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Exercise Series 4

Issue date: 10th/12th October 2022

1. Exercise

Consider the half-space $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}^\top \boldsymbol{x} \geq b\}$ where $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a} \neq \boldsymbol{0}$, and $b \in \mathbb{R}$. Prove that the relative position of the half-space to its defining hyperplane is given by the direction of the vector $\boldsymbol{a}$. Or roughly spoken, that the half-space associated with "$\geq$" is "on the side where the vector $\boldsymbol{a}$ is pointing".

2. Exercise

Consider the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 \leq 2, x_1 + x_2 \leq 4, \boldsymbol{x} \geq \boldsymbol{0}\}$.

a) Transform P into inequality form $P = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{Ax} \leq \boldsymbol{b}\}$ and write down the constraints explicitly. Determine matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$.
b) Show that $\boldsymbol{A}$ has full column rank (and hence, the polyhedron P has vertices).
c) Draw a graphic representation of P. Determine all vertices of P.
d) Determine all defining hyperplanes of P.
e) Determine all possible basic selections, and the corresponding basic solutions to the system $\boldsymbol{Ax} \leq \boldsymbol{b}$. Which basic solutions are feasible, i.e. represent vertices of P ?

3. Exercise

Consider the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 2, x_2 = 1, \boldsymbol{x} \geq \boldsymbol{0}\}$.

a) Transform P into inequality form $P = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{Ax} \leq \boldsymbol{b}\}$ and write down the constraints explicitly. Determine matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$.
b) Show that $\boldsymbol{A}$ has full column rank (and hence, the polyhedron P has vertices).
c) Draw a graphic representation of P. Determine all vertices of P.
d) Determine all defining hyperplanes of P.
e) Determine all possible basic selections and the corresponding basic solutions to the system $\boldsymbol{Ax} \leq \boldsymbol{b}$. Which basic solutions are feasible, i.e. represent vertices of P ?

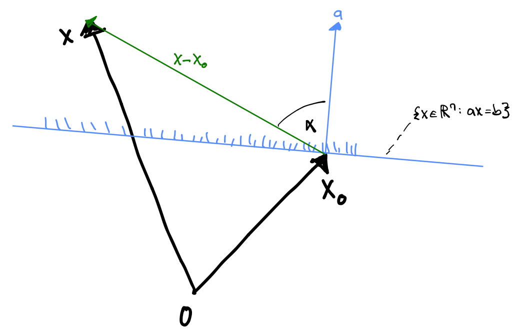
4. Exercise

Polyhedra in *standard form* $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ are very special polyhedra from a geometrical point of view.

It is well-known from elementary linear algebra that the solution set of a linear equation system $\boldsymbol{Ax} = \boldsymbol{b}$ is an *affine set* (also called *flat*). One can visualize flats geometrically as unbounded "flat" shapes of a certain "dimension". In $\mathbb{R}^3$, lines are the 1-dimensional flats, planes are the 2-dimensional flats, and the space $\mathbb{R}^3$ is the only 3-dimensional flat. As special cases, isolated points are regarded as 0-dimensional flats, and the empty set is regarded as an (-1)-dimensional flat. The solution set of the system $\boldsymbol{Ax} = \boldsymbol{b}$ has dimension $n - r$ if $rank(\boldsymbol{A}) = r$, i.e. if the number of linearly independent equations in $\boldsymbol{A}$ is r.

Exercise 1

The point x lies on the same side of the hyperplane $\{x \in \mathbb{R}^n : a^T x = b\}$ as the side to which the vector a is pointing if and only if the angle α between the vectors a and $x - x^0$ lies between 0 and 90 (or between 270 and 360) degrees, i.e. if and only if $\cos(\alpha) \geq 0$. Hence $a^T(x-x^0) = |a| \cdot |x-x^0| \cdot \cos\alpha \geq 0$ which implies
$a^T(x-x^0) = a^T x - a^T x^0 = a^T x - b \geq$, and therefore $a^T x \geq b$.



Exercise 2

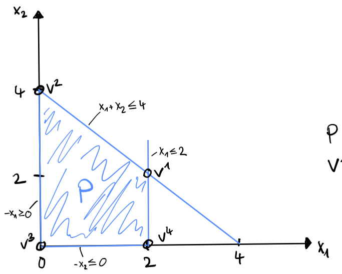
Consider the polyhedron $P = \{x \in \mathbb{R}^2 : x_1 \leq 2,\ x_1 + x_2 \leq 4,\ x \geq 0\}$

a) Inequality form $P = \{x \in \mathbb{R}^2 : x_1 \leq 2,\ x_1 + x_2 \leq 4,\ -x_1 \leq 0,\ -x_2 \leq 0\}$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

b) We just have two pivot elements (see above) and therefore A has full column rank, i.e. $\operatorname{rank}(A) = n = 2$

c)



