MASTER OF SCIENCE
IN ENGINEERING

Analysis of Sequential Data
Signal Processing Basics

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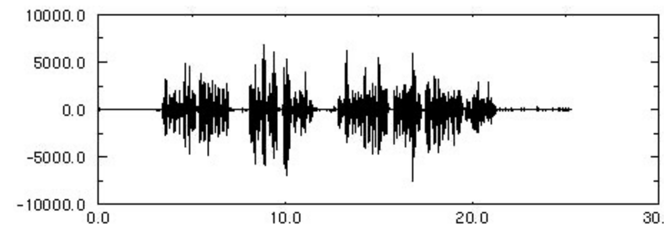
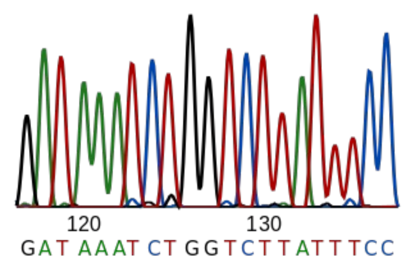
Objectives

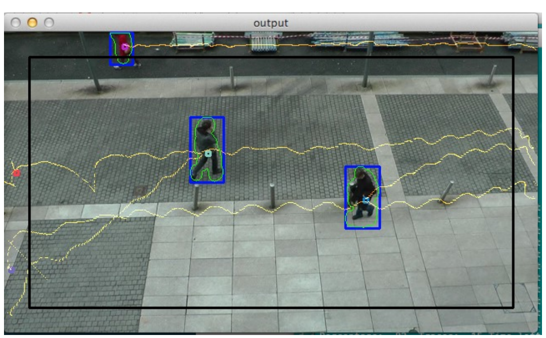
- You know what a digital signal is
- You know different types of digital signals
- You know basic terms of signal processing

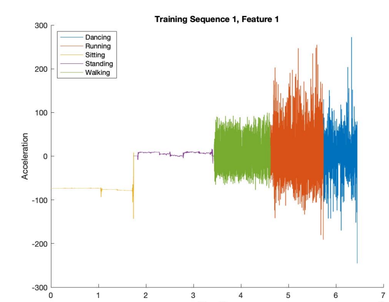
Sequential Data

- Examples
  - Music
  - Speech
  - Text
  - Video
  - Sensor signals, mesurements
  - DNA sequence
  - Financial data



