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Exercises Series 2

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Exercise 1

Consider the introductory problem “Vehicle dispatching in a car rental company” from the lecture notes.

a) Implement the linear optimization model from the lecture in Excel. Use the distances coming from the attached separate file.

b) Write down the mathematical formulation of an extended optimization problem which can handle dispatching problems with various vehicle types. (You don’t need to implement the extended model in Excel.)

c) Write down the optimization problem (not extended) in the explicit form (i.e. with actual numbers and without sums and vectors) for an instance with 3 car hire locations. The data is given as follows:

		1	2	3
Current stock	a_i	7	5	9
Requested stock	b_j	4	11	6

Distances

c_{ij}	1	2	3
1	0 1,1	20 1,2	35 1,3
2	20 2,1	0 2,2	19 2,3
3	35 3,1	24 3,2	0 3,3

Exercise 2

Consider the introductory problem “Product mixture in an oil refinery” from the lecture notes.

a) Formulate the problem as a linear optimization problem for the general case with a set $I = \{1, \ldots, m\}$ describing the raw fuels and the set $J = \{1, \ldots, n\}$ describing the jet fuels. Specify in detail the used sets, parameters, variables, further constraints and the objective function.

b) Write down the optimization problem in the explicit form (i.e. with actual numbers and without sums and vectors) for the instance with four raw fuel types and two jet fuel types mentioned in the lecture notes.

c) Implement the optimization model for the instance from the lecture notes in Excel, and determine an optimal solution.

Exercise 1

a) see excel sheet

b) We can basically treat every vehicle typ as an individual independent transportation like the one in a)

c) We want to optimize the following expression

$$\min\ x_{11} + 20x_{12} + 35x_{13} + 20x_{21} + x_{22} + 19x_{23} + 35x_{31} + 24x_{32} + x_{33}$$

with the following constraints

$$x_{11} + x_{12} + x_{13} \leq 7$$
$$x_{21} + x_{22} + x_{23} \leq 5$$
$$x_{31} + x_{32} + x_{33} \leq 9$$
$$x_{11} + x_{21} + x_{31} \geq 4$$
$$x_{12} + x_{22} + x_{32} \geq 11$$
$$x_{13} + x_{23} + x_{33} \geq 6$$
$$x_{11} + \dots + x_{33} \geq 0$$ // you can not have a negative number of cars

Exercise 2

→ see script for problem "Product mixture in an oil refinery"

a) raw fuel i: x_i

amounts of raw fuels in jet fuel j: y_{ij}

TODO

Price raw fuel types   Price Avgas A   Price Avgas B

b) $$\max\ 122.81(x_1+x_2+x_3+x_4) + 175.04(y_{11}+y_{21}+y_{31}+y_{41}) + 152.68(y_{12}+y_{22}+y_{32}+y_{42})$$

I
$$107y_{11} + 93y_{21} + 87y_{31} + 108y_{41} \geq 100(y_{11}+y_{21}+y_{31}+y_{41})$$ The mixture of fuels must have at least $PN \geq 100$ for jet fuel Avgas A
$$107y_{12} + 93y_{22} + 87y_{32} + 108y_{42} \geq 91(y_{12}+y_{22}+y_{32}+y_{42})$$ Same but for Avgas B with $PN \geq 91$

II
$$5y_{11} + 8y_{21} + 4y_{31} + 21y_{41} \leq 7(y_{11}+y_{21}+y_{31}+y_{41})$$
$$5y_{12} + 8y_{22} + 4y_{32} + 21y_{42} \leq 7(y_{12}+y_{22}+y_{32}+y_{42})$$
again same as above but now for $RVP \leq 7$

$$x_1 + y_{11} + y_{12} \leq 3'814$$
$$x_2 + y_{21} + y_{22} \leq 2'666$$
$$x_3 + y_{31} + y_{32} \leq 4'016$$
$$x_4 + y_{41} + y_{42} \leq 1'300$$
sum of fuels can not be bigger than the max. production capacity

$$x_1, x_2, x_3, x_4 \geq 0$$
$$y_{11}, y_{12}, y_{21}, y_{22}, y_{31}, y_{32}, y_{41}, y_{42} \geq 0$$
Amounts of fuels can't be negative

I and II can be re-written as follows

I'
$$(107-100)y_{11} + (93-100)y_{21} + (87-100)y_{31} + (108-100)y_{41} \geq 0$$
$$(107-100)y_{12} + (93-100)y_{22} + (87-100)y_{32} + (108-100)y_{42} \geq 0$$

