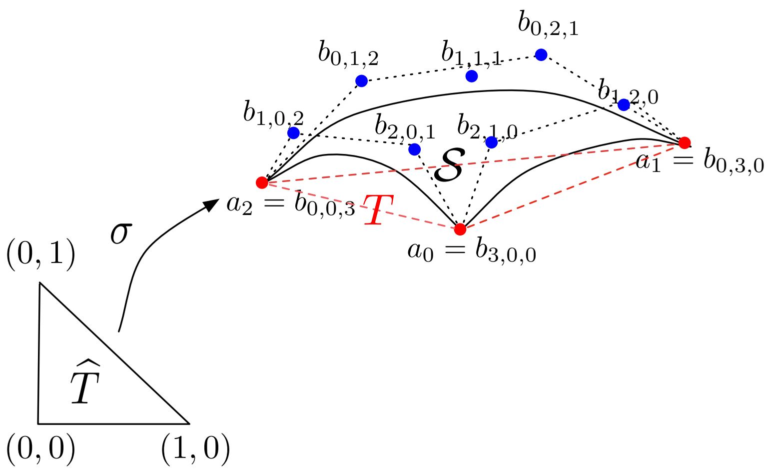
Chapter IV

SURFACES

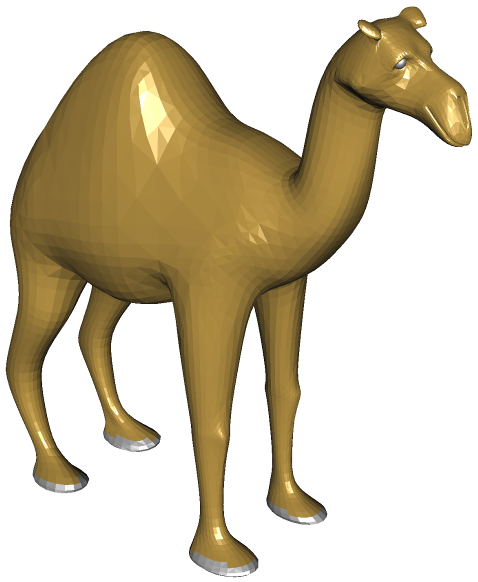
REPRESENTATION, APPROXIMATION

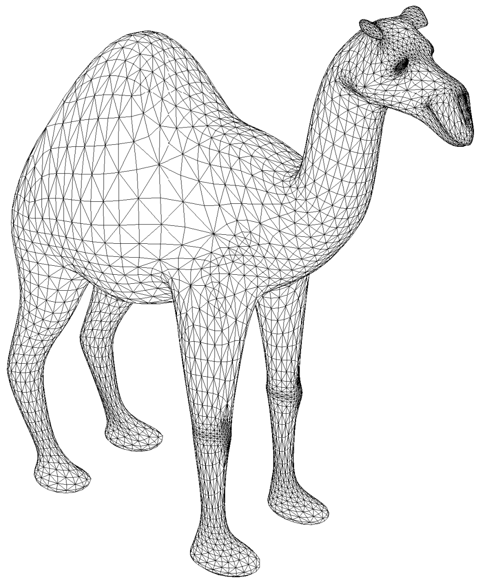


Section 4.4

Discrete Surfaces

Discrete and continuous





continuous representation
(differential) geometry
topology

discrete representation
computational / digital geometry
computational topology

Discretization Approximation Interpolation

Discrete and continuous

Let us start by asking ourselves a series of questions . . .

1. continuous setting vs. discrete setting: 2 sides of the same ? *

2. how deep is the dividing line between continuous and discrete mathematics ?
   - continuous mathematics is classical, well established, with a rich variety of applications.
   - discrete mathematics grew out of puzzles and then is often identified with specific application areas like computer science or operations research.
   - basic structures and methods of both sides of our science are quite different:
     - continuous mathematicians use limits;
     - discrete mathematicians use induction.