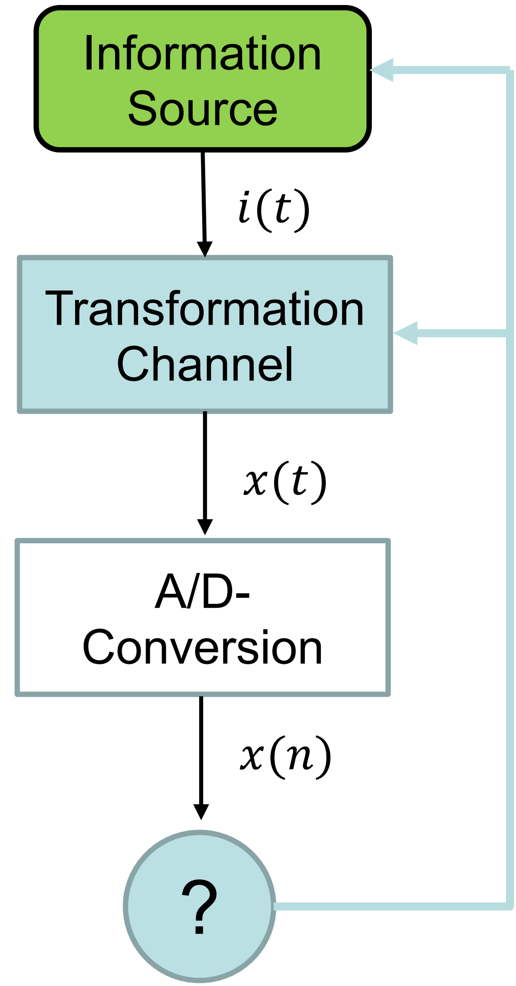






Digital Signal

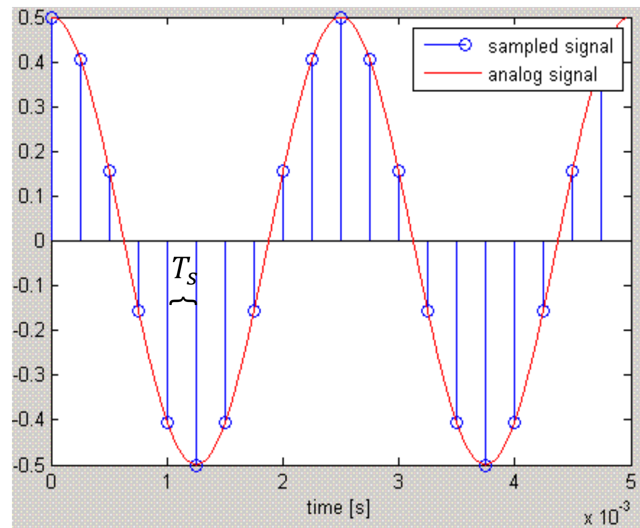
- Very common type of sequential data
- Digital signal
  - Often results from sampling an analog signal $x(t), -\infty \leq t \leq \infty$
  - Sequence of numbers $x(n)$, $-\infty < n < \infty,\ n \in \mathbb{N}, x \ \in \ \mathbb{Q}$
  - $x(n)$ is a discrete function of n
- Digital Signal Processing
  - Wants to find out from $x(n)$
    - Something about the information $i(t)$
    - Something about the Transformation Channel



Digital Signal

- A/D-Conversion
  - Take samples of analog signal $x(t)$ at regular times T_S
  - Digitize analog value $x(t)$ into an integer value $x(n)$

- Sampling
  - Sampling period T_S $[s]$
  - Sampling frequency $f_s = \frac{1}{T_S} [Hz, s^{-1}]$
  - $x(n) = x(t = nT_S)$



Sampling period Ts = 0.25ms (from figure)

Sampling frequency = 1/Ts = 1/(0.25 * 10^{-3} s)

= 4000 Hz

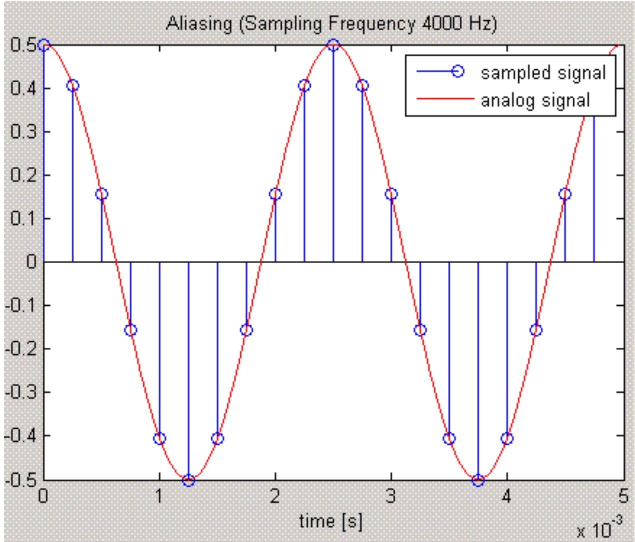
Sampling and Aliasing

- Sampling rate must be high enough:

$$\boxed{f_{s_{min}} > 2 * f_h}$$

f_h: highest frequency in analog signal.

