9

Dyson Brownian Motion

In this chapter we would like to start our investigation of the addition of random matrices, which will lead to the theory of so-called "free matrices". This topic has attracted substantial interest in recent years and will be covered in the next chapters.

We will start by studying how a fixed large matrix (random or not) is modified when one adds a Wigner matrix. The elements of a Wigner matrix are Gaussian random numbers and, as we saw in the previous chapter, each of them can be written as a sum of Gaussian numbers with smaller variance. By pushing this reasoning to the limit we can write the addition of a Wigner matrix as a continuous process of addition of infinitesimal Wigner matrices. Such a matrix Brownian motion process, viewed through the lens of eigenvalues and eigenvectors, is called a Dyson Brownian motion (DBM) after the physicist Freeman Dyson who first introduced it in 1962.

9.1 Dyson Brownian Motion I: Perturbation Theory

9.1.1 Perturbation Theory: A Short Primer

We begin by recalling how perturbation theory works for eigenvalues and eigenvectors. This is a standard topic in elementary quantum mechanics, but is not necessarily well known in other circles. Let us consider a matrix $\mathbf{H} = \mathbf{H}_0 + \epsilon \mathbf{H}_1$, where $\mathbf{H}_0$ is a real symmetric matrix whose eigenvalues and eigenvectors are assumed to be known, and $\mathbf{H}_1$ is a real symmetric matrix that gives the perturbation, with ϵ a small parameter. (In quantum mechanics, $\mathbf{H}_0$ and $\mathbf{H}_1$ are complex Hermitian, but the final equations below are the same in both cases.)

Suppose $\lambda_{i,0}$, $1 \leq i \leq N$, are the eigenvalues of $\mathbf{H}_0$ and $\mathbf{v}_{i,0}$, $1 \leq i \leq N$, are the corresponding eigenvectors. We assume that the perturbed eigenvalues and eigenvectors are given by the series expansion (in ϵ):

$$\lambda_i = \lambda_{i,0} + \sum_{k=1}^{\infty} \epsilon^k \lambda_{i,k}, \quad \mathbf{v}_i = \mathbf{v}_{i,0} + \sum_{k=1}^{\infty} \epsilon^k \mathbf{v}_{i,k}, \tag{9.1}$$

with the constraint that

$$\|\mathbf{v}_i\| = \|\mathbf{v}_{i,0}\| = 1, \quad 1 \leq i \leq N. \tag{9.2}$$

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Since the quantity $\|\mathbf{v}_i\|$ is constant, its first order variation with respect to ϵ must be zero. This constraint gives that $\mathbf{v}_{i,1} \perp \mathbf{v}_{i,0}$.

We assume that the $\lambda_{i,0}$ are all different, i.e. we consider non-degenerate perturbation theory. Then, plugging (9.1) into

$$\mathbf{H}\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{9.3}$$

and matching the left and right hand side term by term in powers of ϵ, one obtains

$$\lambda_i = \lambda_{i,0} + \epsilon(\mathbf{H}_1)_{ii} + \epsilon^2 \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{|(\mathbf{H}_1)_{ij}|^2}{\lambda_{i,0} - \lambda_{j,0}} + O(\epsilon^3), \tag{9.4}$$

where $(\mathbf{H}_1)_{ij} := \mathbf{v}_{j,0}^T \mathbf{H}_1 \mathbf{v}_{i,0}$, and

$$\mathbf{v}_i = \mathbf{v}_{i,0} + \epsilon \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(\mathbf{H}_1)_{ij}}{\lambda_{i,0} - \lambda_{j,0}} \mathbf{v}_{j,0} + O(\epsilon^2). \tag{9.5}$$

Notice that the first order correction to $\mathbf{v}_i$ is indeed perpendicular to $\mathbf{v}_{i,0}$ as it does not have a component in that direction.

9.1.2 A Stochastic Differential Equation for Eigenvalues

Next we use the above formulas to derive the so-called Dyson Brownian motion (DBM), which gives the evolution of eigenvalues of a random matrix plus a Wigner ensemble whose variance grows linearly with time. Let $\mathbf{M}_0$ be the initial matrix (random or not), and $\mathbf{X}_1$ be a unit Wigner matrix that is independent of $\mathbf{M}_0$. Then we study, using (9.4), the eigenvalues of

$$\mathbf{M} = \mathbf{M}_0 + \sqrt{\mathrm{d}t}\, \mathbf{X}_1, \tag{9.6}$$

where $\mathrm{d}t$ is a small quantity which will be interpreted as a differential time step.

The derivation of the SDE for eigenvalues is much simpler if we use the rotational invariance of the Wigner ensemble. The matrix $\mathbf{X}_1$ has the same law in any basis, we therefore choose to express it in the diagonal basis of $\mathbf{M}_0$. In order to do so we must work with the exact rotationally invariant Wigner ensemble where the diagonal variance is twice the off-diagonal variance.

First, for the first order term (in terms of ϵ), we have

$$(\mathbf{X}_1)_{ii} := \mathbf{v}_{i,0}^T \mathbf{X}_1 \mathbf{v}_{i,0} \sim \mathcal{N}\left(0, \frac{2}{N}\right). \tag{9.7}$$

Note that the $(\mathbf{X}_1)_{ii}$ are independent for different i's.

