Adrian Beer QF 26
Dien Hue Vu

T 13

a)

$$X_t = \log(S_t) = f(S_t) \quad ; \quad D_x f(x) = \frac{1}{x} \; ; \; D_x^2 f(x) = -x^{-2} \; ;$$

Itô: $dS_t = \underbrace{\mu S_t}_{\mu(t)} dt + \underbrace{\sigma S_t}_{\sigma(t)} dW_t$

$$df(S_t) = dX_t = \left(\frac{1}{S_t}\mu S_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\sigma^2 S_t^2\right)dt + \frac{1}{S_t}\sigma S_t \, dW_t$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t \quad \Rightarrow \quad X_t = \log S(0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$$[X, X]_t = \sigma dW_t \, \sigma dW_t = \cancel{\sigma^2 dt} \; \sigma^2 t$$

b) $V_0(\varphi) = 1$; $\varphi^1_t = \frac{2}{3}\frac{V_t(\varphi)}{S_t} \Rightarrow \varphi^0_t = \frac{1}{3}\frac{V_t(\varphi)}{B_t}$

$V_t(\varphi) = \varphi^1_t S^*_t + \varphi^0_t S^0_t B_t$; $\cancel{dV_\varphi(t) = \varphi^1(t)^* dS}$

$\cancel{= \frac{2}{3} V_t}$

$$dV_t(\varphi) = \varphi^1_t \, dS_t + \varphi^0_t \, dB_t$$

$$= \varphi^1_t(\mu S_t dt + \sigma S_t dW_t) + \varphi^0_t (rB_t dt)$$

$$= \frac{2}{3}\frac{V_t(\varphi)}{S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{3}\frac{V_t(\varphi)}{B_t}(rB_t dt)$$

$$= \frac{2}{3}(V_t(\varphi)\mu \, dt + V_t(\varphi)\sigma \, dW_t) + \frac{1}{3} r \cdot V_t(\varphi) dt$$

$$= \left(\frac{2}{3}\mu V_t(\varphi) + \frac{1}{3} r V_t(\varphi)\right)dt + \frac{2}{3}\sigma V_t(\varphi) dW_t$$

$$= \underbrace{\left(\frac{2}{3}\mu + \frac{1}{3} r\right)}_{\mu_v(t)} V_t(\varphi) dt + \underbrace{\frac{2}{3}\sigma}_{\sigma_v(t)} V_t(\varphi) dW_t = dV_t(\varphi)$$

$\Rightarrow V_t(\varphi)$ is geom. Brownian motion.

Adrian Ber
Dieu Hue Vu
QF26

T14 Explicit pricing formulas in the BS-model

(0) $r=0;\ \sigma=\sqrt{2};\ S(0)=1;\ T>0$

a)
$$f(S(T)) = 3\sqrt{S(T)} + S(T)^{\frac{3}{2}}$$

$V_1(t,S(t))$ to be determined. $V_1(t,S(t)) \overset{r=0}{\underset{1.\text{FTAP}}{=}} E_Q(V_1(T,S(T))\,|\,\mathcal{F}_t)$

(1) $\hookrightarrow\ = E_Q(f(S(T))\,|\,\mathcal{F}_t)$.

In BS we have $S(T) = S(0)\exp\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma W^Q(T)\right)$
$$\overset{(0)}{=} \exp(-T + \underbrace{\sigma}_{\sqrt{2}} W^Q(T))$$

with $W^Q(T)\big|_{\mathcal{F}_t} \overset{Q}{\sim} \mathcal{N}(0,T-t)$.

$$\cancel{\Rightarrow E_Q(f(S(T))\,|\,\mathcal{F}_t) = \frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty}\left[3\cdot\sqrt{\exp\left(-(T-t)+\frac{\sqrt{2}x}{\sqrt{T-t}}\right)} + \exp\left(-(T-t)+\frac{\sqrt{2}x}{\sqrt{T-t}}\right)^{\frac{3}{2}}\right]\cdot e^{-\frac{x^2}{2}}\,dx}$$

$$\cancel{= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(3\sqrt{S(T)} + S(T)^{\frac{3}{2}}\right)d(W^Q(T) - W^Q(t))}$$

Also: $S(t) = \exp(-t + \sigma W^Q(t))$ s.t.

$$\underline{S(T)} = \exp(-T + \sigma W^Q(T)) = \exp(-t + \sigma W^Q(t) - (T-t) + \sigma(W^Q(T) - W^Q(t)))$$

