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Exercise Series 7

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1. Exercise

Consider the polyhedron $P \subseteq \mathbb{R}^2$ defined by the following constraints:

$$\begin{aligned} -x_1 + 4x_2 &\leq 8 \\ 2x_1 + 2x_2 &\leq 9 \\ 6x_1 - 2x_2 &\leq 11 \\ x_1 + 4x_2 &\geq 4 \\ 2x_1 &\geq 1 \end{aligned}$$

a) Transform these constraints into a system of inequalities of the form $\boldsymbol{Ax} \leq \boldsymbol{b}$.

b) Draw the polyhedron *P*. Indicate in the drawing which constraint corresponds to which "face" of the polyhedron.

c) Draw the integer hull $P_{\mathbb{Z}}$.

d) Construct the integer hull $P_{\mathbb{Z}}$ by successively adding Gomory-Chvátal cuts. Write down clearly for each cutting plane the inequalities and coefficients used to generate it.

e) What is the Chvátal rank of this polyhedron?

2. Exercise (* facultative *)

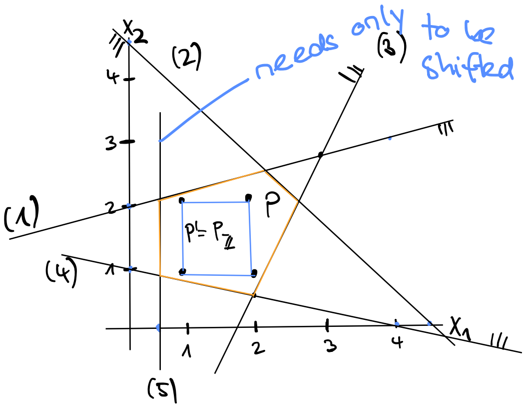
Formulate an ILP model for solving the Sudoku puzzle.

Exercise 1

$-x_1 + 4x_2 \leq 8$
$2x_1 + 2x_2 \leq 9$
$6x_1 - 2x_2 \leq 11$
$x_1 + 4x_2 \geq 4$
$2x_1 \geq 1$

a)
$$-x_1 + 4x_2 \leq 8 \quad (1)$$
$$2x_1 + 2x_2 \leq 9 \quad (2)$$
$$6x_1 - 2x_2 \leq 11 \quad (3)$$
$$-x_1 - 4x_2 \leq -4 \quad (4)$$
$$-2x_1 \leq -1 \quad (5)$$

b)
c)



$$\frac{1}{2} \cdot (-2x_1) \leq -1$$
$$-x_1 \leq -1$$
$$x_1 \geq 1$$

d)
$$u_1(-x_1 + 4x_2) \leq 8u_1$$
$$u_2(2x_1 + 2x_2) \leq 9u_2$$
$$\rightarrow x_1 \underbrace{(-u_1 + 2u_2)}_{\in \mathbb{Z}} + x_2 \underbrace{(4u_1 + 2u_2)}_{\in \mathbb{Z}} \leq \underbrace{8u_1 + 9u_2}_{\notin \mathbb{Z}}$$

$u_1 = \frac{1}{5}$ and $u_2 = \frac{1}{10}$

$$x_1\left(-\frac{1}{5} + \frac{2}{10}\right) + x_2\left(\frac{4}{5} + \frac{2}{10}\right) \leq \frac{8}{5} + \frac{9}{10}$$

$\frac{16}{10} + \frac{9}{10}$, $\frac{25}{10}$

$$x_2 \leq \left\lfloor \frac{5}{2} \right\rfloor = 2$$
$$x_2 \leq 2$$

$$u_1(2x_1 + 2x_2) \leq 9$$
$$u_2(6x_1 - 2x_2) \leq 11$$
$$x_1(2u_1 + 6u_2) + x_2(2u_1 - 2u_2) \leq 9u_1 + 11u_2$$

