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Exercise Series 5

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1. Exercise

Consider the following LP:

$$
\begin{aligned}
\min\ 3x_1 + x_2 & \qquad (0) \\
x_1 + 2x_2 \geq 2 & \qquad (1) \\
-x_1 + x_2 \leq 1 & \qquad (2) \\
x_1 \leq 3 & \qquad (3) \\
x_2 \leq 2 & \qquad (4) \\
x_2 \geq 0 & \qquad (5)
\end{aligned}
$$

a) Draw the solution space, representing a polyhedron in $\mathbb{R}^2$.

b) Solve the LP graphically.

c) Solve the LP with the Simplex Algorithm (as maximization problem in inequality form, cf. lecture notes). Start at vertex $(2,0)^T$, and always choose directions with highest increase of the objective function.

2. Exercise

Consider the following LP:

$$
\begin{aligned}
\max\ x_1 + 3x_2 + 2x_3 & \qquad (0) \\
x_1 + x_2 + x_3 \leq 4 & \qquad (1) \\
x_1 \leq 2 & \qquad (2) \\
x_3 \leq 3 & \qquad (3) \\
3x_2 + x_3 \leq 6 & \qquad (4) \\
x_1 \geq 0 & \qquad (5) \\
x_2 \geq 0 & \qquad (6) \\
x_3 \geq 0 & \qquad (7)
\end{aligned}
$$

a) Draw the solution space, representing a polyhedron in $\mathbb{R}^3$.

b) Solve the LP graphically.

c) Solve the LP with the Simplex Algorithm (as maximization problem in inequality form, cf. lecture notes). Start at vertex $(0,0,0)^T$, and always choose directions with highest increase of the objective function.

Exercise 1

$\min \ 3x_1 + x_2 \quad (0)$

$x_1 + 2x_2 \geq 2 \quad (1)$

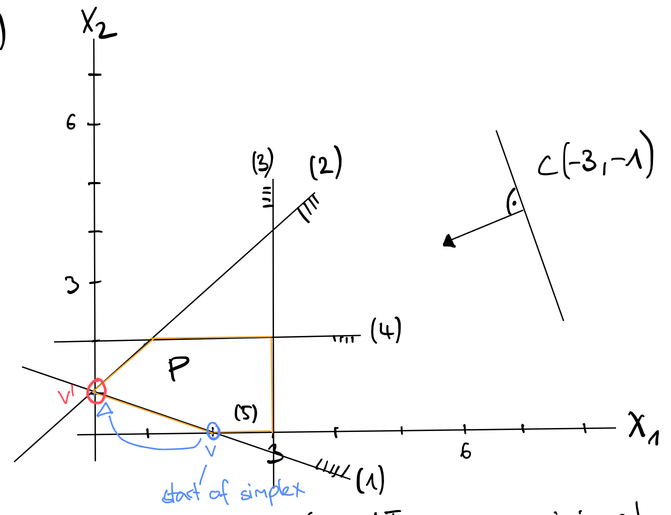
$-x_1 + x_2 \leq 1 \quad (2)$

$x_1 \leq 3 \quad (3)$

$x_2 \leq 2 \quad (4)$

$x_2 \geq 0 \quad (5)$

a)



b) The optimal solution is given by $v = (0,1)^T$ with minimal objective value $-c^Tv = \underline{\underline{1}}$

c) $\max \ -3x_1 - x_2 \quad (0)$

$-x_1 - 2x_2 \leq -2 \quad (1)$

$-x_1 + x_2 \leq 1 \quad (2)$

$x_1 \leq 3 \quad (3)$

$x_2 \leq 2 \quad (4)$

$-x_2 \leq 0 \quad (5)$

$$A = \begin{pmatrix} -1 & -2 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 1 \\ 3 \\ 2 \\ 0 \end{pmatrix}, \quad c = (-3,-1)$$

Iteration 1: The starting point is at vertex $v = (2,0)$. The only basic selection associated to this vertex is $B = \{1,5\}$

$$A_B = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \quad b_B = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \qquad \bar{A} = A_B^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \qquad v = \bar{A} b_B = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{array}{cc} 1 & 5 \end{array}$$

$$u^T = c^T \bar{A} = (-3,-1)\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = (3, -5)$$

$u \geq 0$ is not true, so we continue. We choose $j = 5$

$$d = -\bar{A}_5 = -\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Now we determine λ^*

$$A(v + \lambda d) = Av + \lambda Ad = \begin{pmatrix} -1 & -2 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 & -2 \\ -1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \\ -2 \\ 1 \\ -1 \end{pmatrix} \leq \begin{pmatrix} -2 \\ 1 \\ 3 \\ 2 \\ 0 \end{pmatrix} \begin{array}{l} \\ \lambda \leq 1 \\ \\ \lambda \leq 2 \\ \\ \end{array}$$

$$\min\left\{\frac{1-(-2)}{3}, \frac{2-0}{1}\right\} = \min\{1, 2\} = 1$$

The minimum is obtained at index $k = 2$

$$B' = B - \{j\} \cup \{k\} = \{1,5\} - \{5\} \cup \{2\} = \{1,2\}$$

Iteration 2:

$$A_B = \begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}, \quad b_B = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \bar{A} = A_B^{-1} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad v = \bar{A} b_B = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u^T = c^T \bar{A} = (-3,-1)\begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} = \left(\frac{4}{3}, \frac{5}{3}\right)$$

It is true that $u^T \geq 0^T$. We can stop now.

