T9) $f_L(x) = \frac{\lambda^a}{\Gamma(a)}\left(x+\frac{a}{\lambda}\right)^{a-1}\exp(-\lambda x - a)\,\mathbb{1}_{\{x > -\frac{a}{\lambda}\}}$, $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\,dt$

$$\overset{x \to x+\frac{a}{\lambda}}{=} \frac{\lambda^a}{\Gamma(a)} x^{a-1} \exp\left(-\lambda\left(x-\frac{a}{\lambda}\right)-a\right)\mathbb{1}_{\{x>0\}}$$

$$= \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x + \cancel{a} - \cancel{a}}\,\mathbb{1}_{\{x>0\}}$$

$$f_L(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}\,\mathbb{1}_{\{x>0\}}$$

$$E[x] = \int_0^\infty x \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}\,dx = \int_0^\infty \frac{\lambda^a}{\Gamma(a)} x^a e^{-\lambda x}\,dx = a\lambda^{-1} \underbrace{\int_0^\infty \frac{\lambda^{a+1}}{\Gamma(a+1)} x^{(a+1)-1} e^{-\lambda x}\,dx}_{1}$$

$$E[x] = \boxed{\frac{a}{\lambda} = E[L]}$$

$\Gamma(a+1) = a\cdot\Gamma(a)$, $\Gamma(a+2) = \Gamma((a+1)+1) = (a+1)\Gamma(a+1)$
$= a(a+1)\Gamma(a)$

$$E[x^2] = \int_0^\infty \frac{\lambda^a}{\Gamma(a)} x^2 \cdot x^{a-1} e^{-\lambda x}\,dx = \int_0^\infty \frac{\lambda^a}{\Gamma(a)} x^{a+1} e^{-\lambda x}\,dx = \lambda^{-2}\cdot a(a+1) \underbrace{\int_0^\infty \frac{\lambda^{a+2}}{\Gamma(a+2)} x^{(a+2)-1} e^{-\lambda x}\,dx}_{1}$$

$E[x^2] = \frac{a(a+1)}{\lambda^2}$, $Var(L) = E[x^2] - E(x)^2 = \frac{a(a+1)}{\lambda^2} - \frac{a^2}{\lambda^2} = \cancel{\frac{a^2}{\lambda^2}} + \frac{a}{\lambda^2} - \cancel{\frac{a^2}{\lambda^2}}$
$= \frac{a}{\lambda^2}$

$$\boxed{Var(L) = \frac{a}{\lambda^2}}$$

$\frac{1}{y} = y^{-1} = x = g^{-1}(y) = f_L^{-1}(x)$, $f_L^{-1}(y) = f_x(g^{-1}(y))\left|\frac{\partial}{\partial y} g^{-1}(y)\right|$

$g^{-1}(y) = \frac{1}{y}$, $f_L^{-1}(y) = \frac{\lambda^a}{\Gamma(a)}\left(\frac{1}{y}\right)^{a-1} e^{-\frac{(1/y)}{\lambda^{-1}}}\left|-\frac{1}{y^2}\right|$

$$= \frac{\lambda^a}{\Gamma(a)}\left(\frac{1}{y}\right)^{a-1} e^{-\lambda\left(\frac{1}{y}\right)}\cdot\frac{1}{y^2} = \frac{\lambda^a}{\Gamma(a)}\left(\frac{1}{y}\right)^{a+1} e^{-\lambda\left(\frac{1}{y}\right)} = \frac{\lambda^a}{\Gamma(a)} y^{-a-1} e^{-\lambda\left(\frac{1}{y}\right)}$$

Gammainv $= F^{-1}(x) = \int_0^x \frac{\lambda^a}{\Gamma(a)} y^{-a-1} e^{-\lambda\left(\frac{1}{y}\right)}\,dy$, $t = \frac{\lambda}{y}$, $dt = -\frac{\lambda}{y^2}$

$= \int_\infty^{\frac{\lambda}{x}} \frac{\lambda^a}{\Gamma(a)}\left(\frac{\lambda}{t}\right)^{-a-1} e^{-t}\left(-\frac{\lambda}{t^2}\,dt\right)$, $y = \frac{\lambda}{t}$, $dy = -\frac{\lambda}{t^2}\,dt$

$$= -\int_\infty^{\frac{\lambda}{x}} \frac{\cancel{\lambda^a}}{\Gamma(a)} \cdot \frac{\cancel{\lambda^{-a}}\cdot\cancel{\lambda^{-1}}}{t^{-a-1}} e^{-t}\cdot\frac{\cancel{\lambda}}{t^2}\,dt$$

$$= \int_{\frac{\lambda}{x}}^\infty \frac{1}{\Gamma(a)} t^{a-1} e^{-t}\,dt = \boxed{\frac{\Gamma\left(a, \frac{\lambda}{\alpha}\right)}{\Gamma(a)} = VaR_\alpha(L)} = \frac{\Gamma\left(a,\frac{\lambda}{\alpha}\right)}{\Gamma(a)}$$

T9) $ES_\alpha(L) = \int_{\Gamma(a,\frac{\lambda}{\alpha})}^{\infty} x \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} \, dx$

$$= \int_{\Gamma(a,\frac{\lambda}{\alpha})}^{\infty} \frac{\lambda^a}{\Gamma(a)} x^{a} e^{-\lambda x} \, dx = \frac{a}{\lambda} \int_{\frac{\Gamma(a,\frac{\lambda}{\alpha})}{\Gamma(a)}}^{\infty} \frac{\lambda^{a+1}}{\Gamma(a+1)} x^{(a+1)-1} e^{-\lambda x} \, dx$$

$$= \frac{a}{\lambda} \frac{\Gamma\left(a+1, \lambda \frac{\Gamma(a,\frac{\lambda}{\alpha})}{\Gamma(a)}\right)}{\Gamma(a+1)}$$

$$\boxed{ES_\alpha(L) = \frac{\Gamma\left(a+1, \lambda \frac{\Gamma(a,\frac{\lambda}{\alpha})}{\Gamma(a)}\right)}{\lambda \Gamma(a)}}$$