Christian-Albrechts-Universität zu Kiel **SS 2022**
Mathematisches Seminar
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Computational Finance

Exercises for all participants

T-Exercise 13 (4 points)
For $\mu \in R$ and $\sigma, r > 0$ we consider the Black-Scholes market with bond B and stock price process S which evolve according to

$$\begin{aligned} dB_t &= rB_t dt, & B_0 &= 1, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t, & S_0 &> 0. \end{aligned}$$

(a) Calculate the Itô process representation of the logarithmic stockprice process $X_t := \log(S_t)$ and the associated quadratic variation process $[X, X]_t$.

(b) Consider a self-financing portfolio $\varphi = (\varphi_t^0, \varphi_t^1)_{t \geq 0}$ with initial value $V_0(\varphi) = 1$ that always invests two thirds of the wealth in the stock, i.e. $\varphi_t^1 = \frac{2V_t(\varphi)}{3S_t}$. Show that the value process $V_t(\varphi)$ is a geometric Brownian motion.

T-Exercise 14 (Explicit pricing formulas in the BS-model) (2 + 4 points)
We consider a Black-Scholes model with interest rate $r = 0$, volatility $\sigma = \sqrt{2}$, initial stock price $S(0) = 1$, and maturity $T > 0$.

(a) Show that the fair price $V_1(t, S(t))$ of a European option with payoff

$$f(S(T)) := 3\sqrt{S(T)} + S(T)^{3/2}$$

at maturity equals

$$V_1(t, S(t)) = \exp\Big(-\frac{1}{4}(T-t)\Big) 3\sqrt{S(t)} + \exp\Big(\frac{3}{4}(T-t)\Big) S(t)^{3/2}.$$

(b) Consider an American option with payoff process

$$g(S(t)) := \begin{cases} 4S(t)^{3/4} & \text{if } S(t) < 1, \\ 3\sqrt{S(t)} + S(t)^{3/2} & \text{if } S(t) \geq 1 \end{cases} \tag{1}$$

for $t \leq T$. Show that its fair price $V_2(t)$ equals

$$V_2(t, S(t)) = \begin{cases} g(S(t)) & \text{if } S(t) < e^{-(T-t)}, \\ V_1(t, S(t)) & \text{if } S(t) \geq e^{-(T-t)} \end{cases} \tag{2}$$

for $t \leq T$.

Hint: This is one of the rare examples where an American option price can be computed explicitly in the Black-Scholes model. For the computation recall that $E(e^X) = \exp(\mu + \sigma^2/2)$ for any Gaussian random variable X with mean μ and variance σ^2.

T-Exercise 15 (Digital option in the Black-Scholes model) (4 points)
A *digital call option* with maturity $T > 0$ and strike $K > 0$ is a European option with payoff

$$V(T) = 1_{\{S(T) \geq K\}}.$$

Find a formula for the initial price of a digital call option in the Black-Scholes model, and compute the perfect hedging strategy.

Hint: To find the formula for the price, it is useful to work with the integral representation and not with the Black-Scholes PDE.

T-Exercise 16 (Hedging error in the BS-model) (for math only) (4 points)
Consider a stock with risk-neutral dynamics

$$\begin{aligned} B(t) &= e^{rt}, \\ S(t) &= S(0) \exp\left((r - \tfrac{\sigma^2}{2})t + \sigma W(t) \right). \end{aligned}$$

Denote by $V(t) = v(t, S(t))$ the Black-Scholes price of a European call if the volatility equals $\tilde{\sigma}$ instead of σ, i.e. with

$$\begin{aligned} v(t,x) = {} & x\Phi\left(\frac{\log \frac{x}{K} + r(T-t) + \frac{\tilde{\sigma}^2}{2}(T-t)}{\tilde{\sigma}\sqrt{T-t}} \right) \\ & - Ke^{-r(T-t)}\Phi\left(\frac{\log \frac{x}{K} + r(T-t) - \frac{\tilde{\sigma}^2}{2}(T-t)}{\tilde{\sigma}\sqrt{T-t}} \right). \end{aligned}$$

Suppose that the bank uses the incorrect volatility estimate $\tilde{\sigma}$. It sells a call option for the wrong price $V(0)$, and tries to hedge it with a self-financing portfolio $\varphi = (\varphi_0, \varphi_1)$ containing

$$\varphi_1(t) = \partial_2 v(t, S(t))$$

shares of stock. Determine the Itô process representation of the *observed hedging error* $\varepsilon(t) := V(t) - (V(0) + \int_0^t \varphi_0(s)dB(s) + \int_0^t \varphi_1(s)dS(s))$. What do you observe?

Hint: The computation of φ_0 can be avoided by working with discounted prices.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Thu, 12.05.2022, 08:15