6

Eigenvalues and Orthogonal Polynomials*

In this chapter, we investigate yet another route to shed light on the eigenvalue density of the Wigner and Wishart ensembles. We show (a) that the most probable positions of the Coulomb gas problem coincide with the zeros of Hermite polynomials in the Wigner case, and of Laguerre polynomials in the Wishart case; and (b) that the average (over randomness) of the characteristic polynomials (defined as $\det(z\mathbf{1} - \mathbf{X}_N)$) of Wigner or Wishart random matrices of size N obey simple recursion relations that allow one to express them as, respectively, Hermite and Laguerre polynomials. The fact that the two methods lead to the same result (at least for large N) reflects the fact that eigenvalues fluctuate very little around their most probable positions. Finally we show that for unitary ensembles $\beta = 2$, the expected characteristic polynomial is always an orthogonal polynomial with respect to some weight function related to the matrix potential.

6.1 Wigner Matrices and Hermite Polynomials

6.1.1 Most Likely Eigenvalues and Zeros of Hermite Polynomials

In the previous chapter, we established a general equation for the Stieltjes transform of the most likely positions of the eigenvalues of random matrices belonging to a general orthogonal ensemble, see Eq. (5.34). In the special case of a quadratic potential $V(x) = x^2/2$, this equation reads

$$zg_N(z) - 1 = g_N^2(z) + \frac{g_N'(z)}{N}. \tag{6.1}$$

This ordinary differential equation is of the *Ricatti type*,[1] and can be solved by setting $g_N(z) := \psi'(z)/N\psi(z)$. This yields, upon substitution,

$$\psi''(z) - Nz\psi'(z) + N^2\psi(z) = 0, \tag{6.2}$$

or, with $\psi(z) = \Psi(x = \sqrt{N}z)$,

$$\Psi''(x) - x\Psi'(x) + N\Psi(x) = 0. \tag{6.3}$$

[1] A Ricatti equation is a first order differential equation that is quadratic in the unknown function, in our case $g_N(z)$.

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The solution of this last equation with the correct behavior for large x is the Hermite polynomial of order N. General Hermite polynomials $H_n(x)$ are defined as the nth order polynomial that starts as x^n and is orthogonal to all previous ones under the unit Gaussian measure:[2]

$$\int \frac{\mathrm{d}x}{\sqrt{2\pi}} H_n(x) H_m(x) \mathrm{e}^{-\frac{x^2}{2}} = 0 \text{ when } n \neq m. \tag{6.4}$$

The first few are given by

$$\begin{aligned} H_0(x) &= 1, \\ H_1(x) &= x, \\ H_2(x) &= x^2 - 1, \\ H_3(x) &= x^3 - 3x, \\ H_4(x) &= x^4 - 6x^2 + 3. \end{aligned} \tag{6.5}$$

In addition to the above ODE (6.3), they satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x} H_n(x) = n H_{n-1}(x), \tag{6.6}$$

and the recursion

$$H_n(x) = x H_{n-1}(x) - (n-1) H_{n-2}(x), \tag{6.7}$$

which combined together recovers Eq. (6.3). Hermite polynomials can be written explicitly as

$$H_n(x) = \exp\left[-\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\right] x^n = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{2^m} \frac{n!}{m!\,(n-2m)!} x^{n-2m}, \tag{6.8}$$

where $\lfloor n/2 \rfloor$ is the integer part of $n/2$.

Coming back to Eq. (6.1), we thus conclude that the exact solution for the Stieltjes transform $g_N(z)$ at finite N is

$$g_N(z) = \frac{H_N'(\sqrt{N} z)}{\sqrt{N} H_N(\sqrt{N} z)} \tag{6.9}$$

or, writing $H_N(x) = \prod_{i=1}^N (x - \sqrt{N} h_i^{(N)})$, where $\sqrt{N} h_i^{(N)}$ are the N (real) zeros of $H_N(x)$,

$$g_N(z) = \frac{1}{N} \sum_{i=1}^N \frac{1}{z - h_i^{(N)}}. \tag{6.10}$$

[2] Hermite polynomials can be defined using two different conventions for the unit Gaussian measure. We use here the "probabilists' Hermite polynomials", while the "physicists' convention" uses a Gaussian weight proportional to e^{-x^2}.