Then we study the second order term. We have

$$(\mathbf{X}_1)_{ji} := \mathbf{v}_{j,0}^T \mathbf{X}_1 \mathbf{v}_{i,0} \sim \mathcal{N}\left(0, \frac{1}{N}\right). \tag{9.8}$$

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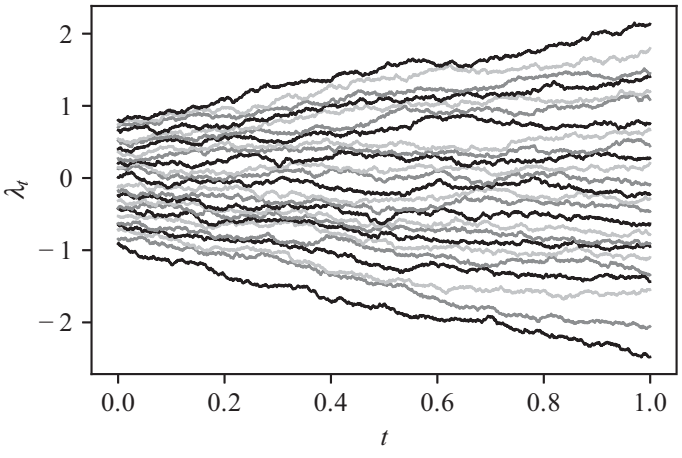

Figure 9.1 A simulation of DBM for an $N = 25$ matrix starting for a Wigner matrix with $\sigma^2 = 1/4$ and evolving for one unit of time.

So $|(\mathbf{X}_1)_{ji}|^2$ is a random variable with mean $1/N$ and with fluctuations around the mean also of order $1/N$. As in Section 8.2.2, one can argue that these fluctuations are negligible when integrated over time in the limit $\mathrm{d}t \to 0$. In other words, $|(\mathbf{X}_1)_{ji}|^2$ can be treated as deterministic.

Now using (9.4), we get that

$$\mathrm{d}\lambda_i = \sqrt{\frac{2}{\beta N}}\mathrm{d}B_i + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\mathrm{d}t}{\lambda_i - \lambda_j}, \tag{9.9}$$

where $\mathrm{d}B_i$ denotes a Brownian increment that comes from the $(\mathbf{X}_1)_{ii}$ term, and we have added a factor β for completeness (equal to 1 for real symmetric matrices). This is the fundamental evolution equation for eigenvalues in a fictitious time that describes how much of a Wigner matrix one progressively adds to the "initial" matrix $\mathbf{M}_0$ (see Fig. 9.1). The astute reader will probably have recognized in the second term a Coulomb force deriving from the logarithmic Coulomb potential $\log|\lambda_i - \lambda_j|$ encountered in Chapter 5.

One can also derive a similar process for the eigenvectors that we give here for $\beta = 1$:

$$\mathrm{d}\mathbf{v}_i = \frac{1}{\sqrt{N}}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\mathrm{d}B_{ij}}{\lambda_i - \lambda_j}\mathbf{v}_j - \frac{1}{2N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\mathrm{d}t}{(\lambda_i - \lambda_j)^2}\mathbf{v}_i, \tag{9.10}$$

where $\mathrm{d}B_{ij} = \mathrm{d}B_{ji}$ $(i \neq j)$ is a symmetric collection of Brownian motions, independent of each other and of the $\{\mathrm{d}B_i\}$ above.

The formulas (9.9) and (9.10) give the Dyson Brownian motion for the stochastic evolution of the eigenvalues and eigenvectors of matrices of the form

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$$\mathbf{M} = \mathbf{M}_0 + \mathbf{X}_t, \tag{9.11}$$

where $\mathbf{M}_0$ is some initial matrix and $\mathbf{X}_t$ is an independent Wigner ensemble with parameter $\sigma^2 = t$. We will call the above matrix process a matrix Dyson Brownian motion.

In our study of large random matrices, we will be interested in the DBM when N is large, but actually the DBM is well defined for any N. As for the Itô lemma, we used the fact that the Gaussian process can be divided into infinitesimal increments and that perturbation theory becomes exact at that scale. We made no assumption about the size of N. We did need a rotationally invariant Gaussian process so the diagonal variance must be twice the off-diagonal one. In the most extreme example of $N = 1$, the eigenvalue of a 1×1 matrix is just the value of its only element. Under DBM it simply undergoes a standard Brownian motion with a variance of 2 per unit time.

Exercise 9.1.1 Variance as a function of time under DBM

Consider the Dyson Brownian motion for a finite N matrix:

$$\mathrm{d}\lambda_i = \sqrt{\frac{2}{N}}\mathrm{d}B_i + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\mathrm{d}t}{\lambda_i - \lambda_j} \tag{9.12}$$

and the function $F(\{\lambda_i\})$ that computes the second moment:

$$F(\{\lambda_i\}) = \frac{1}{N}\sum_{i=1}^{N}\lambda_i^2. \tag{9.13}$$

(a) Write down the stochastic process for $F(\{\lambda_i\})$ using the Itô vectorial formula (8.17). In the case at hand F does not depend explicitly on time and $\sigma_i^2 = 2/N$. You will need to use the following identity:

$$2\sum_{\substack{i,j=1\\ j\neq i}}^{N}\frac{\lambda_i}{\lambda_i - \lambda_j} = \sum_{\substack{i,j=1\\ j\neq i}}^{N}\frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j} = N(N-1). \tag{9.14}$$

(b) Take the expectation value of your equation and show that $F(t) := \mathbb{E}[F(\{\lambda_i(t)\})]$ follows

$$F(t) = F(0) + \frac{N+1}{N}t. \tag{9.15}$$

Do not assume that N is large.

9.2 Dyson Brownian Motion II: Itô Calculus

Another way to derive the Dyson Brownian motion for the eigenvalues is to consider the matrix Brownian motion (9.11) as a Brownian motion of all the elements of the matrix $\mathbf{X}$. We have to treat the diagonal and off-diagonal elements separately because we want to use

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the rotationally invariant Wigner matrix (GOE) with diagonal variance equal to twice the off-diagonal variance. Also, only half of the off-diagonal elements are independent (the matrix $\mathbf{X}$ is symmetric). We have

$$\mathrm{d}X_{kk} = \sqrt{\frac{2}{N}}\mathrm{d}B_{kk} \quad \text{and} \quad \mathrm{d}X_{k\ell} = \sqrt{\frac{1}{N}}\mathrm{d}B_{k\ell} \quad \text{for} \quad k < \ell, \tag{9.16}$$

where $\mathrm{d}B_{kk}$ and $\mathrm{d}B_{k\ell}$ are N and $N(N-1)/2$ independent unit Brownian motions.

