Handout_FactorModels

April 26, 2021

1 A. Online Lecture

1.1 A.1 CAPM

- **Length**: 14min + 6min + 2min = 22min
- **URL**: https://www.youtube.com/watch?v=9YEzUbYjTUY&list=PLyQSjcv8LwAHUlBatCGOm71ey0mv(
- **URL:** https://www.youtube.com/watch?v=e8Hj6UNxZ9M&list=PLyQSjcv8LwAHUlBatCGOm71ey0mvG
- **URL:** https://www.youtube.com/watch?v=Pw9-9hCPx68&list=PLyQSjcv8LwAHUlBatCGOm71ey0mvGY

1.2 A.2 Factor Models: Linear

- **Length**: 11min + 5min + 15min
- **URL**: https://www.youtube.com/watch?v=Xe9E47DNyYM&list=PLyQSjcv8LwAHUlBatCGOm71ey0mv(
- **URL**: https://www.youtube.com/watch?v=Y8fdHN0pChA&list=PLyQSjcv8LwAHUlBatCGOm71ey0mvG
- **URL**: https://www.youtube.com/watch?v=2tuH6no48oc&list=PLyQSjcv8LwAHUlBatCGOm71ey0mvGY

1.3 A.3 Market Efficiency

- **Lengthh**: 10min
- **URL**: https://www.youtube.com/watch?v=y8unppWAdCw&list=PLyQSjcv8LwAHUlBatCGOm71ey0mv(

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2 B. Tutorial and Revision Manual

2.1 B.1 Key Take Aways for the CAPM

- CAPM has three **main messages**

First,

$$ERP_{\mathrm{A}} = \beta_A \times ERP_{\mathrm{M}}$$

Second,

$$\beta_A := \frac{Cov(r_A, r_M)}{Var(r_M)}.$$

Third, the expected risk premium of a stock is unrelated to the stocks' idiosyncratic risk.

The CAPM insights hold if the following **assumptions** hold

- all investors are identical Mean-Variance portfolio optimizers.
- all investors share the same homogeneous expectations about the risk-free rate r_f, the expected holding period return of all assets, μ, and the covariance matrix Σ.

Now, these assumptions imply the following **logical consequences**:

- All investors will come up with the same tangency portfolio
- tangency portfolio must coincide with the aggregate market and hence, can be called 'market portfolio'.
- price of each asset adjusts such that the aggregate demand equals its fixed supply.
- As the market portfolio is a Markowitz efficient portfolio that invests 100% of wealth in the TP portfolio it holds

$$E[r_M] - r_f = \gamma_M \times Var(r_M),$$

- The expected risk premium of an individual security is

$$E[r_A] - r_f = \beta_A \times (E[r_M] - r_f), \ \forall \text{assets A}$$

where β_A, defined as

$$\beta_A = \frac{Cov(r_A, r_M)}{Var(r_M)}$$

measures the amount of systematic risk that asset A contributes to the market's total risk.

Combiningin last three equations reveals

$$E[r_A] - r_f = \gamma_M \times Cov(r_A, r_M); \ \forall \text{assets A}.$$

Notice, 'hedge assets' net present value is priced with a negtive risk premium, i.e.

$$\rho_{A,M} < 0 \rightarrow Cov(r_A, r_M) < 0 \rightarrow E[r_A] < r_f.$$

2.1.1 B.1.2 MBA-like Derivation of the CAPM

Assume all CAPM assumptions hold. As a result it holds,

- market portfolio coincides with tangency portfolio, i.e. $w_{TP} = w_M$
- market risk, σ_M^2, is systematic and undiversifiable
- Every asset A adds only
$$w_A \times Cov(r_A, r_M)$$
to the risk of the market portfolio
- Every asset A contributes
$$w_A \times (E[r_A] - r_f).$$

to the risk premium of the market portfolio

- in equilibrium, all assets pay the same risk-adjusted risk premium, hence,

$$\begin{aligned}\frac{w_A \times (E[r_A] - r_f)}{w_A \times Cov(r_A, r_M)} &= \frac{E[r_M] - r_f}{\sigma_M^2} \\ &= \gamma_M.\end{aligned}$$

Ergo,

$$\begin{aligned}E[r_A] - r_f &= \frac{Cov(r_A, r_M)}{\sigma_M^2} \times (E[r_M] - r_f) \\ &= \beta_A \times (E[r_M] - r_f) \\ &= \gamma_M \times Cov(r_A, r_M).\end{aligned}$$

According to the CAPM, all μ-β combinations line up on a straight line. That line is called 'SML' which stands for 'Security Market Line'.

