16
Products of Many Random Matrices

In this chapter we consider an issue of importance in many different fields: that of products of many random matrices. This problem arises, for example, when one considers the transmission of light in a succession of slabs of different optical indices, or the propagation of an electron in a disordered wire, or the way displacements propagate in granular media. It also appears in the context of chaotic systems when one wants to understand how a small difference in initial conditions "propagates" as the dynamics unfolds. In this context, one usually linearizes the dynamics in the vicinity of the unperturbed trajectory. If one takes stroboscopic snapshots of the system, the perturbation is obtained as the product of matrices (corresponding to the linearized dynamics) applied on the initial perturbation (see Chapter 1). If the phase space of the system is large enough, and the dynamics chaotic enough, one may expect that approximating the problem as a product of large, free matrices should be a good starting point.

16.1 Products of Many Free Matrices

The specific problem we will study is therefore the following: consider the symmetrized product of K matrices, defined as

$$\mathbf{M}_K = \mathbf{A}_K \mathbf{A}_{K-1} \ldots \mathbf{A}_2 \mathbf{A}_1 \mathbf{A}_1^T \mathbf{A}_2^T \ldots \mathbf{A}_{K-1}^T \mathbf{A}_K^T, \tag{16.1}$$

where all $\mathbf{A}_i$ are identically distributed and mutually free, i.e. randomly rotated with respect to one another. We know now that in such a case the S-transforms simply multiply. Noting as $S_i(z)$ the S-transform of $\mathbf{A}_i \mathbf{A}_i^T$, and $S_{\mathbf{M}_K}(z)$ the S-transform of $\mathbf{M}_K$, one has

$$S_{\mathbf{M}_K}(z) = \prod_{i=1}^{K} S_i(z) \equiv S_1(z)^K. \tag{16.2}$$

Now, it is intuitive that all the eigenvalues of $\mathbf{M}_K$ will behave for large K as μ^K, where μ is itself a random variable which we will characterize below. We take this as an assumption and indeed show that the distribution of μ's tends to a well-defined function $\rho_\infty(\mu)$ as $K \to \infty$. Note here a crucial difference with the case of sums of random matrices. If we assume that the eigenvalues of a sum of K free random matrices behave as $K \times \mu$, one can easily establish that the distribution of μ's collapses for large K to $\delta(\mu - \tau(\mathbf{A}))$, with, once again, $\tau(.) = \mathrm{Tr}(.)/N$. For products of random matrices, on the other hand, the distribution of μ remains non-trivial, as we will find below.

Let us compute $S_{\mathbf{M}_K}(z)$ in the large K limit using our ansatz that the eigenvalues of $\mathbf{M}$ are indeed of the form μ^K. We first compute the function $\mathfrak{t}_K(z)$ equal to

$$\mathfrak{t}_K(z) := \int \frac{\mu^K}{z - \mu^K} \rho_\infty(\mu) \mathrm{d}\mu = -\int \frac{1}{1 - z\mu^{-K}} \rho_\infty(\mu) \mathrm{d}\mu. \tag{16.3}$$

Setting $z := u^K$, we see that for $K \to \infty$ there is no contribution to this integral from the region $\mu < u$, whereas the region $\mu > u$ simply yields

$$\mathfrak{t}_K(z) \approx -P_>(z^{1/K}); \qquad P_>(u) := \int_u^\infty \rho_\infty(\mu) \mathrm{d}\mu. \tag{16.4}$$

The next step to get the S-transform is to compute the functional inverse of $\mathfrak{t}_K(z)$. Within the same approximation, this is given by

$$\mathfrak{t}_K^{-1}(z) = \left[P_>^{(-1)}(-z) \right]^K, \tag{16.5}$$

where $P_>^{(-1)}$ is the functional inverse of the cumulative distribution function $P_>$. Finally, by definition,

$$S_{\mathbf{M}_K}(z) := \frac{1+z}{z\mathfrak{t}_K^{-1}(z)} = S_1(z)^K. \tag{16.6}$$

Hence one finds, in the large K limit where $((1+z)/z)^{1/K} \to 1$,

$$P_>^{(-1)}(-z) = \frac{1}{S_1(z)} \Rightarrow P_>(\mu) = -S_1^{(-1)}\left(\frac{1}{\mu}\right) \tag{16.7}$$

and finally $\rho_\infty(\mu) = -P'_>(\mu)$. The final result is therefore quite simple, and entirely depends on the S-transform of $\mathbf{A}_i\mathbf{A}_i^T$.