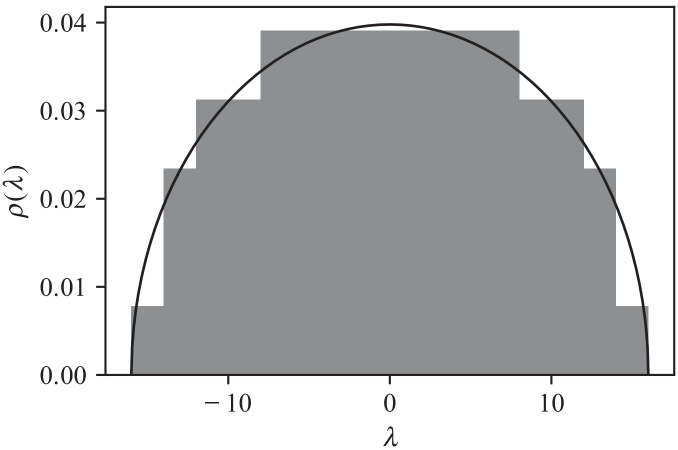

Figure 6.1 Histogram of the 64 zeros of $H_{64}(x)$. The full line is the asymptotic prediction from the semi-circle law.

Comparing with the definition of $g_N(z)$, Eq. (5.30), one concludes that the most likely positions of the Coulomb particles are exactly given by the zeros of Hermite polynomials (scaled by $\sqrt{N}$). This is a rather remarkable result, which holds for more general confining potentials, to which are associated different kinds of orthogonal polynomials. Explicit examples will be given later in this section for the Wishart ensemble, where we will encounter Laguerre polynomials (see also Chapter 7 for Jacobi polynomials).

Since we know that $g_N(z)$ converges, for large N, towards the Stieltjes transform of the semi-circle law, we can conclude that the rescaled zeros of Hermite polynomials are themselves distributed, for large N, according to the same semi-circle law. This classical property of Hermite polynomials is illustrated in Figure 6.1.

6.1.2 Expected Characteristic Polynomial of Wigner Matrices

In this section, we will show that the expected characteristic polynomial of a Wigner matrix $\mathbf{X}_N$, defined as $Q_N(z) := \mathbb{E}[\det(z\,\mathbf{1} - \mathbf{X}_N)]$, is given by the same Hermite polynomial as above. The idea is to write a recursion relation for $Q_N(z)$ by expanding the determinant in minors. Since we will be comparing Wigner matrices of different size, it is more convenient to work with unscaled matrices $\mathbf{Y}_N = \sqrt{N}\mathbf{X}_N$, i.e. symmetric matrices of size N with elements of zero mean and variance 1 (it will turn out that the variance of diagonal elements is actually irrelevant, and so can also be chosen to be 1). We define

$$q_N(z) := \mathbb{E}[\det(z\,\mathbf{1} - \mathbf{Y}_N)]. \tag{6.11}$$

Using $\det(\alpha\mathbf{A}) = \alpha^N \det(\mathbf{A})$, we then have

$$Q_N(z) = N^{-N/2} q_N(\sqrt{N}z). \tag{6.12}$$

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We can compute the first two $q_N(z)$ by hand:

$$q_1(z) = \mathbb{E}[z - \mathbf{Y}_{11}] = z; \qquad q_2(z) = \mathbb{E}\left[(z - \mathbf{Y}_{11})(z - \mathbf{Y}_{22}) - \mathbf{Y}_{12}^2\right] = z^2 - 1. \quad (6.13)$$

To compute the polynomials for $N \geq 3$ we first expand the determinant in minors from the first line. We call $\mathbf{M}_{i,j}$ the ij-minor, i.e. the determinant of the submatrix of $z\,\mathbf{1} - \mathbf{Y}_N$ with the line i and column j removed:

$$\begin{aligned} \det(z\,\mathbf{1} - Y_N) &= \sum_{i=1}^{N} (-1)^{i+1}(z\delta_{i1} - \mathbf{Y}_{1i})\mathbf{M}_{1,i} \\ &= z\mathbf{M}_{1,1} - \mathbf{Y}_{11}\mathbf{M}_{1,1} + \sum_{i=2}^{N} (-1)^i \mathbf{Y}_{1i}\mathbf{M}_{1,i}. \end{aligned} \quad (6.14)$$

We would like to take the expectation value of this last expression. The first two terms are easy: the minor $\mathbf{M}_{1,1}$ is the same determinant with a Wigner matrix of size $N-1$, so $\mathbb{E}[\mathbf{M}_{1,1}] \equiv q_{N-1}(z)$; the diagonal element $\mathbf{Y}_{11}$ is independent from the rest of the matrix and its expectation is zero.

For the other terms in the sum, the minor $\mathbf{M}_{i,1}$ is not independent of $\mathbf{Y}_{i1}$. Indeed, because $\mathbf{X}_N$ is symmetric, the corresponding submatrix contains another copy of $\mathbf{Y}_{1i}$. Let us then expand $\mathbf{M}_{1,i}$ itself on the ith row, to make the other term $\mathbf{Y}_{i1}$ appear explicitly. For $i \neq 1$, we have

$$\begin{aligned} \mathbf{M}_{i,1} &= \sum_{j=1,\, j\neq i}^{N} (-1)^{i-j}\mathbf{Y}_{ij}\mathbf{M}_{1i,ij} \\ &= (-1)^{i-1}\mathbf{Y}_{i1}\mathbf{M}_{1i,i1} + \sum_{j=2,\, j\neq i}^{N} (-1)^{i-j}\mathbf{Y}_{ij}\mathbf{M}_{1i,ij}, \end{aligned} \quad (6.15)$$

where $\mathbf{M}_{ij,kl}$ is the "sub-minor", with rows i,j and columns k,l removed.