*L. Lovász, *Discrete and Continuous: Two sides of the same ?*, Geom. Funct. Anal., 359-382, (2000).

Discrete and continuous

3. what are the levels of interaction between continuous and discrete mathematics ?

   (a) use finite to approximate infinite, i.e., discretize a continuous structure (Riemann integral, triangulating a manifold in homotopy theory, ...),
   - continuous structures are often cleaner, more symmetric, and richer than their discrete counterparts (e.g. regular grids vs. Euclidean space),
   - a natural and powerful method to study discrete structures is to *embed* them in the continuous world;

   (b) the key step in the proof of a purely discrete result is the application of a purely continuous theorem, or vice versa;

   (c) key progress in discrete mathematics achieved through use of complex methods in analysis;

   (d) numerical analysis or discrepancy theory illustrate the connections between discrete and continuous;

   (e) the same phenomenon appears in both the continuous and discrete setting.

Polygonal meshes

- Digital geometry is concerned with mathematical models and algorithms for analyzing and manipulating geometric data.
  Typical operations include notably:
  - surface reconstruction from point samples,
  - filtering operations for noise removal,
  - geometry analysis,
  - shape simplification, and
  - geometric modeling and interactive design.

- Polygonal meshes, and especially triangulations, are very important shape representations and their importance is unarguably increasing.
  There are several reasons for this:
  - the ability of GPUs to render several millions of triangles ($>$ 100M) in real-time,
  - scanning and sensing devices provide ever larger triangle meshes,
  - efficient algorithms for manipulating triangulations have been developed (recently).

Digital geometry

- there is no consensus on the most appropriate way to estimate simple geometric attributes such as normal vectors and curvatures on discrete surfaces;
- many surface-oriented applications require an approximation of the first and second order properties, with as much accuracy as possible;
  For example, accurate curvature and normals estimates are essential to the problem of surface denoising;
- a triangulation is often the only *"reliable"* approximation of the continuous surface at hand;
- unfortunately, since meshes are piecewise linear surfaces, the notion of continuous normal vectors or curvatures is non trivial;
- discrete operators satisfying appropriate discrete versions of continuous properties would guarantee reliable numerical behavior for many applications using meshes.

Geometric properties

- we are interested in geometric quantities which are continuous, in a certain sense.
- if $\mathbb{R}^n$ is endowed with its standard scalar product, any metric quantity associated to Σ is geometric with respect to the group of rigid motions (e.g. the diameter, the area of the convex hull of a set of points on Σ);
- if we want to evaluate a quantity $Q(\Sigma)$ defined on Σ, but if we can only approximate Σ by Σ', we would like to evaluate the quantity $Q(\Sigma')$, and hope that the result is close to $Q(\Sigma)$.
  - we would like to have:

$$\text{if} \quad \lim_{n\to\infty} \Sigma_n = \Sigma, \quad \text{then} \quad \lim_{n\to\infty} Q(\Sigma_n) = Q(\Sigma).$$

- this claim is incomplete since we did not specify the topology on the set of subsets Σ of $\mathbb{R}^3$. The simplest one is the Hausdorff topology.

Differential geometry

Let Σ be a surface (2-manifold) embedded in $\mathbb{R}^3$, described by an arbitrary parametrization $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

- At each point $x \in \Sigma$, the surface can be locally approximated by its tangent plane $T_x\Sigma$ *orthogonal* to the normal vector $n(x)$;
- local bending of the surface is measured by curvatures;
- for every unit direction e_θ in $T_x\Sigma$, the normal curvature $\kappa_N(\theta)$ is defined as the curvature of the curve that belongs to both the surface itself and the plane containing both $n(x)$ and e_θ. The two principal curvatures κ_1 and κ_2 of the surface S, with their associated orthogonal directions e_1 and e_2 are the extremum values of all the normal curvatures.
- The mean curvature κ_H is defined as the average of the normal curvatures:

$$\kappa_H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_N(\theta)\, \mathrm{d}\theta\,.$$

Differential geometry reminder

- the normal curvature can be expressed in terms of the principal curvatures (Euler):

$$\kappa_N(\theta) = \kappa_1 \cos^2(\theta) + \kappa_2 \sin^2(\theta),$$

  which leads to the well-known definition: $\kappa_H = (\kappa_1 + \kappa_2)/2$.