A simple case is when $\mathbf{A}_i\mathbf{A}_i^T$ is a large Wishart matrix, with parameter $q \leq 1$. In this case $S_1(z) = (1 + qz)^{-1}$, from which one easily works out that $\rho_\infty(\mu) = 1/q$ for $\mu \in (1-q, 1)$ and zero elsewhere (see Fig. 16.1 for an illustration).

In many cases of interest, the eigenvalue spectrum of $\mathbf{A}_i\mathbf{A}_i^T$ has some symmetries, coming from the underlying physical problem one is interested in. For example, when our chaotic system is invariant under time reversal (like the dynamics of a Hamiltonian system), each eigenvalue λ must come with its inverse λ^{-1}. A simple example of a spectrum with such a symmetry is the free log-normal, further discussed in the next section. It is defined from its S-transform, given by

$$S^0_{\mathrm{LN}}(z) = \mathrm{e}^{-a(z+\frac{1}{2})}, \tag{16.8}$$

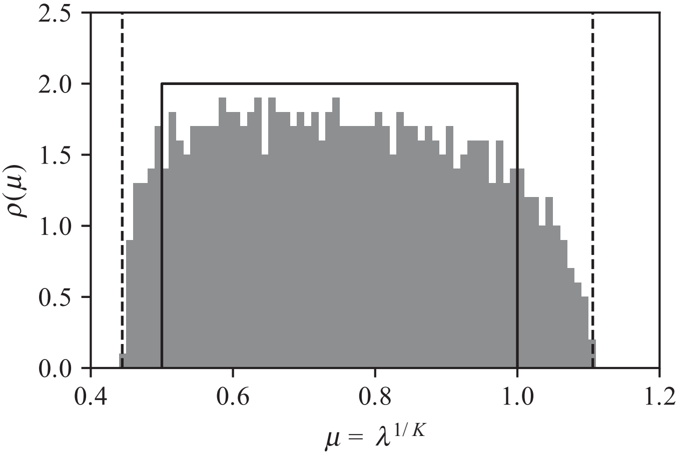

Figure 16.1 Sample density of $\mu = \lambda^{1/K}$ for the free product of $K = 40$ white Wishart matrices with $q = 1/2$ and $N = 1000$. The dark line corresponds to the asymptotic density ($K \to \infty$), which is constant between $1 - q$ and 1 and zero elsewhere. The two dashed vertical lines give the exact positions of the edges of the spectrum ($\mu_- = 0.44$ and $\mu_+ = 1.10$) for $K = 40$, as computed in Exercise 16.1.1.

where the parameter a is related to the trace of the corresponding matrices equal to $e^{a/2}$. Multiplying K such matrices together leads to eigenvalues of the form μ^K, with

$$P_>(\mu) = -\left(S^0_{\rm LN}\right)^{(-1)}\left(\frac{1}{\mu}\right) = \frac{1}{2} - \frac{\log\mu}{a}, \tag{16.9}$$

corresponding to

$$\rho_\infty(\mu) = -P'_>(\mu) = \frac{1}{a\mu}, \qquad \mu \in (e^{-a/2}, e^{a/2}), \tag{16.10}$$

and zero elsewhere. One can explicitly check that μ^{-1} has the same probability distribution function as μ.

One often describes the eigenvalues of large products of random matrices in terms of the Lyapunov exponents Λ, defined as the eigenvalues of

$$\Lambda = \lim_{K\to\infty} \frac{1}{K} \log \mathbf{M}_K. \tag{16.11}$$

Therefore the Lyapunov exponents are simply related to the μ's above as $\Lambda \equiv \log\mu$. For the free log-normal example, the distribution of Λ is found to be uniform between $-a/2$ and $a/2$.