P has the following vertices:
$v^1 = (2,2)^T$, $v^2 = (0,4)^T$, $v^3 = (0,0)^T$, $v^4 = (2,0)^T$

d) P has the following four defining hyperplanes
$H_1 = \{x \in \mathbb{R}^2 : x_1 = 2\}$, $H_2 = \{x \in \mathbb{R}^2 : x_1 + x_2 = 4\}$, $H_3 = \{x \in \mathbb{R}^2 : -x_1 = 0\}$, $H_4 = \{x \in \mathbb{R}^2 : -x_2 = 0\}$

e) There are $\binom{m}{n} = \binom{4}{2} = 6$ possibilities to choose $n=2$ rows of A. However, not all of them correspond to basic selections (since the chosen rows may be linearly dependent):

$B_1 = \{1,2\}$, $A_{B_1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $x_{B_1} = A_{B_1}^{-1} b_{B_1} = (2,2)^T$, feasible, corresponds to vertex $v^1 = (2,2)^T$

$B_2 = \{1,3\}$, $A_{B_2} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, not a basic selection since rows 1 and 3 are linearly dependent

$B_3 = \{1,4\}$, $A_{B_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $x_{B_3} = A_{B_3}^{-1} b_{B_3} = (2,0)^T$, feasible, corresponds to vertex $v^4 = (2,0)^T$

$B_4 = \{2,3\}$, $A_{B_4} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $x_{B_4} = A_{B_4}^{-1} b_{B_4} = (0,4)^T$, feasible, corresponds to vertex $v^2 = (0,4)^T$

$B_5 = \{2,4\}$, $A_{B_5} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $x_{B_5} = A_{B_5}^{-1} b_{B_5} = (4,0)^T$, infeasible, does not correspond to a vertex!

$B_6 = \{3,4\}$, $A_{B_6} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $x_{B_6} = A_{B_6}^{-1} b_{B_6} = (0,0)^T$, feasible, corresponds to vertex $v^3 = (0,0)^T$

Note that linear dependence of rows 1 and 3 in selection B_2 corresponds to the fact that the hyperplanes H_1 and H_3 are parallel in the picture.

Exercise 3

Consider the polyhedron $P = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 2,\ x^2 = 1,\ x \geq 0\}$

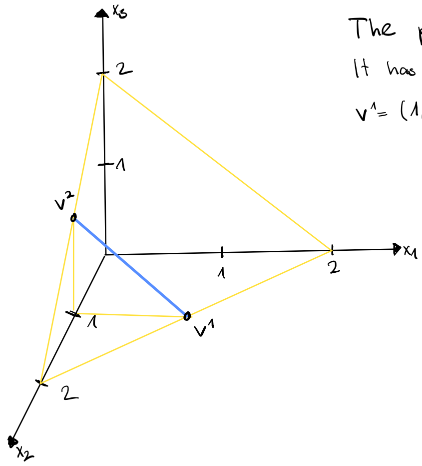
a) $P = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 \leq 2,\ -x_1 - x_2 - x_3 \leq -2,\ x^2 \leq 1,\ -x^2 \leq -1,\ -x_1 \leq 0,\ -x_2 \leq 0,\ -x_3 \leq 0\}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

b) Looking at the last 3 equations shows us that A has full column rank, i.e. $\operatorname{rank}(A) = n = 3$

they are spanning the whole space

c)



The polyhedron P corresponds to the line segment joining v^1 and v^2.

It has the following two vertices:

$v^1 = (1,1,0)^T$, $v^2 = (0,1,1)^T$

d) P has a total of 7 defining hyperplanes, whereby two pairs of hyperplanes are actually identical:

$H_1 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 2\} = H_2 = \{x \in \mathbb{R}^3 : -x_1 - x_2 - x_3 = -2\}$

$H_3 = \{x \in \mathbb{R}^3 : x_2 = 1\} = H_4 = \{x \in \mathbb{R}^3 : -x_2 = -1\}$

$H_5 = \{x \in \mathbb{R}^3 : -x_1 = 0\}$

$H_6 = \{x \in \mathbb{R}^3 : -x_2 = 0\}$

$H_7 = \{x \in \mathbb{R}^3 : -x_3 = 0\}$

Note that all feasible points of P are lying on hyperplane $H_1 = H_2$ and on hyperplane $H_3 = H_4$. This reduces the "dimension" of the polyhedron P to 1, i.e. it corresponds to a line segment.

The solution set of the constraints $\boldsymbol{Ax} = \boldsymbol{b}$ is a flat, hence it is unbounded and has no "faces" and no "corners". Faces and corners of the polyhedron P occur only where the corresponding flat penetrates the faces of the orthant $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \boldsymbol{0}\}$, i.e. for every face and every corner, at least one equation of the form $x_i = 0,\ i \in \{1, \ldots, n\}$ has to be fulfilled.