Each eigenvalue λ_i is a (complicated) function of all the matrix elements of $\mathbf{X}$. We can use the vectorial form of Itô's lemma (8.17) to write a stochastic differential equation (SDE) for $\lambda_i(\mathbf{X})$:

$$\mathrm{d}\lambda_i = \sum_{k=1}^{N} \frac{\partial \lambda_i}{\partial X_{kk}} \sqrt{\frac{2}{N}}\mathrm{d}B_{kk} + \sum_{\substack{k=1 \\ \ell=k+1}}^{N} \frac{\partial \lambda_i}{\partial X_{k\ell}} \sqrt{\frac{1}{N}}\mathrm{d}B_{k\ell} + \sum_{k=1}^{N} \frac{\partial^2 \lambda_i}{\partial X_{kk}^2} \frac{\mathrm{d}t}{N} + \sum_{\substack{k=1 \\ \ell=k+1}}^{N} \frac{\partial^2 \lambda_i}{\partial X_{k\ell}^2} \frac{\mathrm{d}t}{2N}. \tag{9.17}$$

The key is to be able to compute the following partial derivatives:

$$\frac{\partial \lambda_i}{\partial X_{kk}}; \qquad \frac{\partial \lambda_i}{\partial X_{k\ell}}; \qquad \frac{\partial^2 \lambda_i}{\partial X_{kk}^2}; \qquad \frac{\partial^2 \lambda_i}{\partial X_{k\ell}^2}, \tag{9.18}$$

where $k < \ell$.

Since $\mathbf{X}_t$ is rotational invariant, we can rotate the basis such that $\mathbf{X}_0$ is diagonal:

$$\mathbf{X}_0 = \mathrm{diag}(\lambda_1(0), \ldots, \lambda_N(0)). \tag{9.19}$$

In order to compute the partial derivatives above, we can consider adding one small element to the matrix $\mathbf{X}_0$, and compute the corresponding change in eigenvalues. We first perturb diagonal elements and later we will deal with off-diagonal elements.

A shift of the kth diagonal entry of $\mathbf{X}_0$ by δX_{kk} affects λ_i with $i = k$ in a linear fashion but leaves all other eigenvalues unaffected:

$$\lambda_i \to \lambda_i + \delta X_{kk}\delta_{ki}. \tag{9.20}$$

Thus we have

$$\frac{\partial \lambda_i}{\partial X_{kk}} = \delta_{ik}; \qquad \frac{\partial^2 \lambda_i}{\partial X_{kk}^2} = 0. \tag{9.21}$$

Next we discuss how a perturbation in an off-diagonal entry of $\mathbf{X}_0$ can affect the eigenvalues. A perturbation of the $(k\ell)$ entry by $\delta X_{k\ell} = \delta X_{\ell k}$ entry leads to the following matrix:

$$\mathbf{X}_0 + \delta\mathbf{X} = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_k & & \delta X_{k\ell} & & \\ & & & \ddots & & & \\ & & \delta X_{k\ell} & & \lambda_\ell & & \\ & & & & & \ddots & \\ & & & & & & \lambda_N \end{pmatrix}. \tag{9.22}$$

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Since this matrix is block diagonal (after a simple permutation), the eigenvalues in the $N-2$ other 1×1 blocks are not affected by the perturbation, so trivially

$$\frac{\partial \lambda_i}{\partial X_{k\ell}} = 0; \qquad \frac{\partial^2 \lambda_i}{\partial X_{k\ell}^2} = 0, \qquad \forall i \neq k, \ell. \tag{9.23}$$

On the other hand, the eigenvalues of the block

$$\begin{pmatrix} \lambda_k & \delta X_{k\ell} \\ \delta X_{k\ell} & \lambda_\ell \end{pmatrix} \tag{9.24}$$

are modified and and exact diagonalization gives

$$\lambda_\pm = \frac{\lambda_k + \lambda_\ell}{2} \pm \frac{\lambda_k - \lambda_\ell}{2} \sqrt{1 + \frac{4(\delta X_{k\ell})^2}{(\lambda_k - \lambda_\ell)^2}}. \tag{9.25}$$

We can expand this result to second order in $\delta X_{k\ell}$ to find

$$\lambda_k \to \lambda_k + \frac{(\delta X_{k\ell})^2}{\lambda_k - \lambda_\ell} \quad \text{and} \quad \lambda_\ell \to \lambda_\ell + \frac{(\delta X_{k\ell})^2}{\lambda_\ell - \lambda_k}. \tag{9.26}$$

We thus readily see that the first partial derivative of λ_i with respect to any off-diagonal element is zero:

$$\frac{\partial \lambda_i}{\partial X_{k\ell}} = 0 \text{ for } k < \ell. \tag{9.27}$$

For the second derivative, on the other hand, we obtain

$$\frac{\partial^2 \lambda_i}{\partial X_{k\ell}^2} = \frac{2\delta_{ik}}{\lambda_i - \lambda_\ell} + \frac{2\delta_{i\ell}}{\lambda_i - \lambda_k} \text{ for } k < \ell. \tag{9.28}$$

Of the two terms on the right hand side, the first term exists only if $\ell > i$ while the second term is only present when $k < i$. So, for a given i, only $N-1$ terms of the form $2/(\lambda_i - \lambda_j)$ are present (note that the problematic term $i = j$ is absent). Putting everything back into Eq. (9.17), we find, with $\beta = 1$ here,

$$\mathrm{d}\lambda_i = \sqrt{\frac{2}{\beta N}} \mathrm{d}B_i + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mathrm{d}t}{\lambda_i - \lambda_j}, \tag{9.29}$$

where $\mathrm{d}B_i$ are independent Brownian motions (the old $\mathrm{d}B_{kk}$ for $k = i$). We have thus precisely recovered Equation (9.9) using Itô's calculus.

