T-Ex 03

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QF 13

a)

$$V_n = \frac{V_{n-1}}{2}\left(1+r_n^{(1)}\right) + \frac{V_{n-1}}{2}\left(1+r_n^{(2)}\right) = \frac{V_{n-1}}{2}\left(2+r_n^{(1)}+r_n^{(2)}\right)$$

$$= \frac{\left(\frac{V_{n-2}}{2}\left(2+r_{n-1}^{(1)}+r_{n-1}^{(2)}\right)\right)}{2}\left(2+r_n^{(1)}+r_n^{(2)}\right) = \frac{V_{n-2}}{2\cdot 2}\left(2+r_{n-1}^{(1)}+r_{n-1}^{(2)}\right)\left(2+r_n^{(1)}+r_n^{(2)}\right)$$

$$= \frac{V_{n-n}}{2^n}\prod_{i=1}^{n}\left(1+r_i^{(1)}+1+r_i^{(2)}\right) = \frac{V_0}{2^n}\prod_{i=1}^{n}\left(e^{X_n^{(1)}}+e^{X_n^{(2)}}\right) = \frac{V_0}{2^n}\prod_{i=1}^{n}\left(e^{Z_n^{(1)}-Z_{n-1}^{(1)}}+e^{Z_n^{(2)}-Z_{n-1}^{(2)}}\right)$$

$$= f(Z_n \mid \mathcal{F}_{n-1})$$

$$\boxed{1+r_n^{(1)} = \frac{S_n^{(1)}}{S_{n-1}^{(1)}} = e^{Z_n^{(1)}-Z_{n-1}^{(1)}} = e^{X_n^{(1)}}}$$

b) $$X_{n+1}^{(1)} = Z_{n+1}^{(1)} - Z_n^{(1)} = \log\left(\frac{S_{n+1}^{(1)}}{S_n^{(1)}}\right)$$

$$\rightsquigarrow X_{n+1}^{(2)} = \log\left(\frac{S_{n+1}^{(2)}}{S_n^{(2)}}\right)$$

c) $$L_{n+1} = -V_{n+1} + V_n = -\frac{V_n}{2}\left(e^{X_{n+1}^{(1)}}+e^{X_{n+1}^{(2)}}\right) + V_n$$

$$= V_n\left(1-\frac{e^{X_{n+1}^{(1)}}}{2}-\frac{e^{X_{n+1}^{(2)}}}{2}\right)$$

$$\Rightarrow \ell_{[n]}(x) = V_n\left(1-\frac{e^{x_1}}{2}-\frac{e^{x_2}}{2}\right)$$

d) $$L_{n+1}^{\Delta} = -\sum_{i=1}^{2} D_i f(Z_n \mid \mathcal{F}_{n-1}) X_{n+1}^{(i)} = -\sum_{i=1}^{2}\frac{V_{n-1}}{2}\left(e^{Z_n^{(i)}-Z_{n-1}^{(i)}}\right)X_{n+1}^{(i)}$$

$$\ell_{[n]}(x) = -\sum_{i=1}^{2}\frac{V_{n-1}}{2}\left(e^{Z_n^{(i)}-Z_{n-1}^{(i)}}\right)x_i$$