Ch3: The forecasters’ toolbox

Analysis of Sequential Data

MSE Data Science

Credits

Original slides published by Rob Hyndman:

https://robjhyndman.com/teaching/

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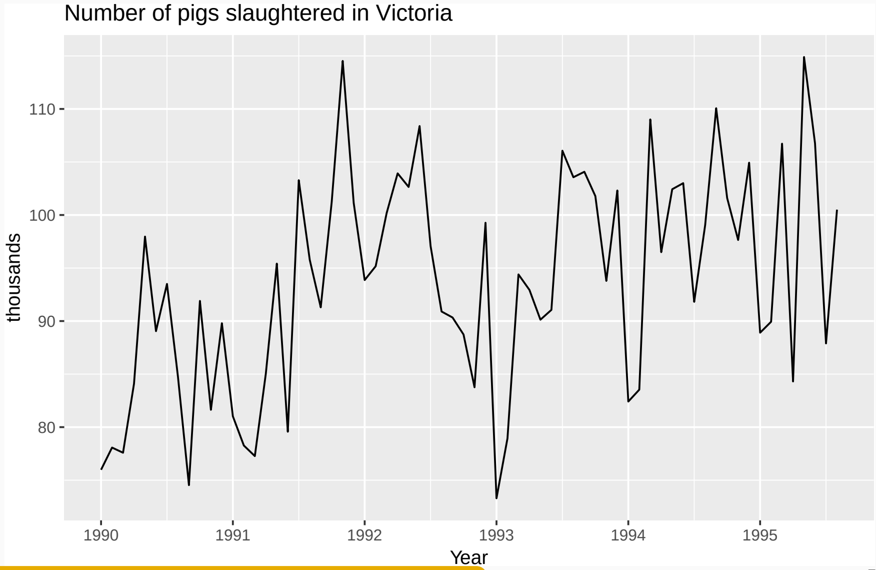
Outline

Some simple forecasting methods

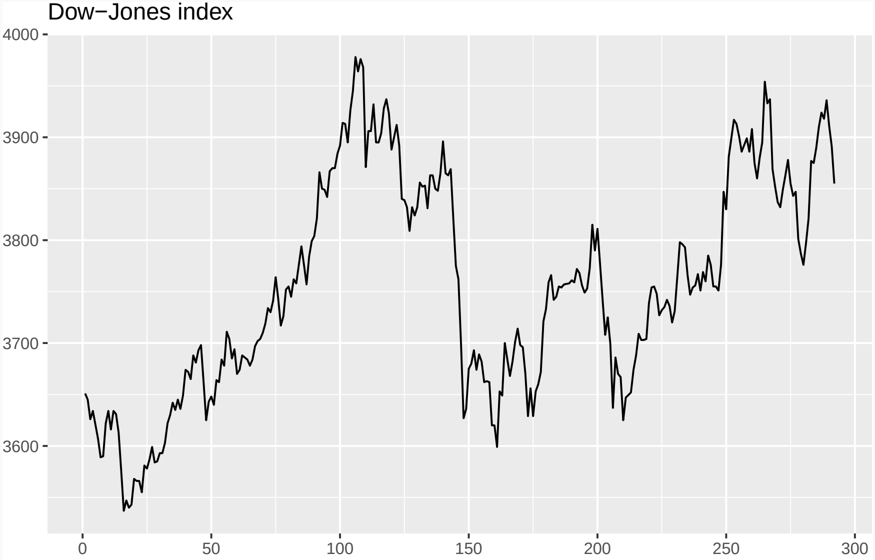
How would you forecast these data?

Some simple forecasting methods



How would you forecast these data?

Some simple forecasting methods



How would you forecast these data?

Some simple forecasting methods

Average method

- Forecast of all future values is equal to mean of historical data $\{y_1, \dots, y_T\}$.
- Forecasts: $\hat{y}_{T+h|T} = \bar{y} = (y_1 + \cdots + y_T)/T$

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Naïve method

- Forecasts equal to last observed value.
- Forecasts: $\hat{y}_{T+h|T} = y_T$.
- Consequence of efficient market hypothesis.

Some simple forecasting methods

Average method

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Naïve method

- Forecasts equal to last observed value.
- Forecasts: $\hat{y}_{T+h|T} = y_T$.
- Consequence of efficient market hypothesis.

