7

The Jacobi Ensemble*

So far we have encountered two classical random matrix ensembles, namely Wigner and Wishart. They are, respectively, the matrix equivalents of the Gaussian and the gamma distribution. For example a 1×1 Wigner matrix is a single Gaussian random number and a 1×1 Wishart is a gamma distributed number (see Eq. (4.16) with $N = 1$). We also saw in Chapter 6 that these ensembles are intimately related to classical orthogonal polynomials, respectively Hermite and Laguerre. The Gaussian distribution and its associated Hermite polynomials appear very naturally in contexts where the underlying variable is unbounded above and below. Gamma distributions and Laguerre polynomials appear in problems where the variable is bounded from below (e.g. positive variables). Variables that are bounded both from above and from below have their own natural distribution and associated classical orthogonal polynomials, namely the beta distribution and Jacobi polynomials.

In this chapter, we introduce a third classical random matrix ensemble: the Jacobi ensemble. It is the random matrix equivalent of the beta distribution (and hence often called matrix variate beta distribution). It will turn out to be strongly linked to Jacobi orthogonal polynomials.

Jacobi matrices appear in multivariate analysis of variance and hence the Jacobi ensemble is sometimes called the MANOVA ensemble. An important special case of the Jacobi ensemble is the arcsine law which we already encountered in Section 5.5, and will again encounter in Section 15.3.1. It is the law governing Coulomb repelling eigenvalues with no external forces save for two hard walls. It also shows up in simple problems of matrix addition and multiplications for matrices with only two eigenvalues.

7.1 Properties of Jacobi Matrices

7.1.1 Construction of a Jacobi Matrix

A beta-distributed random variable $x \in (0, 1)$ has the following law:

$$P_{c_1,c_2}(x) = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(c_1 + c_2)} x^{c_1-1}(1-x)^{c_2-1}, \tag{7.1}$$

where $c_1 > 0$ and $c_2 > 0$ are two parameters characterizing the law.

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To generalize (7.1) to matrices, we could define $\mathbf{J}$ as a matrix generated from a beta ensemble with a matrix potential that tends to $V(x) = -\log(P_{c_1,c_2}(x))$ in the large N limit. Although this is indeed the result we will get in the end, we would rather use a more constructive approach that will give us a sensible definition of the matrix $\mathbf{J}$ at finite N and for the three standard values of β.

A beta(c_1, c_2) random number can alternatively be generated from two gamma-distributed variables:

$$x = \frac{w_1}{w_1 + w_2}, \qquad w_{1,2} \sim \text{Gamma}(c_{1,2}, 1). \tag{7.2}$$

The same relation can be rewritten as

$$x = \frac{1}{1 + w_1^{-1} w_2}. \tag{7.3}$$

This is the formula we need for our matrix generalization. An unnormalized white Wishart with $T = cN$ is the matrix generalization of a Gamma$(c, 1)$ random variable. Combining two such matrices as above will give us our Jacobi random matrix $\mathbf{J}$. One last point before we proceed, we need to symmetrize the combination $w_1^{-1} w_2$ to yield a symmetric matrix. We choose $\sqrt{w_2}\, w_1^{-1} \sqrt{w_2}$ which makes sense as Wishart matrices (like gamma-distributed numbers) are positive definite.

We can now define the Jacobi matrix. Let $\mathbf{E}$ be the symmetrized product of a white Wishart and the inverse of another independent white Wishart, both without the usual $1/T$ normalization:

$$\mathbf{E} = \widetilde{\mathbf{W}}_2^{1/2} \widetilde{\mathbf{W}}_1^{-1} \widetilde{\mathbf{W}}_2^{1/2}, \quad \text{where} \quad \widetilde{\mathbf{W}}_{1,2} := \mathbf{H}_{1,2} \mathbf{H}_{1,2}^T. \tag{7.4}$$

The two matrices $\mathbf{H}_{1,2}$ are rectangular matrices of standard Gaussian random numbers with aspect ratio $c_1 = T_1/N$ and $c_2 = T_2/N$ (note that the usual aspect ratio is $q = c^{-1}$). The standard Jacobi matrix is defined as

$$\mathbf{J} = (\mathbf{1} + \mathbf{E})^{-1}. \tag{7.5}$$

A Jacobi matrix has all its eigenvalues between 0 and 1. For the matrices $\widetilde{\mathbf{W}}_{1,2}$ to make sense we need $c_{1,2} > 0$. In addition, to ensure that $\widetilde{\mathbf{W}}_1$ is invertible we need to impose $c_1 > 1$. It turns out that we can relax that assumption later and the ensemble still makes sense for any $c_{1,2} > 0$.