Assets on the SML are 'fairly priced'. Their price equals the sum of all expected future cashflows, discounted by μ. And according to the CAPM, μ is determined according to the linear SML relation. Notice, an overvalued asset would have a μ that is below the SML. Its discount rate is too low, relative to the CAPM, and hence its price is too high; which means the asset is overvalued.

2.1.2 B.1.3 CAPM - Risk Analysis

The CAPM allows to decompose an asset's return variance σ_i^2 as

$$\sigma_i^2 = \beta_i^2 \times \sigma_M^2 \; + \sigma_{\epsilon,i}^2$$

where the l.h.s. captures total risk of asset i, $\sigma_{\epsilon,i}^2$ captures the amount of asset specific risk and $\beta_i^2 \times \sigma_M^2$ is asset i's amount of systematic risk.

In addition, remember, only $\beta_i^2 \times \sigma_M^2$ is compensated by a risk premium; $\sigma_{\epsilon,i}^2$ is not. Graphically, that is represented by the Security Market Line.

2.2 B.2 Linear Single Factor Model (Single Index Model)

A single-index model is basically the empirical analog of the CAPM. It assumes that returns of asset i and j co-move because both assets have exposure to the same underlying risk factor; here it would be the market return. Using equations, one could write

$$\begin{aligned}
\forall i, j, j \neq i: \quad & r_i - r_f = \alpha_i + \beta_i \times (r_M - r_f) + \epsilon_i; \\
& r_j - r_f = \alpha_j + \beta_j \times (r_M - r_f) + \epsilon_j; \\
& \quad \text{with } E[\epsilon_i] = E[\epsilon_j] = E[\epsilon_i \epsilon_j] = E[\epsilon_i r_M] = E[\epsilon_j r_M] = 0. \\
& \quad \text{with } [\alpha_i, \alpha_j, \beta_i, \beta_j] \in \mathcal{R}^4.
\end{aligned}$$

This single-index model implies the same expected return - beta relationship as the Security Market Line in the CAPM, i.e.

$$E[r_i] - r_f = \alpha_i + \beta_i \times (E[r_M] - r_f)$$

Let's have a closer look at why different assets co-move in a single-index model. We do that through some simple equation-based manipulations.

$$Cov(r_i - r_f, r_j - r_f) = Cov(\alpha_i + \beta_i(r_M - r_f) + \epsilon_i,\ \alpha_j + \beta_j(r_M - r_f) + \epsilon_j)$$
$$= \beta_i \times \beta_j \times \sigma_M^2$$

because

$$\epsilon_i \perp \epsilon_j \perp r_M.$$

This says that two assets co-move only because both have a beta-exposure to the common risk factor r_M.

We can also look directly at the pairwise correlations that a single-index model induces

$$Corr(r_i - r_f, r_j - r_f) = \frac{Cov(r_i - r_f, r_j - r_f)}{\sigma_i \times \sigma_j}$$

$$= \frac{\beta_i \times \beta_j \times \sigma_M^2}{\sigma_i \times \sigma_j}$$

Let's multiply the nominator and denominator by σ_M^2 which gives us

$$Corr(r_i - r_f, r_j - r_f) = \frac{\beta_i \times \sigma_M^2}{\sigma_i \times \sigma_M} \times \frac{\beta_j \times \sigma_M^2}{\sigma_j \times \sigma_M}$$

$$= Corr(r_i, r_M) \times Corr(r_j, r_M).$$

The last equation states that the correlation of two assets coincides with the product of the respective correlation coefficients with the single-factor.

The single-index model has also a very practical advantage for portfolio optimization. Using a single-index model reduces the required number of estimates for μ and Σ in a Markowitz portfolio selection process. Let's consider a portfolio with $n = 50$ assets. Assuming that their returns follow a single-index model means one has to estimate 152 input parameters. These are still quite a lot, but roughly a 90% reduction relative to the 1325 parameter estimates that a full Mean-Variance portfolio optimization input list requires.

But as expected, a dimensionality reduction of that size comes potentially at a cost. The Mean-Variance portfolio with $\hat{\mu}$ and $\hat{\Sigma}$ being estimated by a single-index model can be substantially inferior to a full covariance Markowitz approach. Hence, it is important to ensure that the factor model that you use implies that statistically speaking, firm-specific return shocks are uncorrelated with each other.

Using the full covariance matrix Σ as in the Mean-Variance portfolio selection approach of Markowitz is superior to the single-index model approximation, if all entries of Σ can be estimated with high precision. But, in practice, it is difficult to estimate pairwise correlations in an accurate way. It is therefore to be expected that a low dimensional factor model will be less prone to the 'Garbage In, Garbage Out' problem. In order to be really sure, one needs to do careful backtesting and an accurate assessment of the adequacy of the factor model at hand.

