12

Free Random Matrices

In the last chapter, we introduced the concept of freeness rather abstractly, as the proper non-commutative generalization of *independence* for usual random variables. In the present chapter, we explain why large, randomly rotated matrices behave as free random variables. This justifies the use of R-transforms and S-transforms to deal with the spectrum of sums and products of large random matrices. We also revisit the abstract central limit theorem of the previous chapter (Section 11.2.4) in the more concrete case of sums of randomly rotated matrices.

12.1 Random Rotations and Freeness

12.1.1 Statement of the Main Result

Recall the definition of freeness. A and B are free if for any set of traceless polynomials $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ the following equality holds:

$$\tau\left(p_1(A)q_1(B)p_2(A)q_2(B)\ldots p_n(A)q_n(B)\right) = 0. \tag{12.1}$$

In order to make the link with large matrices we will consider $\mathbf{A}$ and $\mathbf{B}$ to be large symmetric matrices and $\tau(\mathbf{M}) := 1/N\,\mathrm{Tr}(\mathbf{M})$. The matrix $\mathbf{A}$ can be diagonalized as $\mathbf{U}\Lambda\mathbf{U}^T$ and $\mathbf{B}$ as $\mathbf{V}\Lambda'\mathbf{V}^T$. A traceless polynomial $p_i(\mathbf{A})$ can be diagonalized as $\mathbf{U}\Lambda_i\mathbf{U}^T$, where $\mathbf{U}$ is the same orthogonal matrix as for $\mathbf{A}$ itself and $\Lambda_i = p_i(\Lambda)$ is some traceless diagonal matrix, and similarly for $q_i(\mathbf{B})$. Equation (12.1) then becomes

$$\tau\left(\Lambda_1\mathbf{O}\Lambda_1'\mathbf{O}^T\Lambda_2\mathbf{O}\Lambda_2'\mathbf{O}^T\ldots\Lambda_n\mathbf{O}\Lambda_n'\mathbf{O}^T\right) = 0, \tag{12.2}$$

where we have introduced $\mathbf{O} = \mathbf{U}^T\mathbf{V}$ as the orthogonal matrix of basis change rotating the eigenvectors of $\mathbf{A}$ into those of $\mathbf{B}$.

As we argue below, in the large N limit Eq. (12.2) always holds true when averaged over the orthogonal matrix $\mathbf{O}$ and whenever matrices Λ_i and Λ_i' are traceless. We also expect that in the large N limit Eq. (12.2) becomes self-averaging, so a single matrix $\mathbf{O}$ behaves as the average over all such matrices. Hence, two large symmetric matrices whose eigenbases are randomly rotated with respect to one another are essentially free. For example, Wigner matrices $\mathbf{X}$ and white Wishart matrices $\mathbf{W}$ are rotationally invariant, meaning that the

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matrices of their eigenvectors are random orthogonal matrices. We conclude that for N large, both $\mathbf{X}$ and $\mathbf{W}$ are free with respect to any matrix independent from them, in particular they are free from any deterministic matrix.

12.1.2 Integration over the Orthogonal Group

We now come back to the central statement that in the large N limit the average over $\mathbf{O}$ of Eq. (12.2) is zero for traceless matrices Λ_i and Λ_i'. In order to compute quantities like

$$\left\langle \tau\left(\Lambda_1 \mathbf{O} \Lambda_1' \mathbf{O}^T \Lambda_2 \mathbf{O} \Lambda_2' \mathbf{O}^T \dots \Lambda_n \mathbf{O} \Lambda_n' \mathbf{O}^T\right)\right\rangle_{\mathbf{O}}, \tag{12.3}$$

one needs to understand how to compute the following moments of rotation matrices, averaged over the Haar (flat) measure over the orthogonal group $O(N)$:

$$I(\mathbf{i},\mathbf{j},n) := \left\langle \mathbf{O}_{i_1 j_1} \mathbf{O}_{i_2 j_2} \dots \mathbf{O}_{i_{2n} j_{2n}} \right\rangle_{\mathbf{O}}. \tag{12.4}$$

The general formula has been worked out quite recently and involves the *Weingarten functions*. A full discussion of these functions is beyond the scope of this book, but we want to give here a brief account of the structure of the result. When $N \to \infty$, the leading term is quite simple: one recovers the Wick's contraction rules, as if $\mathbf{O}_{i_1 j_1}$ were independent random Gaussian variables with variance $1/N$. Namely,

$$I(\mathbf{i},\mathbf{j},n) = N^{-n} \sum_{\text{pairings } \pi} \delta_{i_{\pi(1)} i_{\pi(2)}} \delta_{j_{\pi(1)} j_{\pi(2)}} \dots \delta_{i_{\pi(2n-1)} i_{\pi(2n)}} \delta_{j_{\pi(2n-1)} j_{\pi(2n)}} + O(N^{-n-1}). \tag{12.5}$$

Note that all pairings of the i-indices are the same as those of the j-indices. For example, for $n = 1$ and $n = 2$ one has explicitly, for $N \to \infty$,

$$N \left\langle \mathbf{O}_{i_1 j_1} \mathbf{O}_{i_2 j_2} \right\rangle_{\mathbf{O}} = \delta_{i_1 i_2} \delta_{j_1 j_2} \tag{12.6}$$

and

$$\begin{aligned} N^2 \left\langle \mathbf{O}_{i_1 j_1} \mathbf{O}_{i_2 j_2} \mathbf{O}_{i_3 j_3} \mathbf{O}_{i_4 j_4} \right\rangle_{\mathbf{O}} &= \delta_{i_1 i_2} \delta_{j_1 j_2} \delta_{i_3 i_4} \delta_{j_3 j_4} \\ &+ \delta_{i_1 i_3} \delta_{j_1 j_3} \delta_{i_2 i_4} \delta_{j_2 j_4} \\ &+ \delta_{i_1 i_4} \delta_{j_1 j_4} \delta_{i_2 i_3} \delta_{j_2 j_3}. \end{aligned} \tag{12.7}$$

The case $n = 1$ is exact and has no subleading corrections in N, so we can use it to compute

$$\begin{aligned} \left\langle \tau(\Lambda_1 \mathbf{O} \Lambda_1' \mathbf{O}^T) \right\rangle_{\mathbf{O}} &= N^{-1} \sum_{i,j=1}^{N} (\Lambda_1)_i \left\langle \mathbf{O}_{ij} (\Lambda_1')_j \mathbf{O}^T_{ji} \right\rangle_{\mathbf{O}} \\ &= N^{-2} \sum_{i,j=1}^{N} (\Lambda_1)_i (\Lambda_1')_j \\ &= \tau(\Lambda_1) \tau(\Lambda_1'). \end{aligned} \tag{12.8}$$

(Recall that $\tau(\mathbf{A})$ is equal to $N^{-1} \operatorname{Tr} \mathbf{A}$.) Clearly the result is zero when $\tau(\Lambda_1) = \tau(\Lambda_1') = 0$, as required by the freeness condition.

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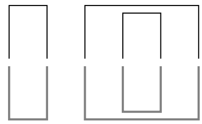

Figure 12.1 Number of loops $\ell(\pi,\pi)$ for $\pi = (1,2)(3,6)(4,5)$. The bottom part of the diagram (thick gray) corresponds to the first partition π and the top part (black) to the second partition, here also equal to π. In this example there are three loops each of size 2.

Now, using only the leading Wick terms for $n = 2$ and after some index contractions and manipulations, one would obtain

$$\lim_{N\to\infty} \langle \tau(\Lambda_1 \mathbf{O}\Lambda'_1 \mathbf{O}^T \Lambda_2 \mathbf{O}\Lambda'_2 \mathbf{O}^T)\rangle_{\mathbf{O}} = \tau(\Lambda_1\Lambda_2)\tau(\Lambda'_1)\tau(\Lambda'_2) + \tau(\Lambda_1)\tau(\Lambda_2)\tau(\Lambda'_1\Lambda'_2). \tag{12.9}$$

However, this cannot be correct. Take for example $\Lambda_1 = \Lambda_2 = \mathbf{1}$, for which $\tau(\Lambda_1 \mathbf{O}\Lambda'_1 \mathbf{O}^T \Lambda_2 \mathbf{O}\Lambda'_2 \mathbf{O}^T) = \tau(\Lambda'_1\Lambda'_2)$ exactly, whereas the formula above adds an extra term $\tau(\Lambda'_1)\tau(\Lambda'_2)$. The correct formula actually reads

$$\begin{aligned}\lim_{N\to\infty} \langle \tau(\Lambda_1 \mathbf{O}\Lambda'_1 \mathbf{O}^T \Lambda_2 \mathbf{O}\Lambda'_2 \mathbf{O}^T)\rangle_{\mathbf{O}} &= \tau(\Lambda_1\Lambda_2)\tau(\Lambda'_1)\tau(\Lambda'_2) + \tau(\Lambda_1)\tau(\Lambda_2)\tau(\Lambda'_1\Lambda'_2) \\ &\quad - \tau(\Lambda_1)\tau(\Lambda_2)\tau(\Lambda'_1)\tau(\Lambda'_2),\end{aligned} \tag{12.10}$$

which again is zero whenever all individual traces are zero (i.e. the freeness condition).