7.1.2 Joint Law of the Elements

The joint law of the elements of a Jacobi matrix for $\beta = 1, 2$ or 4 is given by

$$P_\beta(\mathbf{J}) = c_J^{\beta,T_1,T_2} [\det(\mathbf{J})]^{\beta(T_1-N+1)/2-1} [\det(\mathbf{1} - \mathbf{J})]^{\beta(T_2-N+1)/2-1}, \tag{7.6}$$

$$c_J^{\beta,T_1,T_2} = \prod_{j=1}^{N} \frac{\Gamma(1+\beta/2)\Gamma(\beta(T_1+T_2-N+j)/2)}{\Gamma(1+\beta j/2)\Gamma(\beta(T_1-N+j)/2)\Gamma(\beta(T_2-N+j)/2)}, \tag{7.7}$$

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over the space of matrices of the proper symmetry such that both $\mathbf{J}$ and $\mathbf{1} - \mathbf{J}$ are positive definite.

To obtain this result, one needs to know the law of Wishart matrices (Chapter 4) and the law of a matrix given the law of its inverse (Chapter 1).

Here is the derivation in the real symmetric case. We first write the law of the matrix $\mathbf{E}$ by realizing that for a fixed matrix $\widetilde{\mathbf{W}}_1$, the matrix $\mathbf{E}/T_2$ is a Wishart matrix with $T = T_2$ and true covariance $\widetilde{\mathbf{W}}_1^{-1}$. From Eq. (4.16), we thus have

$$P(\mathbf{E}|\mathbf{W}_1) = \frac{(\det \mathbf{E})^{(T_2-N-1)/2}(\det \widetilde{\mathbf{W}}_1)^{T_2/2}}{2^{NT_2/2}\Gamma_N(T_2/2)} \exp\left[-\frac{1}{2}\operatorname{Tr}\left(\mathbf{E}\widetilde{\mathbf{W}}_1\right)\right]. \tag{7.8}$$

The matrix $\widetilde{\mathbf{W}}_1/T_1$ is itself a Wishart with probability

$$P\left(\widetilde{\mathbf{W}}_1\right) = \frac{(\det \widetilde{\mathbf{W}}_1)^{(T_1-N-1)/2}}{2^{NT_1/2}\Gamma_N(T_1/2)} \exp\left[-\frac{1}{2}\operatorname{Tr}\left(\widetilde{\mathbf{W}}_1\right)\right]. \tag{7.9}$$

Averaging Eq. (7.8) with respect to $\widetilde{\mathbf{W}}_1$ we find

$$\begin{aligned} P(\mathbf{E}) = {} & \frac{(\det \mathbf{E})^{(T_2-N-1)/2}}{2^{N(T_1+T_2)/2}\Gamma_N(T_1/2)\Gamma_N(T_2/2)} \\ & \times \int \mathrm{d}\mathbf{W}(\det \mathbf{W})^{(T_1+T_2-N-1)/2} \exp\left[-\frac{1}{2}\operatorname{Tr}\left((\mathbf{1}+\mathbf{E})\mathbf{W}\right)\right]. \end{aligned} \tag{7.10}$$

We can perform the integral over $\mathbf{W}$ by realizing that $\mathbf{W}/T$ is a Wishart matrix with $T = T_1 + T_2$ and true covariance $\mathbf{C} = (\mathbf{1} + \mathbf{E})^{-1}$, see Eq. (4.16). We just need to introduce the correct power of $\det \mathbf{C}$ and numerical factors to make the integral equal to 1. Thus,

$$P(\mathbf{E}) = \frac{\Gamma_N((T_1+T_2)/2)}{\Gamma_N(T_1/2)\Gamma_N(T_2/2)}(\det \mathbf{E})^{(T_2-N-1)/2}(\det(\mathbf{1}+\mathbf{E}))^{-(T_1+T_2)/2}. \tag{7.11}$$

The miracle that for any N one can integrate exactly the product of a Wishart matrix and the inverse of another Wishart matrix will appear again in the Bayesian theory of SCM (see Section 18.3).