9.3 The Dyson Brownian Motion for the Resolvent

9.3.1 A Burgers' Equation for the Stieltjes Transform

Consider a matrix $\mathbf{M}_t$ that undergoes a DBM starting from a matrix $\mathbf{M}_0$. At each time t, $\mathbf{M}_t$ can be viewed as the sum of the matrix $\mathbf{M}_0$ and a Wigner matrix $\mathbf{X}_t$ of variance t:

$$\mathbf{M}_t = \mathbf{M}_0 + \mathbf{X}_t. \tag{9.30}$$

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In order to characterize the spectrum of the matrix $\mathbf{M}_t$, one should compute its Stieltjes transform $\mathfrak{g}(z)$, which is the expectation value of the large N limit of function $g_N(z)$, defined by

$$g_N(z,\{\lambda_i\}) := \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i}. \tag{9.31}$$

g_N is thus a function of all eigenvalues $\{\lambda_i\}$ that undergo DBM while z is just a constant parameter. Since the eigenvalues evolve with time g_N is really a function of both z and t. We can use Itô's lemma to write a SDE for $g_N(z,\{\lambda_i\})$. The ingredients are, as usual now, the following partial derivatives:

$$\frac{\partial g_N}{\partial \lambda_i} = \frac{1}{N}\frac{1}{(z-\lambda_i)^2}; \qquad \frac{\partial^2 g_N}{\partial \lambda_i^2} = \frac{2}{N}\frac{1}{(z-\lambda_i)^3}. \tag{9.32}$$

We can now apply Itô's lemma (8.17) using the dynamical equation for eigenvalues (9.9) to find

$$\mathrm{d}g_N = \frac{1}{N}\sqrt{\frac{2}{N}}\sum_{i=1}^{N}\frac{\mathrm{d}B_i}{(z-\lambda_i)^2} + \frac{1}{N^2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\frac{\mathrm{d}t}{(z-\lambda_i)^2(\lambda_i-\lambda_j)} + \frac{2}{N^2}\sum_{i=1}^{N}\frac{\mathrm{d}t}{(z-\lambda_i)^3}. \tag{9.33}$$

We now massage the second term to arrive at a symmetric form in i and j. In order to do so, note that i and j are dummy indices that are summed over, so we can rename $i \to j$ and vice versa and get the same expression. Adding the two versions and dividing by 2, we get that this term is equal to

$$\begin{aligned}
&\frac{1}{N^2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\frac{\mathrm{d}t}{(z-\lambda_i)^2(\lambda_i-\lambda_j)} = \frac{1}{2N^2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\left[\frac{\mathrm{d}t}{(z-\lambda_i)^2(\lambda_i-\lambda_j)} + \frac{\mathrm{d}t}{(z-\lambda_j)^2(\lambda_j-\lambda_i)}\right]\\
&=\frac{1}{2N^2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\frac{(2z-\lambda_i-\lambda_j)\mathrm{d}t}{(z-\lambda_i)^2(z-\lambda_j)^2} = \frac{1}{N^2}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\frac{\mathrm{d}t}{(z-\lambda_i)(z-\lambda_j)^2}\\
&=\frac{1}{N^2}\sum_{i,j=1}^{N}\frac{\mathrm{d}t}{(z-\lambda_i)(z-\lambda_j)^2} - \frac{1}{N^2}\sum_{i=1}^{N}\frac{\mathrm{d}t}{(z-\lambda_i)^3} = -g_N\frac{\partial g_N}{\partial z}\mathrm{d}t - \frac{1}{N^2}\sum_{i=1}^{N}\frac{\mathrm{d}t}{(z-\lambda_i)^3}.
\end{aligned} \tag{9.34}$$

Note that very similar manipulations have been used in Section 5.2.2. Thus we have

$$\begin{aligned}
\mathrm{d}g_N &= \frac{1}{N}\sqrt{\frac{2}{N}}\sum_{i=1}^{N}\frac{\mathrm{d}B_i}{(z-\lambda_i)^2} - g_N\frac{\partial g_N}{\partial z}\mathrm{d}t + \frac{1}{N^2}\sum_{i=1}^{N}\frac{\mathrm{d}t}{(z-\lambda_i)^3}\\
&= \frac{1}{N}\sqrt{\frac{2}{N}}\sum_{i=1}^{N}\frac{\mathrm{d}B_i}{(z-\lambda_i)^2} - g_N\frac{\partial g_N}{\partial z}\mathrm{d}t + \frac{1}{2N}\frac{\partial^2 g_N}{\partial z^2}\mathrm{d}t.
\end{aligned} \tag{9.35}$$

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Taking now the expectation of this equation (such that the dB_i term vanishes), we get

$$\mathbb{E}[dg_N(z)] = -\mathbb{E}\left[g_N(z)\frac{\partial g_N(z)}{\partial z}\right]dt + \frac{1}{2N}\mathbb{E}\left[\frac{\partial^2 g_N(z)}{\partial z^2}\right]dt. \tag{9.36}$$

This equation is exact for any N. We can now take the $N \to \infty$ limit, where $g_N(z) \to \mathfrak{g}_t(z)$. Using the fact that the Stieltjes transform is self-averaging, we get a PDE for the time dependent Stieltjes transform $\mathfrak{g}_t(z)$:

$$\frac{\partial \mathfrak{g}_t(z)}{\partial t} = -\mathfrak{g}_t(z)\frac{\partial \mathfrak{g}_t(z)}{\partial z}. \tag{9.37}$$

Equation (9.37) is called the inviscid Burgers' equation. Such an equation can in fact develop singularities, so it often needs to be regularized by a viscosity term, which is in fact present for finite N: it is precisely the last term of Eq. (9.36). Equation (9.37) can be exactly solved using the methods of characteristics. This will be the topic of Section 10.1 in the next chapter.