12.1.3 Beyond Wick Contractions: Weingarten Functions

Where does the last term in Eq. (12.10) come from? The solution to this puzzle lies in the fact that some subleading corrections to Eq. (12.7) also contribute to the trace we are computing: summing over indices from 1 to N can prop up some subdominant terms and make them contribute to the final result. Hence we need to know a little more about the Weingarten functions. This will allow us to conclude that the freeness condition holds for arbitrary n. The general Weingarten formula reads

$$I(\mathbf{i},\mathbf{j},n) = \sum_{\text{pairings } \pi,\sigma} W_n(\pi,\sigma)\delta_{i_{\pi(1)}i_{\pi(2)}}\delta_{j_{\sigma(1)}j_{\sigma(2)}}\ldots\delta_{i_{\pi(2n-1)}i_{\pi(2n)}}\delta_{j_{\sigma(2n-1)}j_{\sigma(2n)}}, \tag{12.11}$$

where now the pairings π of i's and σ of j's do not need to coincide. The Weingarten functions $W_n(\pi,\sigma)$ can be thought of as matrices with pairings as indices. They are given by the pseudo-inverse[1] of the matrices $M_n(\pi,\sigma) := N^{\ell(\pi,\sigma)}$, where $\ell(\pi,\sigma)$ is the number of loops obtained when superposing π and σ. For example, when $\pi = \sigma$ one finds n loops, each of length 2, see Figure 12.1. While when $\pi \neq \sigma$ the number of loops is always less than n ($\ell(\pi,\sigma) < n$), see Figure 12.2. At large N, the diagonal of the matrix M_n dominates and the matrix is always invertible. By expanding in powers of $1/N$, we see that its inverse W_n, whose elements are the Weingarten functions, has an N^{-n} behavior

[1] The pseudo-inverse of M is such that $WMW = W$ and $MWM = M$. When M is invertible, $W = M^{-1}$. If M is diagonalizable, the eigenvalues of W are the reciprocal of those of M with the rule $1/0 \to 0$.

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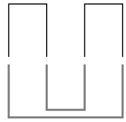

Figure 12.2 Number of loops $\ell(\pi,\sigma)$ for $\pi = (1,4)(2,3)$ and $\sigma = (1,2)(3,4)$. In this example there is only one loop.

on the diagonal and off-diagonal terms are at most N^{-n-1}. More generally, one has the following beautiful expansion:

$$W_n(\pi,\sigma) = N^{\ell(\pi,\sigma)-2n} \sum_{g=0}^{\infty} \Omega_g(\pi,\sigma) N^{-g}, \tag{12.12}$$

where the coefficients $\Omega_g(\pi,\sigma)$ depend on properties of certain "geodesic paths" in the space of partitions. The bottom line of this general expansion formula is that non-Wick contractions are of higher order in N^{-1}.

As an illustration, let us come back to the missing term in Eq. (12.9) when one restricts to Wick contractions, for which the number of loops $\ell(\pi,\pi) = 2$. The next term corresponds to $\ell(\pi,\sigma) = 1$ for which $W_2(\pi,\sigma) \sim -N^{-3}$ (see Exercise 12.1.1). Consider the pairings $i_1 = i_4$, $i_2 = i_3$ and $j_1 = j_2$, $j_3 = j_4$, for which $\ell(\pi,\sigma) = 1$ (see Fig. 12.2). Such pairings do not add any constraint on the $2n = 4$ free indices that appear in Eq. (12.3); summing over them thus yields $\tau(\Lambda_1)\tau(\Lambda_2)\tau(\Lambda_1')\tau(\Lambda_2')$, with a -1 coming from the Weingarten function $W_2(\pi,\sigma)$. Hence we recover the last term of Eq. (12.10).

Exercise 12.1.1 Exact Weingarten functions at $n = 2$

There are three possible pair partitions of four elements. They are shown on Figure 3.2. If we number these π_1,π_2 and π_3, M_2 is a 3×3 matrix whose elements are equal to $N^{\ell(\pi_i,\pi_j)}$.

(a) By trying a few combinations of the three pairings, convince yourself that for $n = 2$, $\ell(\pi_i,\pi_j) = 2$ if $i = j$ and 1 otherwise.

(b) Build the matrix M_2. For $N > 1$ it is invertible, find its inverse W_2. Hint: use an ansatz for W_2 with a variable a on the diagonal and b off-diagonal. For $N = 1$ find the pseudo-inverse of M_2.

(c) Finally show that (when $N > 1$) of the nine pairs of pairings, the three Wick contractions (diagonal elements) have

$$W_2^{\text{Wick}} = \frac{N+1}{N^3 + N^2 - 2N} \overset{N\to\infty}{\to} N^{-2}, \tag{12.13}$$

and the six non-Wick parings (off-diagonal) have

$$W_2^{\text{non-Wick}} = -\frac{1}{N^3 + N^2 - 2N} \overset{N\to\infty}{\to} -N^{-3}. \tag{12.14}$$

For $N = 1$, all nine Weingarten functions are equal to $1/9$.

(d) The expression $\langle \tau(\mathbf{O}\mathbf{O}^T\mathbf{O}\mathbf{O}^T)\rangle$ is always equal to 1. Write it as a sum over four indices and expand the expectation value over orthogonal matrices as nine terms each containing two Dirac deltas multiplied by a Weingarten function. Each sum of delta terms gives a power of N; find these for all nine terms and using your result from (c), show that indeed the Weingarten functions give $\langle \tau(\mathbf{O}\mathbf{O}^T\mathbf{O}\mathbf{O}^T)\rangle = 1$ for all N.

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12.1.4 Freeness of Large Matrices

We are now ready to show that all expectations of the form (12.3) are zero. Let us look at them more closely:

$$\left\langle \tau\left(\Lambda_1 \mathbf{O} \Lambda_1' \mathbf{O}^T \Lambda_2 \mathbf{O} \Lambda_2' \mathbf{O}^T \dots \Lambda_n \mathbf{O} \Lambda_n' \mathbf{O}^T\right)\right\rangle_{\mathbf{O}}$$
$$= \frac{1}{N} \sum_{\mathbf{ij}} I(\mathbf{i},\mathbf{j},n)[\Lambda_1]_{i_{2n} i_1} [\Lambda_2]_{i_2 i_3} \dots [\Lambda_n]_{i_{2n-2} i_{2n-1}} [\Lambda_1']_{j_1 j_2} [\Lambda_2']_{j_3 j_4} \dots [\Lambda_n']_{j_{2n-1} j_{2n}}. \tag{12.15}$$

The object $I(\mathbf{i},\mathbf{j},n)$ contains all possible pairings of the i indices and all those of the j indices with a Weingarten function as its prefactor. The i and j indices never mix. We concentrate on i pairings. Each pairing will give rise to a product of normalized traces of Λ_i matrices. For example, the term

$$\tau(\Lambda_5)\tau(\Lambda_4 \Lambda_1 \Lambda_2)\tau(\Lambda_3 \Lambda_6) \tag{12.16}$$

would appear for $n = 6$. Each normalized trace introduces a factor of N when going from Tr(.) to τ(.). Since by hypothesis $\tau(\Lambda_i) = 0$, the maximum number of non-zero traces is $\lfloor n/2 \rfloor$, e.g.

$$\tau(\Lambda_1 \Lambda_3)\tau(\Lambda_5 \Lambda_6)\tau(\Lambda_2 \Lambda_4). \tag{12.17}$$

The maximum factor of N that can be generated is therefore $N^{\lfloor n/2 \rfloor}$. Applying the same reasoning to the j pairing, and using the fact that the Weingarten function is at most $O(N^{-n})$, we find

$$\left|\left\langle \tau\left(\Lambda_1 \mathbf{O} \Lambda_1' \mathbf{O}^T \Lambda_2 \mathbf{O} \Lambda_2' \mathbf{O}^T \dots \Lambda_n \mathbf{O} \Lambda_n' \mathbf{O}^T\right)\right\rangle_{\mathbf{O}}\right| \le O\left(N^{-1+2\lfloor n/2 \rfloor - n}\right) \overset{N\to\infty}{\to} 0. \tag{12.18}$$

Using the same arguments one can shown that large unitary invariant complex Hermitian random matrices are free. In this case one should consider an integral of unitary matrices in Eq. (12.4). The result is also given by Eq. (12.11) where the functions $W_n(\pi,\sigma)$ are now unitary Weingarten functions. They are different than the orthogonal Weingarten functions presented above but they share an important property in the large N limit, namely

$$W_n(\pi,\sigma) = \begin{cases} O(N^{-n}) \text{ if } \pi = \sigma, \\ O(N^{-n-k}) \text{ if } \pi \neq \sigma \text{ for some } k \ge 1, \end{cases} \tag{12.19}$$

which was the property needed in our proof of freeness of large rotationally invariant symmetric matrices.

