10

Addition of Large Random Matrices

In this chapter we seek to understand how the eigenvalue density of the sum of two large random matrices **A** and **B** can be obtained from their individual densities. In the case where, say, **A** is a Wigner matrix **X**, the Dyson Brownian motion formalism of the previous chapter allows us to swiftly answer that question. We will see that a particular transform of the density of **B**, called the R-transform, appears naturally. We then show that the R-transform appears in the more general context where the eigenbases of **A** and **B** are related by a random rotation matrix **O**. In this case, one can construct a Fourier transform for matrices, which allows us to define the analog of the generating function for random variables. As in the case of IID random variables, the logarithm of this matrix generating function is additive when one adds two randomly rotated, large matrices. The derivative of this object turns out to be the R-transform, leading to the central result of the present chapter (and of the more abstract theory of free variables, see Chapter 11): the R-transform of the sum of two randomly rotated, large matrices is equal to the sum of R-transforms of each individual matrix.

10.1 Adding a Large Wigner Matrix to an Arbitrary Matrix

Let $\mathbf{M}_t = \mathbf{M}_0 + \mathbf{X}_t$ be the sum of a large matrix $\mathbf{M}_0$ and a large Wigner matrix $\mathbf{X}_t$, such that the variance of each element grows as t. This defines a Dyson Brownian motion as described in the previous chapter, see Eq. (9.11). We have shown in Section 9.3.1 that in this case the Stieltjes transform $\mathfrak{g}_t(z)$ of $\mathbf{M}_t$ satisfies the Burgers' equation:

$$\frac{\partial \mathfrak{g}_t(z)}{\partial t} = -\mathfrak{g}_t(z)\frac{\partial \mathfrak{g}_t(z)}{\partial z}, \tag{10.1}$$

with initial condition $\mathfrak{g}_0(z) := \mathfrak{g}_{M_0}(z)$. We now proceed to show that the solution of this Burgers' equation can be simply expressed using an $\mathbf{M}_0$ dependent function: its R-transform.

Using the so-called method of characteristics, one can show that

$$\mathfrak{g}_t(z) = \mathfrak{g}_0(z - t\mathfrak{g}_t(z)). \tag{10.2}$$

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If the method of characteristics is unknown to the reader, one can verify that (10.2) indeed satisfies Eq. (10.1) for any function $\mathfrak{g}_0(z)$. Indeed, let us compute $\partial_t \mathfrak{g}_t(z)$ and $\partial_z \mathfrak{g}_t(z)$ using Eq. (10.2):

$$\partial_t \mathfrak{g}_t(z) = \mathfrak{g}_0'(z - t\mathfrak{g}_t(z))\,[-\mathfrak{g}_t(z) - t\partial_t \mathfrak{g}_t(z)] \Rightarrow \partial_t \mathfrak{g}_t(z) = -\frac{\mathfrak{g}_t(z)\mathfrak{g}_0'(z - t\mathfrak{g}_t(z))}{1 + t\mathfrak{g}_0'(z - t\mathfrak{g}_t(z))}, \quad (10.3)$$

and

$$\partial_z \mathfrak{g}_t(z) = \mathfrak{g}_0'(z - t\mathfrak{g}_t(z))\left[1 - t\partial_z \mathfrak{g}_t(z)\right] \Rightarrow \partial_z \mathfrak{g}_t(z) = \frac{\mathfrak{g}_0'(z - t\mathfrak{g}_t(z))}{1 + t\mathfrak{g}_0'(z - t\mathfrak{g}_t(z))}, \quad (10.4)$$

such that Eq. (10.1) is indeed satisfied.

Example: Suppose $\mathbf{M}_0 = 0$. Then we have $\mathfrak{g}_0(z) = z^{-1}$. Plugging into (10.2), we obtain that

$$\mathfrak{g}_t(z) = \frac{1}{z - t\mathfrak{g}_t(z)}, \quad (10.5)$$

which is the self-consistent Eq. (2.35) in the Wigner case with $\sigma^2 = t$. Indeed, if we start with the zero matrix, then $\mathbf{M}_t = \mathbf{X}_t$ is just a Wigner with parameter $\sigma^2 = t$.

Back to the general case, we denote as $\mathfrak{z}_t(g)$ the inverse function[1] of $\mathfrak{g}_t(z)$. Now fix $g = \mathfrak{g}_t(z) = \mathfrak{g}_0(z - tg)$ and $z = \mathfrak{z}_t(g)$, we apply the function $\mathfrak{z}_0$ to g and get

$$\begin{aligned} \mathfrak{z}_0(g) &= z - tg = \mathfrak{z}_t(g) - tg, \\ \mathfrak{z}_t(g) &= \mathfrak{z}_0(g) + tg. \end{aligned} \quad (10.6)$$

The inverse of the Stieltjes transform of $\mathbf{M}_t$ is given by the inverse of that of $\mathbf{M}_0$ plus a simple shift tg. If we know $\mathfrak{g}_0(z)$ we can compute its inverse $\mathfrak{z}_0(g)$ and thus easily obtain $\mathfrak{z}_t(g)$, which after inversion hopefully recovers $\mathfrak{g}_t(z)$.

Example: Suppose $\mathbf{M}_0$ is a Wigner matrix with variance σ^2. We first want to compute the inverse of $\mathfrak{g}_0(z)$; to do so we use the fact that $\mathfrak{g}_0(z)$ satisfies Eq. (2.35), and we get that

$$\mathfrak{z}_0(g) = \sigma^2 g + \frac{1}{g}. \quad (10.7)$$

Then, by (10.6), we get that

$$\mathfrak{z}_t(g) = \mathfrak{z}_0(g) + tg = \left(\sigma^2 + t\right) g + \frac{1}{g}, \quad (10.8)$$

which is the inverse Stieltjes transform for Wigner matrices with variance $\sigma^2 + t$. In other words $\mathfrak{g}_t(z)$ satisfies the Wigner equation (2.35) with σ^2 replaced by $\sigma^2 + t$. This result is not surprising, each element of the sum of two Wigner matrices is just the sum of Gaussian random variables. So $\mathbf{M}_t$ is itself a Wigner matrix with the sum of the variances as its variance.

[1] We will discuss in Section 10.4 the invertibility of the function $\mathfrak{g}(z)$.

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We can now tackle the more general case when the initial matrix is not necessarily Wigner. Call $\mathbf{B} = \mathbf{M}_t$ and $\mathbf{A} = \mathbf{M}_0$. Then by (10.6), we get

$$\mathfrak{z}_{\mathbf{B}}(g) = \mathfrak{z}_{\mathbf{A}}(g) + tg = \mathfrak{z}_{\mathbf{A}}(g) + \mathfrak{z}_{\mathbf{X}_t}(g) - \frac{1}{g}. \tag{10.9}$$

We now define the R-transform as

$$R(g) := \mathfrak{z}(g) - \frac{1}{g}. \tag{10.10}$$

Note that the R-transform of a Wigner matrix of variance t is simply given by

$$R_{\mathbf{X}}(g) = tg. \tag{10.11}$$

This definition allows us to rewrite Eq. (10.9) above as a nice additive relation between R-transforms:

$$R_{\mathbf{B}}(g) = R_{\mathbf{A}}(g) + R_{\mathbf{X}_t}(g). \tag{10.12}$$

In the next section we will generalize this law of addition to (large) matrices $\mathbf{X}$ that are not necessarily Wigner. The R-transform will prove to be a very powerful tool to study large random matrices. Some of its properties are left to be derived in Exercises 10.1.1 and 10.1.2 and will be further discussed in Chapter 15. We finish this section by computing the R-transform of a white Wishart matrix. Remember that its Stieltjes transform satisfies Eq. (4.37), i.e.

$$qz\mathfrak{g}^2 - (z - 1 + q)\mathfrak{g} + 1 = 0, \tag{10.13}$$

which can be written in terms of the inverse function $\mathfrak{z}(g)$:

$$\mathfrak{z}(g) = \frac{1}{1 - qg} + \frac{1}{g}. \tag{10.14}$$

From which we can read off the R-transform:

$$R_{\mathbf{W}}(g) = \frac{1}{1 - qg}. \tag{10.15}$$

Exercise 10.1.1 Taylor series for the R-transform

Let $\mathfrak{g}(z)$ be the Stieltjes transform of a random matrix $\mathbf{M}$:

$$\mathfrak{g}(z) = \tau\left((z\mathbf{1} - \mathbf{M})^{-1}\right) = \int_{\mathrm{supp}\{\rho\}} \frac{\rho(\lambda)\mathrm{d}\lambda}{z - \lambda}. \tag{10.16}$$