Writing $\mathbf{E}_+ = \mathbf{E} + \mathbf{1}$ we find

$$P(\mathbf{E}_+) = \frac{\Gamma_N((T_1+T_2)/2)}{\Gamma_N(T_1/2)\Gamma_N(T_2/2)}(\det(\mathbf{E}_+ - \mathbf{1}))^{(T_2-N-1)/2}(\det \mathbf{E}_+)^{-(T_1+T_2)/2}. \tag{7.12}$$

Finally we want $\mathbf{J} := \mathbf{E}_+^{-1}$. The law of the inverse of a symmetric matrix $\mathbf{A} = \mathbf{M}^{-1}$ of size N is given by (see Section 1.2.7)

$$P_{\mathbf{A}}(\mathbf{A}) = P_{\mathbf{M}}(\mathbf{A}^{-1})\det(\mathbf{A})^{-N-1}. \tag{7.13}$$

Hence,

$$P(\mathbf{J}) = \frac{\Gamma_N((T_1+T_2)/2)}{\Gamma_N(T_1/2)\Gamma_N(T_2/2)}\left(\det(\mathbf{J}^{-1} - \mathbf{1})\right)^{(T_2-N-1)/2}(\det \mathbf{J})^{(T_1+T_2)/2-N-1}. \tag{7.14}$$

Using $\det(\mathbf{J}^{-1} - \mathbf{1})\det(\mathbf{J}) = \det(\mathbf{1} - \mathbf{J})$ we can reorganize the powers of the determinants and get

$$P(\mathbf{J}) = \frac{\Gamma_N((T_1+T_2)/2)}{\Gamma_N(T_1/2)\Gamma_N(T_2/2)}(\det(\mathbf{1}-\mathbf{J}))^{(T_2-N-1)/2}(\det \mathbf{J})^{(T_1-N-1)/2}, \tag{7.15}$$

which is equivalent to Eq. (7.6) for $\beta = 1$.

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7.1.3 Potential and Stieltjes Transform

The Jacobi ensemble is a beta ensemble satisfying Eq. (5.2) with matrix potential

$$V(x) = -\frac{T_1 - N + 1 - 2/\beta}{N}\log(x) - \frac{T_2 - N + 1 - 2/\beta}{N}\log(1-x). \tag{7.16}$$

The derivative of the potential, in the large N limit, is given by

$$V'(x) = \frac{c_1 - 1 - (c_1 + c_2 - 2)x}{x(x-1)}. \tag{7.17}$$

The function $V'(x)$ is not a polynomial or a Laurent polynomial but the function $x(x-1)V'(x)$ is a degree one polynomial. With a slight modification of the argument of Section 5.2.2, we can show that $x(x-1)\Pi(x) = r + sx$ is a degree one polynomial. In the large N limit we then have

$$\mathfrak{g}(z) = \frac{c_1 - 1 - (c_+ - 2)z + \sqrt[\oplus]{c_+^2 z^2 - 2(c_1 c_+ + c_-)z + (c_1 - 1)^2}}{2z(z-1)}, \tag{7.18}$$

where we have used the fact that we need $s = 0$ and $r = 1 + c_1 + c_2$ to get a $1/z$ behavior at infinity; we used the shorthand $c_\pm = c_2 \pm c_1$.