Seasonal naïve method

- Forecasts equal to last value from same season.
- Forecasts: $\hat{y}_{T+h|T} = y_{T+h-m(k+1)}$, where m = seasonal period and k is the integer part of $(h-1)/m$.

Some simple forecasting methods

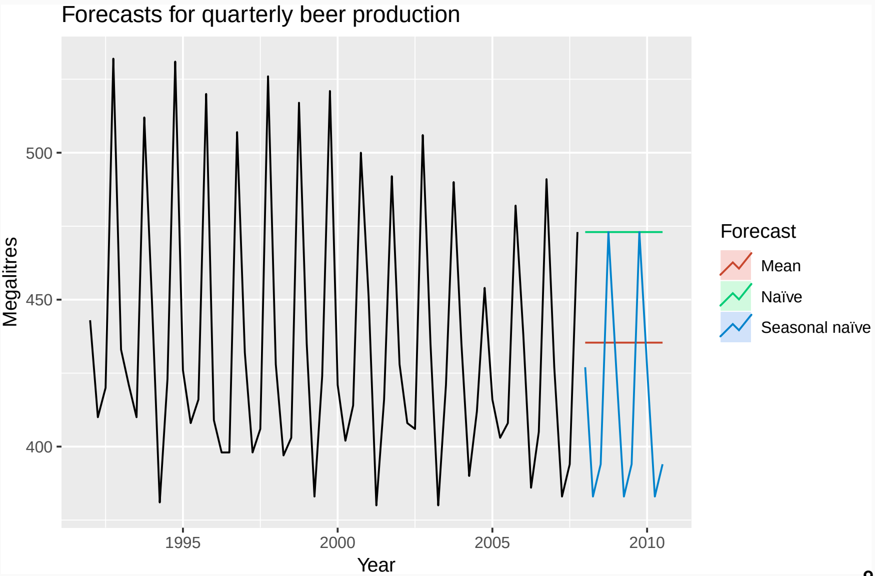
Drift method

- Forecasts equal to last value plus average change.
- Forecasts:

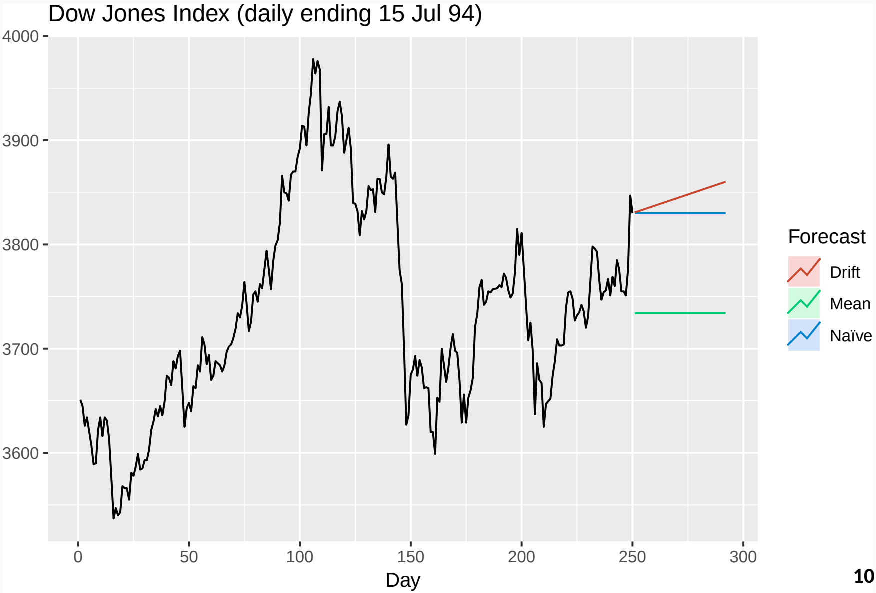
$$\begin{aligned}\hat{y}_{T+h|T} &= y_T + \frac{h}{T-1}\sum_{t=2}^{T}(y_t - y_{t-1})\\ &= y_T + \frac{h}{T-1}(y_T - y_1).\end{aligned}$$

- Equivalent to extrapolating a line drawn between first and last observations.