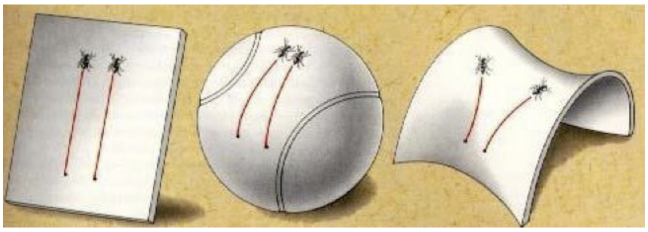
- the Gaussian curvature κ_G is defined as the product of the two principle curvatures: $\kappa_G = \kappa_1\kappa_2$;
- the mean and Gaussian curvatures represent important local properties of a surface. Lagrange noticed that $\kappa_H = 0$ is the Euler-Lagrange equation for surface area minimization. This provides a direct relation between surface area minimization and mean curvature flow:

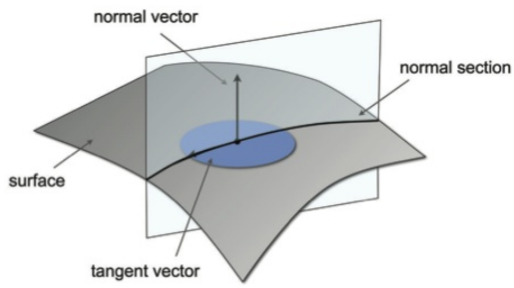
$$2\kappa_H\, n(x) = \lim_{diam(A)\rightarrow 0} \frac{\nabla A(x)}{A(x)}$$

  where $A(x)$ is a infinitesimal area of around a point x on Σ of diameter $diam(A)$.

Differential geometry reminder

- intrinsic geometry of a surface can be intuitively perceived by 2D creatures that live on the surface without knowledge of the third dimension.
  - Gauss' famous Theorema Egregium states that the Gaussian curvature is invariant under local isometries (intrinsic).
  - Gaussian curvature can be determined directly from the first fundamental form.
  - mean curvature is not invariant under isometries, but depends on the embedding.





[Note that the term intrinsic is often also used to denote independence of a particular parameterization.]

Differential geometry reminder

- Laplace-Beltrami operator Δ_Σ extends the Laplace operator to functions defined on Σ.
  - for a given function f defined on a manifold surface Σ, the Laplace-Beltrami is defined as:

    $$\Delta_\Sigma f = \mathrm{div}_\Sigma \nabla_\Sigma f,$$

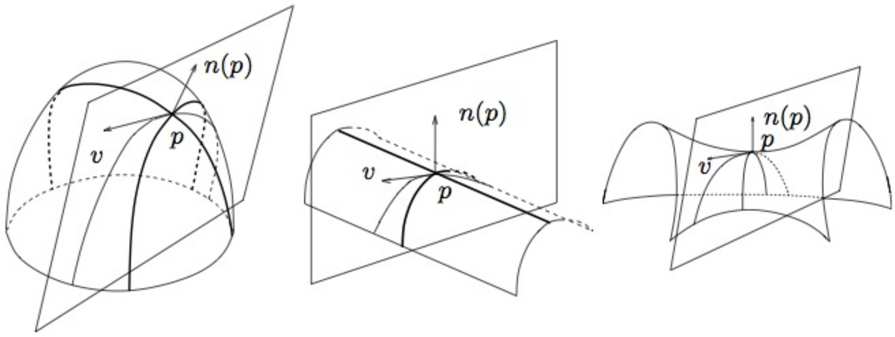
    which requires a suitable definition of divergence and gradient operators on manifolds (see Berger for more details).
  - applied to the coordinate function x of the surface, the Laplace-Beltrami operator evaluates to the *mean curvature normal*:

    $$\Delta_\Sigma x = -2\kappa_H\, n(x).$$

  - Δ_Σ is an intrinsic property which depends on the metric of the manifold, i.e., the first fundamental form.

