Christian-Albrechts-Universität zu Kiel **WT 2022/23**
Mathematisches Seminar
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Risk Management

Exercise Sheet 4

C-Exercise 12 (2 points)

(a) Write a *Python* function

```
empirical_cdf(n),
```

which simulates n i.i.d. standard normal random variables and plots the corresponding cdf. Test the function for $n = 10$, $n = 100$ and $n = 1000$ each and plot the cdf of a standard normal distribution in the same graphics in order to compare it to its empirical cdfs.

(b) Compute the empirical value at risk and the empirical expected shortfall at level $\alpha = 90\%$ and $\alpha = 97.5\%$ of your generated data and compare them to their theoretical values.

Please comment your solution.
Useful *Python* commands: `numpy.sort`

T-Exercise 13 (4 points)
Suppose that losses $L_n, n = 1, \dots, 250$ are i.i.d. with continuous cdf F. Determine the minimal threshold t such that

$$\mathbb{P}(\max\{L_1, \dots, L_{250}\} > t) < 0.05.$$

Determine t more explicitly if the L_n are

(a) normally distributed with mean 0 and variance $\sigma^2 > 0$,

(b) exponentially distributed with parameter $\lambda > 0$.

T-Exercise 14M (for mathematicians only) (4 points)
Let $(L_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $E(L_1^2) < \infty$ and strictly increasing, continuous cdf F. Denote by F_n the empirical cdf of $L_1, ..., L_n$. Show that for all $\alpha \in (0, 1)$ we have

$$\mathrm{ES}_\alpha(F_n) \xrightarrow{P} \mathrm{ES}_\alpha(L_1) \quad \text{as} \quad n \to \infty\ .$$

C-Exercise 15 (6 points)
In this exercise we want to use a ARCH(2) model to estimate *value at risk* and *expected shortfall* of a portfolio loss. For this purpose we assume that our one dimensional risk factor changes $(X_n)_{n\in\mathbb{N}}$ follow the ARCH(2) model:

$$\begin{aligned} X_n &= \sigma_n Y_n, \\ \sigma_n^2 &= \alpha_0 + \alpha_1 X_{n-1}^2 + \alpha_2 X_{n-2}^2, \end{aligned}$$

with parameters $\vartheta = (\alpha_0, \alpha_1, \alpha_2) \in \Theta := (0,\infty)^3$ and iid standard normal random variables $(Y_n)_{n\in\mathbb{N}}$. We write

$$f_{\vartheta,X_1,X_2}^{(X_3,\dots,X_n)} : \mathbb{R}^n \to \mathbb{R}$$

for the common pdf of $(X_3,\dots,X_n)$ under the probability measure P_ϑ.

(a) Write a *Python* -function

```
y= log_likelihood_ARCH2(theta,x),
```

which returns the log likelihood function

$$L_n(\vartheta, x) = \log\left(f_{\vartheta,X_1,X_2}^{(X_3,\dots,X_n)}(x_3,\dots,x_n)\right)$$

evaluated at the parameter $\vartheta = (\alpha_0, \alpha_1, \alpha_2)$ and given historical risk factor changes $x = (x_1,\dots,x_n) \in \mathbb{R}^n$.

(b) Write a *Python* -function

```
theta_hat=estimates_ARCH2(x),
```

which computes the Maximum Likelihood estimates $\widehat{\vartheta}$ for given historical risk factor changes $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ in the ARCH(2) model.
Hint: Use `scipy.optimize.minimize` on $(0,5]^4$*. As initial guess use* $(\alpha_0,\alpha_1,\alpha_2) = [0.01, 0.2, 0.8]$ *and the empirical standard deviation for* σ_1*.*

(c) Write a *Python* -function

```
[VaR,ES]=VaR_ES_ARCH2_MC(k,m,l,alpha,x),
```

that computes the *value at risk* and *expected shortfall* estimates for the k-period loss operator $l : \mathbb{R} \to \mathbb{R}$, level $\alpha \in (0,1)$ and given historical risk factor changes $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ using the Monte-Carlo method with $m \in \mathbb{N}$ simulations.

(d) Compute the logarithmic returns $(x_2,\dots,x_{8368})$ of the DAX time series, that we use as risk factor changes. Compute for the last trading day in the time series the 5 day ahead estimates of *value at risk* and *expected shortfall* at level $\alpha = 0.98$ using $k = 1000$ simulations.

Please comment your solution.

Please comment your solution. Please include your name(s) as comment in the beginning of the file.

Submit until:	Wednesday, 23.11.2022, 12:00
Discussion in tutorial:	Monday, 28.11.2022 and Tuesday, 29.11.2022