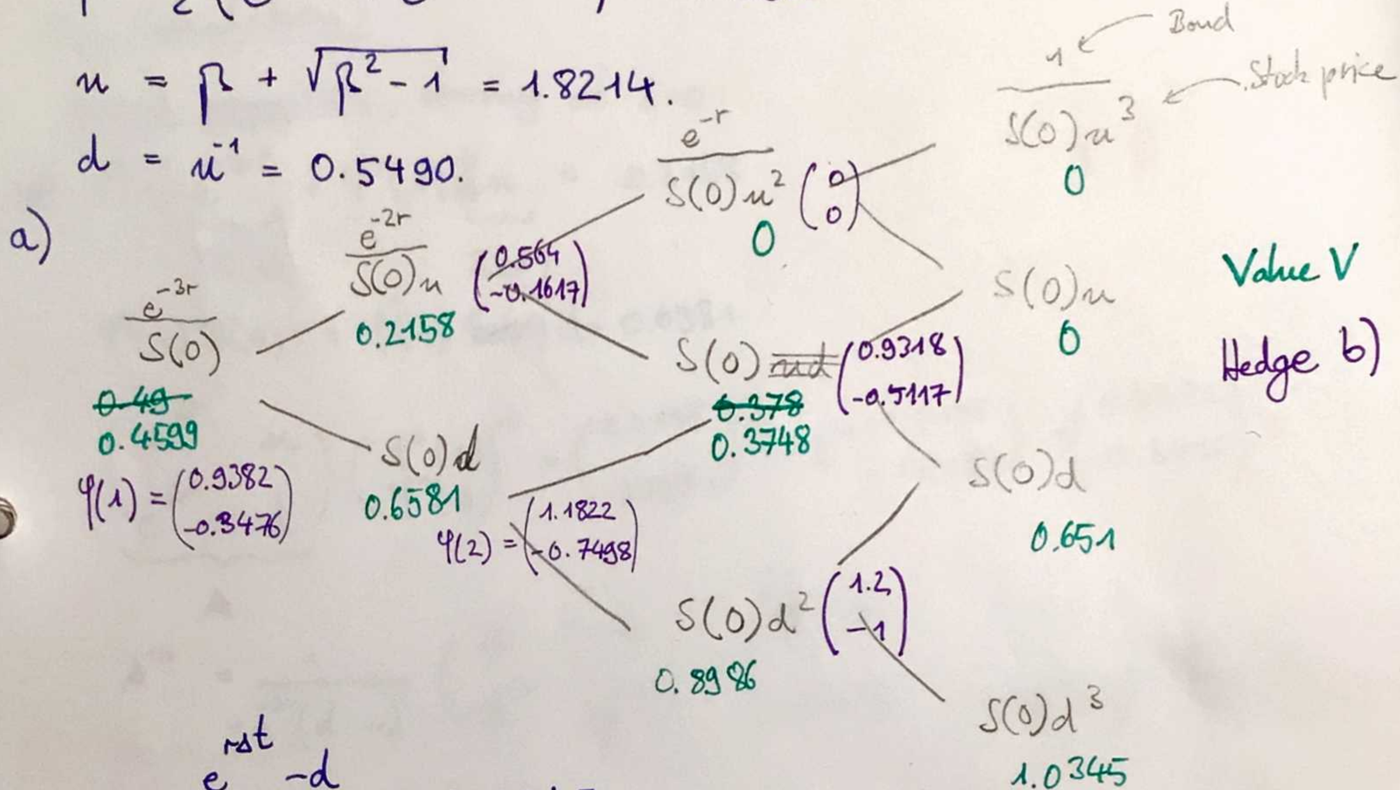
$K = 1.2 \quad M = 3 \quad S(0) = 1 \quad \sigma^2 = 0.3 \quad r = 0.05 \quad T = 3 \quad \Delta t = 1$

$$\beta = \frac{1}{2}\left(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t}\right) = 1.1852.$$

$$u = \beta + \sqrt{\beta^2 - 1} = 1.8214.$$

$$d = u^{-1} = 0.5490.$$

a)