12.2 R-Transforms and Resummed Perturbation Theory

In this section, we want to explore yet another route to obtain the additivity of R-transforms, which makes use of perturbation theory and of the mixed cumulant calculus introduced in the last chapter, Section 11.3.6, exploited in a concrete case.

We want to study the average Stieltjes transform of $\mathbf{A} + \mathbf{B}^{\mathrm{R}}$ where $\mathbf{B}^{\mathrm{R}}$ is a randomly rotated matrix $\mathbf{B}$: $\mathbf{B}^{\mathrm{R}} := \mathbf{OBO}^T$. We thus write

$$\mathfrak{g}(z) := \left\langle \tau\left((z\mathbf{1} - \mathbf{A} - \mathbf{B}^{\mathrm{R}})^{-1}\right)\right\rangle_{\mathbf{O}} := \tau_{\mathrm{R}}\left((z\mathbf{1} - \mathbf{A} - \mathbf{B}^{\mathrm{R}})^{-1}\right), \tag{12.20}$$

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where τ_{R} is meant for both the normalized trace τ and the average over the Haar measure of the rotation group. We now formally expand the resolvent in powers of $\mathbf{B}^{\mathrm{R}}$. Introducing $\mathbf{G}_A = (z\mathbf{1} - \mathbf{A})^{-1}$, one has

$$\mathfrak{g}(z) = \tau_{\mathrm{R}}(\mathbf{G}_A) + \tau_{\mathrm{R}}(\mathbf{G}_A\mathbf{B}^{\mathrm{R}}\mathbf{G}_A) + \tau_{\mathrm{R}}(\mathbf{G}_A\mathbf{B}^{\mathrm{R}}\mathbf{G}_A\mathbf{B}^{\mathrm{R}}\mathbf{G}_A) + \cdots . \tag{12.21}$$

Now, since in the large N limit $\mathbf{G}_A$ and $\mathbf{B}^{\mathrm{R}}$ are free, we can use the general tracial formula, Eq. (11.74), noting that all mixed cumulants (containing both $\mathbf{G}_A$ and $\mathbf{B}^{\mathrm{R}}$) are identically zero.

In order to proceed, one needs to introduce three types of mixed moments where $\mathbf{B}^{\mathrm{R}}$ appears exactly n times:

$$m_n^{(1)} := \tau_{\mathrm{R}}(\mathbf{G}_A\mathbf{B}^{\mathrm{R}}\mathbf{G}_A \ldots \mathbf{G}_A\mathbf{B}^{\mathrm{R}}\mathbf{G}_A), \qquad m_n^{(2)} := \tau_{\mathrm{R}}(\mathbf{B}^{\mathrm{R}}\mathbf{G}_A \ldots \mathbf{G}_A\mathbf{B}^{\mathrm{R}}) \tag{12.22}$$

and

$$m_n^{(3)} := \tau_{\mathrm{R}}(\mathbf{B}^{\mathrm{R}}\mathbf{G}_A \ldots \mathbf{B}^{\mathrm{R}}\mathbf{G}_A) = \tau_{\mathrm{R}}(\mathbf{G}_A\mathbf{B}^{\mathrm{R}} \ldots \mathbf{G}_A\mathbf{B}^{\mathrm{R}}). \tag{12.23}$$

Note that $m_0^{(1)} \equiv \mathfrak{g}_A(z)$ and $m_0^{(2)} = m_0^{(3)} = 0$. We also introduce, for full generality, the corresponding generating functions:

$$\tilde{M}^{(a)}(u) = \sum_{n=0}^{\infty} m_n^{(a)} u^n, \qquad a = 1, 2, 3. \tag{12.24}$$

Note however that we will only be interested here in $\mathfrak{g}(z) := \tilde{M}^{(1)}(u = 1)$ (cf. Eq. (12.21)).

Let us compute $m_n^{(1)}$ using the same method as in Section 11.3.5, i.e. expanding in the size ℓ of the group to which the first $\mathbf{G}_A$ belongs (see Eq. (11.71)):

$$m_n^{(1)} = \sum_{\ell=1}^{n+1} \kappa_{G_A,\ell} \prod_{\substack{k_1,k_2,\ldots,k_\ell \geq 0 \\ k_1+k_2+\cdots+k_\ell = n-\ell}} m_{k_1}^{(2)} m_{k_2}^{(2)} \ldots m_{k_{\ell-1}}^{(2)} m_{k_\ell}^{(3)}, \tag{12.25}$$

where $n \geq 1$ and $\kappa_{G_A,\ell}$ are the free cumulants of $\mathbf{G}_A$. Similarly,

$$m_n^{(2)} = \sum_{\ell=1}^{n} \kappa_{B,\ell} \prod_{\substack{k_1,k_2,\ldots,k_\ell \geq 0 \\ k_1+k_2+\cdots+k_\ell = n-\ell}} m_{k_1}^{(1)} m_{k_2}^{(1)} \ldots m_{k_{\ell-1}}^{(1)} m_{k_\ell}^{(3)} \tag{12.26}$$

and

$$m_n^{(3)} = \sum_{\ell=1}^{n+1} \kappa_{B,\ell} \prod_{\substack{k_1,k_2,\ldots,k_\ell \geq 0 \\ k_1+k_2+\cdots+k_\ell = n-\ell}} m_{k_1}^{(1)} m_{k_2}^{(1)} \ldots m_{k_\ell}^{(1)}, \tag{12.27}$$

where $\kappa_{B,\ell}$ are the free cumulants of $\mathbf{B}$. Multiplying both sides of these equations by u^n and summing over n leads to, respectively,

$$\tilde{M}^{(1)}(u) = \mathfrak{g}_A(z) + \sum_{\ell=1}^{\infty} \kappa_{G_A,\ell} u^\ell [\tilde{M}^{(2)}(u)]^{\ell-1} \tilde{M}^{(3)}(u), \tag{12.28}$$

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and

$$\tilde{M}^{(2)}(u) = \sum_{\ell=1}^{\infty} \kappa_{B,\ell} u^{\ell} [\tilde{M}^{(1)}(u)]^{\ell-1} \tilde{M}^{(3)}(u), \qquad \tilde{M}^{(3)}(u) = \sum_{\ell=1}^{\infty} \kappa_{B,\ell} u^{\ell} [\tilde{M}^{(1)}(u)]^{\ell}. \tag{12.29}$$

Recalling the definition of R-transforms as a power series of cumulants, we thus get

$$\tilde{M}^{(1)}(u) = \mathfrak{g}_A(z) + u\tilde{M}^{(3)}(u) R_{G_A}\left(u\tilde{M}^{(2)}(u)\right) \tag{12.30}$$

and

$$\tilde{M}^{(2)}(u) = u\tilde{M}^{(3)}(u) R_B\left(u\tilde{M}^{(1)}(u)\right), \qquad \tilde{M}^{(3)}(u) = u\tilde{M}^{(1)}(u) R_B\left(u\tilde{M}^{(1)}(u)\right). \tag{12.31}$$

Eliminating $\tilde{M}^{(2)}(u)$ and $\tilde{M}^{(3)}(u)$ and setting $u = 1$ then yields the following relation:

$$\mathfrak{g}(z) = \mathfrak{g}_A(z) + \mathfrak{g}(z)\, R_B(\mathfrak{g}(z))\, R_{G_A}\left(\mathfrak{g}(z) R_B^2(\mathfrak{g}(z))\right). \tag{12.32}$$

In order to rewrite this result in more familiar terms, let us consider the case where $\mathbf{B} = b\mathbf{1}$, in which case $R_B(z) \equiv b$ and, since $\mathbf{A}+\mathbf{B} = \mathbf{A}+b\mathbf{1}$, $\mathfrak{g}(z) \equiv \mathfrak{g}_A(z-b)$. Hence, for arbitrary b, R_{G_A} must obey the relation

$$R_{G_A}(b^2\mathfrak{g}(z)) = \frac{\mathfrak{g}(z) - \mathfrak{g}(z+b)}{b\mathfrak{g}(z)}. \tag{12.33}$$

Now, if for a fixed z we set $b = R_B(\mathfrak{g}(z))$, we find that Eq. (12.32) is obeyed provided $\mathfrak{g}(z) = \mathfrak{g}_A(z - R_B(\mathfrak{g}(z)))$, i.e. precisely the subordination relation Eq. (11.80).

12.3 The Central Limit Theorem for Matrices

In the last chapter, we briefly discussed the extension of the CLT for non-commuting variables. We now restate the result in the context of random matrices, with a special focus on the preasymptotic (cumulant) corrections to the Wigner distribution.

