Zurich University of Applied Sciences
zh aw School of Engineering
IDP Institute of Data Analysis and Process Design

MSE | MASTER OF SCIENCE IN ENGINEERING

Optimization, HS22

Dr. Peter Fusek (fuse@zhaw.ch)
Adrian Willi

Exercise Series 3

Issue date: 3rd/5th October 2022

Exercise 1

If $f:\mathbb{R}^2 \to \mathbb{R}$ is a linear function, i.e. if $f(x) = c_1x_1 + c_2x_2$ for some $c \in \mathbb{R}^2$, the level sets are parallel lines in $\mathbb{R}^2$ and the vector c (the gradient of f) is orthogonal to these lines.

a) Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ and let $\boldsymbol{c} = (3,-2)^\mathsf{T}$. Draw the level sets corresponding to levels 6 and 12. What is the connection between the level sets of f and the vector $\boldsymbol{c}$?

b) Consider the function $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ and let $\boldsymbol{c} = (1,2,1)^\mathsf{T}$. Draw the level sets corresponding to levels 2 and 3. What is the connection between the level sets of f and the vector $\boldsymbol{c}$?

Exercise 2

The following LP is given:

$$\begin{aligned} \max\ x_1 + 3x_2 + 2x_3 \\ x_2 + x_3 \le 2 \\ x_1 - 2x_2 \le -2 \\ x_1, x_2, x_3 \ge 0 \end{aligned}$$

Write this LP in the following possible notations:

- In the vector form in „row notation“: $\max\ \boldsymbol{c}^\mathsf{T}\boldsymbol{x},\ \boldsymbol{a}^i\boldsymbol{x} \le b_i,\ i = 1,\ldots,m,\ \boldsymbol{x} \ge \boldsymbol{0}$
- In the vector form in „column notation“: $\max\ \boldsymbol{c}^\mathsf{T}\boldsymbol{x},\ \sum_{j=1}^{n} \boldsymbol{A}_j x_j \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}$
- In „matrix notation“: $\max\ \boldsymbol{c}^\mathsf{T}\boldsymbol{x},\ \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}$

Exercise 3

Consider the following LP in general form:

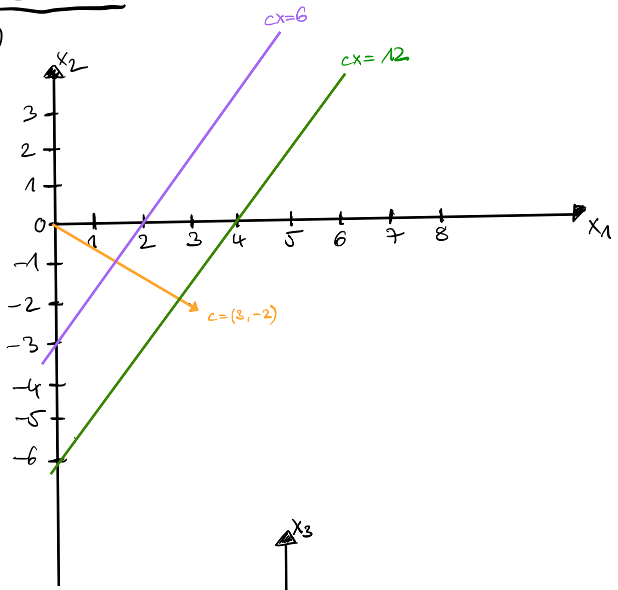
$$\begin{aligned} \min\ x_1 - 2x_2 - 3x_3 \\ x_1 + 2x_2 + 4x_3 \ge 12 \\ x_1 - x_2 + x_3 = 2 \\ x_1 + 2x_2 + x_3 \le 14 \\ x_1 \ge 0 \\ x_2\ \text{free} \\ x_3 \le 0 \end{aligned}$$

Transform the LP into the following forms:

- Maximizing problem in canonical form
- Minimizing problem in canonical form
- Minimizing problem in standard form
- Maximizing problem in inequality form.