- Otherwise aliasing occurs

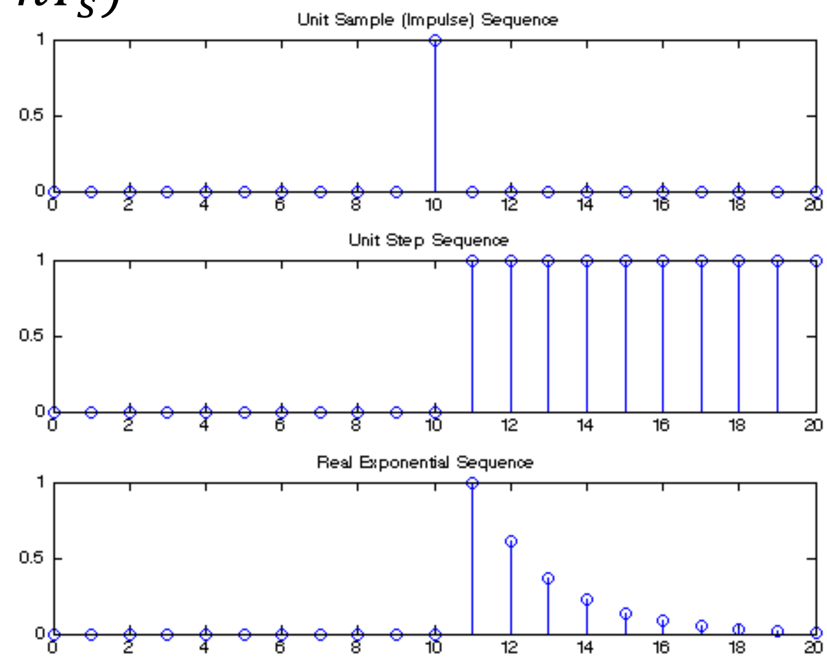


- Digitization introduces (small) quantization errors

Characteristics of Digital Signals

- Discrete in time $x(n) = x(t = nT_s)$
- Discrete in amplitude

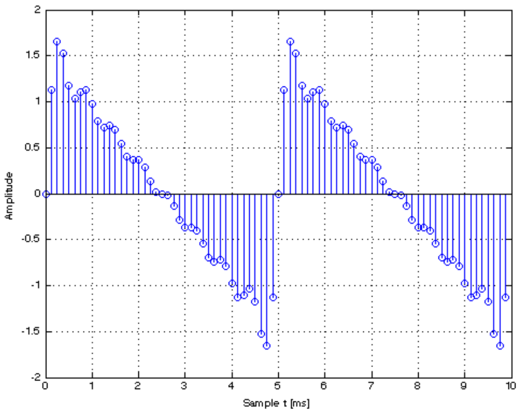
- Important types of (digital) signals
  - Non stationary signals
    - Unit Sample (Impulse)
    - Unit Step
    - Real Exponential

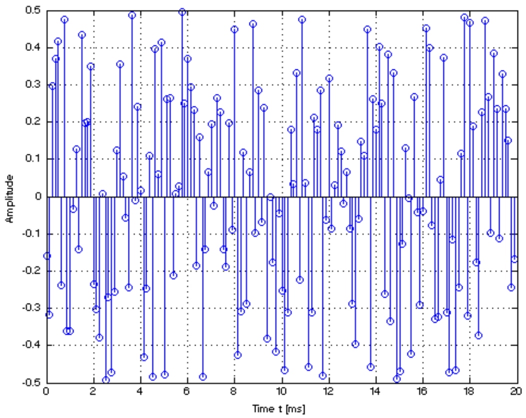


Characteristics of Digital Signals

- Discrete in time $x(n) = x(t = nT_s)$
- Discrete in amplitude

- Important types of (digital) signals
  - Stationary signals
    - Don't change characteristics in time
    - Periodic
    - Aperiodic: noise signals





Sinusoidal Signal

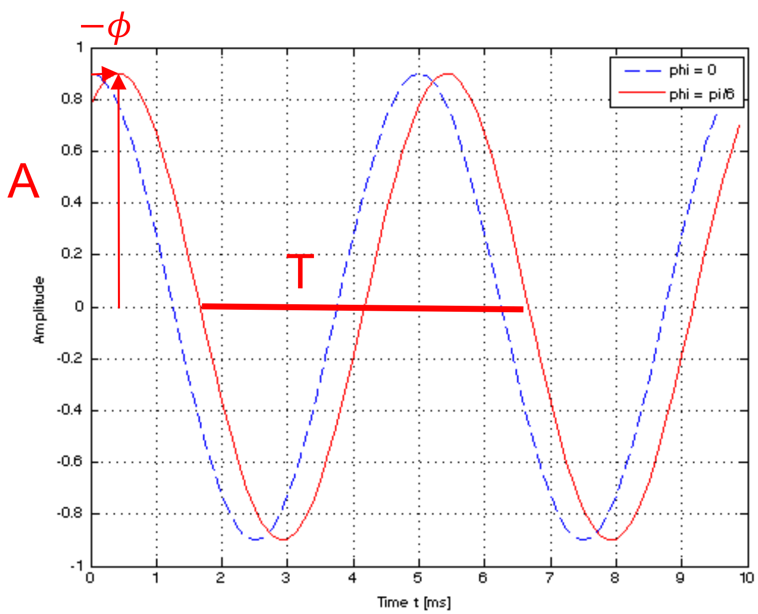
- Basic signal of all periodic signals

$$x(t) = A\ \cos(\omega t + \ \Phi)$$

  - Amplitude A
  - Phase Φ
  - Angular frequency ω [rad/s]

$$\omega = \frac{2\,\pi}{T} = 2\pi f$$

f: linear frequency [Hz = s^{-1}]
T: period [s]