Differential geometry reminder

- Given a vector $v \in T_x\Sigma$ to Σ at x, the derivative of $n(x)$ in the direction v is orthogonal to $n(x)$. The derivative $D_x n$ of n at x thus defines the Weingarten endomorphism;
- the associated quadratic form is called the second fundamental form;
- applied to a unit vector $v \in T_x\Sigma$, it yields the signed curvature of the section of the surface by the plane spanned by $n(x)$, v, and passing through x.
- the principal directions correspond to the values of v where the second fundamental form is maximal or minimal (3 cases).



Second fundamental form of a surface. The sign of the Gaussian curvature (determinant of $D_x n$) yields 3 cases: elliptic (>0), parabolic ($=0$) and hyperbolic (<0).

Triangulations

- triangulated surfaces are the 3-dimensional equivalent of polygons in two dimensions.
- a triangulated surface is a set of triangles, $\mathcal{S} = \{T_1, ..., T_N\}$, each T_k is a 3-tuple of vertices, $T_k = (v_{k1}, v_{k2}, v_{k3})$.
- we also consider the set of all *distinct* vertices in the triangulation, $\mathcal{V} = \{v_1, ..., v_N\}$, and the set of all distinct edges (pairs of vertices belonging to the same face), $\mathcal{E} = \{e_1, ..., e_Q\}$.
- the order of the vertices in each face is important and defines its orientation, which is invariant up to a cyclic permutation of the vertices.
- we only consider regular triangulations (conforming), which are such that the intersection between two faces is either empty or an edge.
- the number

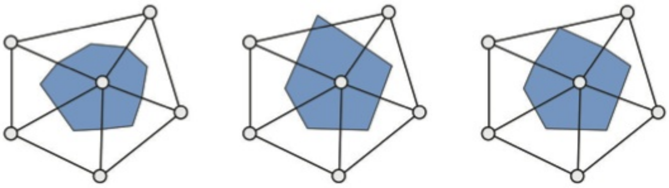
$$\chi = |\mathcal{V}| - |\mathcal{E}| + |\mathcal{S}|$$

is a topological invariant of the surface called the Euler characteristic.

Discrete properties

We consider a surface Σ embedded in $\mathbb{R}^3$, which is only known through a *triangulation* or a mesh $\mathcal{S} = (T_i)_{i=1,\dots,N_S}$.

- the smooth definitions of normal, curvatures need to be reformulated for C^0 surfaces like triangulation $\mathcal{S}$;
- the goal is then to compute approximations of the differential properties of this underlying surface directly from the mesh data;
- the general idea is to compute discrete differential properties as spatial averages over a local neighborhood $B(x)$ of a point x on $\mathcal{S}$.
  - often, x coincide with a mesh vertex,
  - the size of the local neighborhood critically affects the stability and accuracy of the discrete operators.



example of neighborhoods (barycentric, Voronoï, mixed)

Triangulations (2)

We consider a surface Σ embedded in $\mathbb{R}^3$, which is only known through a *triangulation* or a mesh $\mathcal{S} = (T_i)_{i=1,\dots,N_S}$.

