Dual Simplex Algorithm

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Tableaus in the Simplex Algorithm

- To each tableau is associated not only a basis of the initial primal problem but also a basis of the dual problem
- The values of the primal basic variables can be read in the last column of the tableau
- The values of the dual basic solution can be read in the last row of the tableau

$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad\quad \boldsymbol{x_D} \quad\quad & \quad\quad \boldsymbol{x_E} \quad\quad & z & \end{array} \\
\boldsymbol{T_B} = & \begin{array}{|c|c|c||c|}
\hline
\boldsymbol{B}^{-1}\boldsymbol{A} & \boldsymbol{B}^{-1} & \mathbf{0} & \beta \\
\hline
-\gamma_{\boldsymbol{D}} & -\gamma_{\boldsymbol{E}} & 1 & \zeta \\
\hline
\end{array} \\
 & \begin{array}{cc} \boldsymbol{y_E} = (y_{m+1} \dots y_{m+n}) & \boldsymbol{y_D} = (y_1 \dots y_m) \end{array}
\end{array}
$$

Tableaus in the Simplex Algorithm (Cont'd)

- Dual decision variables are associated with slack variables of the primal problem
- Conversely, slack variables of the dual are associated with the primal decision variables
- Primal and dual basic solutions corresponding to the same tableau have the same value and satisfy the complementary slackness conditions (basic variables have null reduced costs !)

$$
\begin{array}{cc}
 & \begin{array}{cccc} \boldsymbol{x_D} & \boldsymbol{x_E} & z & \end{array} \\
\boldsymbol{T_B} = & \begin{array}{|c|c|c||c|} \hline \boldsymbol{B}^{-1}\boldsymbol{A} & \boldsymbol{B}^{-1} & \mathbf{0} & \beta \\ \hline -\gamma_{\boldsymbol{D}} & -\gamma_{\boldsymbol{E}} & 1 & \zeta \\ \hline \end{array} \\
 & \begin{array}{cc} \boldsymbol{y_E} = (y_{m+1} \dots y_{m+n}) & \boldsymbol{y_D} = (y_1 \dots y_m) \end{array}
\end{array}
$$

Tableaus in the Simplex Algorithm (Cont'd)

- In all the tableaus visited by the simplex algorithm, the primal basic solution is always feasible

- The algorithm stops as soon as a feasible dual solution is met and the tableau is optimal

- The optimal tableau contains not only the optimal solution of the initial primal problem but also the optimal solution of its dual problem

$$
\begin{array}{cc}
 & \begin{array}{ccc|c} \quad\boldsymbol{x_D}\quad & \quad\boldsymbol{x_E}\quad & z & \end{array} \\
\boldsymbol{T_B} = & \left|\begin{array}{c|c|c||c}
\boldsymbol{B}^{-1}\boldsymbol{A} & \boldsymbol{B}^{-1} & \mathbf{0} & \beta \\
\hline
-\gamma_{\boldsymbol{D}} & -\gamma_{\boldsymbol{E}} & 1 & \zeta
\end{array}\right| \\
 & \begin{array}{cc} \boldsymbol{y_E} = (y_{m+1} \dots y_{m+n}) & \boldsymbol{y_D} = (y_1 \dots y_m) \end{array}
\end{array}
$$

The Dual Simplex Algorithm

Let's consider the following canonical LP:

$$\begin{array}{llrcrcr} \text{Max} & z = & -x_1 & - & 2x_2 & & \\ \text{s.t.} & & 2x_1 & + & x_2 & \leq & 6 \\ & & -x_1 & - & x_2 & \leq & -4 \\ & & x_1 & , & x_2 & \geq & 0 \end{array}$$

with an initial tableau given by

$$\boldsymbol{T}_0 = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc|cc|c||c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 2 & 1 & 1 & 0 & 0 & 6 \\ -1 & -1 & 0 & 1 & 0 & -4 \\ \hline 1 & 2 & 0 & 0 & 1 & 0 \\ \hline y_3 & y_4 & y_1 & y_2 & & \end{array}$$

$\boldsymbol{T}_0$ is not primal feasible but **dual feasible** !