9.3.2 The Evolution of the Resolvent

Let us now consider the full matrix resolvent of $\mathbf{M}_t$, defined as $\mathbf{G}_t(z) = (z\mathbf{1} - \mathbf{M}_t)^{-1}$. Clearly, the quantity $\mathfrak{g}_t(z)$ studied above is simply the trace of $\mathbf{G}_t(z)$, but $\mathbf{G}_t(z)$ also contains information on the evolution of eigenvectors. Since each element of $\mathbf{G}_t$ depends on all the elements of $\mathbf{M}_t$, one can again use Itô's calculus to derive an evolution equation for $\mathbf{G}_t(z)$. The calculation is more involved because one needs to carefully keep track of all indices. In this technical section, we sketch the derivation of Dyson Brownian motion for the resolvent and briefly discuss the result, which will be used further in Section 10.1.

Since $\mathbf{M}_t = \mathbf{M}_0 + \mathbf{X}_t$, the Itô lemma gives

$$dG_{ij}(z) = \sum_{k,\ell=1}^{N} \frac{\partial G_{ij}}{\partial M_{k\ell}} dX_{k\ell} + \frac{1}{2}\sum_{k,\ell,m,n=1}^{N} \frac{\partial^2 G_{ij}}{\partial M_{k\ell}\partial M_{mn}} d\big[X_{k\ell}X_{mn}\big], \tag{9.38}$$

where the last term denotes the covariation of $X_{k\ell}$ and X_{mn}, and we have considered M_{kl} and M_{lk} to be independent variables following 100% correlated Brownian motions. Next, we compute the derivatives

$$\frac{\partial G_{ij}}{\partial M_{k\ell}} = \frac{1}{2}\big[G_{ik}G_{j\ell} + G_{jk}G_{i\ell}\big], \tag{9.39}$$

from which we deduce the second derivatives

$$\frac{\partial^2 G_{ij}}{\partial M_{k\ell}\partial M_{mn}} = \frac{1}{4}\big[(G_{im}G_{kn} + G_{im}G_{kn})\,G_{j\ell} + \cdots\big], \tag{9.40}$$

where we have not written the other GGG products obtained by applying Eq. (9.39) twice. Now, using the properties of the Brownian noise, the quadratic covariation reads

$$d\big[X_{k\ell}X_{mn}\big] = \frac{dt}{N}\Big(\delta_{km}\delta_{\ell n} + \delta_{kn}\delta_{\ell m}\Big), \tag{9.41}$$

so that we get from (9.38) and taking into account symmetries:

$$dG_{ij}(z,t) = \sum_{k,\ell=1}^{N} G_{ik}G_{j\ell}dX_{k\ell} + \frac{1}{N}\sum_{k,\ell=1}^{N}\Big(G_{ik}G_{\ell k}G_{\ell j} + G_{ik}G_{kj}G_{\ell\ell}\Big)dt. \tag{9.42}$$

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If we now take the average over the Brownian motions $\mathrm{d}X_{k\ell}$, we find the following evolution for the average resolvent:

$$\partial_t \mathbb{E}[\mathbf{G}_t(z)] = \mathrm{Tr}\,\mathbf{G}_t(z)\,\mathbb{E}[\mathbf{G}_t^2(z)] + \frac{1}{N}\mathbb{E}[\mathbf{G}_t^3(z)]. \tag{9.43}$$

Now, one can notice that

$$\mathbf{G}^2(z,t) = -\partial_z \mathbf{G}(z,t); \qquad \mathbf{G}^3(z,t) = \frac{1}{2}\partial_{zz}^2 \mathbf{G}(z,t), \tag{9.44}$$

which hold even before averaging. By sending $N \to \infty$, we then obtain the following matrix PDE for the resolvent:

$$\partial_t \mathbb{E}[\mathbf{G}_t(z)] = -\mathfrak{g}_t(z)\,\partial_z \mathbb{E}[\mathbf{G}_t(z)], \quad \text{with} \quad \mathbb{E}[\mathbf{G}_0(z)] = \mathbf{G}_{\mathbf{M}_0}(z). \tag{9.45}$$

Note that this equation is *linear* in $\mathbf{G}_t(z)$ once the Stieltjes transform $\mathfrak{g}_t(z)$ is known. Taking the trace of Eq. (9.45) immediately leads back to the Burgers' equation (9.37) for $\mathfrak{g}_t(z)$ itself, as expected.

9.4 The Dyson Brownian Motion with a Potential

9.4.1 A Modified Langevin Equation for Eigenvalues

In this section, we modify Dyson Brownian motion by adding a potential such that the stationary state of these interacting random walks coincides with the eigenvalue measure of β-ensembles, namely

$$P(\{\lambda_i\}) = Z_N^{-1} \exp\left\{ -\frac{\beta}{2}\left[\sum_{i=1}^{N} N V(\lambda_i) - \sum_{\substack{i,\,j=1 \\ j\neq i}}^{N} \log|\lambda_i - \lambda_j| \right] \right\}. \tag{9.46}$$

The general vectorial Langevin equation (8.36) leading to such an equilibrium with $\sigma^2 = 2/N$ immediately gives us the following DBM in a potential $V(\lambda)$:

$$\mathrm{d}\lambda_k = \sqrt{\frac{2}{N}}\mathrm{d}B_k + \left(\frac{1}{N}\sum_{\substack{\ell=1 \\ \ell\neq k}}^{N} \frac{\beta}{\lambda_k - \lambda_\ell} - \frac{\beta}{2}V'(\lambda_k) \right)\mathrm{d}t, \tag{9.47}$$

which recovers Eq. (9.9) in the absence of a potential. See Figure 9.2 for an illustration.

