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QF 26

T15

$$V(T) = 1_{\{S(T)>K\}} \qquad \hat{V}(t) = \frac{V(t)}{B(t)} = E_Q(\hat{V}(T)|\mathcal{F}_t)$$

$$= \frac{1}{B(T)} E_Q(1_{\{S(T)>K\}}|\mathcal{F}_t)$$

$$\frac{(1)}{B(T)} \cdot \mathbb{P}_Q(S(T)>K)$$

$$\hookrightarrow \hat{V}(t) = \frac{1}{B(T)} \int_K^\infty f(x)dx \text{"} \quad \cancel{\text{Find } f(x) \text{ mot}}:$$

$$S(T) = S(t)\exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(\underbrace{W^Q(T)-W^Q(t)}_{\sim \mathcal{N}(0,T-t)})\right) > K$$

$$\Leftrightarrow \ln\left(\frac{K}{S(t)}\right) < \left(r-\frac{\sigma^2}{2}\right)(T-t) + \sigma(W^Q(T)-W^Q(t))$$

$$\Leftrightarrow \underbrace{\frac{\ln\left(\frac{K}{S(t)}\right) - \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}_{:=\theta} < Z \sim \mathcal{N}(0,1)$$

$$\Rightarrow \mathbb{P}_Q(S(T)>K|\mathcal{F}_t) = 1-\phi(\theta) \Rightarrow \hat{V}(t) = e^{-rT}(1-\phi(\theta))$$

$$\Rightarrow V(t) = e^{-r(T-t)}(1-\phi(\theta)) = v(t,x) \longrightarrow V(0) = e^{-rT}(1-\phi(\theta_{t=0}))$$

$$\varphi_1 = \partial_x v(t,x) = \partial_\theta(1-\phi(\theta))\cdot\partial_x\theta \cdot e^{-r(T-t)}$$

$$= -e^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} \cdot \left(-\frac{1}{x\sigma\sqrt{T-t}}\right) = e^{-r(T-t)} \frac{1}{x\sigma\sqrt{T-t}} \phi'(\theta)$$

$$\varphi_0 = \frac{V(t) - S(t)\varphi_1(t)}{B(t)} = \frac{e^{-r(T-t)}(1-\phi(\theta)) - S(t)e^{-r(T-t)}\frac{1}{x\sigma\sqrt{T-t}}\phi'(\theta)}{e^{rt}}$$

$$= e^{-rT}\left(1-\phi(\theta) - S(t)\frac{\phi'(\theta)}{x\sigma\sqrt{T-t}}\right).$$