We can now take the expectation value of Eq. (6.14) by noting that $\mathbf{Y}_{1i}$ is independent of all the terms in Eq. (6.15) except the first one. We also realize that $\mathbf{M}_{1i,i1}$ is the same determinant with a Wigner matrix of size $N-2$ that is now independent of $\mathbf{Y}_{1i}$, so we have $\mathbb{E}[\mathbf{M}_{1i,i1}] = q_{N-2}(z)$. Putting everything together we get

$$q_N(z) := \mathbb{E}[\det(z\,\mathbf{1} - \mathbf{Y}_N)] = zq_{N-1}(z) - (N-1)q_{N-2}(z). \quad (6.16)$$

We recognize here precisely the recursion relation (6.7) that defines Hermite polynomials.

How should this result be interpreted? Suppose for one moment that the positions of the eigenvalues λ_i of $\mathbf{Y}_N$ were not fluctuating from sample to sample, and fixed to their most likely values λ_i^*. In this case, the expectation operator would not be needed and one would have

$$g_N(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log Q_N(z) = \frac{H_N'(\sqrt{N}z)}{\sqrt{N}H_N(\sqrt{N}z)}, \quad (6.17)$$

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recovering the result of the previous section. What is somewhat surprising is that

$$\mathbb{E}\left[\prod_{i=1}^{N}(z-\lambda_i)\right] \equiv \prod_{i=1}^{n}(z-\lambda_i^*) \tag{6.18}$$

even when fluctuations are accounted for. In particular, in the limit $N \to \infty$, the average Stieltjes transform should be computed from the average of the logarithm of the characteristic polynomial:

$$\mathfrak{g}(z) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}z} \log\det(z\,\mathbf{1} - \mathbf{X}_N)\right]; \tag{6.19}$$

but the above calculation shows that one can compute the logarithm of the average characteristic polynomial instead. The deep underlying mechanism is the eigenvalue spectrum of random matrices is rigid – fluctuations around most probable positions are small.

Exercise 6.1.1 Hermite polynomials and moments of the Wigner

Show that (for $n \geq 4$)

$$Q_N(x) = x^n - \frac{n-1}{2}x^{n-2} + \frac{(n-1)(n-2)(n-3)}{8n}x^{n-4} + O\left(x^{n-6}\right), \tag{6.20}$$

therefore

$$g_N(z) = \frac{1}{z} - \frac{n-1}{N}\frac{1}{z^3} - \frac{(n-1)(2n-3)}{N}\frac{1}{z^5} + O\left(\frac{1}{z^7}\right), \tag{6.21}$$

so in the large N limit we recover the first few terms of the Wigner Stieltjes transform

$$\mathfrak{g}(z) = \frac{1}{z} - \frac{1}{z^3} - \frac{2}{z^5} + O\left(\frac{1}{z^7}\right). \tag{6.22}$$

6.2 Laguerre Polynomials

6.2.1 Most Likely Characteristic Polynomial of Wishart Matrices

Similarly to the case of Wigner matrices, the Stieltjes transform of the most likely positions of the Coulomb charges in the Wishart ensemble can be written as $g_N(z) := \psi'(z)/N\psi(z)$, where $\psi(z)$ is a monic polynomial of degree N satisfying

$$\psi''(x) - NV'(x)\psi'(x) + N^2\Pi_N(x)\psi(x) = 0, \tag{6.23}$$

with

$$NV'(x) = \frac{N - T - 1 + 2\beta^{-1}}{x} + T, \tag{6.24}$$

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and, using Eq. (5.32),

$$\Pi_N(x) = \frac{1}{N}\sum_{k=1}^{N}\frac{V'(x) - V'(\lambda_k^*)}{x - \lambda_k^*} = \frac{c_N}{x}, \tag{6.25}$$

where

$$c_N = -\frac{N - T - 1 + 2\beta^{-1}}{N^2}\sum_{k=1}^{N}\frac{1}{\lambda_k^*}. \tag{6.26}$$

Writing now $\psi(x) = \Psi(Tx)$ and $u = Tx$, Eq. (6.23) becomes

$$u\Psi''(u) - (N - T - 1 + 2\beta^{-1} + u)\Psi'(u) + \frac{N^2}{T}c_N\Psi(u) = 0. \tag{6.27}$$