Dyson Brownian motion in a potential has many applications. Numerically it can be used to generate matrices for an arbitrary potential and an arbitrary value of β, a task not obvious *a priori* from the definition (9.46). Figure 9.2 shows a simulation of the matrix potential studied in Section 5.3.3. Note that DBM generates the correct density of eigenvalues; it also generates the proper statistics for the joint distribution of eigenvalues.

It is interesting to see how Burgers' equation for the Stieltjes transform, Eq. (9.37), is changed in the case where $V(\lambda) = \lambda^2/2$, i.e. in the standard GOE case $\beta = 1$. Redoing the steps leading to Eq. (9.36) with the extra V' term in the right hand side of Eq. (9.47) modifies the Burgers' equation into

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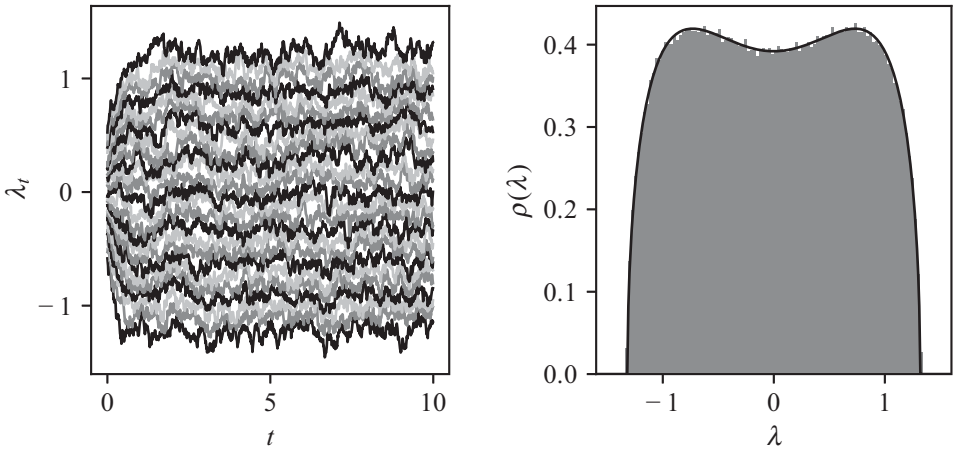

Figure 9.2 (left) A simulation of DBM with a potential for an $N = 25$ matrix starting from a Wigner matrix with $\sigma^2 = 1/10$ and evolving within the potential $V(x) = \frac{x^2}{2} + \frac{x^4}{4}$ for 10 units of time. Note that the steady state is quickly reached (within one or two units of time). (right) Histogram of the eigenvalues for the same process for $N = 400$ and 200 discrete steps per unit time. The histogram is over all matrices from time 3 to 10 (560 000 points). The agreement with the theoretical density (Eq. (5.58)) is very good.

$$\frac{\partial \mathfrak{g}_t(z)}{\partial t} = -\mathfrak{g}_t(z)\frac{\partial \mathfrak{g}_t(z)}{\partial z} + \frac{1}{2}\frac{\partial (z\mathfrak{g}_t(z))}{\partial z}. \tag{9.48}$$

The solution to this equation will be discussed in the next chapter.

More theoretically, DBM can be used in proofs of *local universality*, which is one of the most important results in random matrix theory. Local universality is the concept that many properties of the joint law of eigenvalues do not depend on the specifics of the random matrix in question, provided one looks at them on a scale ξ comparable to the average distance between eigenvalues, i.e. on scales $N^{-1} \lesssim \xi \ll 1$. Many such properties arise from the logarithmic eigenvalue repulsion and indeed depend only on the symmetry class (β) of the model.

Another useful property of DBM is its speed of convergence to the steady state. With time normalized as in Eq. (9.47), global properties (such as the density of eigenvalues) converge in a time of order 1, as we discuss in the next subsection. Local properties on the other hand (e.g. eigenvalue spacing) converge much faster, in a time of order $1/N$, i.e. as soon as the eigenvalues have "collided" a few times with one another.[1]

[1] The time needed for two Brownian motions a distance d apart to meet for the first time is of order d^2/σ^2. In our case $d = 1/N$ (the typical distance between two eigenvalues) and $\sigma^2 = 2/N$. The typical collision time is therefore $(2N)^{-1}$. Note however that eigenvalues actually never cross under Eq. (9.47), but the corresponding eigenvectors strongly mix when such quasi-collisions occur.

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Exercise 9.4.1 Moments under DBM with a potential

Consider the moments of the eigenvalues as a function of time:

$$F_k(t) = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^k(t), \tag{9.49}$$

for eigenvalues undergoing DBM under a potential $V(x)$, Eq. (9.47), in the orthogonal case $\beta = 1$. In this exercise you will need to show (and to use) the following identity:

$$2\sum_{\substack{i,j=1\\ j\neq i}}^{N} \frac{\lambda_i^k}{\lambda_i - \lambda_j} = \sum_{\substack{i,j=1\\ j\neq i}}^{N} \frac{\lambda_i^k - \lambda_j^k}{\lambda_i - \lambda_j} = \sum_{\substack{i,j=1\\ j\neq i}}^{N} \sum_{\ell=0}^{k} \lambda_i^\ell \lambda_j^{k-\ell}. \tag{9.50}$$

(a) Using Itô calculus, write a SDE for $F_2(t)$.

(b) By taking the expectation value of your equation, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[F_2(t)] = 1 - \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} \lambda_i(t) V'(\lambda_i(t))\right] + \frac{1}{N}. \tag{9.51}$$

(c) In the Wigner case, $V'(x) = x$, find the steady-state value of $\mathbb{E}[F_2(t)]$ for any finite N.

