Christian-Albrechts-Universität zu Kiel **SS 2022**
Mathematisches Seminar
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Computational Finance

Exercises for all participants

C-Exercise 29 (Pricing a deep out-of-the-money European call option by Monte-Carlo with importance sampling) (4 points)
Consider a Black-Scholes model with parameters $S(0)$, r, $\sigma > 0$. The goal is to approximate by the Monte-Carlo method the fair price $V(0)$ of an European call option on the stock with strike $K \gg S(0)$ at maturity T.

Write a Python function

```
[V0, CIl, CIr] = BS_EuCall_MC_IS (S0, r, sigma, K, T, mu, N,
                                  alpha)
```

that approximates the price of the European call option via Monte-Carlo based on $N \in \mathbb{N}$ samples and additionally returns the left and right boundary of an asymptotic α-level confidence interval. Use a new random variable $Y \sim N(\mu, 1)$ for the importance sampling method.

Test your function for $S(0) = 110$, $r = 0.03$, $\sigma = 0.2$, $K = 220$, $T = 1$, $N = 10000$, $\alpha = 0.95$ and plot your estimator for $V(0)$ in dependence on μ against the true value.

Hint: Experiment with the range of μ such that you can see visible changes in the variance of your estimator. For the true value use the Black-Scholes formula provided on sheet 02.

C-Exercise 30 (Calculating the Delta/Hedge of a European option via MC methods, infinitesimal perturbation) (4 points)
Consider a Black-Scholes model with parameters r, $\sigma > 0$. The goal is to approximate the hedging position $\varphi_1(t)$ (check equation (3.29)) of a European option with payoff $g(S(T))$ at maturity T by the infinitesimal perturbation method for some time point $t \in [0, T]$ and current stock price $S(t)$. To this end write a Python function

```
phi1_t=EuOptionHedge_BS_MC_IP (St, r, sigma, g, T, t, N)
```

that calculates the hedging position $\varphi_1(t)$.

Test your function for $t = 0, S_0 = 110, r = 0.03, \sigma = 0.2, T = 1, N = 10000, g(x) = (x - 100)^+$ and compare you result to the lecture notes by implementing the formula below (3.30).

Hint: To calculate the derivative needed for the infinitesimal perturbation you may use finite differences or the function `scipy.misc.derivative`.

C-Exercise 31 (Valuation of European options in the Heston model using the Euler method)
Write a Python function

```
[V0, c1, c2] = Heston_EuCall_MC_Euler (S0, r, gamma0, kappa,
                lambda, sigma_tilde, T, g, M, m)
```

that computes the price $V(0)$ of a European option with payoff $g(S_T)$ and maturity T in the Heston model via the Monte-Carlo method using M samples together with the 95%-confidence interval. To this end, approximate the paths by the Euler method with a grid of m equidistant points in time. Therefore first think how the Euler method from Section 5.5 in the lecture notes can be extended for the Heston model, where the model contains a stochastic price and a stochastic volatility process.
Test your function for the European call option using the parameters

$$S0 = 100, \quad r = 0.05, \quad \gamma(0) = 0.2^2, \quad \kappa = 0.5, \quad \lambda = 2.5,$$
$$\tilde{\sigma} = 1, \quad T = 1, \quad g(x) = (x-100)^+, \quad M = 10000, \quad m = 250.$$

T-Exercise 32 (Antithetic variables and monotone functions) (for math only) (4 points)

a) Let $f,g : \mathbb{R} \to \mathbb{R}$ be two increasing functions and X,Y random variables with $\mathbb{E}[f(X)^2]$, $\mathbb{E}[f(Y)^2]$, $\mathbb{E}[g(X)^2]$, $\mathbb{E}[g(Y)^2] < \infty$. Prove

$$\mathbb{E}[f(X)g(X)] + \mathbb{E}[f(Y)g(Y)] \geq \mathbb{E}[f(X)g(Y)] + \mathbb{E}[f(Y)g(X)].$$

b) Use part a) to deduce

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\mathbb{E}[g(X)].$$

c) Use part b) to prove that using antithetic variables reduces the variance if the random number X is mapped to the payoff $f(X)$ using an increasing or a decreasing function f. I.e. show that we have

$$\mathrm{Cov}(f(X), f(-X)) \leq 0$$

for any increasing or decreasing function $f : \mathbb{R} \to \mathbb{R}$ and any random variable X with $\mathbb{E}[f(X)^2], \mathbb{E}[f(-X)^2] < \infty$.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Thu, 16.06.2022, 08:15