Let us consider the following sum of K large, randomly rotated matrices, all assumed to be traceless:

$$\mathbf{M}_K := \frac{1}{\sqrt{K}} \sum_{i=1}^{K} \mathbf{O}_i \mathbf{A}_i \mathbf{O}_i^T, \qquad \operatorname{Tr} \mathbf{A}_i = 0, \tag{12.34}$$

where $\mathbf{O}_i$ are independent, random rotation matrices, chosen with a flat measure over the orthogonal group $O(N)$. We also assume, for simplicity, that all $\mathbf{A}_i$ have the same (arbitrary) moments:

$$\tau(\mathbf{A}_i^{\ell}) \equiv m_{\ell}, \qquad \forall i. \tag{12.35}$$

This means in particular that all $\mathbf{A}_i$'s share the same R-transform:

$$R_{\mathbf{A}_i}(z) \equiv \sum_{\ell=2}^{\infty} \kappa_{\ell} z^{\ell-1}. \tag{12.36}$$

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Using the fact that R-transforms of randomly rotated matrices simply add, together with $R_{\alpha\mathbf{M}}(z) = R_{\mathbf{M}}(\alpha z)$, one finds

$$R_{\mathbf{M}_K}(z) = \sum_{\ell=2}^{\infty} K^{1-\ell/2}\kappa_\ell z^{\ell-1}. \tag{12.37}$$

We thus see that, as K becomes large, all free cumulants except the second one tend to zero, which implies that the limit of $\mathbf{M}_K$ when K goes to infinity is a Wigner matrix, with a semi-circle eigenvalue spectrum.

It is interesting to study the finite K corrections to this result. First, assume that the spectrum of $\mathbf{A}_i$ is not symmetric around zero, such that the skewness $m_3 = \kappa_3 \neq 0$. For large K, the R-transform of $\mathbf{M}_K$ can be approximated as

$$R_{\mathbf{M}_K}(z) \approx \sigma^2 z + \frac{\kappa_3}{\sqrt{K}} z^2 + \cdots . \tag{12.38}$$

In order to derive the corrections to the semi-circle induced by skewness, we posit that the Stieltjes transform can be expanded around the Wigner result $\mathfrak{g}_{\mathbf{X}}(z)$ as

$$\mathfrak{g}_{\mathbf{M}_K}(z) = \mathfrak{g}_{\mathbf{X}}(z) + \frac{\kappa_3}{\sqrt{K}}\mathfrak{g}_3(z) + \cdots \tag{12.39}$$

and assume the second term to be very small. The R-transform, $R_{\mathbf{M}_K}(z)$, can be similarly expanded, yielding

$$R_{\mathbf{X}}\left(\mathfrak{g}_{\mathbf{X}}(z) + \frac{\kappa_3}{\sqrt{K}}\mathfrak{g}_3(z)\right) + \frac{\kappa_3}{\sqrt{K}} R_3(\mathfrak{g}_{\mathbf{X}}(z)) = z \Rightarrow R_3(\mathfrak{g}_{\mathbf{X}}(z)) = -\frac{\mathfrak{g}_3(z)}{\mathfrak{g}'_{\mathbf{X}}(z)} \tag{12.40}$$

or, equivalently,

$$\mathfrak{g}_3(z) = -\mathfrak{g}'_{\mathbf{X}}(z) R_3(\mathfrak{g}_{\mathbf{X}}(z)) = -\mathfrak{g}'_{\mathbf{X}}(z)\mathfrak{g}^2_{\mathbf{X}}(z). \tag{12.41}$$

For simplicity, we normalize the Wigner semi-circle such that $\sigma^2 = 1$, and hence

$$\mathfrak{g}_{\mathbf{X}}(z) = \frac{1}{2}\left(z - \sqrt[\oplus]{\Delta}\right); \qquad \Delta := z^2 - 4. \tag{12.42}$$

One then finds

$$\mathfrak{g}_3(z) = -\frac{1}{4}\left(1 - \frac{z\sqrt[\oplus]{\Delta}}{\Delta}\right)\left(z^2 - 2 - z\sqrt[\oplus]{\Delta}\right). \tag{12.43}$$

The imaginary part of this expression when $z \to \lambda + \mathrm{i}0$ gives the correction to the semi-circle eigenvalue spectrum, and reads, for $\lambda \in [-2, 2]$,

$$\delta\rho(\lambda) = \frac{\kappa_3}{2\pi\sqrt{K}} \frac{\lambda(\lambda^2 - 3)}{\sqrt{4 - \lambda^2}}. \tag{12.44}$$

This correction is plotted in Figure 12.3. Note that it is odd in λ, as expected, and diverges near the edges of the spectrum, around which the above perturbation approach breaks down

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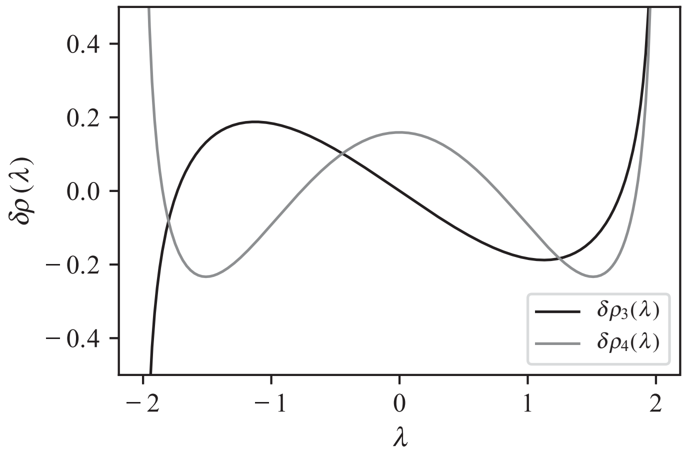

Figure 12.3 First order correction to the Wigner density of eigenvalues for non-zero skewness ($\delta\rho_3(\lambda)$) and kurtosis ($\delta\rho_4(\lambda)$), Eqs. (12.44) and (12.47), respectively.

(because the density of eigenvalues of the Wigner matrix goes to zero). Note that the excess skewness is computed to be

$$\int_{-2}^{2} \lambda^3 \delta\rho(\lambda) d\lambda = \frac{\kappa_3}{\sqrt{K}}, \tag{12.45}$$

as expected.

One can obtain the corresponding correction when $\mathbf{A}_i$ is symmetric around zero, in which case the first correction term comes from the kurtosis. For large K, the R-transform of $\mathbf{M}_K$ can now be approximated as

$$R_{\mathbf{M}_K}(z) \approx \sigma^2 z + \frac{\kappa_4}{K} z^3 + \cdots . \tag{12.46}$$

Following the same path as above, one finally derives the correction to the eigenvalue spectrum, which in this case reads, for $\lambda \in [-2, 2]$,

$$\delta\rho(\lambda) = \frac{\kappa_4}{2\pi K} \frac{\lambda^4 - 4\lambda^2 + 2}{\sqrt{4 - \lambda^2}} \tag{12.47}$$

(see Fig. 12.3). The correction is now even in λ and one can check that

$$\int_{-2}^{2} \delta\rho(\lambda) d\lambda = 0, \tag{12.48}$$

as it should be, since all the mass is carried by the semi-circle. Another check that this result is correct is to compute the excess free kurtosis, given by

$$\int_{-2}^{2} (\lambda^4 - 2\lambda^2) \delta\rho(\lambda) d\lambda = \frac{\kappa_4}{K}, \tag{12.49}$$

again as expected.

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These corrections to the free CLT are the analog of the Edgeworth series for the classical CLT. In full generality, the contribution of the nth cumulant to $\delta\rho(\lambda)$ reads

$$\delta\rho_n(\lambda) = \frac{\kappa_n}{\pi K^{n/2-1}} \frac{T_n\left(\frac{\lambda}{2}\right)}{\sqrt{4-\lambda^2}}, \tag{12.50}$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind.

12.4 Finite Free Convolutions

We saw that rotationally invariant random matrices become asymptotically free as their size goes to infinity. At finite N freeness is not exact, except in very special cases (see Section 12.5). In this section we will discuss operations on polynomials that are analogous to the free addition and multiplication but unfortunately lack the full set of properties of freeness as defined in Chapter 11. When the polynomials are thought of as expected characteristic polynomials of large matrices, finite free addition and multiplication do indeed converge to the free addition and multiplication when the size of the matrix (degree of the polynomial) goes to infinity.