$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.3947.$$

<u>Backward propagation:</u>

$$V(S(0)d^3, t=3) = \max\{0,\ 1.2 - 1\cdot d^3\} = 1.0345$$

$$V(S(0)d, T) = \max\{0,\ 1.2 - 1\cdot d\} = 0.651$$

$$V(S(0)u, T) = (0,\ 1.2 - u)^+ = 0 = V(S(0)u^3, T)$$

$$V(S(0)d^2, 2) = \max\left\{1.2 - d^2,\ \frac{B(2)}{B(3)} E_Q\left(V(S(3),3) \mid \text{down down}\right)\right\}$$

$$= \max\left\{0.8986,\ e^{-0.05}\cdot\left(q\cdot 0.651 + (1-q)1.0345\right)\right\}$$

$$= \max\{\ \cdot\ ,\ 0.8401\} = 0.8986$$

$$V(S(0), 2) = \max\left\{0.2,\ e^{-0.05}\left(q\cdot 0 + (1-q)0.651\right)\right\}$$

$$= \max\{0.2,\ 0.3748\} = 0.3748$$

$$V(S(0)u, 1) = \max\left\{0,\ e^{-0.05}(1-q)\,0.3748\right\} = 0.2158$$

$$V(S(0)d, 1) = \max\left\{1.2 - d,\ e^{-0.05}\left(q\cdot 0.3748 + (1-q)0.8986\right)\right\}$$

$$= \max\{0.651,\ 0.6581\} = 0.6581$$

$$V(S(0),0) = \max\left\{0.2,\ e^{-0.05}\left(q\cdot 0.2158 + (1-q)\, 0.6581\right)\right\}$$

$$= \max\{0.2,\ 0.4599\} = 0.4599$$

b) Calculations:

Forward propagation, starting at $t=0$:

$$\varphi^0(1)\underbrace{e^{-2\cdot r}}_{B(1)} + \varphi^1(1)\underbrace{\cancel{d}u}_{S(1)} = 0.2158$$

$$\varphi^0(1)B(1) + \varphi^1(1)\cancel{S(1)}\,d = 0.6581$$

$$\underbrace{\begin{pmatrix} e^{-2r} & u \\ e^{-2r} & d \end{pmatrix}}_{A}\begin{pmatrix}\varphi^0(1)\\ \varphi^1(1)\end{pmatrix}^{\#} = \begin{pmatrix}0.2158\\ 0.6581\end{pmatrix} \Rightarrow A^{-1}\begin{pmatrix}0.2158\\ 0.6581\end{pmatrix} = \begin{pmatrix}0.9382\\ -0.3476\end{pmatrix} = \varphi(1)_{up}$$

$$A^{-1} = \frac{1}{e^{-2r}(d-u)}\begin{pmatrix} d & -u \\ -e^{-2r} & e^{-2r}\end{pmatrix} = \begin{pmatrix} -0.4769 & 1.582 \\ 0.7859 & -0.7859\end{pmatrix}$$

up:

$$\underbrace{\begin{pmatrix} e^{-r} & u^2 \\ e^{-r} & 1\end{pmatrix}}_{A}\varphi(2) = \begin{pmatrix}0\\ 0.3748\end{pmatrix} \Rightarrow \varphi(2) = \cancel{0.5641}\begin{pmatrix}0.564\\ -0.1617\end{pmatrix}$$

$$A^{-1} = \frac{1}{e^{-r}(1-u^2)}\begin{pmatrix} 1 & -u^2 \\ -e^{-r} & e^{-r}\end{pmatrix} = \begin{pmatrix} -0.4536 & +1.5049 \\ 0.4315 & -0.4315\end{pmatrix}$$

up up: $\varphi(3) = 0$.

down:

$$A = \begin{pmatrix} e^{-r} & 1 \\ e^{-r} & d^2\end{pmatrix} \Rightarrow A^{-1} = \frac{1}{e^{-r}(d^2-1)}\begin{pmatrix} d^2 & -1 \\ -e^{-r} & e^{-r}\end{pmatrix} = \begin{pmatrix} -0.4536 & 1.5048 \\ 1.4314 & -1.4314\end{pmatrix}$$

$$A^{-1}\begin{pmatrix}0.3748\\ 0.8986\end{pmatrix} = \varphi(2) = \begin{pmatrix}1.1822\\ -0.7498\end{pmatrix}$$

down down

$$\begin{pmatrix} 1 & d \\ 1 & d^3 \end{pmatrix} = A \qquad A^{-1} = \frac{1}{d^3 - d} \begin{pmatrix} d^3 & -d \\ -1 & 1 \end{pmatrix} = \begin{bmatrix} -0.4314 & 1.4314 \\ 2.607\cancel{4}4 & -2.6074 \end{bmatrix}$$

$$\varphi(3) = A^{-1} \begin{pmatrix} 0.651 \\ 1.0345 \end{pmatrix} = \cancel{1.189} \begin{pmatrix} 1.2 \\ -1 \end{pmatrix}$$

down up/up down:

$$A = \begin{pmatrix} 1 & u \\ 1 & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{d-u} \begin{pmatrix} d & -u \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -0.4315 & 1.4314 \\ 0.786 & -0.786 \end{pmatrix}$$

$$A^{-1} \begin{pmatrix} 0 \\ 0.651 \end{pmatrix} = \begin{pmatrix} 0.9318 \\ -0.5117 \end{pmatrix}$$