This is the differential equation for the so-called associated Laguerre polynomials $L^{(\alpha)}$ with $\alpha = T - N - 2\beta^{-1}$. It has polynomial solutions of degree N if and only if the coefficient of the $\Psi(u)$ term is an integer equal to N (i.e. if $c_N = T/N$). The solution is then given by

$$\Psi(u) \propto L_N^{(T-N-2\beta^{-1})}(u), \tag{6.28}$$

where

$$L_n^{(\alpha)}(x) = x^{-\alpha}\frac{(\frac{\mathrm{d}}{\mathrm{d}x} - 1)^n}{n!}x^{\alpha+n}. \tag{6.29}$$

Note that associated Laguerre polynomials are orthogonal with respect to the measure $x^\alpha \mathrm{e}^{-x}$, i.e.

$$\int_0^\infty \mathrm{d}x x^\alpha \mathrm{e}^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) = \delta_{nm}\frac{(n+\alpha)!}{n!}. \tag{6.30}$$

Given that the standard associated Laguerre polynomials have as a leading term $(-1)^N/N!\,z^N$ and that $\psi(x)$ is monic, we finally find

$$\psi(x) = \begin{cases} (-1)^N N!\,T^{-N}L^{(T-N-2)}(Tx) & \text{real symmetric} \quad (\beta = 1), \\ (-1)^N N!\,T^{-N}L^{(T-N-1)}(Tx) & \text{complex Hermitian} \quad (\beta = 2). \end{cases} \tag{6.31}$$

Hence, the most likely positions of the Coulomb–Wishart charges are given by the zeros of associated Laguerre polynomials, exactly as the most likely positions of the Coulomb–Wigner charges are given by the zeros of Hermite polynomials.

We should nevertheless check that $c_N = T/N$ is compatible with Eq. (6.26), i.e. that the following equality holds:

$$\frac{T}{N} = \frac{\alpha + 1}{N^2}\sum_{k=1}^{N}\frac{1}{\lambda_k^*}, \tag{6.32}$$

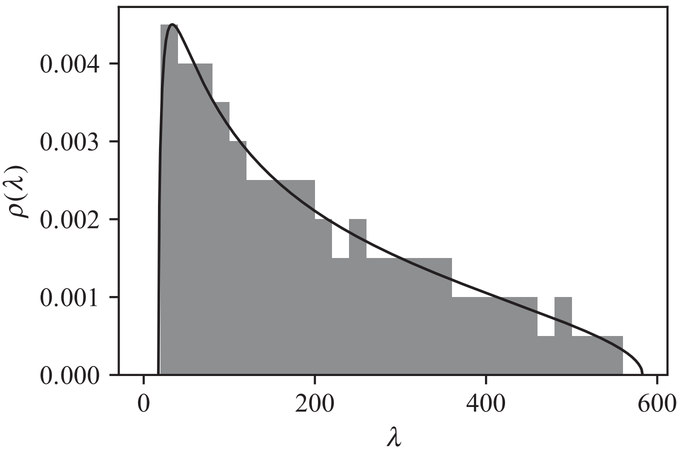

Figure 6.2 Histogram of the 100 zeros of $L_{100}^{(100)}(x)$. The full line is the Marčenko–Pastur distribution (4.43) with $q = \frac{1}{2}$ scaled by a factor $T = 200$.

where the λ_k^* are the zeros of $T^{-1}L(Tx)$, i.e. $\lambda_k^* = \ell_k^{(\alpha)}/T$, where $\ell_k^{(\alpha)}$ are the zeros of the associate Laguerre polynomials $L_N^{(\alpha)}$, which indeed obey the following relation:[3]

$$\frac{1}{N}\sum_{k=0}^{N}\frac{1}{\ell_k^{(\alpha)}} = \frac{1}{\alpha+1}. \tag{6.33}$$

From the results of Section 5.3.1, we thus conclude that the zeros of the Laguerre polynomials $L^{(T-N-2)}(Tx)$ converge to a Wishart distribution with $q = N/T$. Figure 6.2 shows the histogram of zeros of $L_{100}^{(100)}(x)$ with the asymptotic prediction for large N and T. Note that $\alpha \approx N(q^{-1}-1)$ in that limit.

6.2.2 Average Characteristic Polynomial

As in the Wigner case we would like to get a recursion relation for $Q_{q,N}(z) := \mathbb{E}[\det(z\,\mathbf{1} - \mathbf{W}_q^{(N)})]$, where $\mathbf{W}_q^{(N)}$ is a white Wishart matrix of size N and parameter $q = T/N$. This time the recursion will be over T at N fixed. So we keep N fixed (we will drop the (N) index to keep the notation light) and consider an unnormalized white Wishart matrix:

$$\mathbf{Y}_T = \sum_{t=1}^{T}\mathbf{v}_t\mathbf{v}_t^T, \tag{6.34}$$

where $\mathbf{v}_t$ are T N-dimensional independent random vectors uniformly distributed on the sphere. We want to compute

[3] See e.g. Alıcı and Taeli [2015], where other inverse moments of the $\ell_k^{(\alpha)}$ are also derived.