12.4.1 Notations: Roots and Coefficients

Let $p(z)$ be a polynomial of degree N. By the fundamental theorem of algebra, this polynomial will have exactly N roots (when counted with their multiplicity) and can be written as

$$p(z) = a_0 \prod_{i=1}^{N} (z - \lambda_i). \tag{12.51}$$

If $a_0 = 1$ we say that $p(z)$ is monic and if all the λ_i's are real the polynomial is called *real-rooted*. In this section we will only consider real-rooted monic polynomials. Such a polynomial can always be viewed as the characteristic polynomial of the diagonal matrix Λ containing its roots, i.e.

$$p(z) = \det(z\mathbf{1} - \Lambda). \tag{12.52}$$

We can expand the product (12.51) as

$$p(z) = \sum_{k=0}^{N} (-1)^k a_k z^{N-k}. \tag{12.53}$$

Note that we have defined the coefficient a_k as the coefficient of z^{N-k} and *not* that of z^k and that we have included alternating signs $(-1)^k$ in its definition. The reason is that we want a simple link between the coefficients a_k and the roots λ_i. We have

$$a_k = \sum_{\substack{\text{ordered} \\ k\text{-tuples } \mathbf{i}}}^{N} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}, \tag{12.54}$$

$$a_0 = 1, \quad a_1 = \sum_{i=1}^{N} \lambda_i, \quad a_2 = \sum_{\substack{i=1 \\ j=i+1}}^{N} \lambda_i\lambda_j, \quad \ldots, \quad a_N = \prod_{i=1}^{N} \lambda_i. \tag{12.55}$$

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Note that the coefficient a_k is homogeneous to λ^k. From the coefficients a_k we can compute the sample moments[2] of the set of λ_i's. In particular we have

$$\mu(\{\lambda_i\}) = \frac{a_1}{N} \quad \text{and} \quad \sigma^2(\{\lambda_i\}) = \frac{a_1^2}{N} - \frac{2a_2}{N-1}. \tag{12.56}$$

The polynomial $p(z)$ will often be the expected characteristic polynomial of some random matrix $\mathbf{M}$ of size N with joint distribution of eigenvalues $P(\{\mu_i\})$. The coefficients a_i are then multi-linear moments of this joint distribution. If the random eigenvalues μ_i do not fluctuate, we have $\lambda_i = \mu_i$ but in general we should think of the λ_i's as deterministic numbers fixed by the random ensemble considered. The case of independent eigenvalues gives a trivial expected characteristic polynomial, i.e. $p(z) = (z - \mathbb{E}(\mu))^N$ or else $\lambda_i = \mathbb{E}(\mu)\ \forall i$.

Shifts and scaling of the matrix $\mathbf{M}$ can be mapped onto operations on the polynomial $p_{\mathbf{M}}(z)$. For a shift, we have

$$p_{\mathbf{M}+\alpha\mathbf{1}}(z) = p_{\mathbf{M}}(z-\alpha). \tag{12.57}$$

Multiplication by a scalar gives

$$p_{\alpha\mathbf{M}}(z) = \alpha^N p_{\mathbf{M}}(\alpha^{-1}z) \Longleftrightarrow a_k = \alpha^k a_k. \tag{12.58}$$

Finally there is a formula for matrix inversion which is only valid when the eigenvalues are deterministic (e.g. $\mathbf{M} = \mathbf{O}\Lambda\mathbf{O}^T$ with fixed Λ):

$$p_{\mathbf{M}^{-1}}(z) = \frac{z^N}{a_N} p_{\mathbf{M}}(1/z) \Longleftrightarrow a_k = \frac{a_{N-k}}{a_N}. \tag{12.59}$$

A degree N polynomial can always be written as a degree N polynomial of the derivative operator acting on the monomial z^N. We introduce the notation $\hat{p}$ as

$$p(z) =: \hat{p}\,(\mathrm{d}_z)\, z^N \quad \text{the coefficients of } \hat{p} \text{ are} \quad \hat{a}_k = (-1)^N \frac{k!}{N!} a_{N-k}, \tag{12.60}$$

where d_z is a shorthand notation for $\mathrm{d}/\mathrm{d}z$. It will prove useful to compute finite free convolutions.

12.4.2 Finite Free Addition

The equivalent of the free addition for two monic polynomials $p_1(x)$ and $p_2(x)$ of the same degree N is the finite free additive convolution defined as

$$p_1 \boxplus p_2(z) = \left\langle \det[z\mathbf{1} - \Lambda_1 - \mathbf{O}\Lambda_2\mathbf{O}^T] \right\rangle_{\mathbf{O}}, \tag{12.61}$$

where the diagonal matrices $\Lambda_{1,2}$ contain the roots of $p_{1,2}(z)$ all assumed to be real. The averaging over the orthogonal matrix $\mathbf{O}$ is as usual done over the flat (Haar) measure. We could have chosen to integrate $\mathbf{O}$ over unitary matrices or even permutation matrices; the final result would be the same.

As we will show in Section 12.4.5, the additive convolution can be expressed in a very concise form using the polynomials $\hat{p}_{1,2}(x)$:

$$p_1 \boxplus p_2(z) = \hat{p}_1\,(\mathrm{d}_z)\, p_2(z) = \hat{p}_2\,(\mathrm{d}_z)\, p_1(z) = \hat{p}_1\,(\mathrm{d}_z)\, \hat{p}_2\,(\mathrm{d}_z)\, z^N. \tag{12.62}$$

[2] Here we have chosen to normalize the sample variance with a factor $(N-1)^{-1}$. It may seem odd to use the formula suited to when the mean is unknown for computing the variance of deterministic numbers, but this definition will later match that of the finite free cumulant and give the Hermite polynomial the variance of the corresponding Wigner matrix.

It is easy to see that when $p_s(z) = p_1 \boxplus p_2(z)$, $p_s(z)$ is again a monic polynomial of degree N. What is less obvious, but true, is that $p_s(x)$ is also real-rooted. A proof of this is beyond the scope of this book. The additive convolution is bilinear in the coefficients of $p_1(z)$ and $p_2(z)$, which means that the operation commutes with the expectation value. If $p_{1,2}(z)$ are independent random polynomials (for example, characteristic polynomials of independent random matrices) we have a relation for their expected value:

$$\mathbb{E}[p_1 \boxplus p_2(z)] = \mathbb{E}[p_1(z)] \boxplus \mathbb{E}[p_2(z)]. \tag{12.63}$$

The finite free addition can also be written as a relation between the coefficients $a_k^{(s)}$ of $p_s(z)$ and those of $p_{1,2}(z)$:

$$a_k^{(s)} = \sum_{i+j=k} \frac{(N-i)!\,(N-j)!}{N!\,(N-k)!} a_i^{(1)} a_j^{(2)}. \tag{12.64}$$

More explicitly, the first three coefficients are given by

$$\begin{aligned} a_0^{(s)} &= 1, \\ a_1^{(s)} &= a_1^{(1)} + a_1^{(2)}, \\ a_2^{(s)} &= a_2^{(1)} + a_2^{(2)} + \frac{N-1}{N} a_1^{(2)} a_1^{(2)}. \end{aligned} \tag{12.65}$$

From which we can verify that both the sample mean and the variance (Eq. (12.56)) are additive under the finite free addition.

If we call $p_\mu(z) = (z-\mu)^N$ the polynomial with a single root μ so $p_0(z) = z^N$ is the trivial monic polynomial, we have that, under additive convolution with any $p(z)$, $p_0(z)$ acts as a null element and $p_\mu(z)$ acts as a shift:

$$p \boxplus p_0(z) = p(z) \quad \text{and} \quad p \boxplus p_\mu(z) = p(z-\mu). \tag{12.66}$$

Hermite polynomials are stable under this addition:

$$H_N \boxplus H_N(z) = 2^{N/2} H_N(2^{-N/2} z), \tag{12.67}$$

where the factors $2^{N/2}$ compensate the doubling of the sample variance.

The average characteristic polynomials of Wishart matrices can easily be understood in terms the finite free sum. Consider an N-dimensional rank-1 matrix $\mathbf{M} = \mathbf{x}\mathbf{x}^T$ where $\mathbf{x}$ is a vector of IID unit variance numbers. It has one eigenvalue equal to N (on average) and all others are zero, so its average characteristic polynomial is

$$p(z) = z^{N-1}(z-N) = (1-\mathrm{d}_z)\, z^N, \tag{12.68}$$

from which we read that $\hat{p}(\mathrm{d}_z) = (1-\mathrm{d}_z)$. But since an unnormalized Wishart matrix of parameter T is just the free sum of T such projectors, Eq. (12.62) immediately leads to $p_T(z) = (1-\mathrm{d}_z)^T z^N$, which coincides with Eq. (6.41).

12.4.3 A Finite R-Transform

If we look back at Eq. (12.62), we notice that the polynomial $\hat{p}(\mathrm{d}_z)$ behaves like the Fourier transform under free addition. Its logarithm is therefore additive. We need to be a bit careful of what we mean by the logarithm of a polynomial. The function $\hat{p}(\mathrm{d}_z)$ has two important properties: first as a power series in d_z it always starts $1 + O(\mathrm{d}_z)$; second it is defined by its action on Nth order polynomials in z, so only its $N+1$ first terms in its Taylor series matter. We will say that the polynomial is defined modulo d_z^{N+1}, meaning

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that higher order terms are set to zero. When we take the logarithm of $\hat{p}(u)$ we thus mean: apply the Taylor series of the logarithm around 1 and expand up to the power u^N. The resulting function

$$L(u) := -\left[\log \hat{p}(u)\right] \mod u^{N+1} \tag{12.69}$$

is then additive. For average characteristic polynomials of random matrices, it should be related to the R-transform in the large N limit.

Let us examine more closely $L(u)$ in three simple cases: identity, Wigner and Wishart.