II'
$$(5-7)y_{11} + (8-7)y_{21} + (4-7)y_{31} + (21-7)y_{41} \leq 0$$
$$(5-7)y_{12} + (8-7)y_{22} + (4-7)y_{32} + (21-7)y_{42} \leq 0$$

c) see excel sheet

Exercise 3

Consider a base set

$$S = \{\boldsymbol{x} \in \{0,1\}^5 : \sum_{i=1}^{5} x_i = 3\}$$

and a function given by

$$f : S \to \mathbb{R} \text{ with } f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}} \boldsymbol{x}\text{, where } \boldsymbol{c}^{\mathsf{T}} = (5,3,7,1,2).$$

Further, we define on S a neighbourhood notion N as follows:

$$N(\boldsymbol{x}) = \{\boldsymbol{x}' \in S : x'_i = x_{i \bmod 5+1} \text{ for } i = 1,\ldots,5\} \cup$$
$$\{\boldsymbol{x}' \in S : x'_i = x_{(i+3) \bmod 5+1} \text{ for } i = 1,\ldots,5\} \cup \{\boldsymbol{x}\}$$

a) Determine all elements of S and the corresponding function values of f.

b) Determine all local (with regard to N) and all global maximal and minimal solutions of f.

c) Choose any local minimal solution and prove that it is indeed a local minimal solution.

Hint: To understand the terms $x'_i = x_{i \bmod 5+1}$ *and* $x'_i = x_{(i+3) \bmod 5+1}$ *enter some values of* i *into the formulas. The operator "mod" describes the modulo operator, which calculates the remainder when performing integer division. For instance,* $5 \bmod 5 = 0$ *and* $6 \bmod 5 = 1$.

Exercise 4

The *Traveling Salesman Problem* (TSP) is a fundamental optimization problem in the graph theory and belongs to the most researched problems of the Combinatorial Optimization. The task (in a symmetric TSP) is to find a tour with shortest length in an undirected graph with given edge lengths. Here, a tour is a round trip where every vertex of the graph is visited exactly once.

There exist numerous (meta-)heuristics for the TSP; many of them are based on the principle of step-wise improvement of the existing solution (*iterative improvement, local search*). Starting at a current solution (i.e. a tour) $\boldsymbol{x}$ we are looking for a better tour in a *neighbourhood* $N(\boldsymbol{x})$ of the tour $\boldsymbol{x}$; if a better tour is found, it will replace the current tour $\boldsymbol{x}$. The method is repeated until no better tours can be found in the neighbourhood. The final tour is then *locally optimal* regarding the used neighbourhood.

One of the most successful neighbourhood notions is the so-called *2-opt-neighbourhood,* which was developed in 1973 by Shen Lin and Brian Kernighan. Here, a neighbouring tour is constructed by eliminating two arbitrary non-adjacent edges from the tour and by connecting the remaining line segments by two new edges, resulting in a new tour (see example below).

As an example, consider the following undirected graph $G = (V, E)$; the edge lengths are not needed here and hence omitted. We are given the marked tour $\boldsymbol{x}$ in the left picture which visits the vertices 1, 2, 3, 4, 5, 1 in this order. The right picture depicts a neighbouring tour from the neighbourhood $N(\boldsymbol{x})$ which can be obtained by eliminating the edges (2, 3) and (1, 5), and simultaneously inserting the edges (1, 3) and (2, 5).

Exercise 3

a) $S = \{x \in \{0,1\}^5 : \sum_{i=1}^{5} x_i = 3\}$

All elements of S consists of 5 numbers whereby 3 of them are 1 and two of them are 0. All possible combinations of this are elements of S. The solution space consists of exactly 10 solutions because the cardinality of S is given by

$|S| = \binom{5}{3} = \frac{5!}{3!\,(5-3)!} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$

We now just need to put these values into function f to get the corresponding function values. All elements and values of the function are listed below.

$f: S \to \mathbb{R}$ with $f(x) = c^T x$, where $c^T = (5,3,7,1,2)$

b) The neighborhood $N(x)$ is defined as the union of three sets where the last set just contains the vector x itself.