We saw that the power series of $g(z)$ around $z = \infty$ is given by the moments of $\mathbf{M}$ ($m_n := \tau(\mathbf{M}^n)$):

$$g(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}} \text{ with } m_0 \equiv 1. \tag{10.17}$$

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Call $\mathfrak{z}(g)$ the functional inverse of $\mathfrak{g}(z)$ which is well defined in a neighborhood of $g = 0$. And define $R(g)$ as

$$R(g) = \mathfrak{z}(g) - 1/g. \tag{10.18}$$

(a) By writing the power series of $R(g)$ near zero, show that $R(g)$ is regular at zero and that $R(0) = m_1$. Therefore the power series of $R(g)$ starts at g^0:

$$R(g) = \sum_{n=1}^{\infty} \kappa_n g^{n-1}. \tag{10.19}$$

(b) Now assume $m_1 = \kappa_1 = 0$ and compute κ_2, κ_3 and κ_4 as a function of m_2, m_3 and m_4 in that case.

Exercise 10.1.2 Scaling of the R-transform

Using your answer from Exercise 2.3.1: If $\mathbf{A}$ is a random matrix drawn from a well-behaved ensemble with Stieltjes transform $\mathfrak{g}_{\mathbf{A}}(z)$ and R-transform $R_{\mathbf{A}}(g)$, what is the R-transform of the random matrices $\alpha\mathbf{A}$ and $\mathbf{A} + b\mathbf{1}$ where α and b are non-zero real numbers?

Exercise 10.1.3 Sum of symmetric orthogonal and Wigner matrices

Consider as in Exercise 1.2.4 a random symmetric orthogonal matrix $\mathbf{M}$ and a Wigner matrix $\mathbf{X}$ of variance σ^2. We are interested in the spectrum of their sum $\mathbf{E} = \mathbf{M} + \mathbf{X}$.

(a) Given that the eigenvalues of $\mathbf{M}$ are ± 1 and that in the large N limit each eigenvalue appears with weight $\frac{1}{2}$, write the limiting Stieltjes transform $\mathfrak{g}_{\mathbf{M}}(z)$.

(b) $\mathbf{E}$ can be thought of as undergoing Dyson Brownian motion starting at $\mathbf{E}(0) = \mathbf{M}$ and reaching the desired $\mathbf{E}$ at $t = \sigma^2$. Use Eq. (10.2) to write an equation for $\mathfrak{g}_{\mathbf{E}}(z)$. This will be a cubic equation in $\mathfrak{g}$.

(c) You can obtain the same equation using the inverse function $\mathfrak{z}_{\mathbf{M}}(g)$ of your answer in (a). Show that

$$\mathfrak{z}_{\mathbf{M}}(g) = \frac{1 + \sqrt{1 - 4g^2}}{2g}, \tag{10.20}$$

where one had to pick the root that makes $\mathfrak{z}(g) \sim 1/g$ near $g = 0$.

(d) Using Eq. (10.6), write $z_{\mathbf{E}}(g)$ and invert this relation to obtain an equation for $\mathfrak{g}_{\mathbf{E}}(z)$. You should recover the same equation as in (b).

(e) Eigenvalues of $\mathbf{E}$ will be located where your equation admits non-real solutions for real z. First look at $z = 0$; the equation becomes quadratic after factorizing a trivial root. Find a criterion for σ^2 such that the equation admits non-real solutions. Compare with your answer in Exercise 1.2.4 (b).

(f) At $\sigma^2 = 1$, the equation is still cubic but is somewhat simpler. A real cubic equation of the form $ax^3 + bx^2 + cx + d = 0$ will have non-real solutions iff $\Delta < 0$ where $\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$. Using this

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criterion show that for $\sigma^2 = 1$ the edges of the eigenvalue spectrum are given by $\lambda = \pm 3\sqrt{3}/2 \approx \pm 2.60$.

(g) Again at $\sigma^2 = 1$, the solution near $g(0) = 0$ can be expanded in fractional powers of z. Show that we have

$$g(z) = z^{1/3} + O(z), \text{ which implies } \rho(x) = \frac{\sqrt{3}}{2}\sqrt[3]{|x|}, \tag{10.21}$$

for x near zero.

(h) For $\sigma^2 = 1/2, 1$ and 2, solve numerically the cubic equation for $\mathfrak{g}_{\mathbf{E}}(z)$ for $z = x$ real and plot the density of eigenvalues $\rho(x) = |\operatorname{Im}(\mathfrak{g}_{\mathbf{E}}(x))|/\pi$ for one of the complex roots if present.

10.2 Generalization to Non-Wigner Matrices

10.2.1 Set-Up

In the previous section, we derived a formula for the Stieltjes transform of the sum of a Wigner matrix and an arbitrary matrix. We would like to find a generalization of this result to a larger class of matrices.

Take two $N \times N$ matrices: $\mathbf{A}$, with eigenvalues $\{\lambda_i\}_{1\leq i\leq N}$ and eigenvectors $\{\mathbf{v}_i\}_{1\leq i\leq N}$, and $\mathbf{B}$, with eigenvalues $\{\mu_i\}_{1\leq i\leq N}$ and eigenvectors $\{\mathbf{u}_i\}_{1\leq i\leq N}$. Then the eigenvalues of $\mathbf{C} = \mathbf{B} + \mathbf{A}$ will in general depend in a complicated way on the overlaps between the eigenvectors of $\mathbf{B}$ and the eigenvectors of $\mathbf{A}$. In the trivial case where $\mathbf{v}_i = \mathbf{u}_i$ for all i, we have that the eigenvalues of $\mathbf{B} + \mathbf{A}$ are simply given by $\nu_i = \lambda_i + \mu_i$. However, this is neither generic nor very interesting.

One important property of Wigner matrices is that their eigenvectors are Haar distributed, that is, the matrix of eigenvectors is distributed uniformly in the group $O(N)$ and each eigenvector is uniformly distributed on the unit sphere S^{N-1}. Thus, when N is large, it is very unlikely that any one of them will have a significant overlap with the eigenvectors of $\mathbf{B}$. This is the property that we want to keep in our generalization. We will study what happens for general matrices $\mathbf{B}$ and $\mathbf{A}$ when their eigenvectors are random with respect to one another. We will define this relative randomness notion (called "freeness") more precisely in the next chapter. Here, to ensure the randomness of the eigenvectors, we will apply a random rotation to the matrix $\mathbf{A}$ and define the free addition as

$$\mathbf{C} = \mathbf{B} + \mathbf{O}\mathbf{A}\mathbf{O}^T, \tag{10.22}$$

where $\mathbf{O}$ is a Haar distributed random orthogonal matrix. Then it is easy to see that $\mathbf{O}\mathbf{A}\mathbf{O}^T$ is rotational invariant since $\mathbf{O}'\mathbf{O}$ is also Haar distributed for any fixed $\mathbf{O}' \in O(N)$.

10.2.2 Matrix Fourier Transform

We saw in Section 8.1 that the function $H_X(t) = \log \mathbb{E} \exp(\mathrm{i}tX)$ is additive when one adds independent scalar variables. When X is a matrix, it is plausible that t should also be a matrix $\mathbf{T}$, but in the end we need to take the exponential of a scalar, so a possible candidate would be

$$I(\mathbf{X},\mathbf{T}) := \left\langle \exp\left(\frac{N}{2} \operatorname{Tr} \mathbf{TOXO}^T\right)\right\rangle_{\mathbf{O}}. \tag{10.23}$$

The notation $\langle\cdot\rangle_{\mathbf{O}}$ means that we average over all orthogonal matrices $\mathbf{O}$ (with a flat weight) normalized such that $\langle 1\rangle_{\mathbf{O}} = 1$. This defines the Haar measure on the group of orthogonal matrices. Equation (10.23) defines the so-called Harish-Chandra–Itzykson–Zuber (HCIZ) integral.[2] Note that by definition, $I(\mathbf{O}_1\mathbf{X}\mathbf{O}_1^T,\mathbf{T}) = I(\mathbf{X},\mathbf{O}_1\mathbf{T}\mathbf{O}_1^T) = I(\mathbf{X},\mathbf{T})$ for an arbitrary rotation matrix $\mathbf{O}_1$. This means that $I(\mathbf{X},\mathbf{T})$ only depends on the eigenvalues of $\mathbf{X}$ and $\mathbf{T}$.