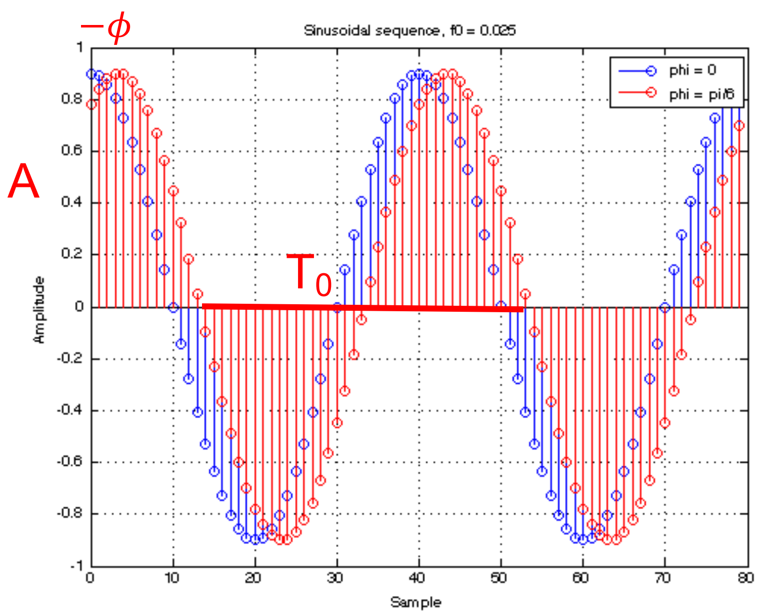
Sinusoidal Sequence

f_s= 8 kHz, -> T_s=1/f_s=0.125ms

T_0 = 40 (from figure)
T = $T_0 T_s$=40 0.125ms = 5ms
f = 1/T = 200 Hz

- Sinusoidal sequence

$$x(n) = A \cos(\omega_0 n + \Phi)$$



- Amplitude A
- Phase Φ
- Normalized angular frequency ω_0 [rad]

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

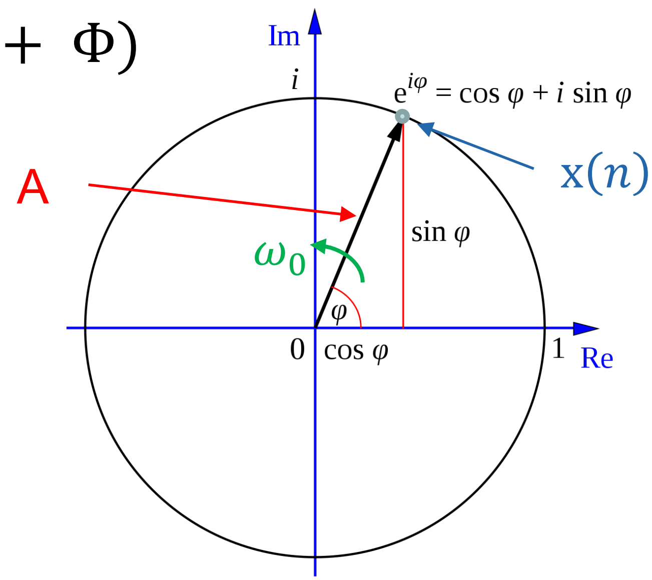
f_0: normalized linear frequency $\frac{f}{f_s}$ [0..1]

T_0: normalized period $\frac{T}{T_s}$ (≥ 1)

Sinusoidal Sequence

- Complex exponential sequence

$$x(n) = A\, e^{i\overbrace{(\omega_0 n + \Phi)}^{\varphi(n)}}$$
$$= A \cos(\omega_0 n + \Phi) + i\, A \sin(\omega_0 n + \Phi)$$



- Amplitude A
- Phase Φ
- Normalized angular frequency ω_0

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

f_0: normalized linear frequency $\frac{f}{f_S}$ [0..1]

T_0: normalized period $\frac{T}{T_S} (\geq 1)$

Periodic Sequences

- With period N (integer) for which the following holds true:

  $x(n) = x(n+N)$, for all n

- Each periodic sequence with period N can be represented by a sum of N complex exponential functions (Discrete Fourier Series):

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{i\overbrace{(2\pi/N)k}^{\omega_0(k)}n}$$