2.3 B.3 Linear Multi-Factor Models (Multi-Index Models)

Factor models aim to characterize variations in systematic risk by a set of factors.

F captures the deviation of the common factor from its expected value. To simplify the exposition, we treat right now F has a single factor. A single-factor model would therefore postulate the following return dynamic for asset i

$$r_i - E[r_i] = \beta_i \times F + \epsilon_i$$

where

$$\begin{aligned} E[\epsilon_i] &= 0 \\ E[\epsilon_i F] &= 0 \\ E[\epsilon_i \epsilon_j] &= 0, \quad \forall i \neq j. \end{aligned}$$

The next more general linear factor model is a two-factor model. As an example, consider we want to test whether business cycle risk and interest rate risk is part of the economy's systematic risk factor. So, we say that

$$GDP \quad \text{and} \quad IR$$

capture unexpected innovations to GDP growth and unexpected innovations to the one-month nominal interest rate. The respective 2-factor model for asset i would read

$$
\begin{aligned}
r_i - E[r_i] = \beta_{i,GDP} \times GDP + \beta_{i,IR} \times IR + \epsilon_i, \\
E[\epsilon_i] = 0;\ E[\epsilon_i GDP] = 0;\ E[\epsilon_i IR] = 0;\ E[\epsilon_i \epsilon_j] = 0\, \forall i \neq j.
\end{aligned}
$$

Both beta coefficients in the last equation are also called 'factor sensitivities'.

Notice, a multi-factor model for asset returns induces a multi-factor security market line, i.e.

$$E[r_i] = r_f + \beta_{i,GDP} \times ERP_{GDP} + \beta_{i,IR} \times ERP_{IR}$$

where $ERP_{{GDP}} $ stands for the expected holding period premium for a one unit exposure to GDP risk and ERP_{IR} is the expected holding period premium for a one unit exposure to interest rate risk.

2.4 B.4 Arbitrage Pricing Theory (APT)

The Arbitrage Pricing Theory, or APT, as it is usually called, is an empirical way to derive a CAPM-like security market line for well diversified portfolios. The APT requires three assumptions

- **(i) All asset returns follow a factor model**
- **(ii) There are sufficiently many assets to invest into such that asset specific risk can be diversified away**
- **(iii) Asset markets are arbitrage-free.**

Let's talk about the intuition of the APT. In order to keep notation at a minimum, we pretend that the systematic risk factor F was one-dimensional. Assumption **(i)** implies that a portfolio made up of N assets follows a factor model. Let's look at the equations that support this claim

$$
\begin{aligned}
r_p - E[r_p] &= \beta_p \times F + \epsilon_p, \quad Var(\epsilon_p) = \sigma^2_{\epsilon,p} \\
E[r_p] &:= \sum_{i=1}^{N} w_i \times E[r_i] \\
\beta_p &:= \sum_{i=1}^{N} w_i \times \beta_i \\
\sigma_p^2 &:= \beta_p^2 \times \sigma_F^2 + \sigma^2_{\epsilon,p} \\
\sigma^2_{\epsilon,p} &:= \sum_{i=1}^{N} w_i^2 \times \sigma^2_{\epsilon,i}, \quad b/c\ \epsilon_i \perp \epsilon_j\ for\ i \neq j.
\end{aligned}
$$

Assumption **(ii)** implies that N is sufficiently large, so that it holds

$$\sigma^2_{\epsilon,p} \to 0.$$

So, a well-diversified portfolio has the following factor structure

$$r_p = E[r_p] + \beta_p \times F.$$

Assumption **(iii)** ensures that market prices of all N assets are arbitrage-free. As arbitrage is ruled out, it means prices adjust freely so that diversified portfolios with the same factor exposure pay also the same expected risk premium. This coincides with a Security Market Line for well diversified portfolios.

Notice, if you choose

$$F = r_M - E[r_M]$$

then you get for the expected return of a well-diversified portfolio

$$\mu_p = r_f + \beta_p(\mu_M - r_f)$$

which coincides with the CAPM Security Market Line for portfolios. ... So what have we just accomplished? Well, we have detected a second way to derive a Security Market Line. The CAPM derives the Security Market Line based on mean-variance portfolio optimizing investors. The APT derives the Security Market Line based on an arbitrage-free capital market with many assets with returns that follow a linear factor model.