Verify the above-mentioned statements by graphically representing the following polyhedra in standard form:

- $P = \{\boldsymbol{x} \in \mathbb{R}^3 : 4x_1 + 3x_2 + 3x_3 = 12,\ \boldsymbol{x} \geq \boldsymbol{0}\}$
- $Q = \{\boldsymbol{x} \in \mathbb{R}^3 : 4x_1 + 3x_2 + 3x_3 = 12, x_3 = 2,\ \boldsymbol{x} \geq \boldsymbol{0}\}$

5. Exercise

Consider the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 - x_2 \leq 1, \boldsymbol{x} \geq \boldsymbol{0}\}$.

a) Transform *P* into inequality form $P = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{Ax} \leq \boldsymbol{b}\}$ and write down the constraints explicitly. Determine matrix ***A*** and vector ***b***.

b) Transform *P* into standard form $P' = \{\boldsymbol{x}' \in \mathbb{R}^3 : \boldsymbol{A}'\boldsymbol{x}' = \boldsymbol{b}', \boldsymbol{x}' \geq \boldsymbol{0}\}$ and write down the constraints explicitly. Determine vector $\boldsymbol{x}'$, matrix $\boldsymbol{A}'$ and vector $\boldsymbol{b}'$.

c) Draw a graphic representation of *P* and P', and compare both pictures.

d) The point $\boldsymbol{x} = (1,1)^\top$ is feasible in *P*. Determine the point $\boldsymbol{x}' \in P'$ corresponding to ***x***.

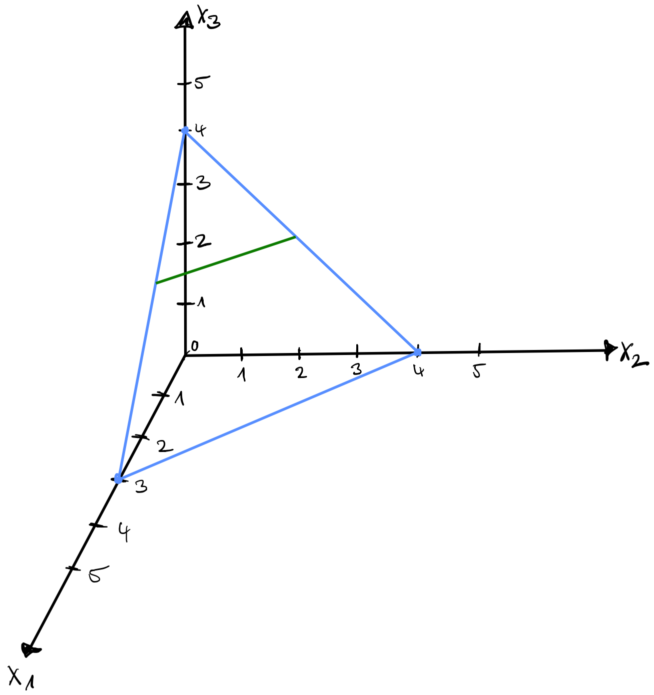
e) Determine all vertices of *P* and *P'*. Compare the vertices of the two polyhedra. What is the connection between the two polyhedra?

Exercise 4

$P = \{x \in \mathbb{R}^3 : 4x_1 + 3x_2 + 3x_3 = 12,\ x \geq 0\}$

$Q = \{x \in \mathbb{R}^3 : 4x_1 + 3x_2 + 3x_3 = 12,\ x_3 = 2,\ x \geq 0\}$

The two polyhedra P and Q are drawn below. P is a ("2-dimensional") section of a plane, and Q is a ("1-dimensional") line segment.



Exercise 5

Consider the polyhedron $P = \{x \in \mathbb{R}^2 : x_1 - x_2 \leq 1,\ x \geq 0\}$

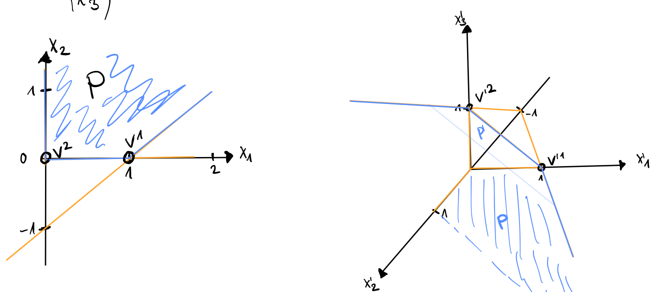
a) $P = \{x \in \mathbb{R}^2 : x_1 - x_2 \leq 1,\ -x_1 \leq 0,\ -x_2 \leq 0\}$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix},\quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b) $P' = \{x \in \mathbb{R}^3 : x_1' - x_2' + x_3' = 1,\ x_1' \geq 0,\ x_2' \geq 0,\ x_3' \geq 0\}$

$$x' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix},\quad A = (1\ \ -1\ \ 1),\quad b = (1)$$

c)



d) The point $x' \in P'$ corresponding to the point $x = (1,1)^T \in P$ is given by:

$x' = (x_1', x_2', x_3')' = (x_1, x_2, b_1 - a^{1'}x)' = (x_1, x_2, 1 - (x_1 - x_2))' = (1, 1, 1)'$

e) P has two vertices: $v^1 = (1,0)^T$, $v^2 = (0,0)^T$

P' has two vertices as well: $v'^1 = (1,0,0)^T$, $v'^2 = (0,0,1)^T$

The connection between the points can be described as follows:

$x = (x_1, x_2)^T \in P \Leftrightarrow x' = (x_1', x_2', x_3')^T = (x_1, x_2, b_1 - a^{1'}x)^T \in P'$