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$$q_{T,N}(z) = \mathbb{E}[\det(z\,\mathbf{1} - \mathbf{Y}_T)]. \tag{6.35}$$

The properly normalized expected characteristic polynomial is then given by $Q_{q,N}(z) = T^{-n} q_{T,N}(Tz)$. To construct our recursion relation, we will make use of the Shermann–Morrison formula (Eq. (1.30)), which states that for an invertible matrix $\mathbf{A}$ and vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\det(\mathbf{A} + \mathbf{u}\mathbf{v}^T) = (1 + \mathbf{v}^T \mathbf{A}^{-1}\mathbf{u}) \det \mathbf{A}. \tag{6.36}$$

Applying this formula with $\mathbf{A} = z\,\mathbf{1} - \mathbf{Y}_{T-1}$, we get

$$\det(z\,\mathbf{1} - \mathbf{Y}_T) = (1 - \mathbf{v}_T^T (z\,\mathbf{1} - \mathbf{Y}_{T-1})^{-1}\mathbf{v}_T) \det(z\,\mathbf{1} - \mathbf{Y}_{T-1}). \tag{6.37}$$

The vector $\mathbf{v}_T$ is independent of the rest of $\mathbf{Y}_{T-1}$, so taking the expectation value with respect to this last vector we get

$$\mathbb{E}_{\mathbf{v}_T}\left[\mathbf{v}_T^T (z\,\mathbf{1} - \mathbf{Y}_{T-1})^{-1}\mathbf{v}_T\right] = \mathrm{Tr}[(z\,\mathbf{1} - \mathbf{Y}_{T-1})^{-1}]. \tag{6.38}$$

Now, using once again the general relation (easily derived in the basis where $\mathbf{A}$ is diagonal),

$$\mathrm{Tr}[(z\,\mathbf{1} - \mathbf{A})^{-1}] \det(z\,\mathbf{1} - \mathbf{A}) = \frac{\mathrm{d}}{\mathrm{d}z} \det(z\,\mathbf{1} - \mathbf{A}), \tag{6.39}$$

with $\mathbf{A} = \mathbf{Y}_{T-1}$, we can take the expectation value of Eq. (6.37). We obtain

$$q_{T,N}(z) = \left(1 - \frac{\mathrm{d}}{\mathrm{d}z}\right) q_{T-1,N}(z). \tag{6.40}$$

To start the recursion relation, we note that $\mathbf{Y}_0$ is the N-dimensional zero matrix for which $q_0(z) = z^N$. Hence,[4]

$$q_{T,N}(z) = \left(1 - \frac{\mathrm{d}}{\mathrm{d}z}\right)^T z^N. \tag{6.41}$$

If we apply an extra $\left(1 - \frac{\mathrm{d}}{\mathrm{d}z}\right)$ to Eq. (6.41), we get the following recursion relation:

$$q_{T+1,N}(z) = q_{T,N}(z) - N q_{T,N-1}(z), \tag{6.42}$$

which is similar to the classic "three-point rule" for Laguerre polynomials:

$$L_N^{(\alpha+1)}(x) = L_N^{(\alpha)}(x) + L_{N-1}^{(\alpha+1)}(x). \tag{6.43}$$

This allows us to make the identification

$$q_{T,N}(z) = (-1)^N N!\, L_N^{(T-N)}(z). \tag{6.44}$$

The correctly normalized average characteristic polynomial finally reads:

$$Q_{T,N}(z) = (-1)^N T^{-N} N!\, L_N^{(T-N)}(Tz). \tag{6.45}$$

[4] This relation will be further discussed in the context of finite free convolutions, see Chapter 12.

Hence, the average characteristic polynomial of real Wishart matrices is a Laguerre polynomial, albeit with a slightly different value of α compared to the one obtained in Eq. (6.31) above ($\alpha = T - N$ instead of $\alpha = T - N - 2$). The difference however becomes small when $N, T \to \infty$.