From the large z limit of (7.18) one can read off the normalized trace (or the average eigenvalue):

$$\tau(\mathbf{J}) = \frac{c_1}{c_1 + c_2}, \tag{7.19}$$

which is equal to one-half when $c_1 = c_2$. For $c_1 > 1$ and $c_2 > 1$, there are no poles, and eigenvalues exist only when the argument of the square-root is negative. The density of eigenvalues is therefore given by (see Fig. 7.1)

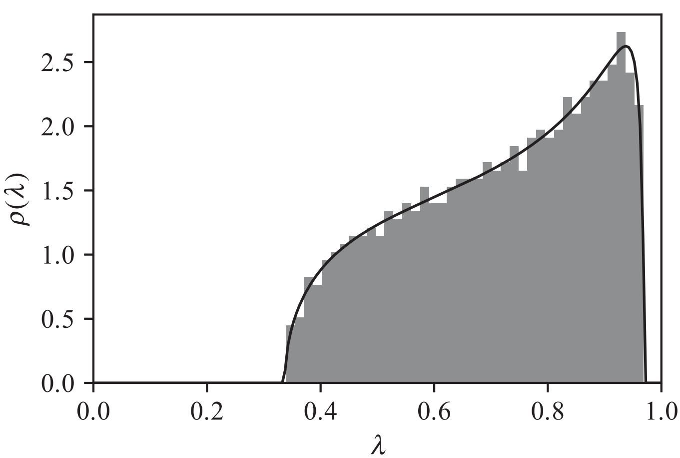

Figure 7.1 Density of eigenvalues for the Jacobi ensemble with $c_1 = 5$ and $c_2 = 2$. The histogram is a simulation of a single $N = 1000$ matrix with the same parameters.

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$$\rho(\lambda) = c_+ \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{2\pi\lambda(1-\lambda)}, \tag{7.20}$$

where the edges of the spectrum are given by

$$\lambda_\pm = \frac{c_1 c_+ + c_- \pm 2\sqrt{c_1 c_2 (c_+ - 1)}}{c_+^2}. \tag{7.21}$$

For $0 < c_1 < 1$ or $0 < c_2 < 1$, Eq. (7.18) will have Dirac deltas at $z = 0$ or $z = 1$, depending on cases (see Exercise 7.1.1).

In the symmetric case $c_1 = c_2 = c$, we have explicitly

$$\mathfrak{g}(z) = \frac{(c-1)(1-2z) + \sqrt[\oplus]{c^2(2z-1)^2 - c(c-2)}}{2z(z-1)}. \tag{7.22}$$

The density for $c \geq 1$ has no Dirac mass and is given by

$$\rho(\lambda) = c \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\pi\lambda(1-\lambda)}, \tag{7.23}$$

with the edges given by

$$\lambda_\pm = \frac{1}{2} \pm \frac{\sqrt{2c-1}}{2c}. \tag{7.24}$$

Note that the distribution is symmetric around $\lambda = 1/2$ (see Fig. 7.2).

As $c \to 1$, the edges tend to 0 and 1 and we recover the arcsine law:

$$\rho(\lambda) = \frac{1}{\pi\sqrt{\lambda(1-\lambda)}}. \tag{7.25}$$

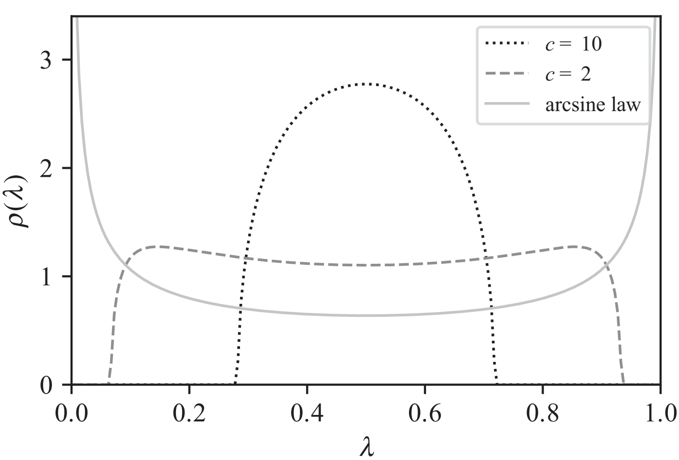

Figure 7.2 Density of eigenvalues for a Jacobi matrix in the symmetric case ($c_1 = c_2 = c$) for $c = 20, 2$ and 1. The case $c = 1$ is the arcsine law.