Now consider $\mathbf{C} = \mathbf{B} + \mathbf{O}_1\mathbf{A}\mathbf{O}_1^T$ with a random $\mathbf{O}_1$. For large matrix sizes, the eigenvalue spectrum of $\mathbf{C}$ will turn out not to depend on the specific choice of $\mathbf{O}_1$, provided it is chosen according to the Haar measure. Therefore, one can average $I(\mathbf{C},\mathbf{T})$ over $\mathbf{O}_1$ and obtain

$$I(\mathbf{C},\mathbf{T}) = \left\langle \exp\left(\frac{N}{2} \operatorname{Tr} \mathbf{TO}(\mathbf{B} + \mathbf{O}_1\mathbf{A}\mathbf{O}_1^T)\mathbf{O}^T\right)\right\rangle_{\mathbf{O},\mathbf{O}_1} = I(\mathbf{B},\mathbf{T})I(\mathbf{A},\mathbf{T}), \tag{10.25}$$

where we have used that $\mathbf{OO}_1 = \mathbf{O}'$ is a random rotation independent from $\mathbf{O}$ itself. Hence we conclude that $\log I$ is additive in this case, as is the logarithm of the characteristic function in the scalar case.

For a general matrix $\mathbf{T}$, the HCIZ integral is quite complicated, as will be further discussed in Section 10.5. Fortunately, for our purpose we can choose the "Fourier" matrix $\mathbf{T}$ to be rank-1 and in this case the integral can be computed. A symmetric rank-1 matrix can be written as

$$\mathbf{T} = t\,\mathbf{v}\mathbf{v}^T, \tag{10.26}$$

where t is the eigenvalue and $\mathbf{v}$ is a unit vector. We will show that the large N behavior of $I(\mathbf{T},\mathbf{B})$ is given, in this case, by

$$I(\mathbf{T},\mathbf{B}) \approx \exp\left(\frac{N}{2} H_{\mathbf{B}}(t)\right), \tag{10.27}$$

for some function $H_{\mathbf{B}}(t)$ that depends on the particular matrix $\mathbf{B}$.

[2] The HCIZ can be defined with an integral over orthogonal, unitary or symplectic matrices. In the general case it is defined as

$$I_\beta(\mathbf{X},\mathbf{T}) := \left\langle \exp\left(\frac{N\beta}{2} \operatorname{Tr} \mathbf{XOTO}^\dagger\right)\right\rangle_{\mathbf{O}}, \tag{10.24}$$

with beta equal to 1, 2 or 4 and $\mathbf{O}$ is averaged over the corresponding group. The unitary $\beta = 2$ case is the most often studied, for which some explicit results are available.

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More formally we define

$$H_{\mathbf{B}}(t) = \lim_{N\to\infty} \frac{2}{N} \log \left\langle \exp\left(\frac{tN}{2} \operatorname{Tr} \mathbf{v}\mathbf{v}^T \mathbf{O}\mathbf{B}\mathbf{O}^T\right)\right\rangle_{\mathbf{O}}. \tag{10.28}$$

If $\mathbf{C} = \mathbf{B} + \mathbf{A}$ where $\mathbf{A}$ is randomly rotated with respect to $\mathbf{B}$, the precise statement is that

$$H_{\mathbf{C}}(t) = H_{\mathbf{B}}(t) + H_{\mathbf{A}}(t), \tag{10.29}$$

i.e. the function H is additive. We now need to relate this function to the R-transform encountered in the previous section.

10.3 The Rank-1 HCIZ Integral

To get a useful theory, we need to have a concrete expression for this function $H_{\mathbf{B}}$. Without loss of generality, we can assume $\mathbf{B}$ is diagonal (in fact, we can diagonalize B and absorb the eigenmatrix into the orthogonal matrix $\mathbf{O}$ we integrate over). Moreover, for simplicity we assume that $t > 0$. Then $\mathbf{O}^T\mathbf{T}\mathbf{O}$ can be regarded as proportional to a random projector:

$$\mathbf{O}^T\mathbf{T}\mathbf{O} = \boldsymbol{\psi}\boldsymbol{\psi}^T, \tag{10.30}$$

with $\|\boldsymbol{\psi}\|^2 = t$ and $\boldsymbol{\psi}/\|\boldsymbol{\psi}\|$ uniformly distributed on the unit sphere. Then we make a change of variable $\boldsymbol{\psi} \to \boldsymbol{\psi}/\sqrt{N}$, and calculate

$$Z_t(\mathbf{B}) = \int \frac{\mathrm{d}^N\boldsymbol{\psi}}{(2\pi)^{N/2}} \delta\left(\|\boldsymbol{\psi}\|^2 - Nt\right) \exp\left(\frac{1}{2}\boldsymbol{\psi}^T\mathbf{B}\boldsymbol{\psi}\right), \tag{10.31}$$

where we have added a factor of $(2\pi)^{-N/2}$ for later convenience. Because $Z_t(\mathbf{B})$ is not properly normalized (i.e. $Z_t(0) \neq 1$), we will need to normalize it to compute $I(\mathbf{T},\mathbf{B})$:

$$\left\langle \exp\left(\frac{N}{2} \operatorname{Tr} \mathbf{T}\mathbf{O}\mathbf{B}\mathbf{O}^T\right)\right\rangle_{\mathbf{O}} = \frac{Z_t(\mathbf{B})}{Z_t(0)}. \tag{10.32}$$

10.3.1 A Saddle Point Calculation

We can now express the Dirac delta as an integral over the imaginary axis:

$$\delta(x) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}zx}}{2\pi}\mathrm{d}z = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{-zx/2}}{4\mathrm{i}\pi}\mathrm{d}z.$$

Now let Λ be a parameter larger than the maximum eigenvalue of $\mathbf{B}$: $\Lambda > \lambda_{\max}(\mathbf{B})$. We introduce the factor

$$1 = \exp\left(-\frac{\Lambda\left(\|\boldsymbol{\psi}\|^2 - Nt\right)}{2}\right),$$

since $\|\boldsymbol{\psi}\|^2 = Nt$. Then, absorbing Λ into z, we get that

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$$Z_t(\mathbf{B}) = \int_{\Lambda-\mathrm{i}\infty}^{\Lambda+\mathrm{i}\infty} \frac{\mathrm{d}z}{4\pi} \int \frac{\mathrm{d}^N \boldsymbol{\psi}}{(2\pi)^{N/2}} \exp\left(-\frac{1}{2}\boldsymbol{\psi}^T(z-\mathbf{B})\boldsymbol{\psi} + \frac{Nzt}{2}\right). \tag{10.33}$$

We can now perform the Gaussian integral over the vector $\boldsymbol{\psi}$:

$$\begin{aligned} Z_t(\mathbf{B}) &= \int_{\Lambda-\mathrm{i}\infty}^{\Lambda+\mathrm{i}\infty} \frac{\mathrm{d}z}{4\pi} \det(z-\mathbf{B})^{-1/2} \exp\left(\frac{Nzt}{2}\right) \\ &= \int_{\Lambda-\mathrm{i}\infty}^{\Lambda+\mathrm{i}\infty} \frac{\mathrm{d}z}{4\pi} \exp\left[\frac{N}{2}\left(zt - \frac{1}{N}\sum_k \log(z-\lambda_k(\mathbf{B}))\right)\right], \end{aligned} \tag{10.34}$$

where $\lambda_k(\mathbf{B})$, $1 \le k \le N$, are the eigenvalues of $\mathbf{B}$. Then we denote

$$F_t(z,\mathbf{B}) := zt - \frac{1}{N}\sum_k \log(z-\lambda_k(\mathbf{B})). \tag{10.35}$$