The Dual Simplex Algorithm (Cont'd)

Let's try to solve the dual problem !

- In $\boldsymbol{T}_0$, the dual objective function (to **minimize** !) can be written as $w = \boldsymbol{yb}$
- We need to increase a dual decision variable associated with an element $b_i < 0$ in order to decrease w (since $w = \boldsymbol{yb}$)
- The only candidate is $b_2 = -4$, the primal variable x_4 exits the primal basis and the dual variable y_2 enters the dual basis

$$\boldsymbol{T}_0 = \begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & z & \end{array} \\ \left[\begin{array}{cc|cc|c||c} 2 & 1 & 1 & 0 & 0 & 6 \\ -1 & -1 & 0 & 1 & 0 & -4 \\ \hline 1 & 2 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{cccc} y_3 & y_4 & y_1 & y_2 \end{array} \end{array}$$

The Dual Simplex Algorithm (Cont'd)

- To keep the dual feasibility, the pivot needs to be selected in a column r that satisfies:

$$\frac{-\gamma_r}{\alpha_{2r}} = \max\left\{\frac{-\gamma_k}{\alpha_{2k}} \mid \alpha_{2k} < 0\right\}$$

- As $-\gamma_1/\alpha_{21} = -1$ and $-\gamma_2/\alpha_{22} = -2$, we need to pivot on α_{21} and x_1 enters the basis and replaces x_4

$$\begin{array}{c} \\ \boldsymbol{T}_0 = \\ \\ \end{array} \begin{array}{cccc|c||c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 2 & 1 & 1 & 0 & 0 & 6 \\ \mathbf{-1} & -1 & 0 & 1 & 0 & -4 \\ \hline 1 & 2 & 0 & 0 & 1 & 0 \\ \hline y_3 & y_4 & y_1 & y_2 & & \end{array}$$

The Dual Simplex Algorithm (Cont'd)

$$\boldsymbol{T}_0 = \begin{array}{cc|cc|c||c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 2 & 1 & 1 & 0 & 0 & 6 \\ \mathbf{-1} & -1 & 0 & 1 & 0 & -4 \\ \hline 1 & 2 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

$$\boldsymbol{T}_1 = \begin{array}{cc|cc|c||c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 0 & -1 & 1 & 2 & 0 & -2 \\ 1 & 1 & 0 & -1 & 0 & 4 \\ \hline 0 & 1 & 0 & 1 & 1 & -4 \\ \hline y_3 & y_4 & y_1 & y_2 & & \end{array}$$

The tableau $\boldsymbol{T}_1$ is still dual feasible but β_1 is negative. So x_3 will exit the primal basis and y_1 will enter the dual basis. The only negative pivot in the first row is $\alpha_{12} = -1$

The Dual Simplex Algorithm (Cont'd)

$$\boldsymbol{T}_1 = \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 0 & \mathbf{-1} & 1 & 2 & 0 & -2 \\ 1 & 1 & 0 & -1 & 0 & 4 \\ \hline 0 & 1 & 0 & 1 & 1 & -4 \end{array}$$

$$\boldsymbol{T}_2 = \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & z & \\ \hline 0 & 1 & -1 & -2 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 & 2 \\ \hline 0 & 0 & 1 & 3 & 1 & -6 \\ y_3 & y_4 & y_1 & y_2 & & \end{array}$$

The tableau $\boldsymbol{T}_2$ is primal and dual feasible. Consequently, it is **optimal**. The primal optimal solution is $x_1^* = x_2^* = 2$ ($x_3^* = x_4^* = 0$) and the optimal dual solution is $y_1^* = 1, y_2^* = 3$ ($y_3^* = y_4^* = 0$). The **value** of the optimal solution is given by $z^* = w^* = -6$.