Let us end this section with an important remark: we have up to now considered products of K matrices with a fixed spectrum, independent of K, which leads to a non-universal distribution of Lyapunov exponents (i.e. a distribution that explicitly depends on the full function $S_1(z)$). Let us now instead assume that these matrices are of the form

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$$\mathbf{A}\mathbf{A}^T = \left(1 + \frac{a}{2K}\right)\mathbf{1} + \frac{\mathbf{B}}{\sqrt{K}}, \tag{16.12}$$

where a is a parameter and $\mathbf{B}$ is traceless and characterized by its second cumulant $b = \tau(\mathbf{B}^2)$. For large K, $S_1(z)$ can then be expanded as

$$S_1(z) = 1 - \frac{a}{2K} - \frac{b}{K}z + o(K^{-1}). \tag{16.13}$$

Therefore, for large K, the product of such matrices converges to a matrix characterized by

$$S_{\mathbf{M}_K}(z) = \left(1 - \frac{a}{2K} - \frac{b}{K}z\right)^K \to \mathrm{e}^{-a/2-bz}, \tag{16.14}$$

which can be interpreted as a multiplicative CLT for free matrices, since the detailed statistics of $\mathbf{B}$ has disappeared. The choice $b = a$ corresponds to the free log-normal with inversion symmetry S^0_{LN} (see next section).

Exercise 16.1.1 Edges of the spectrum for the free product of many white Wishart matrices

In this exercise, we will compute the edges of the spectrum of eigenvalues of a matrix $\mathbf{M}$ given by the free product of K large white Wishart matrices with parameter q.

(a) The S-transform of $\mathbf{M}$ is simply given by the S-transform of a white Wishart raised to the power K. Using Eq. (11.92), write an equation for the inverse of the T-transform, $\zeta(t)$, of the matrix $\mathbf{M}$. This is a polynomial equation of order $K + 1$.

(b) For odd N, plot $\zeta(t)$ for various K and $0 < q < 1$ and convince yourself that there is always a region of ζ where $\zeta(t) = \zeta$ has no real solution. This region is between a local maximum and a local minimum of $\zeta(t)$. For even N, the argument is more subtle, but the correct branch exists only between the same two extrema.

(c) Differentiate $\zeta(t)$ with respect to t to find an equation for the extrema of $\zeta(t)$. After simplifications and discarding the $t = -1/q$ solution, this equation is quadratic in t^* with two solutions corresponding to the local minimum and maximum. Find the two solutions $t^*_\pm$ and plug these back in your equation for $\zeta(t)$ to find the edges of the spectrum $\lambda_\pm$.

(d) Use your result for $K = 40$ and $q = 0.5$ to verify the edges of the spectrum given in Figure 16.1.

(e) Compute the large K limit of $t^*_\pm$. You should find $t^*_- \to -1$ and $t^*_+ \to (q(K-1))^{-1}$. Show that at large K we have $\lambda_-^{1/K} \to 1 - q$ and $\lambda_+^{1/K} \to 1$.

16.2 The Free Log-Normal

There exists a free version of the log-normal. Its S-transform is given by

$$S_{\mathrm{LN}}(t) = \mathrm{e}^{-a/2-bt}. \tag{16.15}$$

As a two-parameter family, the free log-normal is stable in the sense that the free product of two free log-normals with parameters a_1, b_1 and a_2, b_2 is a free log-normal with parameters $a = a_1 + a_2, b = b_1 + b_2$. The first three free cumulants can be computed from Eq. (16.15):

$$S_{\mathrm{LN}}(t) = \mathrm{e}^{-a/2}\left[1 - bt + \frac{1}{2}b^2t^2\right] + O(t^3). \tag{16.16}$$