$N(x) = \{x' \in S : x'_i = x_{i \bmod 5+1} \text{ for } i = 1,\dots,5\} \cup \{x' \in S : x'_i = x_{(i+3) \bmod 5+1} \text{ for } i = 1,\dots,5\} \cup \{x\}$ Now we insert the values $1,\dots,5$:

$$\begin{aligned}
x'_1 &= x_{1 \bmod 5+1} = x_2 \\
x'_2 &= x_{2 \bmod 5+1} = x_3 \\
x'_3 &= x_{3 \bmod 5+1} = x_4 \\
x'_4 &= x_{4 \bmod 5+1} = x_5 \\
x'_5 &= x_{5 \bmod 5+1} = x_1
\end{aligned}
\qquad
\begin{aligned}
x'_1 &= x_{(1+3) \bmod 5+1} = x_5 \\
x'_2 &= x_{(2+3) \bmod 5+1} = x_1 \\
x'_3 &= x_{(3+3) \bmod 5+1} = x_2 \\
x'_4 &= x_{(4+3) \bmod 5+1} = x_3 \\
x'_5 &= x_{(5+3) \bmod 5+1} = x_4
\end{aligned}$$

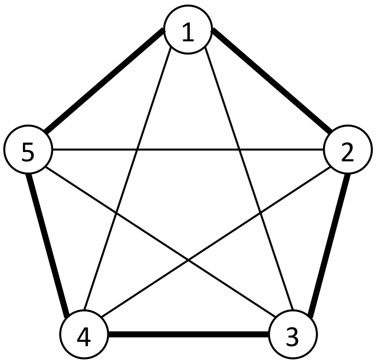
cyclic right shift — cyclic left shift

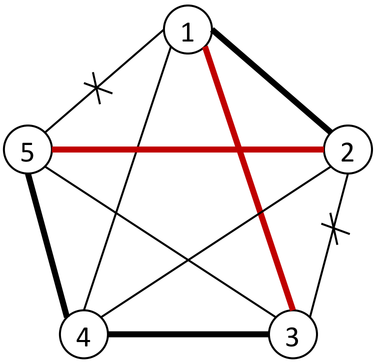
$x = (1,1,1,0,0)^T \to x^{right} = (1,1,0,0,1)^T$ — $x = (1,1,1,0,0)^T = (1,1,0,0,1)^T$

All local/global Max/Min points are listed below in the table.

x:	f(x):	x^{right}:	$f(x^{right})$:	x^{left}:	$f(x^{left})$:	Max:	Min:
(1,1,1,0,0)	15	(0,1,1,1,0)	11	(1,1,0,0,1)	10	global	–
(0,1,1,1,0)	11	(0,0,1,1,1)	10	(1,1,1,0,0)	15	–	–
(0,0,1,1,1)	10	(1,0,0,1,1)	8	(0,1,1,1,0)	11	–	–
(1,0,0,1,1)	8	(1,1,0,0,1)	10	(0,0,1,1,1)	10	–	local
(1,1,0,0,1)	10	(1,1,1,0,0)	15	(1,0,0,1,1)	8	–	–
(1,1,0,1,0)	9	(0,1,1,0,1)	12	(1,0,1,0,1)	14	–	local
(0,1,1,0,1)	12	(1,0,1,1,0)	13	(1,1,0,1,0)	9	–	–
(1,0,1,1,0)	13	(0,1,0,1,1)	6	(0,1,1,0,1)	12	local	–
(0,1,0,1,1)	6	(1,0,1,0,1)	14	(1,0,1,1,0)	13	–	global
(1,0,1,0,1)	14	(1,1,0,1,0)	9	(0,1,0,1,1)	6	local	–

c) When we look at the local minima $x^*(1,0,0,1,1)$ we need to verify that the values in the neighborhood are bigger. This is clearly the case because $8 \leq 10$ and $8 \leq 10$.





Tours in a graph $G = (V, E)$ can be regarded as subsets of edges and can be described by an *incidence vector* $\boldsymbol{x} \in \{0,1\}^E$. The incidence vector is a 0-1-vector which consists of components x_{vw} for all edges (v, w). A component is equal to 1 if the corresponding edge belongs to the tour, otherwise it is 0. The edge order in the incidence vector can be chosen freely but has to be followed consequently. For instance, the current tour in the left picture is given by the incidence vector

$$\boldsymbol{x} = (x_{12}, x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45})^{\mathsf{T}} = (1,0,0,1,1,0,0,1,0,1)^{\mathsf{T}}.$$

a) Write down the set of vertices V and the set of edges E of the graph $G = (V, E)$.

b) Write down all elements (i.e. corresponding incidence vectors) of the neighbourhood $N(\boldsymbol{x})$, where $\boldsymbol{x}$ is the current tour mentioned above.

c) *(* optional *)* Let $S \subseteq \{0,1\}^E$ be the solution space of the TSP, i.e. the set of all incidence vectors of all tours. For the considered example, show that $N(\boldsymbol{x}) = N_\varepsilon(\boldsymbol{x}) \cap S$ for $\varepsilon = 2.1$.