The integral in (10.34) is oscillatory, and by the stationary phase approximation (see Appendix A.1), it is dominated by the point where

$$\partial_z F_t(z,\mathbf{B}) = 0 \Rightarrow t - \frac{1}{N}\sum_k \frac{1}{z-\lambda_k(\mathbf{B})} = t - g_N^{\mathbf{B}}(z) = 0. \tag{10.36}$$

If $g_N^{\mathbf{B}}(z)$ can be inverted then we can express z as $\mathfrak{z}(t)$. For $x > \lambda_{\max}$, $g_N^{\mathbf{B}}(x)$ is monotonically decreasing and thus invertible. So for $t < g_N^{\mathbf{B}}(\lambda_{\max})$, a unique $\mathfrak{z}(t)$ exists and $\mathfrak{z}(t) > \lambda_{\max}$ (see Section 10.4). Since $F_t(z,\mathbf{B})$ is analytic to the right of $z = \lambda_{\max}$, we can deform the contour to reach this point (see Fig. 10.1). Using the saddle point formula (Eq. (A.3)), we have

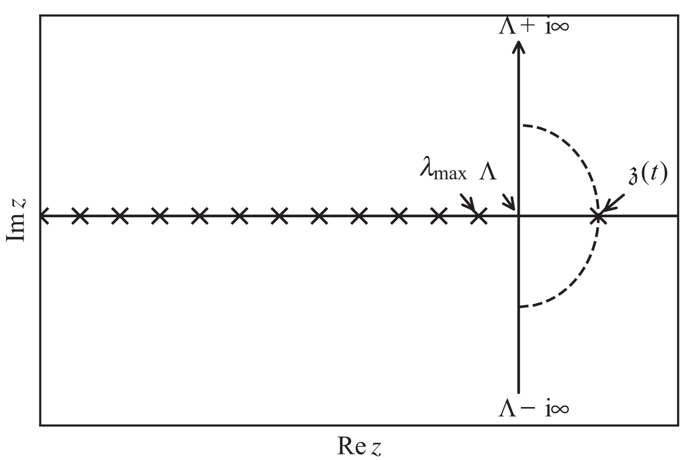

Figure 10.1 Graphical representation of the integral Eq. (10.34) in the complex plane. The crosses represent the eigenvalues of $\mathbf{B}$ and are singular points of the integrand. The integration is from $\Lambda - \mathrm{i}\infty$ to $\Lambda + \mathrm{i}\infty$ where $\Lambda > \lambda_{\max}$. The saddle point is at $z = \mathfrak{z}(t) > \lambda_{\max}$. Since the integrand is analytic right of $\lambda_{\max}$, the integration path can be deformed to go through $\mathfrak{z}(t)$.

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$$Z_t(\mathbf{B}) \sim \frac{\sqrt{4\pi}/(4\pi)}{|N\partial_z^2 F(\mathfrak{z}(t),\mathbf{B})|^{1/2}} \exp\left[\frac{N}{2}\left(\mathfrak{z}(t)t - \frac{1}{N}\sum_k \log(\mathfrak{z}(t)-\lambda_k(\mathbf{B}))\right)\right]$$
$$\sim \frac{1}{2\sqrt{N\pi|g'_{\mathbf{B}}(\mathfrak{z}(t))|}} \exp\left[\frac{N}{2}\left(\mathfrak{z}(t)t - \frac{1}{N}\sum_k \log(\mathfrak{z}(t)-\lambda_k(\mathbf{B}))\right)\right]. \tag{10.37}$$

For the case $\mathbf{B} = 0$, we have $g_{\mathbf{B}}(z) = z^{-1} \Rightarrow \mathfrak{z}(t) = t^{-1}$, so we get

$$Z_t(0) \sim \frac{1}{2t\sqrt{N\pi}} \exp\left[\frac{N}{2}(1+\log t)\right]. \tag{10.38}$$

In the large N limit, the prefactor in front of the exponential does not contribute to $H_{\mathbf{B}}(t)$ and we finally get

$$\lim_{N\to\infty} \frac{2}{N} \log\left\langle \exp\left(\frac{N}{2}\operatorname{Tr}\mathbf{TOBO}^T\right)\right\rangle_{\mathbf{O}} = \mathfrak{z}(t)t - 1 - \log t - \frac{1}{N}\sum_k \log(\mathfrak{z}(t)-\lambda_k(\mathbf{B})). \tag{10.39}$$

By the definition (10.28), we then get that

$$H_{\mathbf{B}}(t) = \mathcal{H}(\mathfrak{z}(t),t), \quad \mathcal{H}(z,t) := zt - 1 - \log t - \frac{1}{N}\sum_k \log(z-\lambda_k(\mathbf{B})). \tag{10.40}$$

10.3.2 Recovering R-Transforms

We found an expression for $H_{\mathbf{B}}(t)$ but in a form that is not easy to work with. But note $\mathcal{H}(z,t)$ comes from a saddle point approximation and therefore its partial derivative with respect to z is zero: $\partial_z\mathcal{H}(\mathfrak{z}(t),t) = 0$. This allows us to compute a much simpler expression for the derivative of $H_{\mathbf{B}}(t)$:

$$\frac{\mathrm{d}H_{\mathbf{B}}(t)}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial z}(\mathfrak{z}(t),t)\frac{\mathrm{d}\mathfrak{z}(t)}{\mathrm{d}t} + \frac{\partial\mathcal{H}}{\partial t}(\mathfrak{z}(t),t) = \frac{\partial\mathcal{H}}{\partial t}(\mathfrak{z}(t),t) = \mathfrak{z}(t) - \frac{1}{t} \equiv R_{\mathbf{B}}(t), \tag{10.41}$$

where $R_{\mathbf{B}}(t)$ denotes the R-transform defined in (10.10) (we have used the very definition of $\mathfrak{z}(t)$ from the previous section). Moreover, from its definition, we trivially have $H_{\mathbf{B}}(0) = 0$. Hence we can write

$$H_{\mathbf{B}}(t) := \int_0^t R_{\mathbf{B}}(x)\mathrm{d}x. \tag{10.42}$$

We already know that H is additive. Thus its derivative, i.e. the R-transform, is also additive:

$$R_{\mathbf{C}}(t) = R_{\mathbf{B}}(t) + R_{\mathbf{A}}(t), \tag{10.43}$$

as is the case when $\mathbf{A}$ is a Wigner matrix. This property is therefore valid as soon as $\mathbf{A}$ is "free" with respect to $\mathbf{B}$, i.e. when the basis that diagonalizes $\mathbf{A}$ is a random rotation of the basis that diagonalizes $\mathbf{B}$.

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The discussion leading to Eq. (10.42) can be extended to the HCIZ integral (Eq. (10.23)), when the rank of the matrix $\mathbf{T}$ is very small compared to N. In this case we get[3]

$$I(\mathbf{T},\mathbf{B}) \approx \exp\left(\frac{N}{2}\sum_{i=1}^{n} H_{\mathbf{B}}(t_i)\right) = \exp\left(\frac{N}{2}\operatorname{Tr} H_{\mathbf{B}}(\mathbf{T})\right), \tag{10.45}$$

where t_i are the n non-zero eigenvalues of $\mathbf{T}$ and with the same $H_{\mathbf{B}}(t)$ as above. When $\mathbf{T}$ has rank-1 we recover that $\operatorname{Tr} H_{\mathbf{B}}(\mathbf{T}) = H_{\mathbf{B}}(t)$, where t is the sole non-zero eigenvalue of $\mathbf{T}$.

The above formalism is based on the assumption that $\mathfrak{g}(z)$ is invertible, which is generally only true when $t = \mathfrak{g}(z)$ is small enough. This corresponds to the case where z is sufficiently large. Recall that the expansion of $g(z)$ at large z has coefficients given by the moments of the random matrix by (2.22). On the other hand, the expansion of $H(t)$ around $t = 0$ will give coefficients called the free cumulants of the random matrix, which are important objects in the study of free probability, as we will show in the next chapter.