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Exercise 7.1.1 Dirac masses in the Jacobi density

(a) Assuming that $c_1 > 1$ and $c_2 > 1$ show that there are no poles in Eq. (7.18) at $z = 0$ or $z = 1$ by showing that the numerator vanishes for these two values of z.

(b) The parameters $c_{1,2}$ can be smaller than 1 (as long as they are positive). Show that in that case $\mathfrak{g}(z)$ can have poles at $z = 0$ and/or $z = 1$ and find the residue at these poles.

7.2 Jacobi Matrices and Jacobi Polynomials

7.2.1 Centered-Range Jacobi Ensemble

The standard Jacobi matrix defined above has all its eigenvalues between 0 and 1. We would like to use another definition of the Jacobi matrix with eigenvalues between -1 and 1. This will make easier the link with orthogonal polynomials. We define the centered-range Jacobi matrix[1]

$$\mathbf{J}_c = 2\mathbf{J} - 1. \tag{7.26}$$

This definition is equivalent to

$$\mathbf{J}_c = \widetilde{\mathbf{W}}_+^{-1/2}(\widetilde{\mathbf{W}}_-)\widetilde{\mathbf{W}}_+^{-1/2}, \text{ where } \widetilde{\mathbf{W}}_\pm = \mathbf{H}_1\mathbf{H}_1^T \pm \mathbf{H}_2\mathbf{H}_2^T, \tag{7.27}$$

with $\mathbf{H}_{1,2}$ as above.

The matrix $\mathbf{J}_c$ is still a member of a beta ensemble satisfying Eq. (5.2) with a slightly modified matrix potential:

$$NV(x) = -(T_1 - N + 1 - 2/\beta)\log(1+x) - (T_2 - N + 1 - 2/\beta)\log(1-x). \tag{7.28}$$

In the large N limit, the density of eigenvalues can easily be obtained from (7.20):

$$\rho(\lambda) = c_+ \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{2\pi(1-\lambda^2)}, \tag{7.29}$$

where the edges of the spectrum are given by

$$\lambda_\pm = \frac{c_-(2-c_+) \pm 4\sqrt{c_1c_2(c_+ - 1)}}{c_+^2}. \tag{7.30}$$

The special case $c_1 = c_2 = 1$ is the centered arcsine law:

$$\rho(\lambda) = \frac{1}{\pi\sqrt{(1-\lambda^2)}}. \tag{7.31}$$

[1] Note that the matrix $\mathbf{J}_c$ is not necessarily centered in the sense $\tau(\mathbf{J}_c) = 0$ but the potential range of its eigenvalues is $[-1, 1]$ centered around zero.

7.2.2 Average Expected Characteristic Polynomial

In Section 6.3.3, we saw that the average characteristic polynomial when $\beta = 2$ is the Nth monic polynomial orthogonal to the weight $w(x) \propto \exp(-NV(x))$. For the centered-range Jacobi matrix we have

$$w(x) = (1+x)^{T_1-N}(1-x)^{T_2-N}. \tag{7.32}$$

The Jacobi polynomials $P_n^{(a,b)}(x)$ are precisely orthogonal to such weight functions with $a = T_2 - N$ and $b = T_1 - N$. (Note that unfortunately the standard order of the parameters is inverted with respect to Jacobi matrices.) Jacobi polynomials satisfy the following differential equation:

$$(1-x^2)y'' + (b - a - (a+b+2)x)y' + n(n+a+b+1)y = 0. \tag{7.33}$$

This equation has polynomial solutions if and only if n is an integer. The solution is then $y \propto P_n^{(a,b)}(x)$.