Exercise 5

Consider the base set

$$S = \{x \in \mathbb{R} : -3 \le x \le 3\}$$

and the function f given by

$$f : S \to \mathbb{R} \text{ and } f(x) = |-x^2 + 4|$$

a) Plot the graph of the function f.

b) Determine all local (with regard to Euclidean neighbourhoods) and global maximal solutions of f and the corresponding function values.

c) Determine all local (with regard to Euclidean neighbourhoods) and global minimal solutions of f and the corresponding function values.

d) Prove that $x^* = 0$ is a local maximal solution with regard to Euclidean neighbourhoods.

Exercise 4

a) $V = \{1, 2, 3, 4, 5\}$, $E = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$

b) Todo

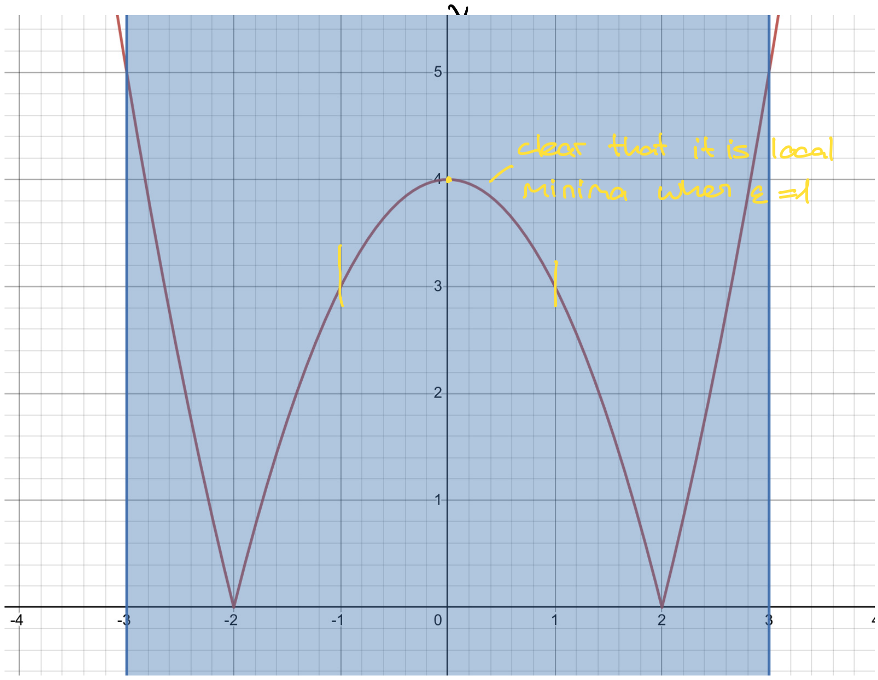
c) optional

Exercise 5

Base set $S = \{x \in \mathbb{R} : -3 \leq x \leq 3\}$

and the function f given by $f: S \to \mathbb{R}$ and $f(x) = |1 - x^2 + 4|$

a)



b)

x^*:	$f(x^*)$:	Local maximum:	Global maximum:
-3	5	yes	yes
0	4	yes	no
3	5	yes	yes

c)

x^*:	$f(x^*)$:	Local minimum:	Global minimum:
-2	0	yes	yes
2	0	yes	yes

d) We look at the solution for $x^* = 0$. We have to show that these exists $\varepsilon > 0$ such that

$$f(x) \leq f(x^*) \text{ for all } x \in N_\varepsilon(x^*) \cap S = \{y \in \mathbb{R} : |y - x^*| < \varepsilon\} \cap S$$

$$= \{y \in \mathbb{R} : x^* - \varepsilon < y < x^* + \varepsilon, -3 \leq y \leq 3\}$$

$$= \{y \in \mathbb{R} : -\varepsilon < y < \varepsilon, -3 \leq y \leq y\}$$

Let $\varepsilon = 1$. Hence we have to show that

$$f(x) = |1 - x^2 + 4| \leq f(x^*) = 4 \qquad \text{for all } x \text{ with } -1 < x < 1$$

But this is obvious; from $-1 < x < 1$ it follows that $0 \leq x^2 < 1$ and $3 < f(x) = |1 - x^2 + 4| \leq 4$.

see also the drawing in the graph above!