With

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i(2\pi/N)kn}$$

DFT Spectrum

- $X(k)\ ;\ 0 \leq k < N$
  - Amplitude of the k-th frequency component in signal $x(n)$
  - Is called the **Discrete Fourier Spectrum (N-point DFT spectrum)** of signal $x(n)$
  - It shows the N frequency components that comprise the periodic signal $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i(2\pi/N)kn} \quad 0 \leq k < N$$

$$X(k) = X(f = \frac{f_s k}{N})$$

DFT Spectrum

- $X(k)$ is equal to the continuous spectrum $X(\omega)$ of $x(n)$ sampled at N equally spaced intervals $(2\pi/N)$ in the range $0 \leq \omega < 2\pi$, i.e. in the lin. frequency range of $0 \leq f < f_s$

$$X(k) = X\left(f = \frac{f_s k}{N}\right); 0 \leq k < N$$

f_s: sampling frequency
N: sampling period

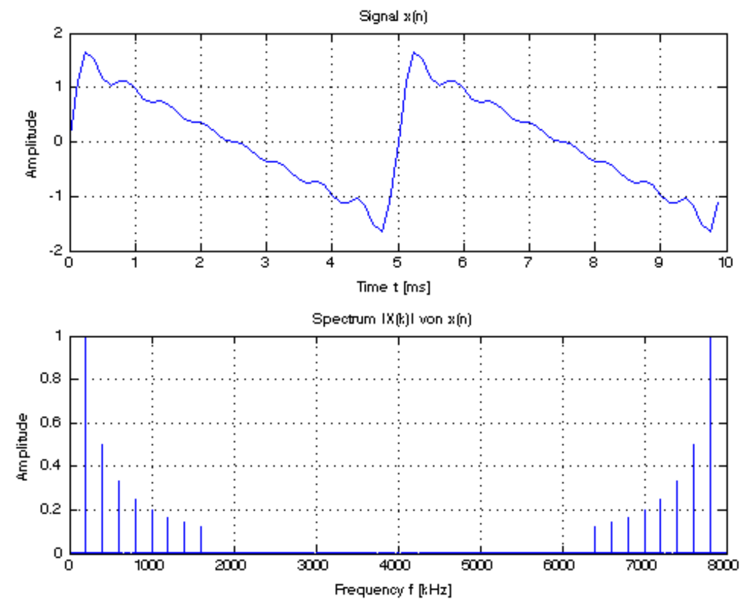
DFT Spectrum

- Example: Spectrum of the periodic signal

$$x(t) = \sum_{m=1}^{8} \frac{1}{m} \cos(m\omega_0 t - {}^{\pi}/_{2})$$

  - $\omega_0 = 2\pi f_0, f_0 = 200\ Hz$
  - $x(n) = x(t)$ sampled at 8kHz

- $|X(k)|$ shows the amplitudes A of the frequency components $f = (\frac{f_s k}{N})$

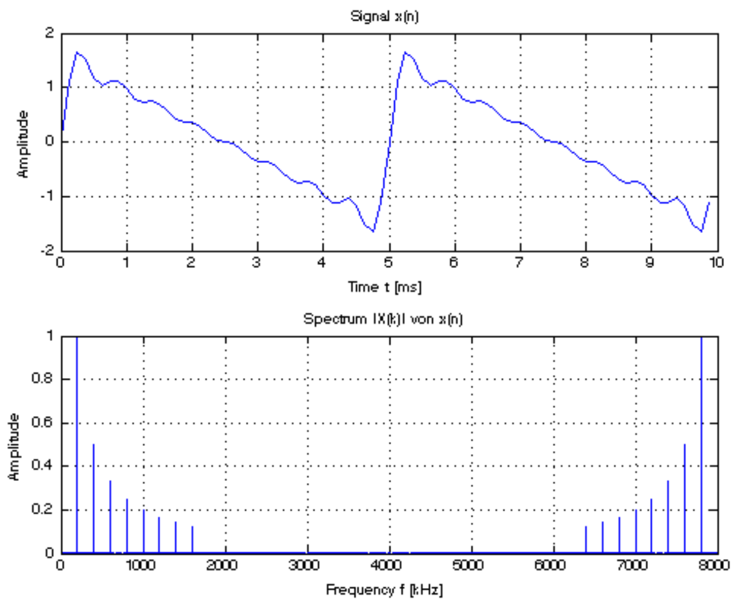


N-point DFT

DFT Spectrum

- Example: Spectrum of the periodic signal

- **Note:**
  - $|X(k)|$ has additional frequency components
    - Original components mirrored at $\frac{f_s}{2}$
    - Due to sampling and Euler formula $\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$