6.3 Unitary Ensembles

In this section we will discuss the average characteristic polynomial for unitary ensembles, ensembles of complex Hermitian matrices that are invariant under unitary transformations. Although this book mainly deals with real symmetric matrices, more is known about complex Hermitian matrices, so we want to give a few general results about these matrices that do not have a known equivalent in the real case. The main reason unitary ensembles are easier to deal with than orthogonal ones has to do with the Vandermonde determinant which is needed to change variables from matrix elements to eigenvalues. Recall that

$$|\det(\Delta(\mathbf{M}))| = |\det \mathbf{V}|^{\beta} \text{ with } \mathbf{V} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_N^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \lambda_3^{N-1} & \dots & \lambda_N^{N-1} \end{pmatrix} \tag{6.46}$$

in the orthogonal case $\beta = 1$, the absolute value sign is needed to get the correct result. In the case $\beta = 2$, $(\det \mathbf{V})^2$ is automatically positive and no absolute value is needed. The absolute value for $\beta = 1$ is very hard to deal with analytically, while for $\beta = 2$ the Vandermonde determinant is a polynomial in the eigenvalues.

6.3.1 Complex Wigner

In Section 6.1.2 we have shown that the expected characteristic polynomial of a unit variance real Wigner matrix is given by the Nth Hermite polynomial properly rescaled. The argument relied on two facts: (i) the expectation value of any element of a Wigner matrix is zero and (ii) all matrix elements are independent, save for

$$\mathbb{E}[\mathbf{W}_{ij}\mathbf{W}_{ji}] = 1/N \text{ for } i \neq j. \tag{6.47}$$

These two properties are shared by complex Wigner matrices. Therefore, the expected characteristic polynomial of a complex Wigner matrix is the same as for a real Wigner of the same size. We have shown that for a real or complex Wigner of size N,

$$Q_N(z) := \mathbb{E}[\det(z\,\mathbf{1} - \mathbf{W})] = N^{-N/2} H_N(\sqrt{N} z), \tag{6.48}$$

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where $H_N(x)$ is the Nth Hermite polynomial, i.e. the Nth monic polynomial orthogonal in the following sense:

$$\int_{-\infty}^{\infty} H_i(x)H_j(x)\mathrm{e}^{-x^2/2} = 0 \text{ when } i \neq j. \tag{6.49}$$

We can actually absorb the factors of N in $Q_N(z)$ in the measure $\exp(-x^2/2)$ and realize that the polynomial $Q_N(z)$ is the Nth monic polynomial orthogonal with respect to the measure

$$w_N(x) = \exp(-Nx^2/2). \tag{6.50}$$

There are two important remarks to be made about the orthogonality of $Q_N(x)$ with respect to the measure $w_N(x)$. First, $Q_N(x)$ is the Nth in a series of orthogonal polynomials with respect to an N-dependent measure. In particular $Q_M(x)$ for $M \neq N$ is an orthogonal polynomial coming from a different measure. Second, the measure $w_N(x)$ is exactly the weight coming from the matrix potential $\exp(-\beta NV(\lambda)/2)$ for $\beta = 2$ and $V(x) = x^2/2$. In Section 6.3.3, we will see that these two statements are true for a general potential $V(x)$ when $\beta = 2$.

6.3.2 Complex Wishart

A complex white Wishart matrix can be written as a normalized sum of rank-1 complex Hermitian projectors:

$$\mathbf{W} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{v}_t\mathbf{v}_t^{\dagger}, \tag{6.51}$$

where the vectors $\mathbf{v}_t$ are vectors of IID complex Gaussian numbers with zero mean and normalized as

$$\mathbb{E}[\mathbf{v}_t\mathbf{v}_t^{\dagger}] = \mathbf{1}. \tag{6.52}$$

The derivation of the average characteristic polynomial in the Wishart case in Section 6.2.2 only used the independence of the vectors $\mathbf{v}_t$ and the expectation value $\mathbb{E}[\mathbf{v}_t\mathbf{v}_t^{T}] = \mathbf{1}$. So, by replacing the matrix transposition by the Hermitian conjugation in the derivation we can show that the expected characteristic polynomial of a complex white Wishart of size N is also given by a Laguerre polynomial, as in Eq. (6.45). The Laguerre polynomials $L_k^{(T-N)}(x)$ are orthogonal in the sense of Eq. (6.30), with $\alpha = T - N$. As in the Wigner case, we can include the extra factor of T in the orthogonality weight and realize that the expected characteristic polynomial of a real or complex Wishart matrix is the Nth monic polynomial orthogonal with respect to the weight:

$$w_N(x) = x^{T-N}\mathrm{e}^{-Tx} \text{ for } 0 \leq x < \infty. \tag{6.53}$$

This weight function is precisely the single eigenvalue weight, without the Vandermonde term, of a complex Hermitian white Wishart of size N (see the footnote on page 46).

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Note that the normalization of the weight $w_N(x)$ is irrelevant: the condition that the polynomial is monic uniquely determines its normalization. Note as well that the real case is given by the same polynomials, i.e. polynomials that are orthogonal with respect to the *complex* weight, which is different from the real weight.