Comparing with Eq. (15.11), this leads to

$$\begin{aligned} \kappa_1 &= \mathrm{e}^{a/2}, \\ \kappa_2 &= b\mathrm{e}^{a}, \\ \kappa_3 &= \frac{3b^2}{2}\mathrm{e}^{2a}. \end{aligned} \tag{16.17}$$

In the special case $b = a$, the free log-normal S^0_{LN} has the additional property that its matrix inverse has exactly the same law. Indeed, we have shown in Section 11.4.4 that the following general relation holds:

$$S_{\mathbf{M}^{-1}}(t) = \frac{1}{S_{\mathbf{M}}(-t-1)}, \tag{16.18}$$

or, in the free log-normal case with $a = b$,

$$S_{\mathbf{M}^{-1}}(t) = \mathrm{e}^{a/2-b(1+t)} = S_{\mathbf{M}}(t) \tag{16.19}$$

when $b = a$. This implies that the eigenvalue distribution is invariant under $\lambda \to 1/\lambda$ and therefore that $\mathbf{M}$ has unit determinant. Let us study in more detail the eigenvalue spectrum for the symmetric case $a = b$. By looking for the real extrema of

$$\zeta(t) = \frac{t+1}{t}\mathrm{e}^{a(t+1/2)}, \tag{16.20}$$

we can find the points $t_\pm$ where $t(\zeta)$ ceases to be invertible, which in turn give the edges of the spectrum $\lambda_\pm = \zeta(t_\pm)$:

$$t_\pm = \frac{\pm\sqrt{1+\frac{4}{a}} - 1}{2} \tag{16.21}$$

or

$$\lambda_+ = \frac{1}{\lambda_-} = \left[\sqrt{\frac{a}{4}} + \sqrt{1+\frac{a}{4}}\right]^2 \exp\left(\sqrt{a+\frac{a^2}{4}}\right). \tag{16.22}$$

Note that $\lambda_+ = \lambda_- = 1$ when $a = b = 0$, corresponding to the identity matrix. The eigenvalue distribution is symmetric in $\lambda \to 1/\lambda$ so the density $\rho(\ell)$ of $\ell = \log(\lambda)$ is even. Figure 16.2 shows the density of ℓ for $a = 100$.

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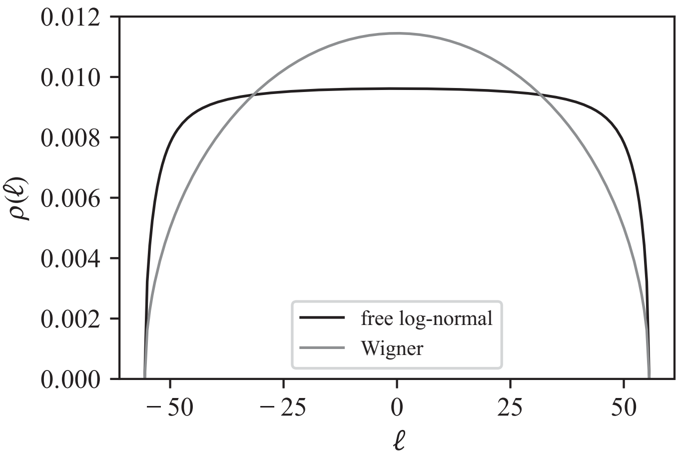

Figure 16.2 Probability density of $\ell = \log(\lambda)$ for a symmetric free log-normal (16.15) with $a = b = 100$ compared with a Wigner semi-circle with the same endpoints. As expected, the distribution is even in ℓ. For $a \lesssim 1$ the density of ℓ is indistinguishable to the eye from a Wigner semi-circle (not shown), whereas for $a \to \infty$ the distribution of ℓ/a tends to a uniform distribution on $[-1/2, 1/2]$.

In the more general case $a \neq b$, the whole distribution of $\ell = \log(\lambda)$ is just shifted by $(a-b)/2$, as expected from the scaling property of the S-transform upon multiplication by a scalar.