10.4 Invertibility of the Stieltjes Transform

The question of the invertibility of the Stieltjes transform arises often enough that it is worth spending some time discussing it. In Section 10.1, we used the inverse of the limiting Stieltjes transform $\mathfrak{g}(z)$ to solve Burgers' equation, which led to the introduction of the R-transform $R(g) = \mathfrak{z}(g) - 1/g$. In Section 10.3.1 we invoked the invertibility of the discrete Stieltjes transform $g_N(z)$ to compute the rank-1 HCIZ integral.

10.4.1 Discrete Stieltjes Transform

Recall the discrete Stieltjes transform of a matrix $\mathbf{A}$ with N eigenvalues $\{\lambda_k\}$:

$$g_N^{\mathbf{A}}(z) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{z-\lambda_k}. \tag{10.46}$$

This function is well defined for any z on the real axis except on the finite set $\{\lambda_k\}$. For $z > \lambda_{\max}$, each of the terms in the sum is positive and monotonically decreasing with z so $g_N^{\mathbf{A}}(z)$ is a positive monotonically decreasing function of z. As $z \to \infty$, $g_N^{\mathbf{A}}(z) \to 0$. By the same argument, for $z < \lambda_{\min}$, $g_N^{\mathbf{A}}(z)$ is a negative monotonically decreasing function of z tending to zero as z goes to minus infinity. Actually, the normalization of $g_N^{\mathbf{A}}(z)$ is such that we have

[3] The same computation can be done for any value of beta, yielding

$$I_\beta(\mathbf{T},\mathbf{B}) \approx \exp\left(\frac{N\beta}{2}\sum_{i=1}^{n} H_{\mathbf{B}}(t_i)\right) = \exp\left(\frac{N\beta}{2}\operatorname{Tr} H_{\mathbf{B}}(\mathbf{T})\right), \tag{10.44}$$

where $I_\beta(\mathbf{T},\mathbf{B})$ is defined in the footnote on page 141 and $\mathbf{T}$ has low rank.

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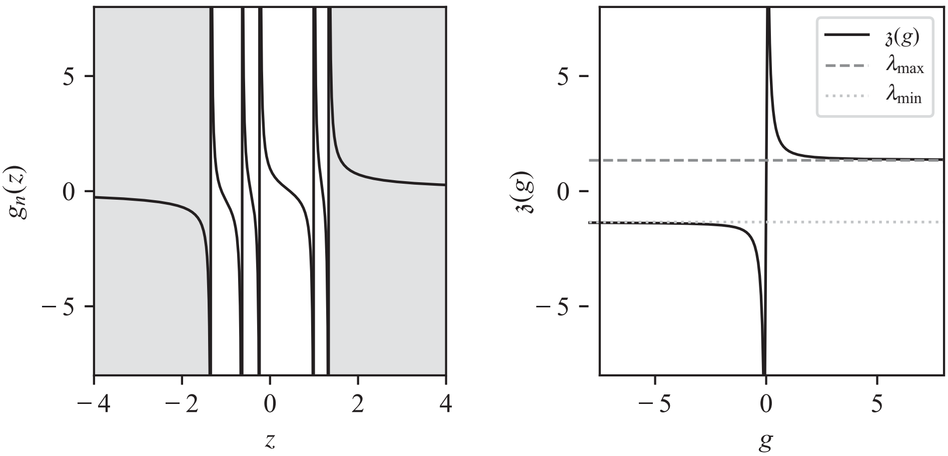

Figure 10.2 (left) A particular $g_N^{\mathbf{A}}(z)$ for $\mathbf{A}$ a Wigner matrix of size $N = 5$ shown for real values of z. The gray areas left of $\lambda_{\min}$ and right of $\lambda_{\max}$ show the values of z for which it is invertible. (right) The inverse function $\mathfrak{z}(g)$. Note that $g(z)$ behaves as $1/z$ near zero and tends to $\lambda_{\min}$ and $\lambda_{\max}$ as g goes to plus or minus infinity respectively.

$$g_N^{\mathbf{A}}(z) = \frac{1}{z} + O\left(\frac{1}{z^2}\right) \quad \text{when} \quad |z| \to \infty. \tag{10.47}$$

For large $|z|$, $g_N^{\mathbf{A}}(z)$ is thus invertible and its inverse behaves as

$$\mathfrak{z}(g) = \frac{1}{g} + \text{regular terms} \quad \text{when} \quad |g| \to 0. \tag{10.48}$$

If we consider values of $g_N^{\mathbf{A}}(z)$ for $z > \lambda_{\max}$, we realize that the function takes all possible positive values once and only once, from the extremely large (near $z = \lambda_{\max}$) to almost zero (when $z \to \infty$). Similarly, all possible negative values are attained when $z \in (-\infty, \lambda_{\min})$ (see Fig. 10.2 left). We conclude that the inverse function $\mathfrak{z}(g)$ exists for all non-zero values of g. The behavior of $g_N^{\mathbf{A}}(z)$ near $\lambda_{\min}$ and $\lambda_{\max}$ gives us the asymptotes

$$\lim_{g \to -\infty} \mathfrak{z}(g) = \lambda_{\min} \quad \text{and} \quad \lim_{g \to \infty} \mathfrak{z}(g) = \lambda_{\max}. \tag{10.49}$$

10.4.2 Limiting Stieltjes Transform

Let us now discuss the inverse function of the limiting Stieltjes transform $\mathfrak{g}(z)$. The limiting Stieltjes transform satisfies Eq. (2.41), which we recall here:

$$\mathfrak{g}(z) = \int_{\mathrm{supp}\{\rho\}} \frac{\rho(x)\mathrm{d}x}{z - x}, \tag{10.50}$$

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where $\rho(\lambda)$ is the limiting spectral distribution and may contain Dirac deltas. We denote $\lambda_\pm$ the edges of the support of ρ. We have that for $z > \lambda_+$, $\mathfrak{g}(z)$ is a positive, monotonically decreasing function of z. Similarly for $z < \lambda_-$, $\mathfrak{g}(z)$ is a negative, monotonically decreasing function of z. From the normalization of $\rho(\lambda)$, we again find that

$$\mathfrak{g}(z) = \frac{1}{z} + O\left(\frac{1}{z^2}\right) \quad \text{when} \quad |z| \to \infty. \tag{10.51}$$

Using the same arguments as for the discrete Stieltjes transform, we have that the inverse function $\mathfrak{z}(g)$ exists for small arguments and behaves as

$$\mathfrak{z}(g) = \frac{1}{g} + \text{regular terms} \quad \text{when} \quad |g| \to 0. \tag{10.52}$$

The behavior of $\mathfrak{g}(z)$ at $\lambda_\pm$ can be different from that of $g_N(z)$ at its extreme eigenvalues. The points $\lambda_\pm$ are singular points of $\mathfrak{g}(z)$. If the density near λ_+ goes to zero as $\rho(\lambda) \sim (\lambda_+ - \lambda)^\theta$ for some $\theta > 0$ (typically $\theta = 1/2$) then the integral (10.50) converges at $z = \lambda_+$ and $g_+ := \mathfrak{g}(\lambda_+)$ is a finite number. For $z < \lambda_+$ the function $\mathfrak{g}(z)$ has a branch cut and is ill defined for z on the real axis. The point $z = \lambda_+$ is an essential singularity of $\mathfrak{g}(z)$. The function is clearly no longer invertible for $z < \lambda_+$. Similarly, if $\rho(\lambda)$ grows as a positive power near λ_-, then $\mathfrak{g}(z)$ is invertible up to the point $g_- := \mathfrak{g}(\lambda_-)$.

If the density $\rho(\lambda)$ does not go to zero at one of its edges (or if it has a Dirac delta), the function $\mathfrak{g}(z)$ diverges at that edge. We may still define $g_\pm = \lim_{z\to\lambda_\pm} \mathfrak{g}(z)$ if we allow $g_\pm$ to be plus or minus infinity.

In all cases, the inverse function $\mathfrak{z}(g)$ exists in the range $g_- \le g \le g_+$, with the property

$$\mathfrak{z}(g_\pm) = \lambda_\pm. \tag{10.53}$$

In the unit Wigner case, we have $\lambda_\pm = \pm 2$ and $g_\pm = \pm 1$ and the inverse function $\mathfrak{z}(g)$ only exists between -1 and 1 (see Fig. 10.3).