6.3.3 General Potential $V(x)$

The average characteristic polynomial for a matrix of size N in a unitary ensemble with potential $V(\mathbf{M})$ is given by

$$Q_N(x) := \mathbb{E}[\det(z\,\mathbf{1} - \mathbf{M})], \tag{6.54}$$

which we can express via the joint law of the eigenvalues of $\mathbf{M}$:

$$Q_N(z) \propto \int \mathrm{d}^N\mathbf{x} \prod_{k=1}^{N}(z - x_k)\,\Delta^2(\mathbf{x})\mathrm{e}^{-N\sum_{k=1}^{N} V(x_k)}, \tag{6.55}$$

where $\Delta(\mathbf{x})$ is the Vandermonde determinant:

$$\Delta(\mathbf{x}) = \prod_{k<\ell}(x_\ell - x_k). \tag{6.56}$$

We do not need to worry about the normalization of the above expectation value as we know that $Q_N(z)$ is a monic polynomial of degree N. In other words, the condition $Q_N(z) = z^N + O(z^{N-1})$ is sufficient to properly normalize $Q_N(z)$. The first step of the computation is to combine one of the two Vandermonde determinants with the product of $(z - x_k)$:

$$\Delta^2(\mathbf{x})\prod_{k=1}^{N}(z - x_k) = \Delta(\mathbf{x})\prod_{k<\ell}(x_\ell - x_k)\prod_{k=1}^{N}(z - x_k) \equiv \Delta(\mathbf{x})\Delta(\mathbf{x}; z), \tag{6.57}$$

where $\Delta(\mathbf{x}; z)$ is a Vandermonde determinant of $N+1$ variables, namely the N variables x_k and the extra variable z.

The second step is to write the determinants in the Vandermonde form:

$$\Delta(\mathbf{x}) = \det\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_N \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_N^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \dots & x_N^{N-1} \end{pmatrix}. \tag{6.58}$$

We can add or subtract to any line a multiple of any other line and not change the above determinant. By doing so we can transform all monomials x_ℓ^k into a monic polynomial of degree k of our choice, so we have

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$$\Delta(\mathbf{x}) = \det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ p_1(x_1) & p_1(x_2) & p_1(x_3) & \dots & p_1(x_N) \\ p_2(x_1) & p_2(x_2) & p_2(x_3) & \dots & p_2(x_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{N-1}(x_1) & p_{N-1}(x_2) & p_{N-1}(x_3) & \dots & p_{N-1}(x_N) \end{pmatrix}. \tag{6.59}$$

We will choose the polynomials $p_n(x)$ to be the monic polynomials orthogonal with respect to the measure $w(x) = e^{-NV(x)}$, this will turn out to be extremely useful. We can now perform the integral of the vector $\mathbf{x}$ in the following expression:

$$Q_N(z) \propto \int d^N\mathbf{x}\, \Delta(\mathbf{x})\Delta(\mathbf{x}; z) e^{-N\sum_{k=1}^N V(x_k)}. \tag{6.60}$$

If we expand the two determinants $\Delta(\mathbf{x})$ and $\Delta(\mathbf{x}; z)$ as signed sums over all permutations and take their product, we realize that in each term each variable x_k will appear exactly twice in two polynomials, say $p_n(x_k)$ and $p_m(x_k)$, but by orthogonality we have

$$\int dx_k\, p_n(x_k) p_m(x_k) e^{-NV(x_k)} = Z_n \delta_{mn}, \tag{6.61}$$

where Z_n is a normalization constant that will not matter in the end. The only terms that will survive are those for which every x_k appears in the same polynomial in both determinants. For this to happen, the variable z must appear as $p_N(z)$, the only polynomial not in the first determinant. So this trick allows us to conclude with very little effort that

$$Q_N(z) \propto p_N(z). \tag{6.62}$$

But since both $Q_N(z)$ and $p_N(z)$ are monic, they must be equal. We have just shown that for a Hermitian matrix $\mathbf{M}$ of size N drawn from a unitary ensemble with potential $V(x)$,

$$\mathbb{E}[\det(z\,\mathbf{1} - \mathbf{M})] = p_N(z), \tag{6.63}$$

where $p_N(x)$ is the Nth monic orthogonal polynomial with respect to the measure $e^{-NV(x)}$.