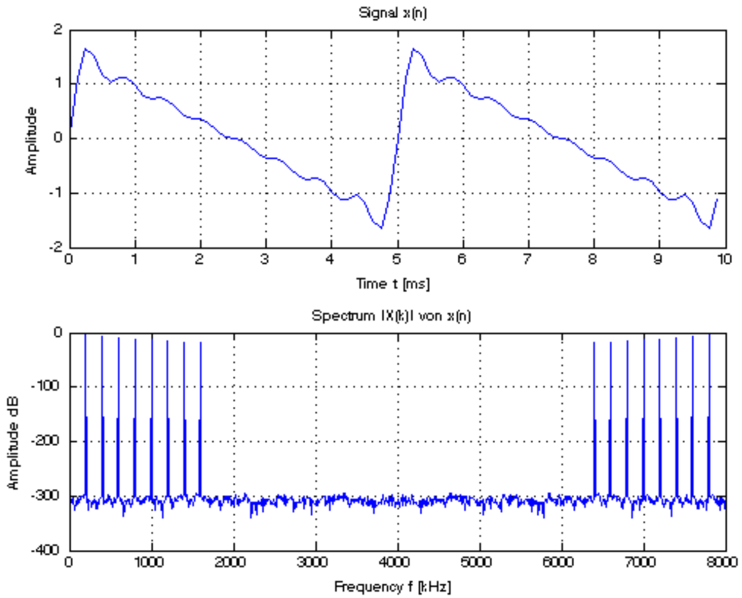


DFT dB-Spectrum

- Amplitudes in the dB-spectrum are normalized to a reference amplitude
- The ratio is depicted in Decibel:

  $$A[dB] = 20\log(\frac{A}{A_{ref}})$$

- 20 dB -> 10 times higher
  40 dB -> 100 times (!)
- A_{ref} may be the highest amplitude, e.g.



IDFT

- Inverse Discrete Fourier Transform
  - $x(n)$ can be calculated from $X(k)$ using the IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{i(2\pi/N)kn} \qquad , 0 \leq n < N$$

  - DFT and IDFT form a transformation (analysis/synthesis) pair:
    - Frequency domain (DFT):

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i(2\pi/N)kn} \qquad , 0 \ \leq k < N$$

    - Time domain (IDFT):

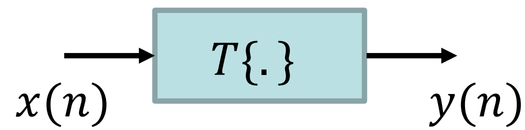
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{i(2\pi/N)kn} \qquad , 0 \leq n < N$$

Important Properties of the DFT

Property	Time Domain $x(n)$	Frequency Domain (DFT $X(k)$)
Periodicity	$x(n) = x((n))_N$	$X(k) = X((k))_N$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(k) + bX_2(k)$
Convolution	$x_1(n) * x_2((n))_N$	$X_1(k)X_2(k)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{N}(X_1(k) * X_2\big((k)\big)_N)$

Signal Transformation

- In general: $y(n) = T\{x(n)\}$
  - T maps input sequence $x(n)$ to output sequence $y(n)$
  - Examples
    - Delay system
      $y(n) = x(n-d)$
    - Moving average
      $$y(n) = \frac{1}{2M+1} \sum_{k=-M}^{M} x(k)$$
    - ARIMA system



Signal Transformation
LTI Sytems

- Linear Time-Invariant Systems (LTI)
  - linear:
    $T\{a_1x_1(n) + a_2x_2(n)\}$
    $= a_1T\{x_1(n)\} + a_2T\{x_2(n)\}$
    $= a_1\ y_1(n) + a_2y_2(n)$
  - Time-invariant
    if $y(n) = T\{x(n)\}$ then
    $y(n-d) = T\{x(n-d)\}$

$$\longrightarrow \boxed{T\{.\}} \longrightarrow$$

- Linear

$$a_1x_1(n) + a_2x_2(n) \xrightarrow{T\{.\}} a_1y_1(n) + a_2y_2(n)$$

- Time-invariant

$$x(n-d) \xrightarrow{T\{.\}} y(n-d)$$

Signal Transformation LTI Sytems

- Linear Time-Invariant Systems (LTI)
  - Examples:
    - Delay system
      $y(n) = x(n-d)$
    - Moving average
      $$y(n) = \frac{1}{2M+1} \sum_{k=-M}^{M} x(k)$$
    - ARIMA systems
  - Non-LTI examples
    - Median smoother
    - Hard limiter
    - Full/Half wave rectifier