Some simple forecasting methods



Some simple forecasting methods



Some simple forecasting methods

- Mean: `meanf(y, h=20)`
- Naïve: `naive(y, h=20)`
- Seasonal naïve: `snaive(y, h=20)`
- Drift: `rwf(y, drift=TRUE, h=20)`

Some simple forecasting methods

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Your turn

- Use these four functions to produce forecasts for `goog` and `auscafe`.
- Plot the results using `autoplot()`.

Outline

1 Some simple forecasting methods

2 Residual diagnostics

3 Evaluating forecast accuracy

4 Prediction intervals

Fitted values

- $\hat{y}_{t|t-1}$ is the forecast of y_t based on observations $y_1, \dots, y_t$.
- We call these "fitted values".
- Sometimes drop the subscript: $\hat{y}_t \equiv \hat{y}_{t|t-1}$.
- Often not true forecasts since parameters are estimated on all data.

For example:

- $\hat{y}_t = \bar{y}$ for average method.
- $\hat{y}_t = y_{t-1} + (y_T - y_1)/(T - 1)$ for drift method.

Forecasting residuals

Residuals in forecasting: difference between observed value and its fitted value: $e_t = y_t - \hat{y}_{t|t-1}$.

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Assumptions

1. $\{e_t\}$ uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
2. $\{e_t\}$ have mean zero. If they don't, then forecasts are biased.

Forecasting residuals

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Assumptions

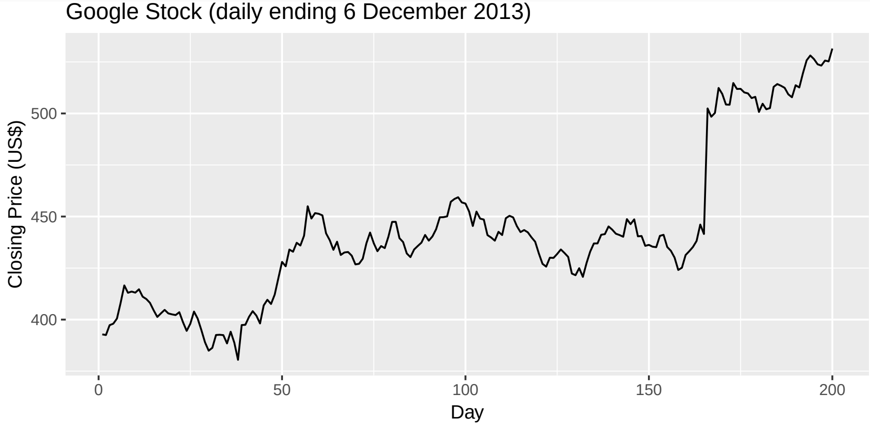
1. $\{e_t\}$ uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
2. $\{e_t\}$ have mean zero. If they don't, then forecasts are biased.

Useful properties (for prediction intervals)

3. $\{e_t\}$ have constant variance.
4. $\{e_t\}$ are normally distributed.

Example: Google stock price

```r
autoplot(goog200) +
  xlab("Day") + ylab("Closing Price (US$)") +
  ggtitle("Google Stock (daily ending 6 December 2013)")
```



Example: Google stock price

Naïve forecast:

$$\hat{y}_{t|t-1} = y_{t-1}$$

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$$e_t = y_t - y_{t-1}$$

Example: Google stock price

Naïve forecast:

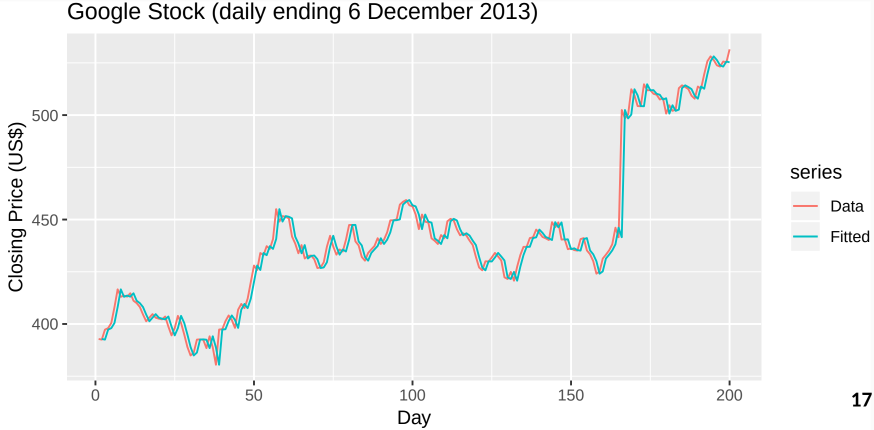
$$\hat{y}_{t|t-1} = y_{t-1}$$

$$e_t = y_t - y_{t-1}$$

Note: e_t are one-step-forecast residuals

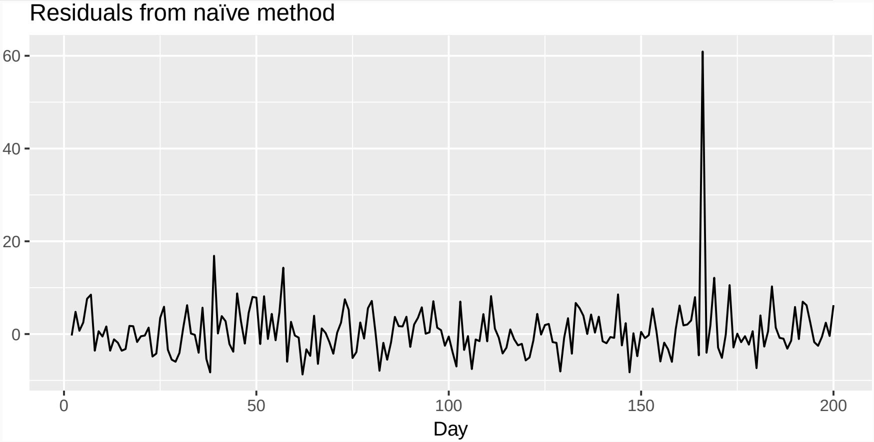
Example: Google stock price

```r
fits <- fitted(naive(goog200))
autoplot(goog200, series="Data") +
  autolayer(fits, series="Fitted") +
  xlab("Day") + ylab("Closing Price (US$)") +
  ggtitle("Google Stock (daily ending 6 December 2013)")
```



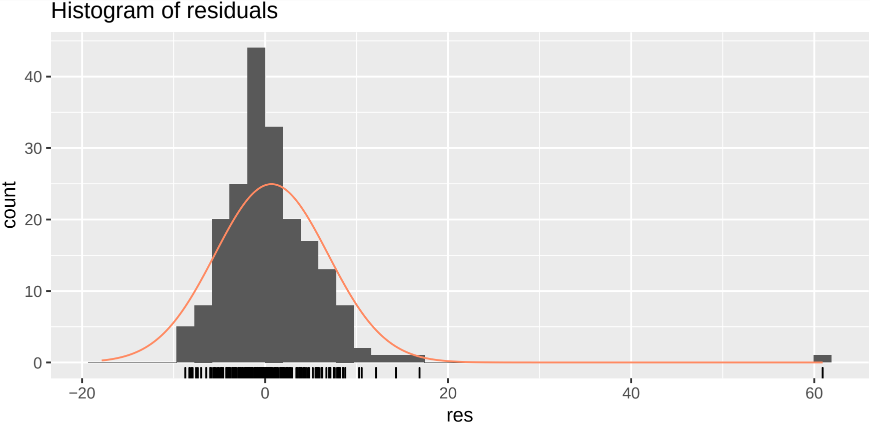
Example: Google stock price

```r
res <- residuals(naive(goog200))
autoplot(res) + xlab("Day") + ylab("") +
  ggtitle("Residuals from naïve method")
```