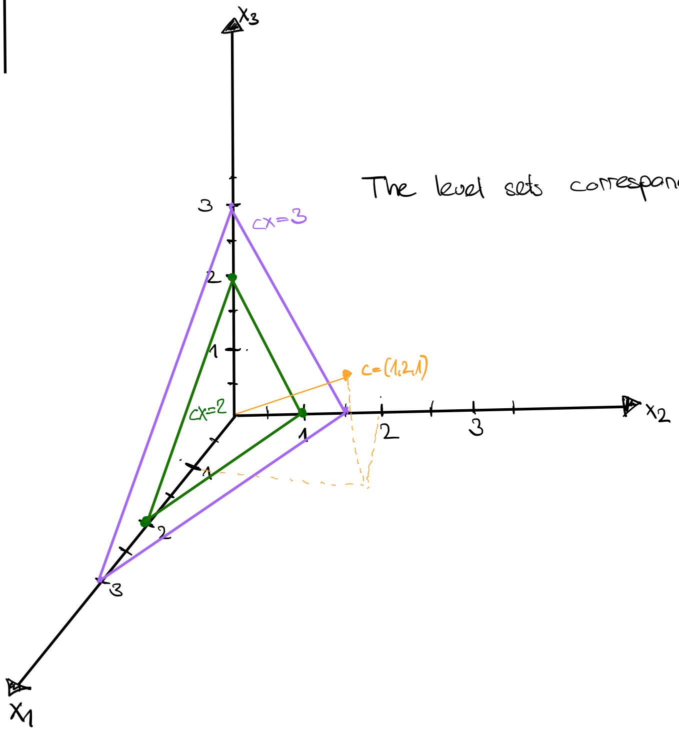
Exercise 1

a)



The direction of c corresponds to the direction of growth of the levels corresponding to the level sets.

b)



The level sets corresponds to parallel planes.

Exercise 2

$\max\ x_1 + 3x_2 + 2x_3$

$x_2 + x_2 \leq 2$

$x_1 - 2x_2 \leq -2$

$x_1, x_2, x_3 \geq 0$

row notation

$\max\ c^T x \qquad \max (1,3,2)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$a^1 x \leq b_1 \qquad (0,1,1)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq 2$

$a^2 x \leq b_2 \qquad (1,-2,0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq -2$

$x \geq 0 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq 0$

column notation

$\max\ c^T x \qquad \max (1,3,2)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\sum_{j=1}^{n} A_j x_j \leq b \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_3 \leq \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$x \geq 0 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq 0$

matrix notation

$\max\ c^T x \qquad \max (1,2,3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$Ax \leq b \qquad \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$x \geq 0 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq 0$

Exercise 3

Replace non-positive variable by non-negative variable $x_j \le 0 \rightsquigarrow x_j := -\bar{x}_j,\ \bar{x}_j \ge 0$

Replace free variable by the difference of two non-negative variables: x_j free $\rightsquigarrow x_j := x_j^+ - x_j^-,\ x_j^+, x_j^- \ge 0$

$$
\begin{aligned}
\min\ & x_1 - 2x_2 - 3x_3 \\
& x_1 + 2x_2 + 4x_3 \ge 12 \\
& x_1 - x_2 + x_3 = 2 \\
& x_1 + 2x_2 + x_3 \le 14 \\
& x_1 \ge 0 \\
& x_2 \text{ free} \\
& x_3 \le 0
\end{aligned}
\qquad
\begin{aligned}
\min\ & x_1 - 2x_2 + 3\bar{x}_3 \\
& x_1 + 2x_2 - 4\bar{x}_3 \ge 12 \\
& x_1 - x_2 - \bar{x}_3 = 2 \\
& x_1 + 2x_2 - \bar{x}_3 \le 14 \\
& x_1 \ge 0 \\
& x_2 \text{ free} \\
& \bar{x}_3 \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\min\ & x_1 - 2(x_2^+ - x_2^-) + 3\bar{x}_3 \\
& x_1 + 2(x_2^+ - x_2^-) - 4\bar{x}_3 \ge 12 \\
& x_1 - (x_2^+ - x_2^-) - \bar{x}_3 = 2 \\
& x_1 + 2(x_2^+ - x_2^-) - \bar{x}_3 \le 14 \\
& x_1 \ge 0 \\
& x_2^+ \ge 0 \\
& x_2^- \ge 0 \\
& \bar{x}_3 \ge 0
\end{aligned}
\quad = \quad
\begin{aligned}
\min\ & x_1 - 2x_2^+ + 2x_2^- + 3\bar{x}_3 \\
& x_1 + 2x_2^+ - 2x_2^- - 4\bar{x}_3 \ge 12 \\
& x_1 - x_2^+ + x_2^- - \bar{x}_3 = 2 \\
& x_1 + 2x_2^+ - 2x_2^- - \bar{x}_3 \le 14 \\
& x_1 \ge 0 \\
& x_2^+ \ge 0 \\
& x_2^- \ge 0 \\
& \bar{x}_3 \ge 0
\end{aligned}
$$