16.3 A Multiplicative Dyson Brownian Motion

Let us now consider the problem of multiplying random matrices close to unity from a slightly different angle. Consider the following iterative construction for $N \times N$ matrices:

$$\mathbf{M}_{n+1} = \mathbf{M}_n^{\frac{1}{2}} \left[(1 + \frac{a\varepsilon}{2})\mathbf{1} + \sqrt{\varepsilon}\mathbf{B}_n \right] \mathbf{M}_n^{\frac{1}{2}}, \tag{16.23}$$

where $\mathbf{B}_n$ is a sequence of identical, free, traceless $N \times N$ matrices and $\varepsilon \ll 1$. Using second order perturbation theory, one can deduce an iteration formula for the eigenvalues $\lambda_{i,n}$ of $\mathbf{M}_n$, which reads

$$\lambda_{i,n+1} = \lambda_{i,n} \left(1 + \frac{a\varepsilon}{2} + \sqrt{\varepsilon}\mathbf{v}_{i,n}^T \mathbf{B}_n \mathbf{v}_{i,n} \right) + \varepsilon \sum_{j \neq i} \frac{\lambda_{i,n}\lambda_{j,n}(\mathbf{v}_{i,n}^T \mathbf{B}_n \mathbf{v}_{j,n})^2}{\lambda_{i,n} - \lambda_{j,n}}, \tag{16.24}$$

where $\mathbf{v}_{i,n}$ are the corresponding eigenvectors. Noting that $\mathbf{M}_n$ and $\mathbf{B}_n$ are mutually free and that $\tau(\mathbf{B}_n) = 0$, one has, in the large N limit (using, for example, Eq. (12.8)),

$$\mathbb{E}[\mathbf{v}_{i,n}^T \mathbf{B}_n \mathbf{v}_{i,n}] = 0; \qquad \mathbb{E}[(\mathbf{v}_{i,n}^T \mathbf{B}_n \mathbf{v}_{j,n})^2] = \frac{b}{N}, \tag{16.25}$$

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where $b := \tau(\mathbf{B}_n^2)$. Choosing $\varepsilon = \mathrm{d}t$, an infinitesimal time scale, we end up with a multiplicative version of the Dyson Brownian motion in fictitious time t:

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = \frac{a}{2}\lambda_i + \frac{b}{N}\sum_{j\neq i}\frac{\lambda_i\lambda_j}{\lambda_i - \lambda_j} + \sqrt{\frac{b}{N}}\lambda_i\xi_i, \tag{16.26}$$

where ξ_i is a Langevin noise, independent for each λ_i (compare with Eq. (9.9)).

Now, let us consider the "time" dependent Stieltjes transform, defined as usual as

$$\mathfrak{g}(z,t) = \frac{1}{N}\sum_i \frac{1}{z - \lambda_i(t)}. \tag{16.27}$$

Its evolution is obtained as

$$\frac{\partial \mathfrak{g}}{\partial t} = \frac{1}{N}\sum_i \frac{1}{(z-\lambda_i)^2}\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = -\frac{1}{N}\frac{\partial}{\partial z}\sum_i \frac{1}{(z-\lambda_i)}\frac{\mathrm{d}\lambda_i}{\mathrm{d}t}. \tag{16.28}$$

After manipulations very similar to those encountered in Section 9.3.1, and retaining only leading terms in N, one finally obtains

$$\frac{\partial \mathfrak{g}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial z}\left[(2b-a)z\mathfrak{g} - bz^2\mathfrak{g}^2\right]. \tag{16.29}$$

Now, introduce the auxiliary function $h(\ell,t) := \mathrm{e}^{\ell}\mathfrak{g}(\mathrm{e}^{\ell},t) + a/2b - 1$, which obeys

$$\frac{\partial h}{\partial t} = -bh\frac{\partial h}{\partial \ell}. \tag{16.30}$$