$u_1 = \frac{1}{8}$ $u_2 = \frac{1}{8}$

$$x_1 \leq \left\lfloor \frac{5}{2} \right\rfloor = 2$$
$$x_1 \leq 2$$

$$u_1(6x_1 - 2x_2) \leq 11$$
$$u_2(-x_1 - 4x_2) \leq -4$$
$$x_1(6u_1 - u_2) + x_2(-2u_1 - 4u_2) \leq 11u_1 - 4u_2$$

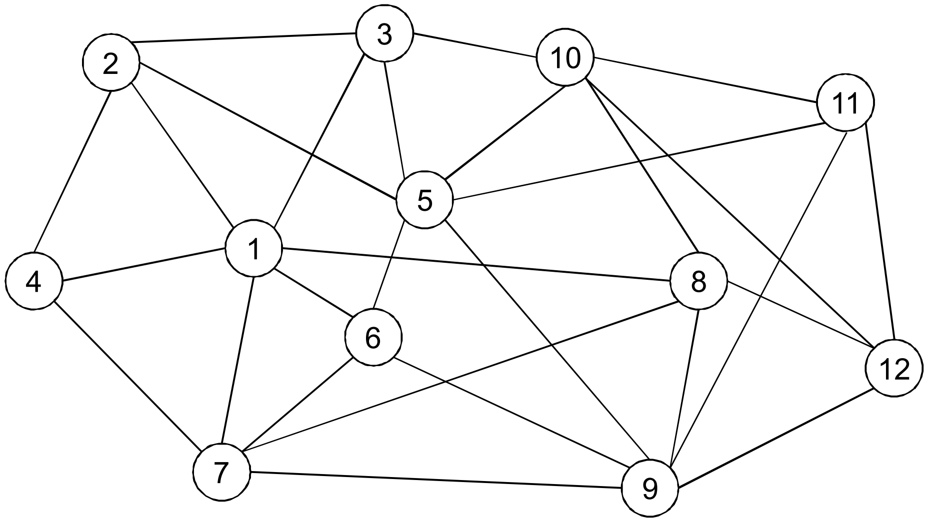
$u_1 = \frac{1}{26}$ and $u_2 = \frac{3}{13}$

$$x_1\left(\frac{6}{26} - \frac{3}{13}\right) + x_2\left(-\frac{2}{26} - \frac{12}{13}\right) \leq \frac{11}{26} - \frac{12}{13}$$

$$-x_2 \leq \left\lfloor -\frac{1}{2} \right\rfloor = -1$$
$$x_2 \geq 1$$

3. Exercise (* facultative *)

Let be given the following undirected graph $G=(V,E)$, where edge weights have been omitted for clarity. Consider the ILP formulation "Version 1" for the problem of the minimum weight spanning tree, with $v_0=9$ as selected special node.



a) Write down the constraint associated to node set $S=\{1,2,5,6,8,12\}$.

b) The edge set $T=\{(1,2),(1,3),(1,7),(1,8),(3,5),(3,10),(5,10),(5,11),(8,10)\}$ does not represent a connector. Hence, at least one constraint of the formulation must be violated. Usually, there are even more than one violated constraints for an infeasible edge set.

   Find at least three constraints that exclude the infeasible edge set *T* from the solution space. Write down the incidence vector of *T*, and show that it does not fulfill the constraints found.

Hint: As mentioned, a connector is an edge set $T\subseteq E$ such that the subgraph (V,T) is connected, i.e. a connector is an edge set connecting all nodes of a graph.

4. Exercise (* facultative *)

Let be given the same graph as in the preceding exercise. Consider now the ILP formulation "Version 2" for the problem of the minimum weight spanning tree

a) Write down the constraints associated to partition $\mathscr{P}=\{\{1,4,5,9\},\{2,6,7\},\{3,8,10,11,12\}\}$.

b) The edge set $T=\{(1,2),(1,3),(1,4),(1,7),(2,3),(5,6),(5,10),(8,10),(9,12),(11,12)\}$ does not represent a connector. Hence, at least one constraint of the formulation must be violated. Usually, there are even more than one violated constraints for an infeasible edge set.

   Find at least three constraints that exclude the infeasible edge set *T* from the solution space. Write down the incidence vector of *T*, and show that it does not fulfill the constraints found.