & & + & x_4 & & \\
\text{s.t.} & & -x_1 & & & + & 2x_3 & - & x_4 & \leq & 0 \\
& & x_1 & & & - & 2x_3 & - & x_4 & \leq & 0 \\
& & 4x_1 & - & x_2 & + & 2x_3 & & & = & 6 \\
& & & & 2x_2 & - & 4x_3 & & & \geq & 4 \\
& & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

b) Let's define $t \in \mathbb{R}$ such that:

$$t \leq 2x_2 - 4$$

$$t \leq 4x_3 + x_1$$

Equivalent LP :

$$
\begin{array}{llrcrcrcrcr}
\text{Max} & z = & x_1 & & & & & + & t & & \\
\text{s.t.} & & x_1 & + & 3x_2 & + & 2x_3 & & & \leq & 9 \\
& & & & 5x_2 & - & x_3 & & & = & 4 \\
& & & & 2x_2 & & & - & t & \geq & 4 \\
& & x_1 & & & + & 4x_3 & - & t & \geq & 0 \\
& & x_1 & , & x_2 & , & x_3 & & & \geq & 0 \\
& & & & & & & & t & \in & \mathbb{R}
\end{array}
$$

c) It is impossible to find a LP equivalent to that problem. Indeed, if we add a variable t to replace expression $|x_1 - x_2|$, the problem is equivalent to maximize $z = x_1 + t$ with the additional constraint that $t \leq |x_1 - x_2|$. It is not possible to express this constraint as a linear one. But we can solve this problem by taking the best solution of two LPs defined as below. Note that $|x_1 - x_2| = x_1 - x_2$ if $x_1 - x_2 \geq 0$ and $|x_1 - x_2| = x_2 - x_1$ if $x_1 - x_2 \leq 0$.

$$\begin{array}{llcrcrcrcl}
\text{Max} & z_1 & = & 2x_1 - x_2 & & & & & & \\
\text{s.t.} & & & x_1 & + & x_2 & & & \leq & 5 \\
& & & & & x_2 & - & x_3 & \leq & 7 \\
& & & 5x_1 & + & 2x_2 & - & 8x_3 & \leq & 5 \\
& & & x_1 & - & x_2 & & & \geq & 0 \\
& & & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
\qquad
\begin{array}{llcrcrcrcl}
\text{Max} & z_2 & = & x_2 & & & & & & \\
\text{s.t.} & & & x_1 & + & x_2 & & & \leq & 5 \\
& & & & & x_2 & - & x_3 & \leq & 7 \\
& & & 5x_1 & + & 2x_2 & - & 8x_3 & \leq & 5 \\
& & & x_1 & - & x_2 & & & \leq & 0 \\
& & & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

The optimal solution is given by:

$$z^* = \max\{z_1^*, z_2^*\}.$$

Problem 3

Transformation of $(\mathbf{A} \mid \mathbf{I})$ to row echelon form:

$$\mathbf{A}_1 = \mathbf{E}_{21}(-1) \cdot (\mathbf{A} \mid \mathbf{I}) = \left(\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 3 & 0 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\mathbf{A}_2 = \mathbf{E}_{31}(-3) \cdot \mathbf{A}_1 = \left(\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & -6 & -8 & -3 & 0 & 1 \end{array}\right)$$

$$\mathbf{A}_3 = \mathbf{E}_2(-\frac{1}{2}) \cdot \mathbf{A}_2 = \left(\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -6 & -8 & -3 & 0 & 1 \end{array}\right)$$

$$\mathbf{A}_4 = \mathbf{E}_{32}(6) \cdot \mathbf{A}_3 = \left(\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 & -3 & 1 \end{array}\right)$$

$$\mathbf{A}_5 = \mathbf{E}_3(-\frac{1}{2}) \cdot \mathbf{A}_4 = \left(\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

Then we row-reduce $\mathbf{A}_5$ in a second step:

$$\mathbf{A}_6 = \mathbf{E}_{23}(-1) \cdot \mathbf{A}_5 = \left(\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

$$\mathbf{A}_7 = \mathbf{E}_{13}(-4) \cdot \mathbf{A}_6 = \left(\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & -6 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

$$\mathbf{A}_8 = \mathbf{E}_{12}(-2)\cdot\mathbf{A}_7 = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

Consequently:

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ \frac{1}{2} & -2 & \frac{1}{2} \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Moreover, we have that:

$$\mathbf{A}^{-1} = \mathbf{E}_{12}(-2)\cdot\mathbf{E}_{13}(-4)\cdot\mathbf{E}_{23}(-1)\cdot\mathbf{E}_{3}(-\frac{1}{2})\cdot\mathbf{E}_{32}(6)\cdot\mathbf{E}_{2}(-\frac{1}{2})\cdot\mathbf{E}_{31}(-3)\cdot\mathbf{E}_{21}(-1)$$

and

$$\mathbf{A} = \mathbf{E}_{21}(1)\cdot\mathbf{E}_{31}(3)\cdot\mathbf{E}_{2}(-2)\cdot\mathbf{E}_{32}(-6)\cdot\mathbf{E}_{3}(-2)\cdot\mathbf{E}_{23}(1)\cdot\mathbf{E}_{13}(4)\cdot\mathbf{E}_{12}(2)$$

Problem 4

a) Gauss elimination: we start with the augmented matrix of the sytem:

$$\left(\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & 5 & 7 \\ 2 & -4 & 5 & 6 & 11 & 10 \\ 3 & -6 & 8 & 10 & 11 & 3 \\ 0 & 0 & 1 & 5 & -2 & -9 \end{array}\right) \begin{array}{l} \\ l_2 - 2l_1 \\ l_3 - 3l_1 \\ \\ \end{array} \rightarrow \left(\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & 5 & 7 \\ 0 & 0 & 1 & 6 & 1 & -4 \\ 0 & 0 & 2 & 10 & -4 & -18 \\ 0 & 0 & 1 & 5 & -2 & -9 \end{array}\right) \begin{array}{l} l_1 - 2l_2 \\ \\ l_3 - 2l_2 \\ l_4 - \frac{1}{2}l_3 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & -12 & 3 & 15 \\ 0 & 0 & 1 & 6 & 1 & -4 \\ 0 & 0 & 0 & -2 & -6 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} l_1 - 6l_3 \\ l_2 + 3l_3 \\ -\frac{1}{2}l_3 \\ \\ \end{array} \rightarrow \left(\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 39 & 75 \\ 0 & 0 & 1 & 0 & -17 & -34 \\ 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The set of solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5 \;\middle|\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 75 \\ 0 \\ -34 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -39 \\ 0 \\ 17 \\ -3 \\ 1 \end{pmatrix}, \quad s,t \in \mathbb{R} \right\}.$$

b) We conclude that $\text{rank}(\mathbf{A}) = 3$.

c) The dimension of the column space is equal to the rank of $\mathbf{A}$. We just need to choose the columns of $\mathbf{A}$ based on the reduced row echelon form to get a basis $\mathcal{B}_c$ of the columns of $\mathbf{A}$:

$$\mathcal{B}_c = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 10 \\ 5 \end{pmatrix} \right\}$$

d) The dimension of the row space is equal to the rank of $\mathbf{A}$. We just need to choose the rows of $\mathbf{A}$ based on the reduced row echelon form to get a basis $\mathcal{B}_c$ of the rows of $\mathbf{A}$:

$$\mathcal{B}_l = \{(1 \quad -2 \quad 2 \quad 0 \quad 5), (2 \quad -4 \quad 5 \quad 6 \quad 11), (3 \quad -6 \quad 8 \quad 10 \quad 11)\}$$