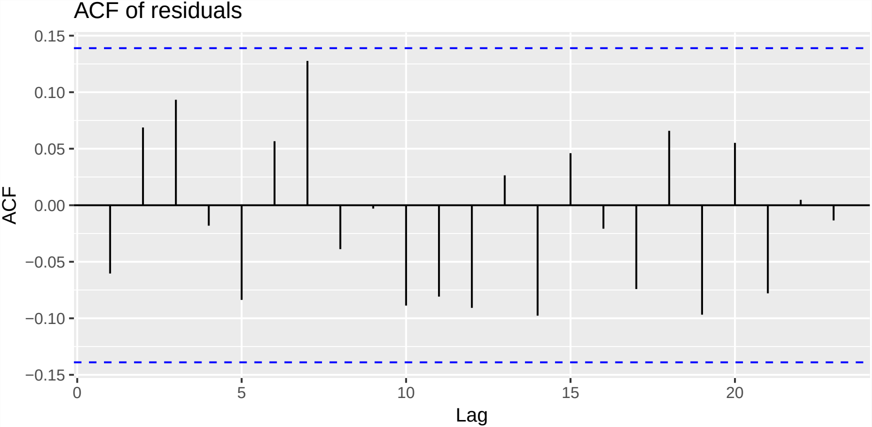
Example: Google stock price

```r
gghistogram(res, add.normal=TRUE) +
  ggtitle("Histogram of residuals")
```



Example: Google stock price

```r
ggAcf(res) + ggtitle("ACF of residuals")
```



ACF of residuals

- We assume that the residuals are white noise (uncorrelated, mean zero, constant variance). If they aren't, then there is information left in the residuals that should be used in computing forecasts.
- So a standard residual diagnostic is to check the ACF of the residuals of a forecasting method.
- We *expect* these to look like white noise.

Statistical test of autocorrelation

Consider a *whole set* of r_k values, and develop a test to see whether the set is significantly different from a zero set.

Ljung-Box test

$$Q^* = T(T+2)\sum_{k=1}^{h}(T-k)^{-1}r_k^2$$

where h is max lag being considered and T is number of observations.

- If each r_k close to zero, Q will be **small**.
- If some r_k values large (positive or negative), Q will be **large**.

Recommended defaults for h

Ljung-Box test

$$Q^* = T(T+2)\sum_{k=1}^{h}(T-k)^{-1}r_k^2$$

where h is max lag being considered and T is number of observations.

- $h = 10$ for non-seasonal data
- $h = 2m$ for seasonal data, where m is the length of the season.

Portmanteau tests

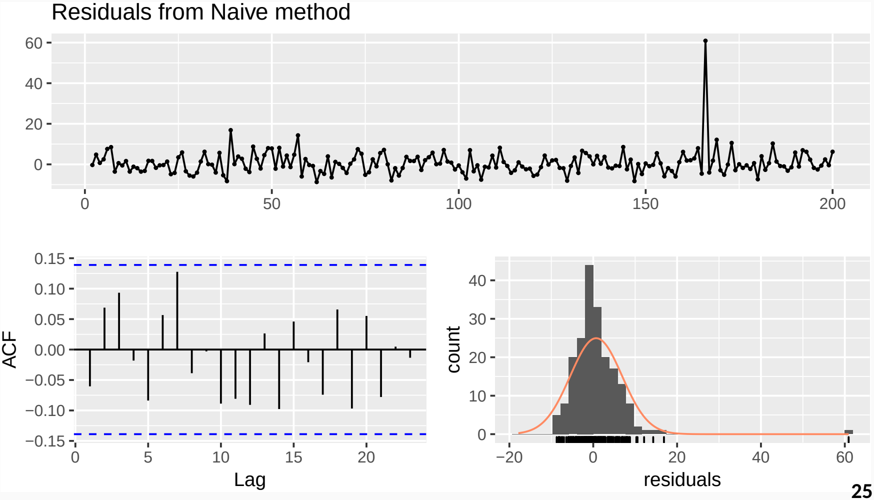
- If data are WN, Q^* has χ^2 distribution with $(h - K)$ degrees of freedom where K = no. parameters in model.
- When applied to raw data, set $K = 0$.
- For the Google example:

```
# lag=h and fitdf=K
Box.test(res, lag=10, fitdf=0, type="Lj")
```

```
##
##  Box-Ljung test
##
## data:  res
## X-squared = 11.031, df = 10, p-value =
## 0.3551
```

checkresiduals function

```r
checkresiduals(naive(goog200))
```



checkresiduals function

```
##
##  Ljung-Box test
##
## data:  Residuals from Naive method
## Q* = 11.031, df = 10, p-value = 0.3551
## Model df: 0.   Total lags used: 10
```

Interpretation

- The test checks the *null hypothesis* that the data are white noise.
- Small p-values lead to rejecting the null hypothesis; they are evidence of significant auto-correlation
- Large p-values lead instead to accepting the null hypothesis.
- Typical threshold decision:
  - p-value $> 0.05 \rightarrow$: accept the null hypothesis (white noise)
  - p-value $< 0.05 \rightarrow$: reject the null hypothesis, concluding that there is a significant autocorrelation.