$$\longrightarrow \boxed{T\{.\}} \longrightarrow$$

- Linear

$$a_1 x_1(n) + a_2 x_2(n) \xrightarrow{T\{.\}} a_1 y_1(n) + a_2 y_2(n)$$

- Time-invariant

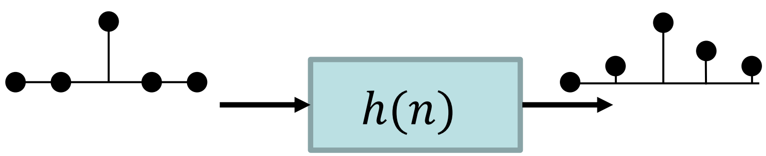
$$x(n-d) \xrightarrow{T\{.\}} y(n-d)$$

Signal Transformation LTI Sytems

- LTI systems are fully characterized by there **impulse response $\boldsymbol{h(n)}$**

$$y(n) = \sum_{k=-\infty}^{\infty} x(n)h(n-k)$$
$$= x(n) * h(n)$$

* denotes the convolution operator



- Impulse response

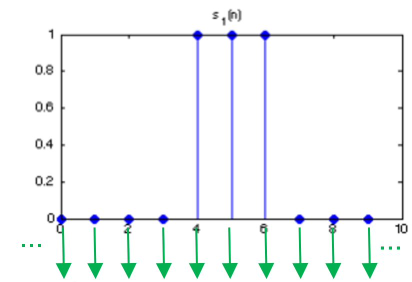
$$\delta(n) \xrightarrow{T\{.\}} h(n)$$

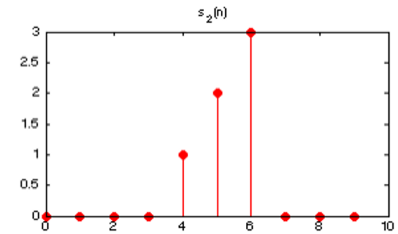
- General response

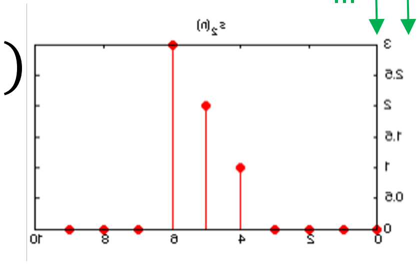
$$x(n) \xrightarrow{T\{.\}} y(n) = x(n) * h(n)$$

Discrete Convolution

- $y(n) = s_1(n) * s_2(n)$
$$= \sum_{k=-\infty}^{\infty} s_1(k) s_2(n-k)$$



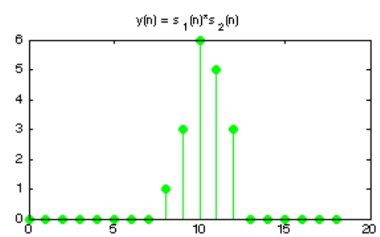




$y(0) = 0$ $y(1)$ = 0 $y(10)$ $y(16)$ = 0

- Circular convolution (modulo N)

$$s_1(n) * s_2((n))_N$$
$$= \sum_{k=0}^{N-1} s_1(k) s_2((n-k))_N$$



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Discrete Convolution Exercise

- $s_1(n)$ = 0, 1, 2, 2, 0
- $s_2(n)$ = 1, 0, 0, 0, 0

- What is the discrete convolution

- $y(n) = s_1(n) * s_2(n)$?

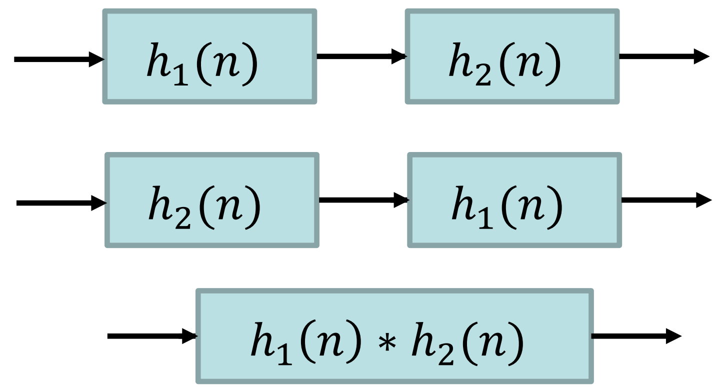
- $y(n) = s_1(n) * s_2(n)$
$$= \sum_{k=-\infty}^{\infty} s_1(k) s_2(n-k)$$