10.4.3 The Inverse Stieltjes Transform for Larger Arguments

In some computations, as in the HCIZ integral, one needs the value of $\mathfrak{z}(g)$ beyond $g_\pm$. What can we say then? First of all, one should not be fooled by spurious solutions of the inversion problem. For example in the Wigner case we know that $\mathfrak{g}(z)$ satisfies

$$g + \frac{1}{g} - z = 0, \tag{10.54}$$

so we would be tempted to write

$$\mathfrak{z}(g) = g + \frac{1}{g} \tag{10.55}$$

for all g. But this is wrong as $g + 1/g$ is not the inverse of $\mathfrak{g}(z)$ for $|g| > 1$ (Fig. 10.3).

The correct way to extend $\mathfrak{z}(g)$ beyond $g_\pm$ is to realize that in most computation, we use $\mathfrak{g}(z)$ as an approximation for $g_N(z)$ for very large N. For $z > \lambda_+$ the function $g_N(z)$

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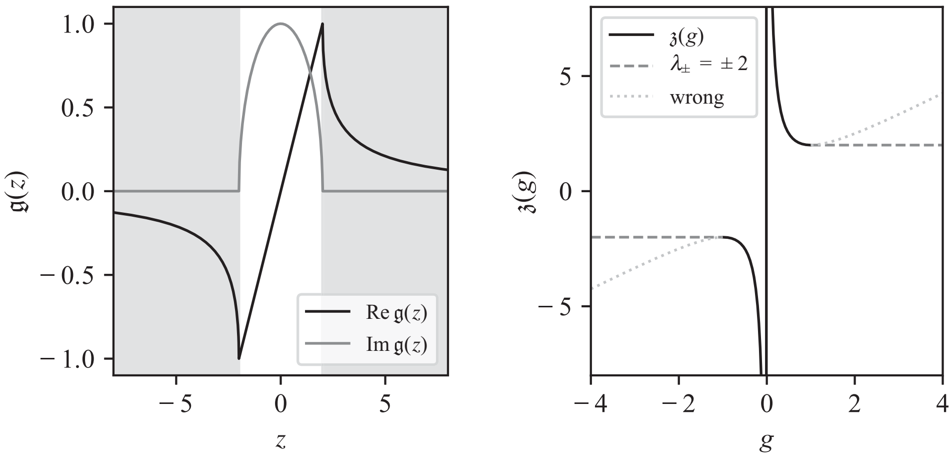

Figure 10.3 (left) The limiting function $\mathfrak{g}(z)$ for a Wigner matrix, a typical density that vanishes at its edges. The function is plotted against a real argument. In the white part of the graph, the function is ill defined and it is shown here for a small negative imaginary part of its argument. In the gray part ($z < \lambda_-$ and $z > \lambda_+$) the function is well defined, real and monotonic. It is therefore invertible. (right) The inverse function $\mathfrak{z}(g)$ only exists for $g_- \le g \le g_+$ and has a $1/g$ singularity at zero. The dashed lines show the extension of $\mathfrak{z}(g)$ to all values of g that are natural when we think of $\mathfrak{g}(z)$ as the limit of $g_N(z)$ with maximal and minimal eigenvalues $\lambda_\pm$. The dotted lines indicate the wrong branch of the solution of $\mathfrak{z}(g) = g + 1/g$.

converges to $\mathfrak{g}(z)$ and this approximation can be made arbitrarily good for large enough N. On the other hand we know that on the support of ρ, $g_N(z)$ does not converge to $\mathfrak{g}(z)$. The former has a series of simple poles at random locations, while the later has typically a branch cut.

At large but finite N, there will be a maximum eigenvalue $\lambda_{\max}$. This eigenvalue is random but to dominant order in N it converges to λ_+, the edge of the spectrum. For z above but very close to λ_+ we should think of $g_N(z)$ as

$$g_N(z) \approx \mathfrak{g}(z) + \frac{1}{N}\frac{1}{z-\lambda_{\max}} \approx \mathfrak{g}(z) + \frac{1}{N}\frac{1}{z-\lambda_+}. \tag{10.56}$$

Because $1/N$ goes to zero, the correction above does not change the limiting value of $\mathfrak{g}(z)$ at any finite distance from λ_+. On the other hand, this correction does change the behavior of the inverse function $\mathfrak{z}(g)$. We now have

$$\lim_{z\to\lambda_+} g_N(z) \to \infty \quad \text{and} \quad \mathfrak{z}(g) = \lambda_+ \text{ for } g > g_+. \tag{10.57}$$

For negative z and negative g, the same argument follows near λ_-. We realize that, while the limiting Stieltjes transform $\mathfrak{g}(z)$ loses all information about individual eigenvalues, its inverse function $\mathfrak{z}(g)$, or really the large N limit of the inverse of the function $g_N(z)$, retains information about the smallest and largest eigenvalues. In Chapter 14 we will study random matrices where a finite number of eigenvalues lie outside the support of ρ. In the large N limit, these eigenvalues do not change the density or $\mathfrak{g}(z)$ but they do show up in the inverse function $\mathfrak{z}(g)$.

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Let us define $z_{\text{bulk}}(g)$, the inverse function of $\mathfrak{g}(z)$ without considering extreme eigenvalues or outliers. In the presence of outliers we have $\lambda_{\max} \geq \lambda_+$ and $g_{\max} := \mathfrak{g}(\lambda_{\max}) \leq \mathfrak{g}(\lambda_+)$ and similarly for $g_{\min}$. With arguments similar to those above we find the following result for the limit of the inverse of $g_N(z)$:

$$\mathfrak{z}(g) = \begin{cases} \lambda_{\min} & \text{for } g \leq g_{\min}, \\ z_{\text{bulk}}(g) & \text{for } g_{\min} < g < g_{\max}, \\ \lambda_{\max} & \text{for } g \geq g_{\max}. \end{cases} \tag{10.58}$$

In the absence of outliers the result still applies with max and min (extreme eigenvalues) replaced by $+$ and $-$ (edge of the spectrum), respectively.

10.4.4 Large t Behavior of I_t

Now that we understand the behavior of $\mathfrak{z}(g)$ for larger arguments we can go back to our study of the rank-1 HCIZ integral. There is indeed an apparent paradox in the result of our computation of $I_t(\mathbf{B})$. For a given matrix $\mathbf{B}$ there are two immediate bounds to $I_t(\mathbf{B}) = Z_t(\mathbf{B})/Z_t(0)$:

$$\exp\left(\frac{Nt\lambda_{\min}}{2}\right) \leq I_t(\mathbf{B}) \leq \exp\left(\frac{Nt\lambda_{\max}}{2}\right), \tag{10.59}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of $\mathbf{B}$, respectively. Focusing on the upper bound, we have

$$H_{\mathbf{B}}(t) \leq t\lambda_{\max}. \tag{10.60}$$

On the other hand, the anti-derivative of the R-transform for a unit Wigner matrix reads

$$R_{\mathbf{W}}(t) = t \longrightarrow H_{\mathbf{W}}(t) = \frac{t^2}{2}, \tag{10.61}$$

whereas $\lambda_{\max} \to \lambda_+ = 2$. One might thus think that the quadratic behavior of $H_{\mathbf{W}}(t)$ violates the bound (10.60) for $t > 4$. We should, however, remember that Eq. (10.42) is in fact only valid for $t < g_+$, the value at which $\mathfrak{g}(z)$ ceases to be invertible. In the absence of outliers, $g_+ = \mathfrak{g}(\lambda_+)$. For a unit Wigner this point is $g_+ = \mathfrak{g}(2) = 1$; the bound is not violated. For $t > g_+$, one can still compute $H_{\mathbf{B}}(t)$ but the result depends explicitly on $\lambda_{\max}$.