- For the identity matrix of size N we have

$$p_{\mathbf{1}}(z) = (z-1)^N = \sum_{k=0}^{N} \binom{N}{k} (-1)^k z^{N-k} = \sum_{k=0}^{N} \frac{(-\mathrm{d}_z)^k}{k!} z^N = \exp(-\mathrm{d}_z) z^N. \tag{12.70}$$

So we find

$$\hat{p}_{\mathbf{1}}(u) = \left[\exp(-u)\right] \mod u^{N+1} \quad \Rightarrow \quad L_{\mathbf{1}}(u) = u. \tag{12.71}$$

- For Wigner matrices, we saw in Chapter 6 that the expected characteristic polynomial of a unit Wigner matrix is given by a Hermite polynomial normalized as

$$p_{\mathbf{X}}(z) = N^{-N/2} H_N(\sqrt{N} z) = \exp\left[-\frac{1}{2N}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2\right] z^N, \tag{12.72}$$

where the right hand side comes from Eq. (6.8). We then have

$$\hat{p}_{\mathbf{X}}(u) = \left[\exp\left(-\frac{u^2}{2N}\right)\right] \mod u^{N+1} \quad \Rightarrow \quad L_{\mathbf{X}}(u) = \frac{u^2}{2N}. \tag{12.73}$$

- For Wishart matrices, Eq. (6.41) expresses the expected characteristic polynomial as a derivative operator acting on the monomial z^N. For a correctly normalized Wishart matrix, we then find the monic Laguerre polynomial:

$$p_{\mathbf{W}}(z) = \left(1 - \frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}z}\right)^T z^N, \tag{12.74}$$

from which we can immediately read off the polynomial $\hat{p}(z)$:

$$\begin{aligned} \hat{p}_{\mathbf{W}}(u) &= \left[\left(1 - \frac{qu}{N}\right)^{N/q}\right] \mod u^{N+1} \\ \Rightarrow L_{\mathbf{W}}(u) &= -\frac{N}{q}\left[\log\left(1 - \frac{qu}{N}\right)\right] \mod u^{N+1}. \end{aligned} \tag{12.75}$$

We notice that in these three cases, the L function is related to the corresponding limiting R-transform of the infinite size matrices as

$$L'(u) = \left[R(u/N)\right] \mod u^{N+1}. \tag{12.76}$$

The equality at finite N holds for these simple cases but not in the general case. But the equality holds in general in the limiting case:

$$\lim_{N \to \infty} L'(Nx) = R(x). \tag{12.77}$$

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12.4.4 Finite Free Product

The free product can also be generalized to an operation on a real-rooted monic polynomial of the same degree. We define

$$p_1 \boxtimes p_2(z) = \left\langle \det[z\mathbf{1} - \Lambda_1 \mathbf{O} \Lambda_2 \mathbf{O}^T] \right\rangle_{\mathbf{O}}, \tag{12.78}$$

where as usual the diagonal matrices $\Lambda_{1,2}$ contain the roots of $p_{1,2}(z)$. As in the additive case, averaging the matrix $\mathbf{O}$ over the permutation, orthogonal or unitary group gives the same result.

We will show in Section 12.4.5 that the result of the finite free product has a simple expression in terms of the coefficient a_k defined by (12.53). When $p_{\mathrm{m}}(z) = p_1 \boxtimes p_2(z)$ we have

$$a_k^{(\mathrm{m})} = \left[\binom{N}{k}\right]^{-1} a_k^{(1)} a_k^{(2)}. \tag{12.79}$$

Note that if $\Lambda_1 = \alpha \mathbf{1}$ is a multiple of the identity with $\alpha \neq 0$ we have

$$p_{\alpha\mathbf{1}} = (z - \alpha)^N \quad \Rightarrow \quad a_k^{(\alpha\mathbf{1})} = \binom{N}{k} \alpha^k. \tag{12.80}$$

Plugging this into Eq. (12.79), we see that the free product with a multiple of the identity multiplies each a_k by α^k which is equivalent to multiplying each root by α. In particular the identity matrix ($\alpha = 1$) is the neutral element for that convolution.

When $p_{\mathrm{m}}(z) = p_1 \boxtimes p_2(z)$, the sample mean of the roots of $p_{\mathrm{m}}(z)$ (Eq. (12.56)) behaves as

$$\mu_{(\mathrm{m})} = \mu_{(1)} \mu_{(2)}.$$

When both means are unity, we have for the sample variance

$$\sigma^2_{(\mathrm{m})} = \sigma^2_{(1)} + \sigma^2_{(2)} - \frac{\sigma^2_{(1)} \sigma^2_{(2)}}{N}. \tag{12.81}$$

At large N the last term becomes negligible and we recover the additivity of the variance for the product of unit-trace free variables (see Eq. (11.85)).

Exercise 12.4.1 Free product of polynomials with roots 0 and 1

Consider an even degree $N = 2M$ polynomial with roots 0 and 1 both with multiplicity M: $p(z) = z^M (z-1)^M$. We will study the finite free product of this polynomial with itself $p_{\mathrm{m}}(z) = p \boxtimes p(z)$. We will study the large N limit of this problem in Section 15.4.2.

(a) Expand the polynomial and write the coefficients a_k in terms of binomial coefficients.

(b) Using Eq. (12.79) show that the coefficients of $p_{\mathrm{m}}(z)$ are given by

$$a_k^{(\mathrm{m})} = \begin{cases} \binom{M}{k}^2 \left[\binom{N}{k}\right]^{-1} & \text{when } k \leq M, \\ 0 & \text{otherwise.} \end{cases} \tag{12.82}$$

(c) The polynomial $p_{\mathrm{m}}(z)$ always has zero as a root. What is its multiplicity?

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(d) The degree M polynomial $q(z) = z^{-M} p_{\mathrm{m}}(z)$ only has non-zero roots. What is its average root?

(e) For $M = 2$, show that the two roots of $q(z)$ are $1/2 \pm 1/\sqrt{12}$.

(f) The case $M = 4$ can be solved by noticing that $q(z)$ is a symmetric function of $(z - 1/2)$. Show that the four roots are given by

$$\lambda_{\pm\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{15 \pm 2\sqrt{30}}{35}}. \tag{12.83}$$

12.4.5 Finite Free Convolutions: Derivation of the Results

We will first study the definition (12.61) when the matrix $\mathbf{O}$ is averaged over the permutation group S_N. In this case we can write

$$p_{\mathrm{s}}(z) := p_1 \boxplus p_2(z) = \frac{1}{N!} \sum_{\substack{\text{permutations} \\ \sigma}} \prod_{i=1}^{N} \left(z - \lambda_i^{(1)} - \lambda_{\sigma(i)}^{(2)} \right). \tag{12.84}$$

The coefficients $a_k^{(\mathrm{s})}$ are given by the average over the permutation group of the coefficients of the polynomial with roots $\{\lambda_i^{(1)} + \lambda_{\sigma(i)}^{(2)}\}$. For example for $a_1^{(\mathrm{s})}$ we have

$$\begin{aligned} a_1^{(\mathrm{s})} &= \frac{1}{N!} \sum_{\substack{\text{permutations} \\ \sigma}} \sum_{i}^{N} \left(\lambda_i^{(1)} + \lambda_{\sigma(i)}^{(2)} \right) \\ &= \sum_{i}^{N} \left(\lambda_i^{(1)} + \lambda_i^{(2)} \right) = a_1^{(1)} + a_1^{(2)}. \end{aligned} \tag{12.85}$$

For the other coefficients a_k, the combinatorics is a bit more involved. Let us first figure out the structure of the result. For each permutation we can expand the product

$$\left(z - \lambda_1^{(1)} - \lambda_{\sigma(1)}^{(2)} \right) \left(z - \lambda_2^{(1)} - \lambda_{\sigma(2)}^{(2)} \right) \cdots \left(z - \lambda_N^{(1)} - \lambda_{\sigma(N)}^{(2)} \right). \tag{12.86}$$

To get a contribution to a_k we need to choose the variable z $(N - k)$ times, we can then choose a $\lambda^{(1)}$ i times and a $\lambda^{(2)}$ $(k - i)$ times. For a given k and for each choice of i, once averaged over all permutations, the product of the $\lambda^{(1)}$ and that of the $\lambda^{(2)}$ must both be completely symmetric and therefore proportional to $a_i^{(1)} a_{k-i}^{(2)}$. We thus have

$$a_k^{(\mathrm{s})} = \sum_{i=0}^{k} C(i,k,N) a_i^{(1)} a_{k-i}^{(2)}, \tag{12.87}$$

where $C(i,k,N)$ are combinatorial coefficients that we still need to determine. There is an easy way to get these coefficients: we can use a case that we can compute directly and match the coefficients. If $\Lambda_2 = \mathbf{1}$, the identity matrix, we have

$$p_{\mathbf{1}}(z) = (z-1)^N \quad \Rightarrow \quad a_k^{(\mathbf{1})} = \binom{N}{k}. \tag{12.88}$$

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For a generic polynomial $p(z)$, the free sum with $p_{\mathbf{1}}(z)$ is given by a simple shift in the argument z by 1:

$$p_{\rm s}(z) = p(z-1) = \sum_{k=0}^{N} (-1)^k a_k^{(1)} (z-1)^{N-k} \quad \Rightarrow \quad a_k^{(\rm s)} = \sum_{i=0}^{k} \binom{N-i}{N-k} a_i^{(1)}. \tag{12.89}$$

Combining Eqs. (12.87) and (12.88) and matching the coefficient to (12.89), we arrive at

$$a_k^{(\rm s)} = \sum_{i=0}^{k} \frac{(N-i)!\,(N-k+i)!}{N!\,(N-k)!} a_i^{(1)} a_{k-i}^{(2)}, \tag{12.90}$$

which is equivalent to Eq. (12.64). For the equivalence with Eq. (12.62) see Exercise 12.4.2.