Your turn

Compute seasonal naïve forecasts for quarterly Australian beer production from 1992.

```
beer <- window(ausbeer, start=1992)
fc <- snaive(beer)
autoplot(fc)
```

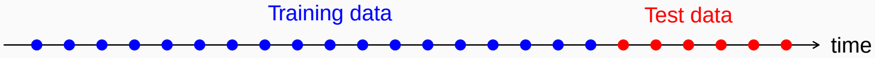
Test if the residuals are white noise.

```
checkresiduals(fc)
```

What do you conclude?

Outline

Training and test sets



- A model which fits the training data well will not necessarily forecast well.
- A perfect fit can always be obtained by using a model with enough parameters.
- Over-fitting a model to data is just as bad as failing to identify a systematic pattern in the data.
- The test set must not be used for *any* aspect of model development or calculation of forecasts.
- Forecast accuracy is based only on the test set.

Forecast errors

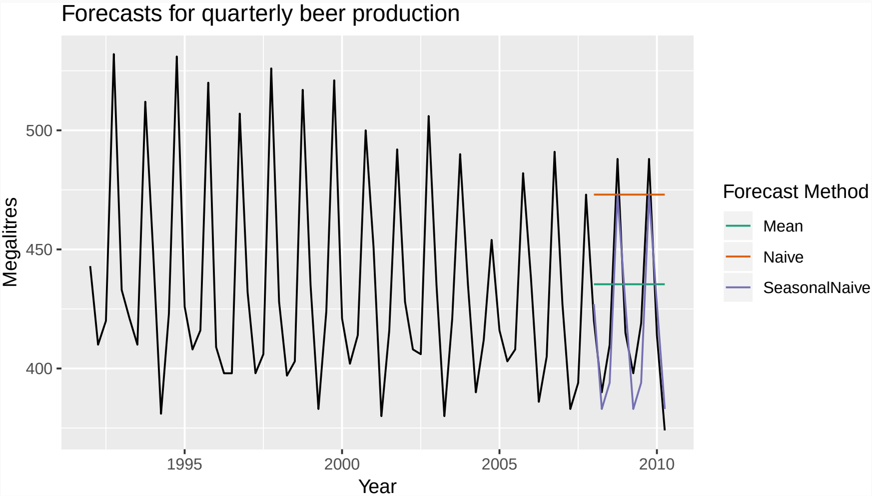
Forecast "error": the difference between an observed value and its forecast.

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T},$$

where the training data is given by $\{y_1, \dots, y_T\}$

- Unlike residuals, forecast errors on the test set involve multi-step forecasts.
- These are *true* forecast errors as the test data is not used in computing $\hat{y}_{T+h|T}$.

Measures of forecast accuracy



Measures of forecast accuracy

$$\begin{aligned} y_{T+h} &= (T+h)\text{th observation}, h = 1, \dots, H \\ \hat{y}_{T+h|T} &= \text{its forecast based on data up to time } T. \\ e_{T+h} &= y_{T+h} - \hat{y}_{T+h|T} \end{aligned}$$

$$\text{MAE} = \text{mean}(|e_{T+h}|)$$

$$\text{MSE} = \text{mean}(e_{T+h}^2) \qquad \text{RMSE} = \sqrt{\text{mean}(e_{T+h}^2)}$$

$$\text{MAPE} = 100\text{mean}(|e_{T+h}|/|y_{T+h}|)$$

Measures of forecast accuracy

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$$

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\end{aligned}
$$

- MAE, MSE, RMSE are all scale dependent.
- MAPE is scale independent but is only sensible if $y_t \gg 0$ for all t, and y has a natural zero.

Measures of forecast accuracy

Mean Absolute Scaled Error

$$\text{MASE} = \text{mean}(|e_{T+h}|/Q)$$

where Q is the MAE of a simple method (naïve or seasonal naïve).