Maximizing problem in canoncial form

$$
\begin{aligned}
\max\ & -x_1 + 2x_2^+ - 2x_2^- - 3\bar{x}_3 \\
& -x_1 - 2x_2^+ + 2x_2^- + 4\bar{x}_3 \le -12 \\
& x_1 - x_2^+ + x_2^- - \bar{x}_3 \le 2 \\
& -x_1 + x_2^+ - x_2^- + \bar{x}_3 \le -2 \\
& x_1 + 2x_2^+ - 2x_2^- - \bar{x}_3 \le 14 \\
& x_1 \ge 0 \\
& x_2^+ \ge 0 \\
& x_2^- \ge 0 \\
& \bar{x}_3 \ge 0
\end{aligned}
$$

Minimizing problem in canoncial form

$$
\begin{aligned}
\min\ & x_1 - 2x_2^+ + 2x_2^- + 3\bar{x}_3 \\
& x_1 + 2x_2^+ - 2x_2^- - 4\bar{x}_3 \ge 12 \\
& x_1 - x_2^+ + x_2^- - \bar{x}_3 \ge 2 \\
& -x_1 + x_2^+ - x_2^- + \bar{x}_3 \ge -2 \\
& -x_1 - 2x_2^+ + 2x_2^- + \bar{x}_3 \ge -14 \\
& x_1 \ge 0 \\
& x_2^+ \ge 0 \\
& x_2^- \ge 0 \\
& \bar{x}_3 \ge 0
\end{aligned}
$$

Minimizing problem in standard form

$$
\begin{aligned}
\min\ & x_1 - 2x_2^+ + 2x_2^- + 3\bar{x}_3 \\
& x_1 + 2x_2^+ - 2x_2^- - 4\bar{x}_3 - x_1^s = 12 \\
& x_1 - x_2^+ + x_2^- - \bar{x}_3 = 2 \\
& x_1 + 2x_2^+ - 2x_2^- - \bar{x}_3 + x_2^s = 14 \\
& x_1 \ge 0 \\
& x_2^+ \ge 0 \\
& x_2^- \ge 0 \\
& \bar{x}_3 \ge 0 \\
& x_1^s \ge 0 \\
& x_2^s \ge 0
\end{aligned}
$$

Maximizing problem in inequality form

$$
\begin{aligned}
\max\ & -x_1 + 2x_2 + 3x_3 \\
& -x_1 - 2x_2 - 4x_3 \le -12 \\
& -x_1 + x_2 - x_3 \le -2 \\
& x_1 - x_2 + x_3 \le 2 \\
& x_1 + 2x_2 + x_3 \le 14 \\
& -x_1 \le 0 \\
& x_3 \le 0
\end{aligned}
$$

Exercise 4

Consider the introductory example "Shift planning in a department store" (see the Script, section 1.2.4)

a) Formulate the mentioned shift planning problem as an integer linear optimization problem in the general case where arbitrary number of time periods and shifts are possible (in the example from Script, the considered time period is one hour).

b) Implement the instance from the Script in Excel and determine the optimal solution using Solver.

Hint:
Denote the set of time periods by $I = \{1,\ldots,m\}$ *and the set of shifts by* $J = \{1,\ldots,n\}$. *Define a 0-1-matrix* $\boldsymbol{A} \in \{0,1\}^{m \times n}$, *where* $a_{ij} = 1$ *if and only if the time period* i *is included in the shift* j. *The constraints can be then written in the form* $\boldsymbol{Ax} \geq \boldsymbol{b}$.