We make the following assumptions on $\mathcal{S}$ and Σ:

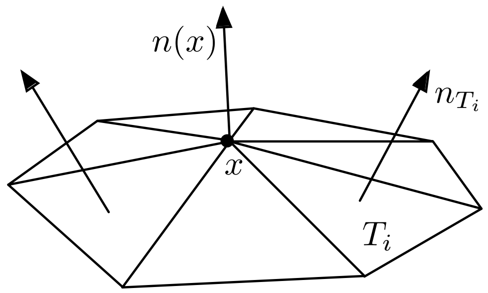
1. the only information available about $\mathcal{S}$ is the set of triangles composing the triangulation and the set $\mathcal{V}$ of vertices;
2. the triangulation $\mathcal{S}$ is conforming, that is, the intersection between any two triangles T_i, T_j, $i \neq j$ is either the empty set, a common vertex or a common edge;
3. the underlying surface Σ is a compact orientable manifold, without boundary;
4. the triangulation $\mathcal{S}$ is endowed with an orientation, *i.e.* the *direct* normal vectors to all the triangles of $\mathcal{S}$ consistently point towards one side of $\mathcal{S}$.

Normal vectors

- visualization usually requires normal vectors per face or per vertex, (e.g. in flat or Phong shading);
- in the neighborhood of a regular vertex x, the approximation of $n(x)$, the unit normal vector to Σ at x is computed by using a weighted sum of the normal vectors to the triangles $\mathcal{B}_{\mathcal{S}}(x)$ of the form:

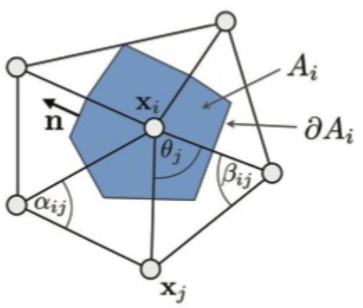
$$n(x) \approx \frac{\sum_{T \in \mathcal{B}_{\mathcal{S}}} \alpha_T \, n_T}{\left| \sum_{T \in \mathcal{B}_{\mathcal{S}}} \alpha_T \, n_T \right|},$$

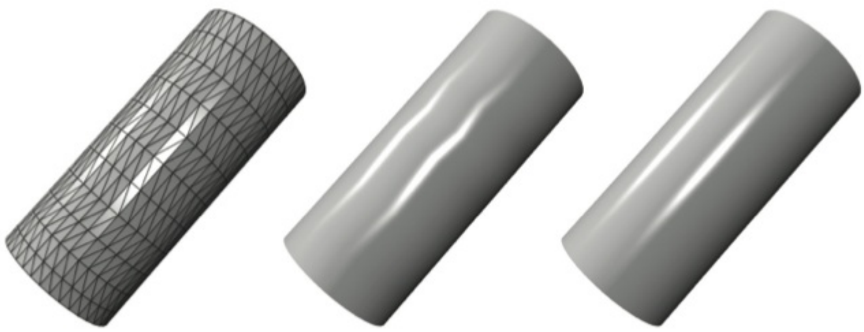
  where α_T are suitable coefficients in $[0, 1]$ such that $\sum_{T \in \mathcal{B}_{\mathcal{S}}} \alpha_T = 1$.



Normal vectors (2)

- there are numerous alternatives for setting the weights α_T:
  - constant weights $\alpha_T = 1$ are efficient to compute but do not consider edge lengths, triangle areas, or angles,
    can give counterintuitive results for irregular meshes;
  - a weighting based on triangle area, i.e., $\alpha_T = |T|$, is particularly efficient to compute but counterintuitive results can occur as well;
  - averaging over sufficiently small geodesic disks: equivalent to weighting by incident triangle angles $\alpha_T = \theta_T$;





Gradients

- we assume that a piecewise linear function f is given at each mesh vertex and is interpolated linearly within each triangle $T = (a_0, a_1, a_2)$:

$$f(x) = \sum_{i=0}^{2} \lambda_i(x)\, f(a_i),$$

where the $\lambda_i(x)$ are the barycentric coordinates of x in T.
λ_i is a polynomial of degree 1 taking the value 1 at a_i and 0 at a_j, $j \neq i$.

- the gradient of f is given by:

$$\nabla f(x) = \sum_{i=0}^{2} \nabla\lambda_i(x)\, f(a_i).$$

Discrete Laplace-Beltrami operator

- uniform discretization of Δ_Σ (Taubin):

$$\Delta_\Sigma f(v_i) = \frac{1}{|\mathcal{B}(v_i)|} \sum_{v_j \in \mathcal{B}(v_i)} f(v_j) - f(v_i),$$

simple and efficient to compute but inacurate for non-uniform triangulations.