The first three Jacobi polynomials are

$$\begin{aligned} P_0^{(a,b)}(x) &= 1, \\ P_1^{(a,b)}(x) &= \frac{(a+b+2)}{2}x + \frac{a-b}{2}, \\ P_2^{(a,b)}(x) &= \frac{(a+b+3)(a+b+4)}{8}(x-1)^2 + \frac{(a+2)(a+b+3)}{2}(x-1) \\ &\quad + \frac{(a+1)(a+2)}{2}. \end{aligned} \tag{7.34}$$

The normalization of Jacobi polynomials is arbitrary but in the standard normalization they are not monic. The coefficient of x^n in $P_n^{(a,b)}(x)$ is

$$a_n = \frac{\Gamma[a+b+2n+1]}{2^n n!\,\Gamma[a+b+n-1]}. \tag{7.35}$$

In summary we have (for $\beta = 2$)

$$\mathbb{E}\left[\det[z\mathbf{1} - \mathbf{J}_\mathrm{c}]\right] = \frac{2^N N!\,\Gamma[T_1 + T_2 + 1 - N]}{\Gamma[T_1 + T_2 + 1]} P_N^{(T_2-N,T_1-N)}(z). \tag{7.36}$$

Note that we must have $T_1 \geq N$ and $T_2 \geq N$.

When $T_1 = T_2 = N$ (i.e. $c_1 = c_2 = 1$, corresponding to the arcsine law), the polynomials $P_N^{(0,0)}(z)$ are called Legendre polynomials $P_N(z)$:[2]

$$\mathbb{E}\left[\det[z\mathbf{1} - \mathbf{Y}_c]\right] = \frac{2^N (N!)^2}{(2N)!} P_N(z). \tag{7.38}$$

[2] Legendre polynomials are defined as the polynomial solution of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P_n(x)}{\mathrm{d}x}\right] + n(n+1)P_n(x) = 0, \qquad P_n(1) = 1. \tag{7.37}$$

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7.2.3 Maximum Likelihood Configuration at Finite N

In Chapter 6, we studied the most likely configuration of eigenvalues for the beta ensemble at finite N. We saw that the finite-N Stieltjes transform of this solution $g_N(z)$ is related to a monic polynomial $\psi(x)$ via

$$g_N(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i} = \frac{\psi'(z)}{N\psi(z)}, \quad \text{where} \quad \psi(x) = \prod_{i=1}^{N}(x-\lambda_i). \tag{7.39}$$

The polynomial $\psi(x)$ satisfies Eq. (6.23) which we recall here:

$$\psi''(x) - NV'(x)\psi'(x) + N^2\Pi_N(x)\psi(x) = 0, \tag{7.40}$$

where the function $\Pi_N(x)$ is defined by Eq. (5.32). For the case of the centered-range Jacobi ensemble we have

$$NV'(x) = \frac{a-b+(a+b+2)x}{1-x^2}, \tag{7.41}$$

where we have anticipated the result by introducing the notation $a = T_2 - N - 2/\beta$ and $b = T_1 - N - 2/\beta$. The function $\Pi_N(x)$ is given by

$$\Pi_N(x) = \frac{r_N + s_N x}{1-x^2}. \tag{7.42}$$

We will see below that the coefficient s_N is zero because of the symmetry $\{c_1, c_2, \lambda_k\} \to \{c_2, c_1, -\lambda_k\}$ of the most likely solution.

The equation for $\psi(x)$ becomes

$$(1-x^2)\psi''(x) + (b-a-(a+b+2)x)\psi'(x) + r_N N^2\psi(x) = 0. \tag{7.43}$$

We recognize the differential equation satisfied by the Jacobi polynomials (7.33). Its solutions are polynomials only if $r_N N = N + a + b + 1$, which implies that $r_N = c_1 + c_2 - 1 + (1-4\beta)/N$. This is consistent with the large N limit $r = c_1 + c_2 - 1$ in Section 7.1.3. The solutions are given by

$$\psi(x) \propto P_N^{(a,b)}(x), \tag{7.44}$$

with the proportionality constant chosen such that $\psi(x)$ is monic.

In the special case $T_1 = T_2 = N + 2/\beta$, i.e. $T = N + 2$ for real symmetric matrices and $T = N + 1$ for complex Hermitian matrices, we have $a = b = 0$ and the polynomials reduce to Legendre polynomials.