It is possible to generalize this result to expectation of products of characteristic polynomials evaluated at K different points z_k, which allows one to study the joint distribution of K eigenvalues. We give here the result without proof.[5] We first define the expectation value of a product of K characteristic polynomials:

$$F_K(z_1, z_2, \dots, z_K) := \mathbb{E}[\det(z_1\mathbf{1} - \mathbf{M}) \det(z_2\mathbf{1} - \mathbf{M}) \dots \det(z_K\mathbf{1} - \mathbf{M})]. \tag{6.64}$$

The multivariate function F_K can be expressed as a determinant of orthogonal polynomials:

$$F_K(z_1, z_2, \dots, z_K) = \frac{1}{\Delta} \det \begin{pmatrix} p_N(z_1) & p_N(z_2) & \dots & p_N(z_K) \\ p_{N+1}(z_1) & p_{N+1}(z_2) & \dots & p_{N+1}(z_K) \\ \vdots & \vdots & \ddots & \vdots \\ p_{N+K-1}(z_1) & p_{N+K-1}(z_2) & \dots & p_{N+K-1}(z_K) \end{pmatrix}, \tag{6.65}$$

[5] See Brézin and Hikami [2011] for a derivation. Note that their formula equivalent to Eq. (6.67) is missing the K-dependent constant factor.

where $\Delta := \Delta(z_1, z_2, \ldots, z_K)$ is the usual Vandermonde determinant and the $p_\ell(x)$ are the monic orthogonal polynomials orthogonal with respect to $e^{-NV(x)}$. When $K = 1$, $\Delta = 1$ by definition and we recover our previous result $F_1(z) = p_N(z)$. When the arguments of F_K are not all different, Eq. (6.65) gives an undetermined result (0/0) but the limit is well defined. A useful case is when all arguments are equal:

$$F_K(z) := F_K(z, z, \ldots, z) = \mathbb{E}[\det(z\mathbf{1} - \mathbf{M})^K]. \tag{6.66}$$

Taking the limit of Eq. (6.65) we find the rather simple result

$$F_K(z) = \frac{1}{\prod_{\ell=0}^{K-1} \ell!} \det \begin{pmatrix} p_N(z) & p'_N(z) & \ldots & p_N^{(K-1)}(z) \\ p_{N+1}(z) & p'_{N+1}(z) & \ldots & p_{N+1}^{(K-1)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ p_{N+K-1}(z) & p'_{N+K-1}(z) & \ldots & p_{N+K-1}^{(K-1)}(z) \end{pmatrix}, \tag{6.67}$$

where $p_\ell^{(k)}(x)$ is the kth derivative of the ℓth polynomial. In particular the average-square characteristic polynomial is given by

$$F_2(z) = p_N(z)p'_{N+1}(z) - p'_N(z)p_{N+1}(z). \tag{6.68}$$

Exercise 6.3.1 Variance of the Characteristic Polynomial of a 2×2 Hermitian Wigner Matrix

(a) Show that the characteristic polynomial of a 2×2 Hermitian Wigner matrix is given by

$$Q_2^{\mathbf{W}}(z) = (z - w_{11})(z - w_{22}) - (w_{12}^{\mathrm{R}})^2 - (w_{12}^{\mathrm{I}})^2, \tag{6.69}$$

where $w_{11}, w_{22}, w_{12}^{\mathrm{R}}$ and w_{12}^{I} are four real independent Gaussian random numbers with variance 1 for the first two and $1/2$ for the other two.

(b) Compute directly the mean and the variance of $Q_2^{\mathbf{W}}(z)$.

(c) Use Eqs. (6.63) and (6.68) and the first few Hermite polynomials given in Section 6.1.1 to obtain the same result, namely $\mathbb{V}[Q_2^{\mathbf{W}}(z)] = 2z^2 + 2$.

Bibliographical Notes

- On orthogonal polynomials, definitions and recursion relations:
  - G. Szegő. *Orthogonal Polynomials*. AMS Colloquium Publications, volume 23. American Mathematical Society, 1975,
  - D. Zwillinger, V. Moll, I. Gradshteyn, and I. Ryzhik, editors. *Table of Integrals, Series, and Products (Eighth Edition)*. Academic Press, New York, 2014,
  - R. Beals and R. Wong. *Special Functions and Orthogonal Polynomials*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016.

- On the relation between Hermite polynomials, log gases and the Calogero model:
  - S. Agarwal, M. Kulkarni, and A. Dhar. Some connections between the classical Calogero-Moser model and the log gas. *Journal of Statistical Physics*, 176(6), 1463–1479, 2019.
- Orthogonal polynomials and random matrix theory:
  - M. L. Mehta. *Random Matrices*. Academic Press, San Diego, 3rd edition, 2004,
  - E. Brézin and S. Hikami. Characteristic polynomials. In *The Oxford Handbook of Random Matrix Theory*. Oxford University Press, Oxford, 2011,
  - P. Deift. *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*. Courant Institute, New York, 1999.
- For identities obeyed by the zeros of Laguerre polynomials:
  - H. Alıcı and H. Taeli. Unification of Stieltjes-Calogero type relations for the zeros of classical orthogonal polynomials. *Mathematical Methods in the Applied Sciences*, 38(14):3118–3129, 2015.