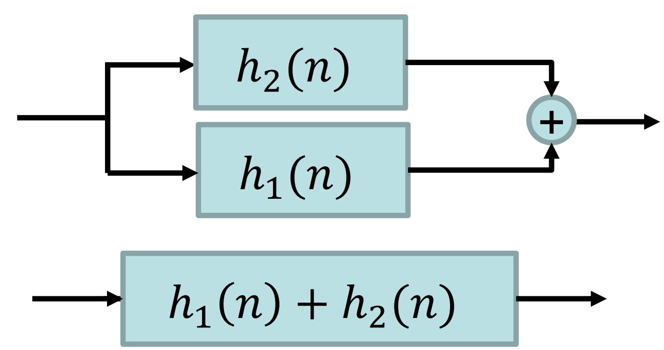
$s_1(n)$ = 0, 1, 2, 2, 0

$s_2(-n)$ = 0, 0, 0, 0, 1

Discrete Convolution

Property	**Equivalences**
Commutative	$x(n) * h(n) \; = \; h(n) * x(n)$
Associative	$x(n) * \left(h_1(n) * h_2(n)\right)$ $= (x(n) * h_1(n)) * h_2(n)$ $= x(n) * h_1(n) * h_2(n)$
Distributive	$x(n) * \left(h_1(n) + h_2(n)\right)$ $= x(n) * h_1(n) + x(n) * h_2(n)$





Frequency Response of LTI systems

- Fourier Transform of $h(n)$:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}$$

  - $H(\omega)$ is called the **frequency response** or **transfer function** of the LTI system
  - $H(\omega)$ is periodic with period 2π

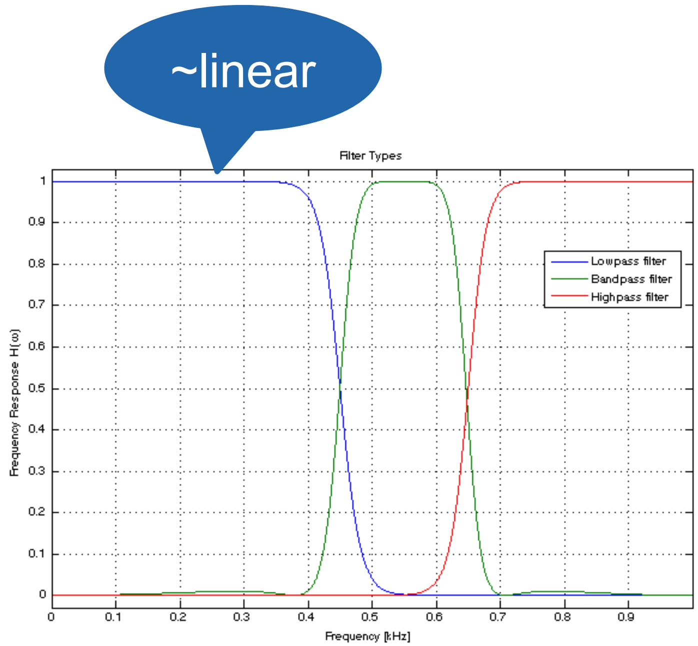
- LTI-systems:

$$y(n) = x(n) * h(n)$$
$$Y(\omega) = X(\omega) H(\omega)$$

  - A frequency component ω of $x(n)$ is transformed by the LTI into a signal with the same frequency amplified/attenuated by $H(\omega)$

Important LTI Systems
Filters

- Filters
  - Attenuate or amplify certain frequencies of the input signal $x(n)$
  - $Y(\omega) = X(\omega)H(\omega)$
  - ~linear in pass band
- Filter types
  - Lowpass filter
  - Highpass filter
  - Bandpass filter
  - Bandstop filter
- Filter design is a discipline of its own



Important LTI Systems
FIR Filters

- FIR-Filters
  - Finite Impulse Response Filter
  - General form:

    $$y(n) = a_1 x(n) + a_2 x(n-1) + a_2 x(n-2) + \ldots + a_p x(n-p)$$

    - p: filter order

  - Depending on coefficients a_i:
    - Highpass filter
    - Lowpass filter
    - Bandpass filter
    - Other filters