Another special case corresponds to Chebyshev polynomials:

$$T_n(x) = P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) \quad \text{and} \quad U_n(x) = P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x), \tag{7.45}$$

where $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of first and second kind respectively. Since $a = T_2 - N - 2/\beta$ and $b = T_1 - N - 2/\beta$, they appear as solutions for $T_1 = T_2 = N + 2/\beta - 1/2$ (first kind) or $T_1 = T_2 = N + 2/\beta + 1/2$ (second kind). These values of $T_1 = T_2$ are not integers but we can still consider the matrix potential

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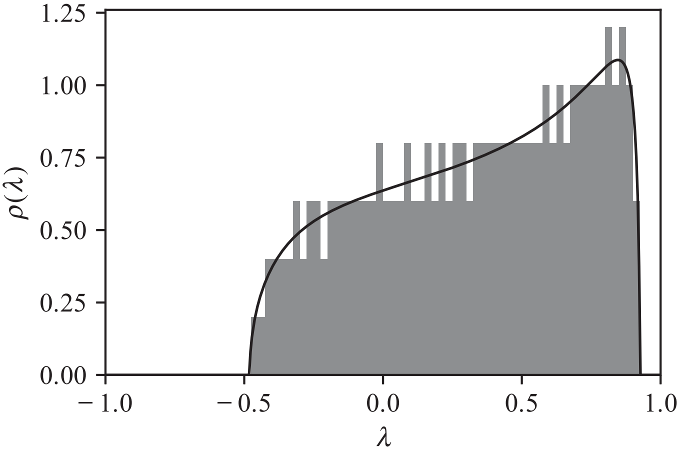

Figure 7.3 Histogram of the 200 zeros of $P_{200}^{(200,600)}(x)$. The full line is the Jacobi eigenvalue density, Eq. (7.29), with $c_1 = 4$ and $c_2 = 2$.

given by Eq. (7.28) without the explicit Wishart matrix construction. In the large N limit, the density of the zeros of the Jacobi polynomials $P_N^{(T_2-N,T_1-N)}(x)$ is given by Eq. (7.29) with $c_{1,2} = T_{1,2}/N$. Figure 7.3 shows a histogram of the zeros of $P_{200}^{(200,600)}(x)$.

When $c_1 \to 1$ and $c_2 \to 1$, the density becomes the centered arcsine law. As a consequence, we have shown that the zeros of Chebyshev (both kinds) and Legendre polynomials (for which $T_1 = T_2 = N + O(1)$) are distributed according to the centered arcsine law in the large N limit.

We have seen that in order for Eq. (7.40) to have polynomial solutions we must have

$$N(1-x^2)\Pi_N(x) = N + a + b + 1, \tag{7.46}$$

where the function $\Pi_N(x)$ is defined from the most likely configuration or, equivalently, the roots of the Jacobi polynomial $P_N^{(a,b)}(z)$ by

$$\Pi_N(x) = \frac{1}{N}\sum_{k=0}^{N} \frac{V'(x) - V'(\lambda_k)}{x - \lambda_k}. \tag{7.47}$$

From these expressions we can find a relationship that roots of Jacobi polynomials must satisfy. Indeed, injecting Eq. (7.41), we find

$$N(1-x^2)\Pi_N(x) = \sum_{k=1}^{N} \frac{(a-b)(x^2-\lambda_k^2) + (a+b+2)(x-\lambda_k)(1+x\lambda_k)}{(1-\lambda_k^2)(x-\lambda_k)}. \tag{7.48}$$

For each k the numerator is a second degree polynomial in x that is zero at $x = \lambda_k$, canceling the $x - \lambda_k$ factor in the denominator, so the whole expression is a first degree polynomial. Equating this expression to Eq. (7.47), we find that the term linear in x of

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this polynomial must be zero and the constant term equal to $N + a + b + 1$, yielding two equations:

$$\frac{1}{N}\sum_{k=1}^{N}\frac{(a+b+2)\lambda_k + (a-b)}{(1-\lambda_k^2)} = 0, \tag{7.49}$$

$$\frac{1}{N}\sum_{k=1}^{N}\frac{(a+b+2) + (a-b)\lambda_k}{(1-\lambda_k^2)} = N + a + b + 1. \tag{7.50}$$