Now suppose we want to average Eq. (12.61) with respect to the orthogonal or unitary group (O(N) or U(N)). For a given rotation matrix $\mathbf{O}$, we can expand the determinant, keeping track of powers of z and of the various $\lambda^{(1)}$ that appear in products containing at most one of each $\lambda_i^{(1)}$. After averaging over the group, the combinations of $\lambda^{(1)}$ must be permutation invariant, i.e. proportional to $a_i^{(1)}$; we then get for the coefficient $a_k^{(\rm s)}$,

$$a_k^{(\rm s)} = \sum_{i=0}^{k} C\left(i, k, N, \left\{\lambda_j^{(2)}\right\}\right) a_i^{(1)}, \tag{12.91}$$

where the coefficients $C\left(i, k, N, \left\{\lambda_j^{(2)}\right\}\right)$ depend on the roots $\lambda_j^{(2)}$. By dimensional analysis, they must be homogeneous to $(\lambda^{(2)})^{k-i}$. Since the expression must be symmetrical in (1) $\leftrightarrow$ (2), it must be of the form (12.87). And since the free addition with the unit matrix is the same for all three groups, Eq. (12.64) must be true in all three cases (S_N, O(N) and U(N)).

We now turn to the proof of Eq. (12.79) for the finite free product. Consider Eq. (12.78), where the matrix $\mathbf{O}$ is averaged over the permutation group S_N. For $p_{\rm m}(z) = p_1 \boxtimes p_2(z)$ we have

$$p_{\rm m}(z) = \frac{1}{N!} \sum_{\substack{\text{permutations} \\ \sigma}} \prod_{i=1}^{N} \left(z - \lambda_i^{(1)} \lambda_{\sigma(i)}^{(2)}\right) := \frac{1}{N!} \sum_{\substack{\text{permutations} \\ \sigma}} p_\sigma(z). \tag{12.92}$$

For a given permutation σ, the coefficients a_k^σ are given by

$$a_k^\sigma = \sum_{\substack{\text{ordered} \\ k\text{-tuples } \mathbf{i}}}^{N} \lambda_{i_1}^{(1)} \lambda_{i_2}^{(1)} \cdots \lambda_{i_k}^{(1)} \lambda_{\sigma(i_1)}^{(2)} \lambda_{\sigma(i_2)}^{(2)} \cdots \lambda_{\sigma(i_k)}^{(2)}. \tag{12.93}$$

After averaging over the permutations σ, we must have that $a_k^{(\rm s)} \propto a_k^{(1)} a_k^{(2)}$. By counting the number of terms in the sum defining each a_k, we realize that the proportionality constant must be one over this number. We then have

$$a_k^{(\rm m)} = \left[\binom{N}{k}\right]^{-1} a_k^{(1)} a_k^{(2)}. \tag{12.94}$$

We could have derived the proportionality constant by requiring that the polynomial $p_{\mathbf{1}}(z) = (z-1)^N$ is the neutral element of this convolution.

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Exercise 12.4.2 Equivalence of finite free addition formulas

For a real-rooted monic polynomial $p_1(z)$ of degree N, we define the polynomial $\hat{p}_1(u)$ as

$$\hat{p}_1(u) = \sum_{k=0}^{N} \frac{(N-k)!}{N!} a_k^{(1)} (-1)^k u^k, \tag{12.95}$$

where the coefficients $a_k^{(1)}$ are given by Eq. (12.53).

(a) Show that

$$p_1(z) = \hat{p}_1\left(\frac{\mathrm{d}}{\mathrm{d}z}\right) z^N. \tag{12.96}$$

(b) For another polynomial $p_2(z)$, show that

$$\hat{p}_1\left(\frac{\mathrm{d}}{\mathrm{d}z}\right) p_2(z) = \sum_{k=0}^{N} (-1)^k \sum_{i=0}^{k} \frac{(N-i)!\,(N-k+i)!}{N!\,(N-k)!} a_i^{(1)} a_{k-i}^{(2)} z^{N-k}, \tag{12.97}$$

which shows the equivalence of Eqs. (12.62) and (12.64).

12.5 Freeness for 2 × 2 Matrices

Certain low-dimensional random matrices can be free. For $N = 1$ (1×1 matrices) all matrices commute and thus behave as classical random numbers. As mentioned in Chapter 11, freeness is trivial for commuting variables as only constants (deterministic variables) can be free with respect to non-constant random variables.

For $N = 2$, there exist non-trivial matrices that can be mutually free. To be more precise, we consider the space of 2×2 symmetric random matrices and define the operator[3]

$$\tau(\mathbf{A}) = \frac{1}{2}\mathbb{E}[\operatorname{Tr}\mathbf{A}]. \tag{12.98}$$

We now consider matrices that have deterministic eigenvalues but random, rotationally invariant eigenvectors. We will see that any two such matrices are free. Since 2×2 matrices only have two eigenvalues we can write these matrices as

$$\mathbf{A} = a\mathbf{1} + \sigma\,\mathbf{O}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{O}^T, \tag{12.99}$$

[3] More formally, we need to consider 2×2 symmetric random matrices with finite moments to all orders. This space is closed under addition and multiplication and forms a ring satisfying all the axioms described in Section 11.1.

where a is the mean of the two eigenvalues and σ their half-difference. The matrix $\mathbf{O}$ is a random rotation matrix which for $N = 2$ only has one degree of freedom and can always be written as

$$\mathbf{O} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{12.100}$$

where the angle θ is uniformly distributed in $[0, 2\pi]$. Note again that we are considering matrices for which a and σ are non-random. If we perform the matrix multiplications and use some standard trigonometric identities we find

$$\mathbf{A} = a\mathbf{1} + \sigma \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{12.101}$$

12.5.1 Freeness of Matrices with Deterministic Eigenvalues

We can now show that any two such matrices $\mathbf{A}$ and $\mathbf{B}$ are free. If we put ourselves in the basis where $\mathbf{A}$ is diagonal, we see that traceless polynomials $p_k(\mathbf{A})$ and $q_k(\mathbf{B})$ are necessarily of the form

$$p_k(\mathbf{A}) = a_k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad q_k(\mathbf{B}) = b_k \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \tag{12.102}$$

for some deterministic numbers a_k and b_k and where θ is the random angle between the eigenvectors of $\mathbf{A}$ and $\mathbf{B}$. We can now compute the expectation value of the trace of products of such polynomials:

$$\tau\left[\prod_{k=1}^{n} p_k(\mathbf{A})q_k(\mathbf{B})\right] = \frac{1}{2}\left(\prod_{k=1}^{n} a_k b_k\right) \mathbb{E}\,\mathrm{Tr}\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}^n. \tag{12.103}$$

We notice that the matrix on the right hand side is a rotation matrix (of angle 2θ) raised to the power n and therefore itself a rotation matrix (of angle $2n\theta$). The average of such a matrix is zero, thus finishing our proof that two rotationally invariant 2×2 matrices with deterministic eigenvalues are free.

As a consequence, we can use the R-transform to compute the average eigenvalue spectrum of $\mathbf{A} + \mathbf{O}\mathbf{B}\mathbf{O}^T$ when $\mathbf{A}$ and $\mathbf{B}$ have deterministic eigenvalues and $\mathbf{O}$ is a random rotation matrix. In particular if $\mathbf{A}$ and $\mathbf{B}$ have the same variance (σ) then this spectrum is given by the arcsine law that we encountered in Section 7.1.3. For positive definite $\mathbf{A}$ we can also compute the spectrum of $\sqrt{\mathbf{A}}\mathbf{O}\mathbf{B}\mathbf{O}^T\sqrt{\mathbf{A}}$ using the S-transform (see Exercises 12.5.1 and 12.5.2).

