A Saddle Point Condition

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Critical Point

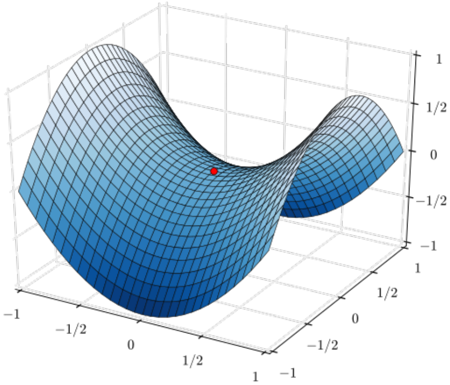
- A critical point or a stationary point of a differentiable function $\mathbb{R}^n \to \mathbb{R}$ is a point where its gradient is null
- We remind that the gradient of f at a point $\boldsymbol{x} = (x_1, \ldots, x_n)$ is the vector $\nabla f(\boldsymbol{x})$ of its partial derivatives:

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f(\boldsymbol{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\boldsymbol{x})}{\partial x_n} \end{pmatrix}$$

- A saddle point is a criticial point which is not a maximum, nor a minimum
- **A critical point can be a local minimum, a local maximum, or a saddle point**

Example of a Saddle Point

A saddle point (in red) on the graph of $z = x^2 - y^2$:



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Definiteness of Symmetric Matrices

- A $n \times n$ symmetric real matrix $\boldsymbol{Q}$ is said to be positive definite (resp. resp. negative definite) if the scalar $\boldsymbol{z}^T\boldsymbol{Q}\boldsymbol{z}$ is > 0 (resp. < 0) for every non-null vector $\boldsymbol{z}$ of n real numbers
- A $n \times n$ symmetric real matrix $\boldsymbol{Q}$ is said to be positive semi-definite (resp. negative semi-definite) if the scalar $\boldsymbol{z}^T\boldsymbol{Q}\boldsymbol{z}$ is ≥ 0 (resp. ≤ 0) for every vector $\boldsymbol{z}$ of n real numbers
- A matrix that is not positive semi-definite and not negative semi-definite is called indefinite
- Positive and negative definite matrices are always **invertible**. This is not the case for positive and negative semi-definite matrices

Conditions for Optimality for Unconstrained Optimization

Theorem (Second Order Optimality Conditions)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable at a point $\bar{\mathbf{x}} \in \mathbb{R}^n$

(1) *(Necessity) If $\bar{\mathbf{x}}$ is a local minimum (resp. local maximum), then $\nabla f(\bar{\mathbf{x}}) = 0$ and $\nabla^2 f(\bar{\mathbf{x}})$ is* ***positive semi-definite*** *(resp.* ***negative semi-definite****)*

(2) *(Sufficiency) If $\nabla f(\bar{\mathbf{x}}) = 0$ and $\nabla^2 f(\bar{\mathbf{x}})$ is* ***positive definite*** *(resp.* ***negative definite****) , then $\bar{\mathbf{x}}$ is a local minimum (resp. local maximum)*

(3) *If $\nabla f(\bar{\mathbf{x}}) = 0$ and $\nabla^2 f(\bar{\mathbf{x}})$ is indefinite, then $\bar{\mathbf{x}}$ is a* ***saddle point***