Itō calculus

A summary of some rules

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1. Linearity of the integral:

$$(H(t) + \tilde{H}(t))dX(t) = H(t)dX(t) + \tilde{H}(t)dX(t),$$

$$(cH(t))dX(t) = c(H(t)dX(t)),$$

$$H(t)d(X + \tilde{X})(t) = H(t)dX(t) + \tilde{H}(t)d\tilde{X}(t),$$

$$H(t)d(cX)(t) = c(H(t)dX(t))$$

2. Integral of differentiable processes:

$$dX(t) = \frac{dX(t)}{dt}dt$$

3. Integration of integrals:

$$H(t)(K(t)dX(t)) = (H(t)K(t))dX(t)$$

4. Integration by parts:

$$\boxed{d(XY)(t) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)}$$

5. Linearity of the covariation:

$$d(X + \tilde{X})(t)dY(t) = dX(t)dY(t) + d\tilde{X}(t)dY(t),$$

$$d(cX)(t)dY(t) = cdX(t)dY(t)$$

6. Symmetry of the covariation:

$$dX(t)dY(t) = dY(t)dX(t)$$

7. If X or Y is of finite variation:

$$dX(t)dY(t) = 0$$

8. Itō's formula (uni- and multivariate):

$$\boxed{df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2}$$

$$\boxed{df(X(t)) = \sum_{i=1}^{d} \partial_i f(X(t))dX_i(t) + \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij} f(X(t))dX_i(t)dX_j(t)}$$

9. The Doléans exponential $d\mathscr{E}(X)(t) = \mathscr{E}(X)(t)dX(t)$:

$$\boxed{\mathscr{E}(X)(t) = \exp\left(X(t) - X(0) - \tfrac{1}{2}[X,X](t)\right)}$$

10. Rules for Wiener processes W:

   - Quadratic variation and covariation of independent Wiener processes W, $\tilde{W}$:

$$\boxed{(dW(t))^2 = dt,}$$

$$dW(t)d\tilde{W}(t) = 0$$

   - Measure change: If the density process Z of $Q \sim P$ is of the form $dZ(t) = Z(t)\sigma(t)dW(t)$,then

$$\boxed{dW(t) = dW^Q(t) + \sigma(t)dt}$$

     for some Q-Wiener process.

   - Moments:

$$E\left(\int_0^t H(s)dW(s)\right) = 0,$$

$$\boxed{\mathrm{Var}(X(t)) = \int_0^t E(\sigma(s)^2)ds}$$

11. Rules for Itō processes $dX(t) = \mu(t)dt + \sigma(t)dW(t)$:

   - Quadratic variation:

$$(dX(t))^2 = \sigma(t)^2 dt$$

   - Itō's formula:

$$\boxed{df(t,X(t)) = \left(\dot{f}(t,X(t)) + f'(t,X(t))\mu(t)\right)dt + \frac{1}{2}f''(t,X(t))\sigma(t)^2 dW(t)}$$

   - Doléans exponential:

$$\boxed{\mathscr{E}(X)(t) = \exp\left(\int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2\right)ds + \int_0^t \sigma(s)dW(s)\right)}$$

   - Stopping at T:

$$dX^T(t) = \mu(t)1_{\{t\leq T\}}dt + \sigma(t)1_{\{t\leq T\}}dW(t)$$