- cotangent formula (Meyer): derived from a mixed finite element / finite volume method.
  - use the divergence theorem for a vector valued function:

$$\int_{A(v_i)} \Delta f(x)dx = \int_{A(v_i)} \text{div } \nabla f(x)dx = \int_{\partial A(v_i)} \nabla f(x)\cdot n(x)ds,$$

  - by integrating over $A(v_i)$, we obtain:

$$\Delta_\Sigma f(v_i) := \frac{1}{2A(v_i)} \sum_{v_j in \mathcal{B}(v_i)} (\cot\alpha_{i,j} + \cot\beta_{i,j})(f(v_j) - f(v_i)).$$

  each triangle defines the local surface metric (surface triangulation = conformal space).

Discrete curvatures

- when applied to coordinate function x, Δ_Σ provides a discrete approximation of the mean curvature normal κ_H:

$$\Delta_\Sigma(x_i) = \frac{1}{2A(v_i)} \sum_{v_j \in \mathcal{B}(v_i)} (\cot \alpha_{i,j} + \cot \beta_{i,j})(x_i - x_j),$$

- and the discrete mean curvature at a vertex v_i is half of the magnitude of this expression:

$$\kappa_H(v_i) = \frac{1}{2}\|\Delta_\Sigma x_i\|,$$

- Gaussian curvature discrete operator can be expressed as (Meyer):

$$\kappa_G(v_i) = \frac{1}{A(v_i)} \left(2\pi - \sum_{v_j \in \mathcal{B}(v_i)} \theta_j \right).$$

- and the principal curvatures can be deduced easily:

$$\kappa_{1,2} = \kappa_H(v_i) \pm \sqrt{\kappa_H(v_i)^2 - \kappa_G(v_i)}.$$

Surface parametrization

- we have discussed previously parametric surfaces, which are mappings from a 2D domain $\Omega \in \mathbb{R}^2$ to a surface $\Sigma \in \mathbb{R}^3$.

- most 3D models in computer graphics are composed of triangles with texture,

- unfortunately, since the images are flat and the 3D model is generally curved, the image needs to be deformed in order to precisely fit the 3D model or conversely the 3D model needs to be flattened to map it onto the image (parametrization).

Surface parametrization

- the matrix of the first fundamental form I with respect to a basis (x_u, x_v) in $T_x\Sigma$ is:

$$I = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

  I is positive definite.

- let $\sigma : U \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$ be a parametrization of a surface. Then σ is:
  - an isometry iif $I = I_2$, preserves lengths
  - conformal iif $I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, where λ is a function on U, preserves angles
  - equiareal iif $\det I = 1$; preserves areas

- Game: try to find such parametrizations ...
  isometric maps (only developable surfaces, characterized by having zero Gaussian curvature). Mercator projection is an example of conformal map, and Lambert azimuthal projection is a equiareal map.

Surface parametrization

Mercator projection (1569) is an example of a conformal parametrization.
Lambert azimuthal projection (1772) is an example of an area preserving parametrization.

Surface parametrization

- convex combination mappings: triangulation has a disk topology,
  - boundary vertices are placed s.t. they form a convex 2d shape,
  - vertices are computed in such a way that

$$x_i = \sum_{j \in \mathcal{B}(v_i)} \lambda_{i,j} x_j \quad \text{with} \quad x_i = (u_i, v_i),$$

    and weights are such that $\sum_{j \in \mathcal{B}(v_i)} \lambda_{i,j} = 1$, and $\lambda_{i,j} > 0$.
  - mapping is one to one (i.e. triangulations do not fold).

