Christian-Albrechts-Universität zu Kiel **SS 2022**
Mathematisches Seminar
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Computational Finance

Exercises for all participants

C-Exercise 17 (Greeks of a European option in the Black-Scholes model) (4 points)
In the 'Material' folder of the OLAT you find a python function

```
V0 = BS_Price_Int (r, sigma, S0, T, g)
```

which computes the price of a European option with payoff $g(S(T))$ at maturity $T > 0$ in a Black-Scholes model with initial stock price $S(0) > 0$, interest rate $r > 0$ and volatility $\sigma > 0$. (This is formula (3.21) from the lecture notes)
The first order greeks for a European option in the Black-Scholes model are given by the first order derivatives

$$\Delta(r,\sigma,S(0),T,g) = \frac{\partial}{\partial S(0)} V_{BS}(r,\sigma,S(0),T,g),$$
$$\nu(r,\sigma,S(0),T,g) = \frac{\partial}{\partial \sigma} V_{BS}(r,\sigma,S(0),T,g),$$
$$\gamma(r,\sigma,S(0),T,g) = \frac{\partial^2}{\partial S(0)\partial S(0)} V_{BS}(r,\sigma,S(0),T,g),$$

where $V_{BS}(r,\sigma,S(0),T,g)$ denotes the Black-Scholes price of the European option.

a) Write a Python function

```
[Delta, vega, gamma]=BS_Greeks_num(r, sigma, S0, T, g ,eps)
```

that computes the greeks described above numerically using the approximations

$$\frac{\partial}{\partial x} f(x,y) \approx \frac{f(x+\varepsilon x,y) - f(x,y)}{\varepsilon x},$$
$$\frac{\partial^2}{\partial x \partial x} f(x,y) \approx \frac{f(x+\varepsilon x,y) - 2f(x,y) + f(x-\varepsilon x,y))}{(\varepsilon x)^2}.$$

For this you can use the function `BS_Price_Int`.

b) Plot $\Delta(r,\sigma,S(0),T,g)$ for the European call with payoff function $g(x) = (x-110)^+$ and parameters $r = 0.05$, $\sigma = 0.3$, $T = 1$ for $S(0) \in [60,140]$. Use $\varepsilon = 0.001$.

C-Exercise 18 (4 points)
Write a Python function

```
V0 = BS_EuCall_Laplace (S0, r, sigma, T, K, R)
```

that computes the initial price of a European call option in the Black-Scholes model via the Laplace transform approach. I.e., implement the formula

$$V(t) = \frac{e^{-r(T-t)}}{\pi} \int_0^\infty \mathrm{Re}\left(\tilde{f}(R+iu)\chi_t(u-iR)\right) \mathrm{d}u$$

from the course. Choose an appropriate R and test your function for

$$S(0) = [50:150], \quad r = 0.03, \quad \sigma = 0.2, \quad T = 1, \quad K = 110$$

Plot your results in a common graph.

Usefull Python commands: `cmath.exp`, `complex`, `scipy.integrate.quad`, `real`

T-Exercise 19QF (Self-quanto call) (for QF only) (4 points)
The self-quanto call with strike K is an option paying $(S(T)-K)^+$ shares of stock at time T, which means that its payoff equals $(S(T)-K)^+S(T)$. Determine the integral transform representation of the fair option price, i.e. the function $\widetilde{f}$ in equation (4.5) of the lecture notes.

T-Exercise 19Math (Laplace transform approach for digital call) (for math only) (4 points)

a) Compute the Laplace transform for the payoff function $g(x) = 1_{\{x \geq K\}}$, $K \in \mathbb{R}$ of a digital call option and determine its domain of convergence.

b) Use the results from part a) to determine the fair value $V_g(0) = e^{-rT}\mathbb{E}_Q[g(S_T)]$ of the digital call in the Black-Scholes model.

T-Exercise 20QF (for QF only) (4 points)
A *Poisson process* N with intensity parameter $\lambda \in \mathbb{R}_+$ is a stochastic process with right-continuous, increasing paths such that for all $s,t \in \mathbb{R}_+$ the increments $N(t+s) - N(t)$ are independent of $N(t)$ and such that $N(t)$ follows a Poisson distribution with parameter λt. For $\rho, \mu \in \mathbb{R}$ and a Poisson process N with intensity parameter $\lambda \in \mathbb{R}_+$, compute the characteristic function of $X(t) := \rho N(t) - \mu t$.

T-Exercise 20Math (Merton's jump diffusion model) (4 points)
In the Merton model the logarithmic stock price is of the form

$$X(t) = X(0) + \mu t + \sigma W(t) + \sum_{j=1}^{N(t)} Y_j$$

where W is a standard Brownian motion, $\mu \in \mathbb{R}$, $\sigma > 0$, $N(t)$ is a Poisson random variable with parameter λt for $\lambda > 0$ and $Y_1, Y_2, \ldots$ are normally distributed with mean $m \in \mathbb{R}$ and variance $s^2 > 0$. Moreover $W, N(t), Y_1, Y_2, \ldots$ are all independent. Calculate the characteristic function

$$\chi_t(u) := E[e^{iuX(t)}].$$

Hints:

(a) *The characteristic function of a normal random variable Y with mean μ and variance σ^2 is given by* $E[\exp(iuY)] = \exp(iu\mu - u^2\sigma^2/2)$.

(b) *We have* $\exp(\sum_{j=1}^{N_t} Y_j) = \sum_{m=0}^{\infty} 1_{\{N(t)=m\}} \exp(\sum_{j=1}^{m} Y_j)$.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Thu, 19.05.2022, 08:15