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f(v) = c^T v = (-3,-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1$$

Exercise 2

$\max \; x_1 + 3x_2 + 2x_3 \quad (0)$

$x_1 + x_2 + x_3 \le 4 \quad (1)$

$x_1 \le 2 \quad (2)$

$x_3 \le 3 \quad (3)$

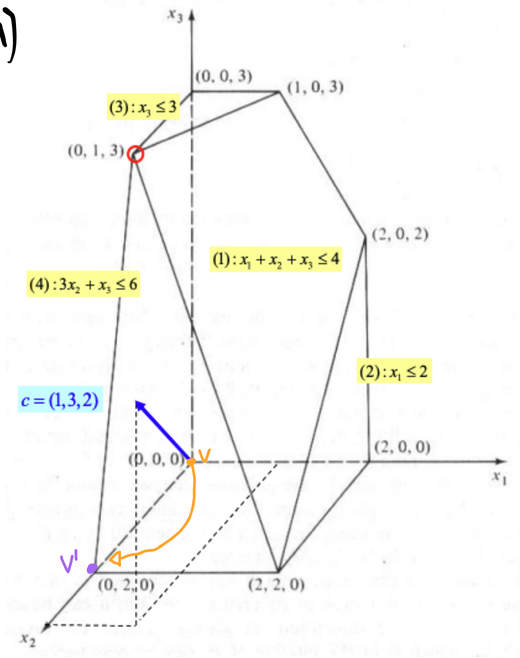
$3x_2 + x_3 \le 6 \quad (4)$

$-x_1 \le 0 \quad (5)$

$-x_2 \le 0 \quad (6)$

$-x_3 \le 0 \quad (7)$

a)



b) The optimal solution is given by $v = (0,1,3)$ with the maximal objective value $c^T v = \begin{pmatrix}1\\3\\2\end{pmatrix}(0,1,3) = \underline{\underline{9}}$

c)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c = (1,3,2)$$

We start at vertex $(0,0,0)$. The only basic selection associated to this vertex is $B = \{5,6,7\}$

Iteration 1:

$$A_B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad b_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \bar{A} = A_B^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad v = \bar{A} b_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$u^T = c^T \bar{A} = c^T A_B^{-1} = (1,3,2)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (-1,-3,-2)$$

$u^T \ge 0^T$ is not true. We choose the component with the highest increase of the objective function. This is the second component. It corresponds to the second index in the basic selection $B = \{5,6,7\}$. Hence, $j = 6$.

$d = -\bar{A}_6 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Determine λ^*

$$A(v + \lambda d) = Av + \lambda Ad = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0 \end{pmatrix} + \lambda \begin{pmatrix} 1\\0\\0\\3\\0\\-1\\0 \end{pmatrix} \le \begin{pmatrix} 4\\2\\3\\6\\0\\0\\0 \end{pmatrix} = b$$

$$\lambda^* = \min\left\{\frac{4-0}{1}, \frac{6-0}{3}\right\} = 2$$

The minimum is obtained at index $k = 4$

$B' = B - \{j\} \cup \{k\} = \{5,6,7\} - \{6\} \cup \{4\} = \{4,5,7\}$

$\rightarrow$ start next iteration etc.

⋮

3. Exercise

Consider the introductory example “Frequency Assignment in Mobile Networks" from the lecture notes. Denote the set of locations by $I = \{1, \ldots, n\}$.

a) Using the given distances l_{ij} , calculate the minimal (absolute) frequency differences d_{ij} between locations $i, j \in I$.

b) Draw an undirected graph $G = (V, E)$ whose vertices V correspond to the locations I, and whose set of edges is given by the pairs of locations $i, j \in I$ with $d_{ij} > 0$, i.e. $E = \{(i, j) \in V^2 : d_{ij} > 0\}$. Set the edge weights equal to d_{ij} for all edges $(i, j) \in E$.

c) Formulate a first obvious model for this optimization problem (it can be non-linear).

   Hint: When formulating the model, use an objective function of the form $\min \max \{x_i : i \in I\}$. *Formulate the constraints with help of the absolute value function* $|\ldots|$.

d) *(* challenging *)* Based on the above model, derive a linear model such that the problem can be formulated as an integer linear program.

4. Exercise

A metal processing company produces two metal alloys consisting of a base metal and three different precious metals. Because of procurement shortages, the used precious metals are available in limited quantities only during the considered production period. The following table shows the available quantities (in kg) of the precious metals as well as the amounts (in kg per tonne) of the precious metals in both alloys.

	Precious metal 1	Precious metal 2	Precious metal 3
Available quantity	12	30	15
Amount in alloy 1	1	6	3
Amount in alloy 2	3	2	2

Since demand for both alloys is very high, we may assume that the produced quantities can be sold completely. The question is what quantities (in tonnes) of alloys to produce in order to maximize the total contribution margin. The individual contribution margins (in 10’000 Fr. per tonne) are 6 for alloy 1, and 5 for alloy 2.

a) Formulate the problem as a LP.

b) Draw the polyhedron representing the solution space of the LP.

c) The objective function has the form $f(\boldsymbol{x}) = \sum_{j \in J} c_j x_j = \boldsymbol{c}^\top \boldsymbol{x}$, where $\boldsymbol{c} \in \mathbb{R}^n$ is the objective function vector.

   Draw the concrete objective function vector for the given instance, and the level sets corresponding to the levels 0 and 15.

d) Determine graphically an optimal solution and the corresponding optimal objective function value.

5. Exercise

Read the paper by Richard Cottle, Ellis Johnson und Roger Wets (in *Notices of the AMS, 2007*) about George B. Dantzig and the Simplex Algorithm.