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T 13

$\boxed{a}$

$$X_t = \log(S_t) = f(S_t) \; ; \; D_x f(x) = \frac{1}{x} \; ; \; D_x^2 f(x) = -x^{-2} \; ;$$

Itô: $dS_t = \underbrace{\mu S_t}_{\mu(t)} dt + \underbrace{\sigma S_t}_{\sigma(t)} dW_t$

$$df(S_t) = dX_t = \left(\frac{1}{S_t}\mu S_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\sigma^2 S_t^2\right)dt + \frac{1}{S_t}\sigma S_t \, dW_t$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \quad \Rightarrow \quad X_t = \log S(0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$$[X, X]_t = \sigma dW_t \, \sigma dW_t = \cancel{\sigma^2 dt} \; \sigma^2 t$$

b)

$$V_0(\varphi) = 1 \; ; \quad \varphi_t^1 = \frac{2}{3}\frac{V_t(\varphi)}{S_t} \Rightarrow \varphi_t^0 = \frac{1}{3}\frac{V_t(\varphi)}{B_t}$$

$$V_t(\varphi) = \varphi_t^1 S_t^* + \varphi_t^0 \cancel{S_t^0} B_t \; ; \quad \cancel{dV_\varphi(t) = \varphi^1(t) dS^*}$$

$$\cancel{= \frac{2}{3} V_t} \quad dV_t(\varphi) = \varphi_t^1 dS_t + \varphi_t^0 dB_t$$

$$= \varphi_t^1 (\mu S_t dt + \sigma S_t dW_t) + \varphi_t^0 (r B_t dt)$$

$$= \frac{2}{3}\frac{V_t(\varphi)}{S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{3}\frac{V_t(\varphi)}{B_t}(r B_t dt)$$

$$= \frac{2}{3}(V_t(\varphi)\mu\, dt + V_t(\varphi)\sigma\, dW_t) + \frac{1}{3} r \cdot V_t(\varphi) dt$$

$$= \left(\frac{2}{3}\mu V_t(\varphi) + \frac{1}{3} r V_t(\varphi)\right) dt + \frac{2}{3}\sigma V_t(\varphi) dW_t$$

$$= \underbrace{\left(\frac{2}{3}\mu + \frac{1}{3} r\right)}_{\mu_V(t)} V_t(\varphi) dt + \underbrace{\frac{2}{3}\sigma}_{\sigma_V(t)} V_t(\varphi) dW_t = dV_t(\varphi)$$

$\Rightarrow V_t(\varphi)$ is geom. Brownian motion.