(d) For a random matrix $\mathbf{X}$ drawn from a generic potential $V(x)$, show that in the large N limit, we have

$$\tau\left[V'(\mathbf{X})\mathbf{X}\right] = 1, \tag{9.52}$$

where τ is the expectation value of the normalized trace defined by (2.1).

(e) Show that this equation is consistent with $\tau(\mathbf{W}) = 1$ for a Wishart matrix whose potential is given by Eq. (5.4).

(f) In the large N limit, find a general expression for $\tau[V'(\mathbf{X})\mathbf{X}^k]$ by writing the steady-state equation for $\mathbb{E}[F_{k+1}(t)]$; you can neglect the Itô term. The first two should be given by

$$\tau[V'(\mathbf{X})\mathbf{X}^2] = 2\tau[\mathbf{X}] \quad \text{and} \quad \tau[V'(\mathbf{X})\mathbf{X}^3] = 2\tau[\mathbf{X}^2] + \tau[\mathbf{X}]^2. \tag{9.53}$$

(g) In the unit Wigner case $V'(x) = x$, show that your relation in (f) is equivalent to the Catalan number inductive relation (3.27), with $\tau(\mathbf{X}^{2m}) = C_m$ and $\tau(\mathbf{X}^{2m+1}) = 0$.

9.4.2 The Fokker–Planck Equation for DBM

From the Langevin equation, Eq. (9.47), one can derive the Fokker–Planck equation describing the time evolution of the joint distribution of eigenvalues, $P(\{\lambda_i\}, t)$. It reads

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$$\frac{\partial P}{\partial t} = \frac{1}{N}\sum_{i=1}^{N}\frac{\partial}{\partial \lambda_i}\left[\frac{\partial P}{\partial \lambda_i} - \mathcal{F}_i P\right], \tag{9.54}$$

where we use P as an abbreviation for $P(\{\lambda_i\},t)$, and, for a quadratic confining potential $V(\lambda) = \lambda^2/2$, a generalized force given by

$$\mathcal{F}_i := \beta \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{\lambda_i - \lambda_j} - \frac{N\beta\lambda_i}{2}. \tag{9.55}$$

The trick now is to introduce an auxiliary function $W(\{\lambda_i\},t)$, defined as

$$P(\{\lambda_i\},t) := \exp\left[\frac{\beta}{4}\sum_{\substack{i,j=1 \\ j\neq i}}^{N}\log|\lambda_i - \lambda_j| - \frac{\beta N}{8}\sum_{i=1}^{N}\lambda_i^2\right] W(\{\lambda_i\},t). \tag{9.56}$$

Then after a little work, one finds the following evolution equation for W:[2]

$$\frac{\partial W}{\partial t} = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial^2 W}{\partial \lambda_i^2} - \mathcal{V}_i W\right], \tag{9.58}$$

with

$$\mathcal{V}_i(\{\lambda_j\}) := \frac{\beta^2 N^2}{16}\lambda_i^2 - \frac{\beta(2-\beta)}{4}\sum_{\substack{j=1 \\ j\neq i}}^{N}\frac{1}{(\lambda_i - \lambda_j)^2} - \frac{N\beta}{4}\left(1 + \frac{\beta(N-1)}{2}\right). \tag{9.59}$$

Looking for W_Γ such that

$$\frac{\partial W_\Gamma}{\partial t} = -\Gamma W_\Gamma, \tag{9.60}$$

one finds that W_Γ is the solution of the following eigenvalue problem:

$$\frac{2}{N}\sum_{i=1}^{N}\left[-\frac{1}{2}\frac{\partial^2 W_\Gamma}{\partial \lambda_i^2} + \frac{1}{2}\mathcal{V}_i W_\Gamma\right] = \Gamma W_\Gamma. \tag{9.61}$$

One notices that Eq. (9.61) is a (real) Schrodinger equation for N interacting "particles" in a quadratic potential, with an interacting potential that depends on the inverse of the square distance between particles. This is called the Calogero model, which happens to be exactly soluble in one dimension, both classically and quantum mechanically. In particular, the whole spectrum of this Hamiltonian is known, and given by[3]

$$\Gamma(n_1, n_2, \ldots, n_N) = \frac{\beta}{2}\left(\sum_{i=1}^{N} n_i - \frac{N(N-1)}{2}\right); \qquad 0 \leq n_1 < n_2 \cdots < n_N. \tag{9.62}$$

[2] One has to use, along the line, the following identity:

$$\sum_{i\neq j\neq k}\frac{1}{\lambda_i - \lambda_j}\frac{1}{\lambda_i - \lambda_k} \equiv 0. \tag{9.57}$$

[3] Because $W_\Gamma(\{\lambda_i\})$ must vanish as two λ's coincide, one must choose the so-called fermionic branch of the spectrum.

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The smallest eigenvalue corresponds to $n_1 = 0, n_2 = 1, \ldots, n_N = N-1$ and is such that $\Gamma \equiv 0$. This corresponds to the equilibrium state of the Fokker–Planck equation:

$$W_0(\{\lambda_i\}) = \exp\left[\frac{\beta}{4}\sum_{\substack{i,j=1\\ j\neq i}}^{N}\log|\lambda_i-\lambda_j| - \frac{\beta N}{8}\sum_{i=1}^{N}\lambda_i^2\right]. \tag{9.63}$$

All other "excited states" have positive Γ's, corresponding to exponentially decaying modes (in time) of the Fokker–Planck equation. The smallest, non-zero value of Γ is such that $n_N = N$, all other values of n_i being unchanged. Hence Γ_1 is equal to $\beta/2$.