For *non*-seasonal time series:

$$Q = (T-1)^{-1}\sum_{t=2}^{T}|y_t - y_{t-1}|$$

Hence MASE is the MAE of the method relative to the MAE of the *naïve*.

Measures of forecast accuracy

Mean Absolute Scaled Error

$$\text{MASE} = \text{mean}(|e_{T+h}|/Q)$$

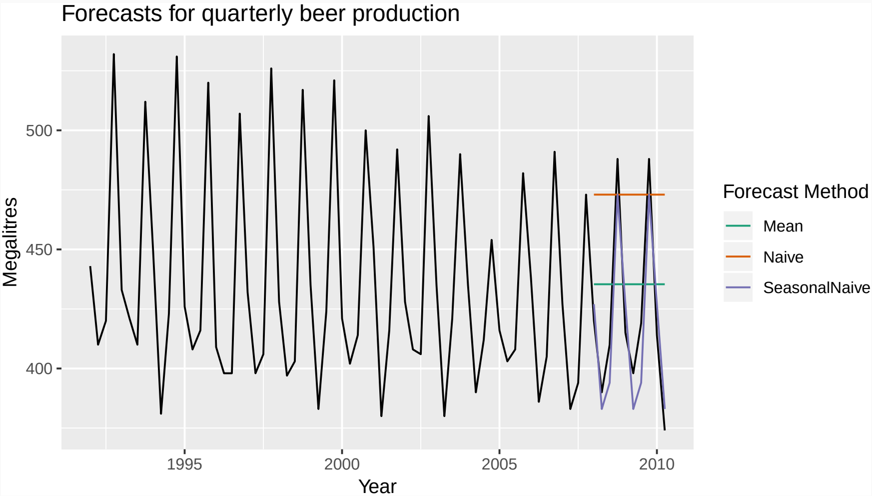
where Q is the MAE of a simple method (naïve or seasonal naïve).

For *seasonal* time series,

$$Q = (T-m)^{-1}\sum_{t=m+1}^{T}|y_t - y_{t-m}|$$

Then MASE is equivalent to MAE relative to a *seasonal naïve* method.

Measures of forecast accuracy



Measures of forecast accuracy

```r
beer2 <- window(ausbeer, start=1992, end=c(2007,4))
beer3 <- window(ausbeer, start=2008)
beerfit1 <- meanf(beer2, h=10)
beerfit2 <- rwf(beer2, h=10)
beerfit3 <- snaive(beer2, h=10)
accuracy(beerfit1, beer3)
accuracy(beerfit2, beer3)
accuracy(beerfit3, beer3)
```

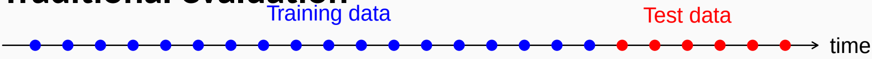
	RMSE	MAE	MAPE	MASE
Mean method	38.45	34.83	8.28	2.44
Naïve method	62.69	57.40	14.18	4.01
Seasonal naïve method	14.31	13.40	3.17	0.94

Poll: true or false?

1. Good forecast methods should have normally distributed residuals.
2. A model with small residuals will give good forecasts.
3. The best measure of forecast accuracy is MAPE.
4. If your model doesn’t forecast well, you should make it more complicated.
5. Always choose the model with the best forecast accuracy as measured on the test set.

Time series cross-validation

Traditional evaluation



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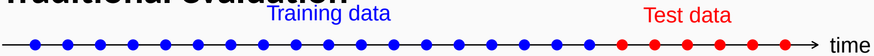

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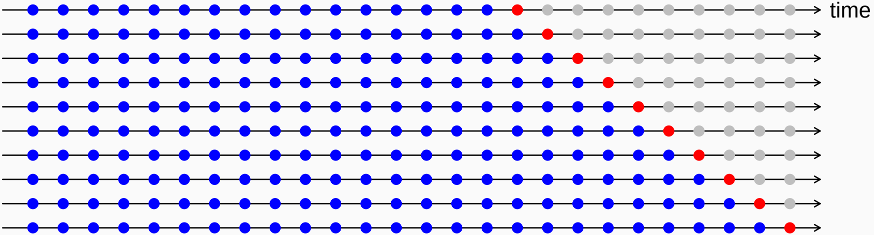