(2) $$= S(t)\exp(-(T-t) + \sigma(W^Q(T) - W^Q(t))).$$

$$\to E_Q(f(S(T))\,|\,\mathcal{F}_t) = E_Q\left(3\sqrt{S(T)} + S(T)^{\frac{3}{2}}\,\middle|\,\mathcal{F}_t\right)$$

$$\overset{(2)}{=} E_Q\left(3\sqrt{S(t)\underbrace{\exp(-(T-t)+\sigma(W^Q(T)-W^Q(t)))}_{:=\delta}} + \left[S(t)\exp(-(T-t)+\sigma(W^Q(T)-W^Q(t)))\right]^{\frac{3}{2}}\,\middle|\,\mathcal{F}_t\right)$$

$$= 3\sqrt{S(t)}\,E_Q(\sqrt{\delta}\,|\,\mathcal{F}_t) + S(t)^{\frac{3}{2}}E_Q(\delta^{\frac{3}{2}}\,|\,\mathcal{F}_t)$$

$$= 3\sqrt{S(t)}\,e^{+\frac{1}{2}(t-T)}\underbrace{E_Q\left(e^{\frac{1}{2}\sigma\sqrt{T-t}Z}\right)}_{\overset{(3)}{=} e^{\frac{1}{4}(T-t)}} + S(t)^{\frac{3}{2}}e^{\frac{3}{2}(t-T)}\underbrace{E_Q\left(e^{\frac{3}{2}\sigma\sqrt{T-t}Z}\right)}_{\overset{(3)}{=} e^{(T-t)\frac{1}{2}\frac{9}{4}}} \quad \text{with } Z\sim\mathcal{N}(0,1)$$

$$= 3\sqrt{S(t)}\exp\left(-\frac{1}{4}(T-t)\right) + S(t)^{\frac{3}{2}}\exp\left(\frac{3}{4}(T-t)\right).$$

(3) using $E(e^Z) = e^{\mu+\frac{\sigma^2}{2}}$

Adrian Beer
Dices Uwe Vu
QF 26

T15

$$V(T) = 1_{\{S(T) > K\}} \qquad \hat{V}(t) = \frac{V(t)}{B(t)} = E_{\mathbb{Q}}(\hat{V}(T) | \mathcal{F}_t)$$

$$= \frac{1}{B(T)} E_{\mathbb{Q}}(1_{\{S(T)>K\}} | \mathcal{F}_t)$$

$$\hookrightarrow \hat{V}(t) = \frac{1}{B(T)} \int_K^\infty f(x)dx \text{ " } \frac{(1)}{B(T)} \cdot \mathbb{P}_{\mathbb{Q}}(S(T) > K) \quad \cancel{\text{Find } f(x) \text{ note}}:$$

$$S(T) = S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\underbrace{(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t))}_{\sim \mathcal{N}(0, T-t)}\right) > K$$

$$\Leftrightarrow \ln\left(\frac{K}{S(t)}\right) < \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t))$$

$$\Leftrightarrow \underbrace{\frac{\ln\left(\frac{K}{S(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}_{:= \theta} < Z \sim \mathcal{N}(0,1)$$

$$\Rightarrow \mathbb{P}_{\mathbb{Q}}(S(T) > K | \mathcal{F}_t) = 1 - \phi(\theta) \Rightarrow \hat{V}(t) = e^{-rT}(1 - \phi(\theta))$$

$$\Rightarrow V(t) = e^{-r(T-t)}(1 - \phi(\theta)) = v(t,x) \longrightarrow V(0) = e^{-rT}(1 - \phi(\theta_{t=0}))$$

$$\varphi_1 = \partial_x v(t,x) = \partial_\theta(1 - \phi(\theta)) \cdot \partial_x \theta \cdot e^{-r(T-t)}$$

$$= -e^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} \cdot \left(-\frac{1}{x\sigma\sqrt{T-t}}\right) = e^{-r(T-t)} \frac{1}{x\sigma\sqrt{T-t}} \phi'(\theta).$$

$$\varphi_0 = \frac{V(t) - S(t)\varphi_1(t)}{B(t)} = \frac{e^{-r(T-t)}(1 - \phi(\theta)) - S(t)e^{-r(T-t)}\frac{1}{x\sigma\sqrt{T-t}}\phi'(\theta)}{e^{rt}}$$

$$= e^{-rT}\left(1 - \phi(\theta) - S(t)\frac{\phi'(\theta)}{x\sigma\sqrt{T-t}}\right).$$