Exercise 12.5.1 Sum of two free 2 × 2 matrices

(a) Consider $\mathbf{A}_1$ a traceless rotationally invariant 2×2 matrix with deterministic eigenvalues $\lambda_\pm = \pm\sigma$. Show that

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$$\mathfrak{g}_{\mathbf{A}}(z) = \frac{z}{z^2 - \sigma^2} \quad \text{and} \quad R(g) = \frac{\sqrt{1 + 4\sigma^2 g^2} - 1}{2g}. \tag{12.104}$$

(b) Two such matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ (with eigenvalues $\pm\sigma_1$ and $\pm\sigma_2$ respectively) are free, so we can sum their R-transforms to find the spectrum of their sum. Show that $\mathfrak{g}_{\mathbf{A}_1+\mathbf{A}_2}(z)$ is given by one of the roots of

$$\mathfrak{g}_{\mathbf{A}_1+\mathbf{A}_2}(z) = \frac{\pm z}{\sqrt{(\sigma_1^4 + \sigma_2^4) - 2(\sigma_1^2 + \sigma_2^2)z^2 + z^4}}. \tag{12.105}$$

(c) In the basis where $\mathbf{A}_1$ is diagonal $\mathbf{A}_1 + \mathbf{A}_2$ has the form

$$\mathbf{A} = \begin{pmatrix} \sigma_1 + \sigma_2 \cos 2\theta & \sigma_2 \sin 2\theta \\ \sigma_2 \sin 2\theta & -\sigma_1 - \sigma_2 \cos 2\theta \end{pmatrix}, \tag{12.106}$$

for a random angle θ uniformly distributed between $[0, 2\pi]$. Show that the eigenvalues of $\mathbf{A}_1 + \mathbf{A}_2$ are given by

$$\lambda_\pm = \pm\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2 \cos 2\theta}. \tag{12.107}$$

(d) Show that the densities implied by (b) and (c) are the same. For $\sigma_1 = \sigma_2$, it is called the arcsine law (see Section 7.1.3).

Exercise 12.5.2 Product of two free 2 × 2 matrices

A rotationally invariant 2×2 matrix with deterministic eigenvalues 0 and $a_1 \geq 0$ has the form

$$A_1 = \mathbf{O} \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix} \mathbf{O}^T. \tag{12.108}$$

Two such matrices are free so we can use the S-transform to compute the eigenvalue distribution of their product.

(a) Show that the T-transform and the S-transform of $\mathbf{A}_1$ are given by

$$\mathfrak{t}(\zeta) = \frac{a_1}{2(\zeta - a_1)} \quad \text{and} \quad S(t) = \frac{2}{a_1}\frac{t+1}{2t+1}. \tag{12.109}$$

(b) Consider another such matrix $\mathbf{A}_2$ with independent eigenvectors and non-zero eigenvalue a_2. Using the multiplicativity of the S-transform, show that the T-transform and the density of eigenvalues of $\sqrt{\mathbf{A}_1}\mathbf{A}_2\sqrt{\mathbf{A}_1}$ are given by

$$\mathfrak{t}_{\mathbf{A}_1\mathbf{A}_2} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\zeta}{\zeta - a_1 a_2}} \tag{12.110}$$

and

$$\rho_{\mathbf{A}_1\mathbf{A}_2}(\lambda) = \frac{1}{2}\delta(\lambda) + \frac{1}{2\pi\sqrt{\lambda(a_1a_2 - \lambda)}} \text{ for } 0 \leq \lambda \leq a_1 a_2, \tag{12.111}$$

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where the delta function in the density indicates the fact that one eigenvalue is always zero.

(c) By directly computing the matrix product in the basis where $\mathbf{A}_1$ is diagonal, show that, in that basis,

$$\sqrt{\mathbf{A}_1}\mathbf{A}_2\sqrt{\mathbf{A}_1} = a_1 a_2 \begin{pmatrix} 0 & 0 \\ 0 & 1-\cos 2\theta \end{pmatrix}, \tag{12.112}$$

where θ is a random angle uniformly distributed between 0 and 2π.

(d) Show that the distribution of the non-zero eigenvalue implied by (b) and (c) is the same. It is the shifted arcsine law.

12.5.2 Pairwise Freeness and Free Collections

For the 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ to be free, they need to have deterministic eigenvalues. The proof above does not work if the eigenvalues of one of these matrices are random. As an illustration, consider three matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as above (2×2 symmetric rotationally invariant matrices with deterministic eigenvalues). Each pair of these matrices is free. On the other hand, the free sum $\mathbf{A} + \mathbf{B}$ has random eigenvalues and it is not necessarily free from $\mathbf{C}$. Actually we can show that it is *not* free with respect to $\mathbf{C}$.

First we show that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ do not form a free collection. For simplicity, we consider them traceless with $\sigma_{\mathbf{A}} = \sigma_{\mathbf{B}} = \sigma_{\mathbf{C}} = 1$. Then one can show that

$$\tau(\mathbf{ABCABC}) = \tau((\mathbf{ABC})^2) = 1, \tag{12.113}$$

violating the freeness condition for three variables. Indeed, let us compute explicitly $\mathbf{ABC}$ in the basis where $\mathbf{A}$ is diagonal. We find

$$\mathbf{ABC} = \begin{pmatrix} \cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi & \cos 2\theta \sin 2\phi - \sin 2\theta \cos 2\phi \\ \cos 2\theta \sin 2\phi - \sin 2\theta \cos 2\phi & -\cos 2\theta \cos 2\phi - \sin 2\theta \sin 2\phi \end{pmatrix}, \tag{12.114}$$

where ϕ is the angle between the eigenvectors of $\mathbf{A}$ and those of $\mathbf{C}$. The matrix $\mathbf{ABC}$ is a non-zero symmetric matrix, so the trace of its square must be non-zero. Actually one finds $(\mathbf{ABC})^2 = \mathbf{1}$.

Another way to see that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are not free as a group is to compute the mixed cumulant $\kappa_6(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{A},\mathbf{B},\mathbf{C})$, given that odd cumulants such as $\kappa_1(\mathbf{A})$ and $\kappa_3(\mathbf{A},\mathbf{B},\mathbf{C})$ are zero and that mixed cumulants involving two matrices are zero (they are pairwise free). The only non-zero term in the moment–cumulant relation for $\tau(\mathbf{ABCABC})$ (see Eq. (11.74)) is

$$\tau(\mathbf{ABCABC}) = \kappa_6(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{A},\mathbf{B},\mathbf{C}) = 1. \tag{12.115}$$

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The matrices **A**, **B** and **C** have therefore at least one non-zero cross-cumulant and cannot be free as a collection.

Now, to show that $\mathbf{A}+\mathbf{B}$ is not free from **C**, we realize that if we expand the sixth cross-cumulant $\kappa_6\,(\mathbf{A}+\mathbf{B},\mathbf{A}+\mathbf{B},\mathbf{C},\mathbf{A}+\mathbf{B},\mathbf{A}+\mathbf{B},\mathbf{C})$, we will encounter the above non-zero cross-cumulant of **A**, **B** and **C**. Indeed, a cross-cumulant is linear in each of its arguments. Since all other terms in this expansion are zero, we find

$$\begin{aligned}&\kappa_6(\mathbf{A}+\mathbf{B},\mathbf{A}+\mathbf{B},\mathbf{C},\mathbf{A}+\mathbf{B},\mathbf{A}+\mathbf{B},\mathbf{C})\\&\quad=\kappa_6\,(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{A},\mathbf{B},\mathbf{C})+\kappa_6\,(\mathbf{B},\mathbf{A},\mathbf{C},\mathbf{A},\mathbf{B},\mathbf{C})\\&\qquad+\kappa_6\,(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{B},\mathbf{A},\mathbf{C})+\kappa_6\,(\mathbf{B},\mathbf{A},\mathbf{C},\mathbf{B},\mathbf{A},\mathbf{C})=4\neq 0.\end{aligned}\tag{12.116}$$

As a consequence, even though 2×2 symmetric rotationally invariant random matrices with deterministic eigenvalues are pairwise free, they do not satisfy the free CLT. If they did, it would imply that 2×2 matrices with a semi-circle spectrum would be stable under addition, which is not the case. Note that Gaussian 2×2 Wigner matrices (which *are* stable under addition) do not have a semi-circle spectrum (see Exercise 12.5.3).

Exercise 12.5.3 Eigenvalue spectrum of 2 × 2 Gaussian matrices

Real symmetric and complex Hermitian Gaussian 2×2 matrices are stable under addition but they are not free. In this exercise we see that their spectrum is not given by a semi-circle law.

(a) Use Eq. (5.22) and the Gaussian potential $V(x)=x^2/2$ to write the joint probability (up to a normalization) of λ_1 and λ_2, the two eigenvalues of a real symmetric Gaussian 2×2 matrix.

(b) To find the eigenvalue density, we need to compute

$$\rho(\lambda_1)=\int_{-\infty}^{\infty}\mathrm{d}\lambda_2 P(\lambda_1,\lambda_2).\tag{12.117}$$

This integral will involve an error function because of the absolute value in $P(\lambda_1,\lambda_2)$. If you have the courage compute $\rho(\lambda)$ leaving the normalization undetermined.

(c) It is easier to do the complex Hermitian case. Use Eq. (5.26) and the same potential to adapt your answer in (a) to the $\beta=2$ case.

(d) The absolute value has now disappeared and the integral in (b) is now much easier. Perform this integral and find the normalization constant. You should obtain

$$\rho(\lambda)=\frac{\lambda^2+\frac{1}{2}}{\sqrt{\pi}}\mathrm{e}^{-\lambda^2}.\tag{12.118}$$

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