Chen, Roll and Ross (1986) have proposed a 5-factor APT with the following specification:

$$\begin{aligned}\forall i \in \{1, ..., N\} \quad r_i - E[r_i] = \beta_{i,IP} \times IP + \beta_{i,EI} \times EI + \beta_{i,UI} \times UI \\ + \beta_{i,CG} \times CG + \beta_{i,GB} \times GB + \epsilon_i \; E[\epsilon_i] = 0, E[\epsilon_i[.]] = 0,\end{aligned}$$

Another prominent APT factor model is the Fama-French (1993) 3-factor model

$$\begin{aligned}\forall i \in \{1, ..., N\} \quad r_i = \alpha_i + \beta_{i,M}(r_m - r_f) + \beta_{i,SMB} \times r_{SMB} \\ + \beta_{i,HML} \times r_{HML} + \epsilon_i\end{aligned}$$

3 B.5 Is the Stock Market Information Efficient?

Notice,

$$r_t = \mu_{t-1} + \epsilon_t$$

The question whether asset markets are information efficiency is a question whether return surprises ϵ_t are only due to the arrival of unpredictable information at time t.

Characteristics of Information Efficient Asset Markets

- expected returns move inline with the asset's systematic risk
- predictable movements in asset prices are only arising because either
  - the market-wide risk premium for the asset's systematic risk changes, and/or
  - the amount of the asset's systematic risk changes

- an asset's average return is a fair compensation for its risk contribution to a well diversified portfolio
- the price of an asset follows only a 'random walk' if either
  - the market-wide risk premium for systematic risk is not predictable, or
  - the amount of an asset's systematic risk is unpredictable
- the return surprise of an asset, measured as the spread between realized return and its expectation, is ONLY driven by incoming information of previously unknown content, also called 'news'.
- the price of an asset jumps immediately upon the arrival of new information, and remains unchanged until new information comes in.

How to test whether an asset market is efficient?

- tests of market efficiency are tests on the asset's return innovations
- tests for market efficiency assume the applied model for expected returns is correct
- hence, tests on market efficiency are joint tests (efficiency + model for expected returns)
- if the return residual is forecastable by old information, markets are not information efficient

Three different forms of market efficiency: - **Weak-Form**: ϵ_t is not driven by $\mathcal{F}_{t-k}, k > 0$ information. So, chart analysis using past prices and any signal that relies on historic prices cannot identify **mis-valued** assets at time t.

- **Semi-Strong Form**: ϵ_t is only driven by newly released public information at time t.
- **Strong Form**: ϵ_t is driven by newly released public information at time t and by information that is only known to insiders, also called private information.

So, to wrap up in one sentence: An asset market is efficient, or information efficient, if variations in ϵ_t are only due to news arriving at time t.

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4 C. Quizzes - Basics

All quizzes are on Ilias

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5 D Exercises - Basics

5.1 D.1 CAPM SML

Write down the CAPM relationship for a project's expected discount rate (all equity financed)

5.2 D.2 CAPM Beta

How is the beta of a firm defined? What does it capture?

5.3 D.3 CAPM and Markowitz

State the CAPM implied expected return - risk relationship for the market portfolio and explain how it relates to the mean-variance portfolio theory

5.4 D.4 Market Risk

What share of the market's return variance is unsystematic?

5.5 D.5 Market Clearing

Why do all assets pay the same risk-adjusted return in equilibrium?

5.6 D.6 Risk-Adjusted Returns

Why do Markowitz investors care for risk-adjusted returns and not returns?

5.7 D.7 CAPM SML

Draw the SML and interpret its meaning and practical relevance and implications

5.8 D.8 CAPM Risk Analysis

Decompose total firm risk as advocated by the CAPM

5.9 D.9 Co-movements in Linear Factor Models

Show that the following holds for a single index model

$$Cov(r_i - r_f, r_j - r_f) = \beta_i \beta_j \sigma^2_{MKT}$$

5.10 D.10 Single Index Model

State and interpret the expected return - beta relationship of a single index model

5.11 D.11 Co-Movement in Single Index Model

Assume BMW stocks have a 0.5 correlation with the DAX, while Daimler has a 0.7 correlation. The volatility of the DAX is assumed to be 25%. How large is the correlation between BMW and Daimler, assuming both stocks follow a single index model with the DAX being the single systematic risk factor.

5.12 D.12 Two-Factor Model

Decompose the variance of a firm, assuming the stock return follows a two-factor model.

5.13 D.13 Assumptions of the APT

State and discuss the assumptions that the Arbitrage Pricing Theory is built on

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6 E. Exercises - Challenging

Prof and students work through one or several challenging exercises during the Prof Cafe.

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7 F. Python Exercise

Python exercise is distributed via Ilias