These equations give us non-trival relations satisfied by the roots of Jacobi polynomials $P_N^{(a,b)}(x)$. If we sum or subtract the two equations above, we finally obtain[3]

$$\frac{1}{N}\sum_{k=1}^{N}\frac{1}{(1-\lambda_k)} = \frac{a+b+N+1}{2(a+1)}, \tag{7.51}$$

$$\frac{1}{N}\sum_{k=1}^{N}\frac{1}{(1+\lambda_k)} = \frac{a+b+N+1}{2(b+1)}. \tag{7.52}$$

7.2.4 Discrete Laplacian in One Dimension and Chebyshev Polynomials

Chebyshev polynomials and the arcsine law are also related via a simple deterministic matrix: the discrete Laplacian in one dimension, defined as

$$\Delta = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{7.53}$$

We will see that the spectrum of $\Delta - \mathbf{1}$ at large N is again given by the arcsine law. One way to obtain this result is to modify the top-right and bottom-left corner elements by adding -1. The modified matrix is then a circulant matrix which can be diagonalized exactly (see Exercise 7.2.1 and Appendix A.3).

We will use a different route which will also uncover the link to Chebyshev polynomials. We will compute the characteristic polynomial $Q_N(z)$ of $\Delta - \mathbf{1}$ for all values of N by induction. The first two are

$$Q_1(z) = z, \tag{7.54}$$

$$Q_2(z) = z^2 - \frac{1}{4}. \tag{7.55}$$

[3] These relations were recently obtained in Alıcı and Taeli [2015].

For $N \geq 3$ we can write a recursion relation by expanding in minors the first line of the determinant of $z\mathbf{1} - \Delta + \mathbf{1}$. The (11)-minor is just $Q_{N-1}(z)$. The first column of the (12)-minor only has one element equal to $1/2$; if we expand this column its only minor is $Q_{N-2}(z)$. We find

$$Q_N(z) = zQ_{N-1}(z) - \frac{1}{4}Q_{N-2}(z). \tag{7.56}$$

This simple recursion relation is similar to that of Chebyshev polynomials $U_N(x)$:

$$U_N(x) = 2xU_{N-1} - U_{N-2}(x). \tag{7.57}$$

The standard Chebyshev polynomials are not monic but have leading term $U_N(x) \sim 2^N x^N$. Monic Chebyshev ($\widetilde{U}_N(x) = 2^{-N}U_N(x)$) in fact precisely satisfy Eq. (7.56). Given our first two polynomials are the monic Chebyshev of the second kind, we conclude that $Q_N(z) = 2^{-N}U_N(z)$ for all N. The eigenvalues of $\Delta - \mathbf{1}$ at size N are therefore given by the zeros of the Nth Chebyshev polynomial of the second kind. In the large N limit those are distributed according to the centered arcsine law. QED.

We will see in Section 15.3.1 that the sum of two random symmetric orthogonal matrices also has eigenvalues distributed according to the arcsine law.

Exercise 7.2.1 Diagonalizing the Discrete Laplacian

Consider $\mathbf{M} = \Delta - \mathbf{1}$ with $-1/2$ added to the top-right and bottom-left corners.

(a) Show that the vectors $[\mathbf{v}_k]_j = e^{i2\pi kj}$ are eigenvectors of $\mathbf{M}$ with eigenvalues $\lambda_k = \cos(2\pi k)$.

(b) Show that the eigenvalue density of $\mathbf{M}$ in the large N limit is given by the centered arcsine law (7.31).

Bibliographical Notes

- For general references on orthogonal polynomials, see
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- On relations obeyed by zeros of Jacobi polynomials, see
  - H. Alıcı and H. Taeli. Unification of Stieltjes-Calogero type relations for the zeros of classical orthogonal polynomials. *Mathematical Methods in the Applied Sciences*, 38(14):3118–3129, 2015.
- On the MANOVA method, see e.g.

  - R. T. Warne. A primer on multivariate analysis of variance (MANOVA) for behavioral scientists. *Practical Assessment, Research & Evaluation*, 19(17):1–10, 2014.
- On the spectrum of Jacobi matrices using Coulomb gas methods, see
  - H. M. Ramli, E. Katzav, and I. P. Castillo. Spectral properties of the Jacobi ensembles via the Coulomb gas approach. *Journal of Physics A: Mathematical and Theoretical*, 45(46):465005, 2012.

  and references therein.