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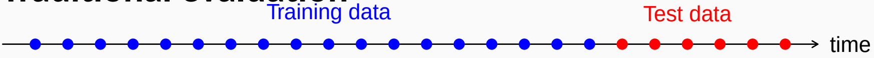

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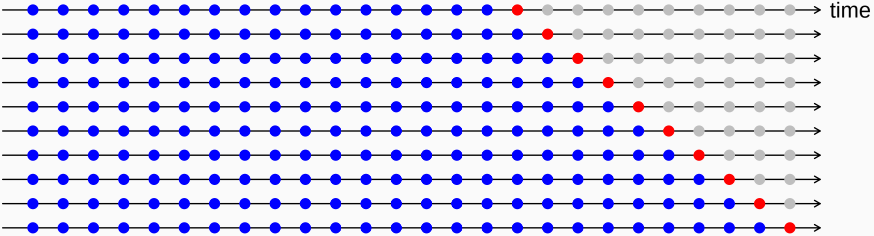

- Forecast accuracy averaged over test sets.
- Also known as “evaluation on a rolling forecasting origin”

tsCV function:

```r
e <- tsCV(goog200, rwf, drift=TRUE, h=1)
sqrt(mean(e^2, na.rm=TRUE))
```

```
## [1] 6.233245
```

```r
sqrt(mean(residuals(rwf(goog200, drift=TRUE))^2,
                                   na.rm=TRUE))
```

```
## [1] 6.168928
```

A good way to choose the best forecasting model is to find the model with the smallest RMSE computed using time series cross-validation.

Outline

Prediction intervals

- A forecast $\hat{y}_{T+h|T}$ is (usually) the mean of the conditional distribution $y_{T+h} \mid y_1, \dots, y_T$.
- A prediction interval gives a region within which we expect y_{T+h} to lie with a specified probability.
- Assuming forecast errors are normally distributed, then a 95% PI is

$$\hat{y}_{T+h|T} \pm 1.96\hat{\sigma}_h$$

  where $\hat{\sigma}_h$ is the st dev of the h-step distribution.
- When $h = 1$, $\hat{\sigma}_h$ can be estimated from the residuals.

Prediction intervals

Naive forecast with prediction interval:

```
res_sd <- sqrt(mean(res^2, na.rm=TRUE))
c(tail(goog200,1)) + 1.96 * res_sd * c(-1,1)
```

```
## [1] 519.3103 543.6462
```

```
naive(goog200, level=95)
```

```
##     Point Forecast    Lo 95    Hi 95
## 201       531.4783 519.3105 543.6460
## 202       531.4783 514.2705 548.6861
## 203       531.4783 510.4031 552.5534
## 204       531.4783 507.1428 555.8138
## 205       531.4783 504.2704 558.6862
## 206       531.4783 501.6735 561.2830
## 207       531.4783 499.2854 563.6711
```

Prediction intervals

- Point forecasts are often useless without prediction intervals.
- Prediction intervals require a stochastic model (with random errors, etc).
- Multi-step forecasts for time series require a more sophisticated approach (with PI getting wider as the forecast horizon increases).

Prediction intervals

Assume residuals are normal, uncorrelated, sd = $\hat{\sigma}$:

Mean forecasts:	$\hat{\sigma}_h = \hat{\sigma}\sqrt{1 + 1/T}$
Naïve forecasts:	$\hat{\sigma}_h = \hat{\sigma}\sqrt{h}$
Seasonal naïve forecasts	$\hat{\sigma}_h = \hat{\sigma}\sqrt{k + 1}$
Drift forecasts:	$\hat{\sigma}_h = \hat{\sigma}\sqrt{h(1 + h/T)}$.

where k is the integer part of $(h - 1)/m$.

Note that when $h = 1$ and T is large, these all give the same approximate value $\hat{\sigma}$.

Prediction intervals

- Computed automatically using: `naive()`, `snaive()`, `rwf()`, `meanf()`, etc.
- Use `level` argument to control coverage.
- Check residual assumptions before believing them.
- Usually too narrow due to unaccounted uncertainty.