- harmonic mapping: solve the previous equation with

$$\lambda_{i,j} = \frac{w_{i,j}}{\sum_{k \in \mathcal{B}(v_i)} w_{i,k}}, \quad w_{i,j} = \frac{1}{2}(\cot \alpha_{i,j} + \cot \beta_{i,j}).$$

- The Dirichlet energy

$$E_D(u) = \frac{1}{2} \int_\Sigma \|\nabla u\| dx$$

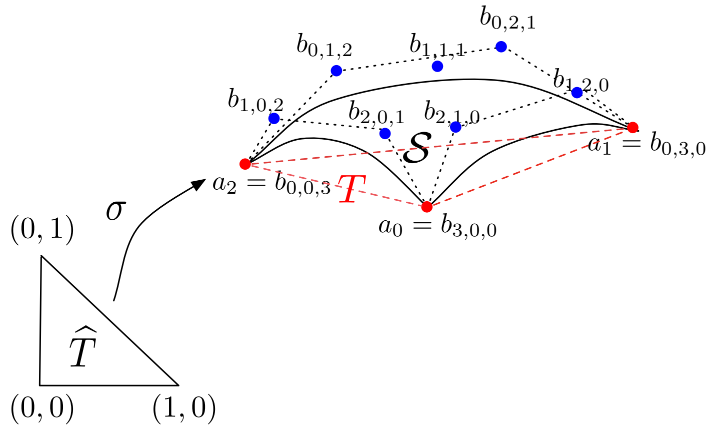
  is minimized by solving $-\Delta_\Sigma(u) = 0$. Such a solution is a harmonic function.

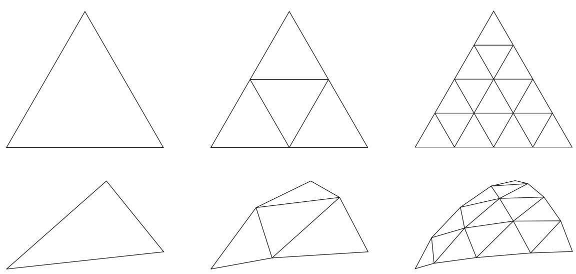
Local parametrization

- motivations:
  - improve the visual quality of existing triangle-based model,
  - improve the geometric approximation.

- idea: replace a surface triangle by a Bézier cubic portion of surface,

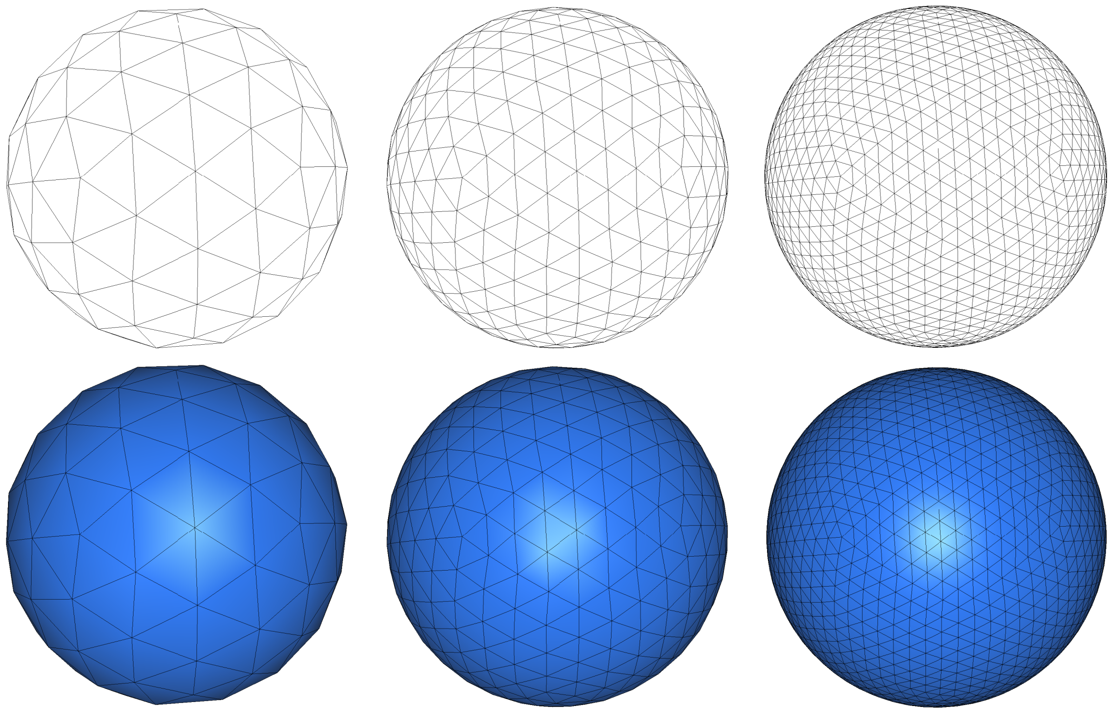
- advantage: compatibility with OpenGL API data structures