Now that we understand the behavior of $\mathfrak{z}(g)$ for larger arguments, including in the presence of outliers, we can extend our result for $H_{\mathbf{B}}(t)$ for large t's. We just need to use Eq. (10.58) into Eq. (10.41):

$$\frac{\mathrm{d}H_{\mathbf{B}}(t)}{\mathrm{d}t} = \begin{cases} R_{\mathbf{B}}(t) & \text{for } t \leq g_{\max} := \mathfrak{g}(\lambda_{\max}), \\ \lambda_{\max} - 1/t & \text{for } t > g_{\max}, \end{cases} \tag{10.62}$$

where the largest eigenvalue $\lambda_{\max}$ can be either the edge of the spectrum λ_+ or a true outlier. We will show in Section 13.3 how this result can also be derived for Wigner matrices using the replica method.

10.5 The Full-Rank HCIZ Integral

We have defined in Eq. (10.23) the HCIZ integral as a generalization of the Fourier transform for matrices, and have seen how to evaluate this integral in the limit $N \to \infty$ when one matrix is of low rank. A generalized HCIZ integral $I_\beta(\mathbf{A}, \mathbf{B})$ can be defined as

$$I_\beta(\mathbf{A},\mathbf{B}) = \int_{G(N)} \mathrm{d}\mathbf{U}\,\mathrm{e}^{\frac{\beta N}{2}\operatorname{Tr}\mathbf{AUBU}^\dagger}, \tag{10.63}$$

where the integral is over the (flat) Haar measure of the compact group $\mathbf{U} \in G(N) = O(N), U(N)$ or $Sp(N)$ in N dimensions and $\mathbf{A},\mathbf{B}$ are arbitrary $N \times N$ symmetric (resp. Hermitian or symplectic) matrices, with, correspondingly, $\beta = 1,2$ or 4. Note that by construction $I_\beta(\mathbf{A},\mathbf{B})$ can only depend on the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, since any change of basis on $\mathbf{B}$ (say) can be reabsorbed in $\mathbf{U}$, over which we integrate. Note also that the Haar measure is normalized, i.e. $\int_{G(N)} \mathrm{d}\mathbf{U} = 1$.

Interestingly, it turns out that in the unitary case $G(N) = U(N)$ ($\beta = 2$), the HCIZ integral can be expressed exactly, for all N, as the ratio of determinants that depend on $\mathbf{A},\mathbf{B}$ and additional N-dependent prefactors. This is the Harish-Chandra–Itzykson–Zuber celebrated result, which cannot be absent from a book on random matrices:

$$I_2(\mathbf{A},\mathbf{B}) = \frac{c_N}{N^{(N^2-N)/2}} \frac{\det\left(\mathrm{e}^{N\nu_i\lambda_j}\right)}{\Delta(\mathbf{A})\Delta(\mathbf{B})}, \tag{10.64}$$

with $\{\nu_i\}$, $\{\lambda_i\}$ the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, $\Delta(\mathbf{A})$, $\Delta(\mathbf{B})$ are the Vandermonde determinants of $\mathbf{A}$ and $\mathbf{B}$, and $c_N = \prod_\ell^{N-1} \ell!$.

Although this result is fully explicit for $\beta = 2$, the expression in terms of determinants is highly non-trivial and quite tricky. For example, the expression becomes degenerate (0/0) whenever two eigenvalues of $\mathbf{A}$ (or $\mathbf{B}$) coincide. Also, as is well known, determinants contain $N!$ terms of alternating signs, which makes their order of magnitude very hard to estimate *a priori*. The aim of this technical section is to discuss how the HCIZ result can be obtained using the Karlin–McGregor equation (9.69). We then use the mapping to the Dyson Brownian motion to derive a large N approximation for the full-rank HCIZ integral in the general case.

10.5.1 HCIZ and Karlin–McGregor

In order to understand the origin of Eq. (10.64), the basic idea is to interpret the HCIZ integrand in the unitary case, $\exp[N \operatorname{Tr}\mathbf{AUBU}^\dagger]$, as a part of the diffusion propagator in the space of Hermitian matrices, and use the Karlin–McGregor formula.

Indeed, adding to $\mathbf{A}$ a sequence of infinitesimal random Gaussian Hermitian matrices of variance $\mathrm{d}t/N$, the probability to end up with matrix $\mathbf{B}$ in a time $t = 1$ is given by

$$P(\mathbf{B}|\mathbf{A}) \propto N^{N^2/2}\,\mathrm{e}^{-N/2\,\operatorname{Tr}(\mathbf{B}-\mathbf{A})^2}, \tag{10.65}$$

where we drop an overall normalization constant in our attempt to understand the structure of Eq. (10.64). The corresponding eigenvalues follow a Dyson Brownian motion, namely

$$\mathrm{d}x_i = \sqrt{\frac{1}{N}}\mathrm{d}B_i + \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\mathrm{d}t}{x_i - x_j}, \tag{10.66}$$

with $x_i(t = 0) = \nu_i$ and $x_i(t = 1) = \lambda_i$. Now, for $\beta = 2$ we can use the Karlin–McGregor equation (9.69) to derive the conditional distribution of the $\{\lambda_i\}$, given by

$$P(\{\lambda_i\}|\{\nu_i\}) = \frac{\Delta(\mathbf{B})}{\Delta(\mathbf{A})} P(\vec{\lambda}, t = 1|\vec{\nu}), \tag{10.67}$$

where $P(\vec{\lambda}, t = 1|\vec{\nu})$ is given by a determinant, Eq. (9.66). With the present normalization, this determinant reads

$$P(\vec{\lambda}, t = 1|\vec{\nu}) = \left(\frac{N}{2\pi}\right)^{N/2} \mathrm{e}^{-\frac{N}{2}(\operatorname{Tr}\mathbf{A}^2 + \operatorname{Tr}\mathbf{B}^2)} \det\left(\mathrm{e}^{N\nu_i\lambda_j}\right). \tag{10.68}$$

Now, the distribution of eigenvalues of $\mathbf{B}$ can be computed from Eq. (10.65). First we make $P(\mathbf{B}|\mathbf{A})$ unitary-invariant by integrating over $U(N)$:

$$\begin{aligned} P(\mathbf{B}|\mathbf{A}) \to \overline{P}(\mathbf{B}|\mathbf{A}) &= N^{N^2/2} \frac{\int_{U(N)} \mathrm{d}\mathbf{U}\, \mathrm{e}^{-N/2\, \operatorname{Tr}(\mathbf{U}\mathbf{B}\mathbf{U}^\dagger - \mathbf{A})^2}}{\Omega_N} \\ &= N^{N^2/2} \mathrm{e}^{-\frac{N}{2}(\operatorname{Tr}\mathbf{A}^2 + \operatorname{Tr}\mathbf{B}^2)} \frac{I_2(\mathbf{A},\mathbf{B})}{\Omega_N}, \end{aligned} \tag{10.69}$$

where $\Omega_N = \int_{U(N)} \mathrm{d}\mathbf{U}$ is the "volume" of the unitary group $U(N)$. This new measure, by construction, only depends on $\{\lambda_i\}$, the eigenvalues of $\mathbf{B}$. Changing variables from $\mathbf{B}$ to $\{\lambda_i\}$ introduces a Jacobian, which in the unitary case is the square of the Vandermonde determinant of $\mathbf{B}$, $\Delta^2(\mathbf{B})$. We thus find a second expression for the distribution of the $\{\lambda_i\}$:

$$P(\{\lambda_i\}|\{\nu_i\}) \propto N^{N^2/2} \Delta^2(\mathbf{B}) \mathrm{e}^{-\frac{N}{2}(\operatorname{Tr}\mathbf{A}^2 + \operatorname{Tr}\mathbf{B}^2)} I_2(\mathbf{A},\mathbf{B}). \tag{10.70}$$

Comparing with Eqs. (10.67) and (10.68) we thus find

$$I_2(\mathbf{A},\mathbf{B}) \propto N^{(N-N^2)/2} \frac{\det\left(\mathrm{e}^{N\nu_i\lambda_j}\right)}{\Delta(\mathbf{A})\Delta(\mathbf{B})}, \tag{10.71}$$

which coincides with Eq. (10.64), up to an overall constant c_N which can be obtained by taking the limit $\mathbf{A} = \mathbf{1}$, i.e. when all the eigenvalues of $\mathbf{A}$ are equal to 1. The limit is singular but one can deal with it in a way similar to the one used by Brézin and Hikami to go from Eq. (6.65) to (6.67). In this limit, the right hand side of Eq. (10.71) reads $\exp(N \operatorname{Tr}\mathbf{B})/c_N$, while the left hand side is trivially equal to $\exp(N \operatorname{Tr}\mathbf{B})$. Hence a factor c_N is indeed missing in the right hand side of Eq. (10.71).