In conclusion, we have explicitly shown that the equilibration time of the DBM in a quadratic potential is equal to $2/\beta$. As announced at the end of the previous section, the density of eigenvalues indeed converges in a time of order unity.

The case $\beta = 2$ is particularly simple, since the interaction term in $\mathcal{V}_i$ disappears completely. We will use this result to show that, in the absence of a quadratic potential, the time dependent joint distribution of eigenvalues, $P(\{\lambda_i\}, t)$, can be expressed, for $\beta = 2$, as a simple determinant: this is the so-called *Karlin–McGregor* representation, see next section.

9.5 Non-Intersecting Brownian Motions and the Karlin–McGregor Formula

Let $p(y,t|x)$ be the probability density that a Brownian motion starting at x at $t = 0$ is at y at time t

$$p(y,t|x) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right), \tag{9.64}$$

where we set $\sigma^2 = 1$. Note that $p(y,t|x)$ obeys the diffusion equation

$$\frac{\partial p(y,t|x)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(y,t|x)}{\partial y^2}. \tag{9.65}$$

We now consider N independent Brownian motions starting at $\mathbf{x} = (x_1, x_2, \ldots x_N)$ at $t = 0$, with $x_1 > x_2 > \ldots > x_N$. The Karlin–McGregor formula states that the probability $P_{KM}(\mathbf{y}, t|\mathbf{x})$ that these Brownian motions have reached positions $\mathbf{y} = (y_1 > y_2 > \ldots > y_N)$ at time t without ever intersecting between 0 and t is given by the following determinant:

$$P_{\mathrm{KM}}(\mathbf{y}, t|\mathbf{x}) = \begin{vmatrix} p(y_1,t|x_1) & p(y_1,t|x_2) & \ldots & p(y_1,t|x_N) \\ p(y_2,t|x_1) & p(y_2,t|x_2) & \ldots & p(y_2,t|x_N) \\ \vdots & \vdots & \ddots & \vdots \\ p(y_N,t|x_1) & p(y_N,t|x_2) & \ldots & p(y_N,t|x_N) \end{vmatrix}. \tag{9.66}$$

One can easily prove this by noting that the determinant involves sums of products of N terms $p(y_i,t|x_j)$, each product Π involving one and only one y_i. Each product Π therefore obeys the multidimensional diffusion equation:

$$\frac{\partial \Pi}{\partial t} = \frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2 \Pi}{\partial y_i^2}. \tag{9.67}$$

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Being the sum of such products, $P_{\rm KM}(\mathbf{y},t|\mathbf{x})$ also obeys the same diffusion equation, as it should since we consider N independent Brownian motions:

$$\frac{\partial P_{\rm KM}(\mathbf{y},t|\mathbf{x})}{\partial t} = \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 P_{\rm KM}(\mathbf{y},t|\mathbf{x})}{\partial y_i^2}, \tag{9.68}$$

The determinant structure also ensures that $P_{\rm KM}(\mathbf{y},t|\mathbf{x}) = 0$ as soon as any two y's are equal. Finally, because the x's and the y's are ordered, $P_{\rm KM}(\mathbf{y},t=0|\mathbf{x})$ obeys the correct initial condition.

Note that the total survival probability $\mathcal{P}(t;\mathbf{x}) := \int \mathrm{d}\mathbf{y}\, P_{\rm KM}(\mathbf{y},t|\mathbf{x})$ decreases with time, since as soon as two Brownian motions meet, the corresponding trajectory is killed. In fact, one can show that $\mathcal{P}(t;\mathbf{x}) \sim t^{-N(N-1)/4}$ at large times.

Now, the probability $P(\mathbf{y},t|\mathbf{x})$ that these N independent Brownian motions end at $\mathbf{y}$ at time t conditional on the fact that the paths never ever intersect, i.e. for any time between $t = 0$ and $t = \infty$, turns out to be given by a very similar formula:

$$P(\mathbf{y},t|\mathbf{x}) = \frac{\Delta(\mathbf{y})}{\Delta(\mathbf{x})} P_{\rm KM}(\mathbf{y},t|\mathbf{x}), \tag{9.69}$$

where $\Delta(\mathbf{x})$ is the Vandermonde determinant $\prod_{i<j}(x_j - x_i)$ (and similarly for $\Delta(\mathbf{y})$).

What we want to show is that $P(\mathbf{y},t|\mathbf{x})$ is the solution of the Fokker–Planck equation for the Dyson Brownian motion, Eq. (9.54), with $\beta = 2$ and in the absence of any confining potential. Indeed, as shown above, $P_{\rm KM}(\mathbf{y},t|\mathbf{x})$ obeys the diffusion equation for N independent Brownian motions with the annihilation boundary condition $P_{\rm KM}(\mathbf{y},t|\mathbf{x}) = 0$ when $y_i = y_j$ for any given pair i, j.

Now compare with the definition Eq. (9.56) of W for the Dyson Brownian motions with $\beta = 2$ and without any confining potential:

$$P(\{\lambda_i\},t) := \exp\left[\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{N} \log|\lambda_i - \lambda_j|\right] W(\{\lambda_i\},t) \equiv \Delta(\{\lambda_i\})W(\{\lambda_i\},t). \tag{9.70}$$

From Eq. (9.58) we see that in the present case $W(\{\lambda_i\},t)$ also obeys the diffusion equation for N independent Brownian motions. Since $P(\{\lambda_i\},t) \sim |\lambda_i - \lambda_j|^2$ when two eigenvalues are close, we also see that $W(\{\lambda_i\},t)$ vanishes linearly whenever two eigenvalues meet, and therefore obeys the same boundary conditions as $P_{\rm KM}(\mathbf{y},t|\mathbf{x})$ with $y_i = \lambda_i$.

The conclusion is therefore that the Dyson Brownian motion without external forces is, for $\beta = 2$, equivalent to N Brownian motions constrained to never ever intersect.

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