hands-on session

Local parametrization



Levels of refinements of a sphere model.

Local parametrization

- we suppose that each triangle $T = a_0 a_1 a_2 \in \mathcal{S}$ accounts for a smooth portion of Σ, which is modeled as a cubic piece of surface $\sigma(\hat{T})$, where

$$\hat{T} := \{(u,v) \in \mathbb{R}^2,\ u \geq 0,\ v \geq 0,\ w := 1 - u - v \geq 0\}$$

is a reference triangle in the plane, and each component of $\sigma : \hat{T} \to \mathbb{R}^3$ is a polynomial of total degree 3 in the two variables $u, v \in \hat{T}$.

- σ can be written under the form of a Bézier cubic polynomial:

$$\forall (u,v) \in \hat{T}, \quad \sigma(u,v) = \sum_{\substack{i,j,k \in \{0,1,2,3\} \\ i+j+k=3}} \frac{3!}{i!\,j!\,k!} w^i\, u^j\, v^k\, b_{i,j,k}\,, \tag{4}$$

where the choice of the control points $b_{i,j,k} \in \mathbb{R}^3$ is dictated by the geometry of the surface Σ.

- this allows for a close and fast evaluation of the Hausdorff distance between the considered triangle $T \in \mathcal{S}$ and the corresponding piece of ideal surface $\sigma(\hat{T}) \in \Sigma$.

Hausdorff distance

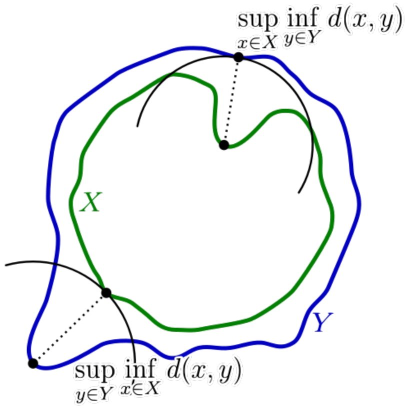
- the distance between a point $x \in \mathbb{R}^n$ and a non-empty bounded subset X of a metric space

$$d(x, X) = \inf_{y \in X} d(x, y),$$

- the Hausdorff distance $d_H(X, Y)$ between two non-empty subsets of a metric space is:

$$d_H(X, Y) = \max\{\sup_{y \in Y} \inf_{x \in X} d(x, y)\,,\ \sup_{x \in X} \inf_{y \in Y} d(x, y)\}$$

The intuition behind Hausdorff distance is to measure "how similar" two sets are in the metric sense.



- hence, $\sigma(\widehat{T})$ is comprised in the convex hull of the control points $b_{i,j,k}$, because for all $(u, v) \in \widehat{T}, \sigma(u, v)$ is a convex combination of the $b_{i,j,k}$. As a consequence, one easily sees that:

$$d_H(T, \sigma(\widehat{T})) \leq \max_{\substack{l=0,1,2 \\ i+j+k=3}} d(a_l, b_{i,j,k}).$$

Hausdorff distance