This is precisely the Burgers' equation (9.37), up to a rescaling of time $t \to bt$. Its solution obeys the following self-consistent equation obtained using the method of characteristics (see Section 10.1):

$$h(\ell,t) = h_0(\ell - bth(\ell,t)); \qquad h_0(\ell) := h(\ell,0) = \frac{1}{1-\mathrm{e}^{-\ell}} + \frac{a}{2b} - 1, \tag{16.31}$$

where we have assumed that at time $t = 0$ the dynamics starts from the identity matrix: $\mathbf{M}_0 = \mathbf{1}$, for which $\mathfrak{g}(z,0) = (z-1)^{-1}$. Hence, with $z = \mathrm{e}^{\ell}$,

$$\mathfrak{g}(z,t) = \frac{1}{z - \mathrm{e}^{t(bz\mathfrak{g}(z,t)+a/2-b)}}. \tag{16.32}$$

Now, let us compare this equation to the one obeyed by the Stieltjes transform of the free log-normal. Injecting $t = z\mathfrak{g}_{\mathrm{LN}} - 1$ in

$$z = \frac{t+1}{tS_{\mathrm{LN}}(t)} \tag{16.33}$$

and using Eq. (16.15), one finds

$$z\mathfrak{g}_{\mathrm{LN}} - 1 = \mathfrak{g}_{\mathrm{LN}}\mathrm{e}^{a/2-b+bz\mathfrak{g}_{\mathrm{LN}}} \to \mathfrak{g}_{\mathrm{LN}} = \frac{1}{z - \mathrm{e}^{bz\mathfrak{g}_{\mathrm{LN}}+a/2-b}}, \tag{16.34}$$

which coincides with Eq. (16.32) for $t = 1$, as it should. For arbitrary times, one finds that the density corresponding to the multiplicative Dyson Brownian motion, Eq. (16.26), is the free log-normal, with parameters ta and tb.

16.4 The Matrix Kesten Problem

The Kesten iteration for scalar random variables appears in many different situations. It is defined by

$$Z_{n+1} = z_n(1 + Z_n), \tag{16.35}$$

where z_n are IID random variables. In the following, we will assume that

$$z_n = 1 + \varepsilon m + \sqrt{\varepsilon}\sigma\eta_n, \tag{16.36}$$

where $\varepsilon \ll 1$ and η_n are IID random variables, of zero mean and unit variance. Setting $Z_n = U_n/\varepsilon$ and expanding to first order in ε, one obtains

$$U_{n+1} = \varepsilon(1 + \varepsilon m + \sqrt{\varepsilon}\sigma\eta_n)\left(1 + \frac{U_n}{\varepsilon}\right) = U_n + \varepsilon m U_n + \sqrt{\varepsilon}\sigma\eta_n U_n + \varepsilon \tag{16.37}$$

or, in the continuous time limit $\mathrm{d}t = \varepsilon$, the following Langevin equation:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = 1 + mU + \sigma\eta U. \tag{16.38}$$

The corresponding Fokker–Planck equation reads

$$\frac{\partial P(U,t)}{\partial t} = -\frac{\partial}{\partial U}\left[(1 + mU)P\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial U^2}\left[U^2 P\right]. \tag{16.39}$$

This process has a stationary distribution provided the drift m is negative. We will thus write $m = -\hat{m}$ with $\hat{m} > 0$. The corresponding stationary distribution $P_{\mathrm{eq}}(U)$ obeys

$$(1 - \hat{m}U)P_{\mathrm{eq}} = \frac{\sigma^2}{2}\frac{\partial}{\partial U}\left[U^2 P\right], \tag{16.40}$$

which leads to

$$P_{\mathrm{eq}}(U) = \frac{2^{\mu}}{\Gamma(\mu)\sigma^{2\mu}}\frac{\mathrm{e}^{-\frac{2}{\sigma^2 U}}}{U^{1+\mu}}; \qquad \mu := 1 + 2\hat{m}/\sigma^2, \tag{16.41}$$

to wit, the distribution of U is an inverse-gamma, with a power-law tail $U^{-1-\mu}$ with a non-universal exponent $\mu = 1 + 2\hat{m}/\sigma^2$.