Equation (10.64) can also be used to obtain an exact formula for the rank-1 HCIZ integral (when $\beta = 2$). The trick is to have one of the eigenvalues of ν_i equal to some non-zero number t and let the $N - 1$ others go to zero. The limit can again be dealt with in the same way as Eq. (6.65). One finally finds

$$I_2(t,\mathbf{B}) = \frac{(N-1)!}{(Nt)^{N-1}} \sum_{j=1}^{N} \frac{\mathrm{e}^{Nt\lambda_j}}{\prod_{k\neq j}(\lambda_j - \lambda_k)}. \tag{10.72}$$

The above formula may look singular at $t = 0$, but we have $\lim_{t\to 0} I_2(t,\mathbf{B}) = 1$ as expected.

10.5.2 HCIZ at Large N: The Euler–Matytsin Equations

We now explain how $I_2(\mathbf{A},\mathbf{B})$ can be estimated for large matrix size, using a Dyson Brownian representation of $\overline{P}(\mathbf{B}|\mathbf{A})$, Eq. (10.66). In terms of these interacting Brownian motions, the question is how to estimate the probability that the $x_i(t)$ start at $x_i(t = 0) =$

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ν_i and end at $x_i(t=1)=\lambda_i$, when their trajectories are determined by Eq. (10.66), which we rewrite as

$$\mathrm{d}x_i = \sqrt{\frac{1}{N}}\mathrm{d}B_i - \partial_{x_i} V\,\mathrm{d}t, \qquad V(\{x_i\}) := -\frac{1}{N}\sum_{i<j}\ln|x_i - x_j|. \tag{10.73}$$

The probability of a given trajectory for the N Brownian motions between time $t=0$ and time $t=1$ is then given by[4]

$$P(\{x_i(t)\}) = Z^{-1}\exp-\left[\frac{N}{2}\int_0^1 \mathrm{d}t\sum_i\left(\dot{x}_i + \partial_{x_i}V\right)^2\right] := Z^{-1}\mathrm{e}^{-N^2 S}, \tag{10.74}$$

where Z is a normalization factor that we will not need explicitly. Expanding the square as $\dot{x}_i^2 + 2\partial_{x_i}V\dot{x}_i + (\partial_{x_i}V)^2$, one can decompose $S=S_1+S_2$ into a total derivative term equal, in the continuum limit, to boundary terms, i.e.

$$S_1 = -\frac{1}{2}\left[\int \mathrm{d}x\mathrm{d}y\rho_C(x)\rho_C(y)\ln|x-y|\right]_{C=A}^{C=B} \tag{10.75}$$

and

$$S_2 := \frac{1}{2N}\int_0^1 \mathrm{d}t\sum_{i=1}^N\left[\dot{x}_i^2 + (\partial_{x_i}V)^2\right]. \tag{10.76}$$

We now look for the "instanton" trajectory that contributes most to the probability P for large N, in other words the trajectory that minimizes S_2. This extremal trajectory is such that the functional derivative of S_2 with respect to all $x_i(t)$ is zero:

$$-2\frac{\mathrm{d}^2x_i}{\mathrm{d}t^2} + 2\sum_{\ell=1}^N \partial^2_{x_i,x_\ell}V\partial_{x_\ell}V = 0, \tag{10.77}$$

which leads, after a few algebraic manipulations, to

$$\frac{\mathrm{d}^2x_i}{\mathrm{d}t^2} = -\frac{2}{N^2}\sum_{\ell\neq i}\frac{1}{(x_i-x_\ell)^3}. \tag{10.78}$$

This can be interpreted as the motion of unit mass particles, accelerated by an *attractive* force that derives from an effective two-body potential $\phi(r) = -(Nr)^{-2}$. The hydrodynamical description of such a fluid, justified when $N\to\infty$, is given by the Euler equations for the density field $\rho(x,t)$ and the velocity field $v(x,t)$:

$$\partial_t\rho(x,t) + \partial_x[\rho(x,t)v(x,t)] = 0 \tag{10.79}$$

and

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = -\frac{1}{\rho(x,t)}\partial_x\Pi(x,t), \tag{10.80}$$

where $\Pi(x,t)$ is the pressure field, which reads, from the "virial" formula for an interacting fluid at temperature T,[5]

[4] We neglect here a Jacobian which is not relevant to compute the leading term of $I_2(\mathbf{A},\mathbf{B})$ in the large N limit.
[5] See e.g. Le Bellac et al. [2004], p. 138.

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$$\Pi = \rho T - \frac{1}{2}\rho \sum_{\ell \neq i} |x_i - x_\ell| \phi'(x_i - x_\ell) \approx -\frac{\rho}{N^2} \sum_{\ell \neq i} \frac{1}{(x_i - x_\ell)^2}, \tag{10.81}$$

because the fluid describing the instanton is at zero temperature, $T = 0$. Now, writing $x_i - x_\ell \approx (i - \ell)/(N\rho)$ and $\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$, one finally finds

$$\Pi(x,t) = -\frac{\pi^2}{3}\rho(x,t)^3. \tag{10.82}$$

Equations (10.79) and (10.80) for ρ and v with Π given by (10.82) are called the Euler–Matytsin equations. They should be solved with the following boundary conditions:

$$\rho(x, t = 0) = \rho_{\mathbf{A}}(x); \qquad \rho(x,t) = \rho_{\mathbf{B}}(x); \tag{10.83}$$

the velocity field $v(x, t = 0)$ should be chosen such that these boundary conditions are fulfilled.

Expressing S_2 in terms of the solution of the Euler–Matytsin equations gives, in the continuum limit,

$$S_2(\mathbf{A},\mathbf{B}) \approx \frac{1}{2}\int \mathrm{d}x \rho(x,t)\left[v^2(x,t) + \frac{\pi^2}{3}\rho^2(x,t)\right]. \tag{10.84}$$

Hence, the probability $P(\{\lambda_i\}|\{\nu_i\})$ to observe the set of eigenvalues $\{\lambda_i\}$ of $\mathbf{B}$ for a given set of eigenvalues ν_i for $\mathbf{A}$ is, in the large N limit, proportional to $\exp[-N^2(S_1 + S_2)]$. Comparing with Eq. (10.70), we get as a final expression for $F_2(\mathbf{A},\mathbf{B}) := -\lim_{N\to\infty} N^{-2} \ln I_2(\mathbf{A},\mathbf{B})$:

$$\begin{aligned} F_2(\mathbf{A},\mathbf{B}) = \frac{3}{4} + S_2(\mathbf{A},\mathbf{B}) - \frac{1}{2}\int \mathrm{d}x\, x^2(\rho_{\mathbf{A}}(x) + \rho_{\mathbf{B}}(x)) \\ + \frac{1}{2}\int \mathrm{d}x\mathrm{d}y\, [\rho_{\mathbf{A}}(x)\rho_{\mathbf{A}}(y) + \rho_{\mathbf{B}}(x)\rho_{\mathbf{B}}(y)] \ln|x - y|. \end{aligned} \tag{10.85}$$

This result was first derived in Matytsin [1994], and proven rigorously in Guionnet and Zeitouni [2002]. Note that this expression is symmetric in $\mathbf{A}, \mathbf{B}$, as it should be, because the solution of the Euler–Matytsin equations for the time reversed path from $\rho_{\mathbf{B}}$ to $\rho_{\mathbf{A}}$ are simply obtained from $\rho(x,t) \to \rho(x, 1 - t)$ and $v(x,t) \to -v(x, 1 - t)$, which leaves $S_2(\mathbf{A},\mathbf{B})$ unchanged.

The whole calculation above can be repeated for the $\beta = 1$ (orthogonal group) or $\beta = 4$ (symplectic group) with the final (simple) result $F_\beta(\mathbf{A},\mathbf{B}) = \beta F_2(\mathbf{A},\mathbf{B})/2$.

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