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T14 Explicit pricing formulas in the BS-model

(0) $r=0$; $\sigma=\sqrt{2}$; $S(0)=1$; $T>0$

a) $$f(S(T)) = 3\sqrt{S(T)} + S(T)^{\frac{3}{2}}$$

$V_1(t,S(t))$ to be determined. $V_1(t,S(t)) \overset{r=0}{\underset{1.\text{FTAP}}{=}} E_Q(V_1(T,S(T))|\mathcal{F}_t)$

(1) $\hookrightarrow = E_Q(f(S(T))|\mathcal{F}_t)$.

In BS we have $S(T) = S(0)\exp\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma W^Q(T)\right)$
$$\overset{(0)}{=} \exp(-T + \underset{\sqrt{2}}{\sigma} W^Q(T))$$

with $W^Q(T)|_{\mathcal{F}_t} \overset{Q}{\sim} \mathcal{N}(0, T-t)$.

$$\cancel{\Rightarrow E_Q(f(S(T))|\mathcal{F}_t) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty}\left[3\cdot\sqrt{\exp(-(T-t)+\sqrt{2}\,x\cdot\sqrt{T-t})} + \exp(-(T-t)+\sqrt{2}\,x\cdot\sqrt{T-t})^{\frac{3}{2}}\right] \cdot e^{-\frac{x^2}{2}}\,dx}$$

$$\cancel{= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(3\sqrt{S(T)} + S(T)^{\frac{3}{2}}\right) d(W^Q(T) - W^Q(t))}$$

Also: $S(t) = \exp(-t + \sigma W^Q(t))$ s.t.

$$S(T) = \exp(-T + \sigma W^Q(T)) = \exp(-t + \sigma W^Q(t) - (T-t) + \sigma(W^Q(T) - W^Q(t))$$

(2) $$= S(t)\exp(-(T-t) + \sigma(W^Q(T) - W^Q(t)))$$

$$\to E_Q(f(S(T))|\mathcal{F}_t) = E_Q\left(3\sqrt{S(T)} + S(T)^{\frac{3}{2}} \,\middle|\, \mathcal{F}_t\right)$$

$$\overset{(2)}{=} E_Q\left(3\sqrt{S(t)\underbrace{\exp(-(T-t)+\sigma(W^Q(T)-W^Q(t))}_{:=\delta}} + \left[S(t)\exp(-(T-t)+\sigma(W^Q(T)-W^Q(t))\right]^{\frac{3}{2}} \,\middle|\, \mathcal{F}_t\right)$$

$$= 3\sqrt{S(t)}\,E_Q(\sqrt{\delta}|\mathcal{F}_t) + S(t)^{\frac{3}{2}} E_Q(\delta^{\frac{3}{2}}|\mathcal{F}_t)$$

$$= 3\sqrt{S(t)}\,e^{+\frac{1}{2}(t-T)}\underbrace{E_Q\left(e^{\frac{1}{2}\sigma\sqrt{T-t}\,Z}|\mathcal{F}_t\right)}_{\overset{(3)}{=} e^{\frac{1}{4}(T-t)}} + S(t)^{\frac{3}{2}} e^{\frac{3}{2}(t-T)} \underbrace{E_Q\left(e^{\frac{3}{2}\sigma\sqrt{T-t}\,Z}\right)}_{\overset{(3)}{=} e^{(T-t)\frac{1}{2}\frac{9}{4}}} \quad \text{with } Z\sim\mathcal{N}(0,1)$$

(3) using $E(e^Z) = e^{\mu+\frac{\sigma^2}{2}}$

$$= 3\sqrt{S(t)}\exp\left(-\frac{1}{4}(T-t)\right) + S(t)^{\frac{3}{2}}\exp\left(\frac{3}{4}(T-t)\right).$$