Now we can generalize the Kesten iteration for symmetric matrices as[1]

$$\mathbf{U}_{n+1} = \varepsilon\sqrt{\mathbf{1} + \frac{\mathbf{U}_n}{\varepsilon}}\left((1 + m\varepsilon)\mathbf{1} + \sqrt{\varepsilon}\sigma\mathbf{B}\right)\sqrt{\mathbf{1} + \frac{\mathbf{U}_n}{\varepsilon}}, \tag{16.42}$$

or

$$\mathbf{U}_{n+1} - \mathbf{U}_n = \varepsilon\left(\mathbf{1} + m\mathbf{U}_n\right) + \sigma\sqrt{\varepsilon}\sqrt{\mathbf{U}_n}\mathbf{B}\sqrt{\mathbf{U}_n}. \tag{16.43}$$

Following the same steps as in the previous section, we obtain a differential equation for the eigenvalues of $\mathbf{U}$ (where we neglect the noise when $N \to \infty$):

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = 1 - \hat{m}\lambda_i + \frac{\sigma^2}{N}\sum_{j\neq i}\frac{\lambda_i\lambda_j}{\lambda_i - \lambda_j}, \tag{16.44}$$

where we again assume that $m < 0$ in order to find a stationary state for our process. The corresponding evolution of the Stieltjes transform reads, for large N,

[1] The results of this section have been obtained in collaboration with T. Gautié and P. Le Doussal.

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$$\frac{\partial \mathfrak{g}}{\partial t} = \frac{\partial}{\partial z}\left[-\mathfrak{g} + (\sigma^2 + \hat{m})z\mathfrak{g} - \frac{1}{2}\sigma^2 z^2 \mathfrak{g}^2\right]. \tag{16.45}$$

If an equilibrium density exists, its Stieltjes transform must obey

$$\frac{1}{2}\sigma^2 z^2 \mathfrak{g}^2 + (1 - (\sigma^2 + \hat{m})z)\mathfrak{g} + C = 0, \tag{16.46}$$

where C is a constant determined by the fact that $z\mathfrak{g} \to 1$ when $z \to \infty$. Hence,

$$C = \frac{1}{2}\sigma^2 + \hat{m}. \tag{16.47}$$

From the second order equation on $\mathfrak{g}$ one gets

$$\mathfrak{g} = \frac{1}{\sigma^2 z^2}\left[((\sigma^2 + \hat{m})z - 1) - \sqrt[\oplus]{\hat{m}^2 z^2 - 2(\sigma^2 + \hat{m})z + 1}\right]. \tag{16.48}$$

As usual, the density of eigenvalues is non-zero when the square-root becomes imaginary. The edges are thus given by the roots of the second degree polynomial inside the square-root, namely

$$\lambda_{\pm} = \frac{\sigma^2 + \hat{m} \pm \sqrt{\sigma^2(\sigma^2 + 2\hat{m})}}{\hat{m}^2}. \tag{16.49}$$

So only when $\hat{m} \to 0$ can the spectrum extend to infinity, with a power-law decay as $\lambda^{-3/2}$. Otherwise, the power law is truncated beyond $2\sigma^2/\hat{m}^2$. Note that, contrary to the scalar Kesten case, the exponent of the power law is universal, with $\mu = 1/2$.

In fact, if one stares at Eq. (16.48), one realizes that the stationary Kesten matrix $\mathbf{U}$ is an inverse-Wishart matrix. Indeed, the eigenvalue spectrum given by Eq. (16.48) maps into the Marčenko–Pastur law, Eq. (4.43), provided one makes the following transformation:

$$\lambda \to x = \frac{2}{\sigma^2 + 2\hat{m}}\frac{1}{\lambda}. \tag{16.50}$$

The parameter q of the Marčenko–Pastur law is then given by

$$q = \frac{\sigma^2}{\sigma^2 + 2\hat{m}} = \frac{1}{\mu} < 1. \tag{16.51}$$

Although not trivial, this result is not so surprising: since Wishart matrices are the matrix equivalent of the scalar gamma distribution, the matrix equivalent of the Kesten variable distributed as an inverse-gamma, Eq. (16.41), is an inverse-Wishart.

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