The Dual Simplex Algorithm (Cont'd)

Primal tableau / **Dual** algo

T_0

x_1	x_2	x_3	x_4	
2	1	1	0	6
-1	-1	0	1	-4
1	2	0	0	0

T_1

0	-1	1	2	-2
1	1	0	-1	4
0	1	0	1	-4

T_2

0	1	-1	-2	2
1	0	1	1	2
0	0	1	3	-6
y_3	y_4	y_1	y_2	

Dual tableau / **Primal** algo

T_0

y_1	y_2	y_3	y_4	
-2	1	1	0	1
-1	1	0	1	2
6	-4	0	0	0

T_1

-2	1	1	0	1
1	0	-1	1	1
-2	0	4	0	4

T_2

0	1	-1	2	3
1	0	-1	1	1
0	0	2	2	6
x_3	x_4	x_1	x_2	

To apply the simplex algorithm, we need to express the dual as a max problem. As *min w* is equivalent to $-max\ -w$, then the optimal dual solution is -6

The Dual Simplex Algorithm (Cont'd)

- If a non-feasible tableau with no pivot is met with the dual simplex algorithm, then it means that the dual is **unbounded** and that the primal problem has **no feasible solution** (weak duality)
- Indeed, in such a situation, we have $b_i < 0$ and $\alpha_{ij} \geq 0 \; \forall \, j$. This corresponds to the following constraint (**impossible** if $x_j \geq 0 \;\; \forall \, j$)

$$0 \leq \sum \alpha_{ij} x_j = b_i < 0$$

The Dual Simplex Algorithm (Cont'd)

Signature of an **unbounded dual tableau**:

x_D		x_E		z	
				0	$*$
$\oplus$ $\dots$	$\dots$	$\dots$	$\dots$ $\oplus$	$\vdots$	$-$
				0	$*$
$\oplus$ $\dots$	$\dots$	$\dots$	$\dots$ $\oplus$	1	$*$

Reminder: $\oplus$ means ≥ 0 and $-$ strictly smaller than 0

The Dual Simplex Algorithm (Phase II)

Input Data: a **dual feasible** tableau
Output: an **optimal** tableau or a **certificate** for the absence of feasible solutions

(1) Choice of the **exiting** variable: choose a row i with $\beta_i < 0$, the basic variable x_j with $j = \sigma(i)$ exits the basis. If it does not exist such variable: STOP, the current tableau is optimal

The Dual Simplex Algorithm (Phase II) (Cont'd)

(2) Choice of the **entering** variable: choose a non-basic column r that maximizes the following ratios:

$$r \in \left\{ k \in \mathcal{N} \mid \frac{-\gamma_k}{\alpha_{ik}} = \max \left\{ \frac{-\gamma_j}{\alpha_{ij}} \mid \alpha_{ij} < 0 \right\} \right\}$$

If it does not exist any entering variable: STOP, the dual is **not bounded** and the primal has **no feasible solution**

(3) Update of the basis and of the tabeau: pivot around α_{ir} and goes back to (1)

Remark: in order to avoid any cycling, we can apply Bland's rule when they are several candidates to enter or to exit the current basis

When to Use the Dual Simplex Algorithm

- When an **initial** tableau is **primal feasible**, then use **Phase II** of the simplex algorithm
- When the **initial** tableau is not **primal feasible**, two possibilities:
  - Use **Phase I** of the simplex algorithm
  - Use the **dual** simplex algorithm **if** the tableau is **dual feasible**

To conclude, Phase I works in **any** case when the initial tableau is not primal feasible. The dual algorithm can only be applied **when** the tableau is **dual feasible**

Phase I vs Phase II

- This algorithm corresponds to the **phase II** of the dual simplex algorithm
- There is also a **phase I** of the dual simplex algorithm
- Phase I consists in finding a **feasible basic dual** solution
- It is not presented in this course