Exercise 5 (* optional *)

Occasionally, one is supposed to consider maximization problems with an objective function in the form

$$f(\boldsymbol{x}) = \min\{\boldsymbol{a}^i \boldsymbol{x} + b_i : i \in I\} \qquad (1)$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{a^i} \in \mathbb{R}^n, b_i \in \mathbb{R}$ for all $i \in I$.

a) Draw the graph of the following function defined on $\mathbb{R}^1$:

$$f(x_1) = \min\left\{\frac{1}{2}x_1 + 1, -\frac{1}{4}x_1 + 4, -x_1 + 10\right\}$$

b) Show that any function of the form (1) is concave on $\mathbb{R}^n$.

Exercise 4

a) Sets:

I Set of time periods, $I = \{1, \dots, m\}$

J Set of shifts, $J = \{1, \dots, n\}$

Parameters:

b_i Demand (number of workers) in time period $i, i \in I$

c_j Costs for shift $j, j \in J$. They are determined by using costs per working hour.

a_{ij} Binary indicator with value 1 if and only if time period i is included in shift $j, i \in I, j \in J$

Variables:

x_j Number of people planned for shift $j, j \in J$

Objective function and constraints:

$$\min \sum_{j \in J} c_j x_j$$

$$\sum_{j \in J} a_{ij} x_j \geq b_i \qquad i \in I$$

$$x_j \in \mathbb{Z}_0 \qquad j \in J$$

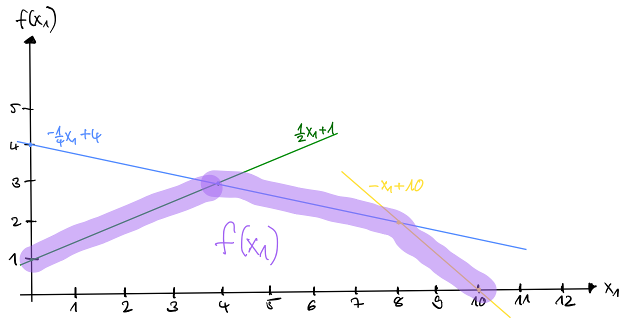
b) see excel sheet

Exercise 5

a) $f(x_1) = \min\left\{\frac{1}{2}x_1 + 1, -\frac{1}{4}x_1 + 4, -x_1 + 10\right\}$

$f(1) = \min\left\{\frac{1}{2} \cdot 1 + 1, -\frac{1}{4} \cdot 1 + 4, -1 + 10\right\}$

$\overset{!}{=} \min\left\{\frac{3}{2}, \frac{15}{4}, 9\right\} = f(1): \frac{1}{2}x_1 + 1$ etc.



b) $f: S \to \mathbb{R}$, where S concave:

$$f(\lambda x^1 + (1-\lambda)x^2) \geq \lambda f(x^1) + (1-\lambda) f(x^2) \qquad \forall x^1, x^2 \in S, \lambda \in [0, 1]$$

Consider the function f given by $f(x) = \min\{a^i x + b_i : i \in I\}$. Take any $x^1, x^2 \in \mathbb{R}^n$ and any $\lambda \in \mathbb{R}$ such that $0 \leq \lambda \leq 1$. We get:

$$\begin{aligned}
f(\lambda x^1 + (1-\lambda)x^2) &= \min\{a^i(\lambda x^1 + (1-\lambda)x^2) + b_i : i \in I\} \\
&\overset{!}{=} \min\{\lambda a^i x^1 + (1-\lambda) a^i x^2 + b_i : i \in I\} \\
&\overset{!}{=} \min\{\lambda(a^i x^1 + b_i) + (1-\lambda)(a^i x^2 + b_i) : i \in I\} \\
&\overset{!}{\geq} \min\{\lambda(a^i x^1 + b_i) : i \in I\} + \min\{(1-\lambda)(a^i x^2 + b_i) : i \in I\} \\
&\overset{!}{=} \lambda \min\{a^i x^1 + b_i : i \in I\} + (1-\lambda) \min\{a^i x^2 + b_i : i \in I\} \\
&\overset{!}{=} \lambda f(x^1) + (1-\lambda) f(x^2)
\end{aligned}$$

In general, functions of the form $f(x) = \min\{a^i x + b_i : i \in I\}$ are called piecewise linear