Christian-Albrechts-Universität zu Kiel **WT 2022/23**
Mathematisches Seminar
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Risk Management

Exercise Sheet 2

C-Exercise 4

Denote by S_n the price of a stock at day t_n, $n \in \mathbb{N}$, and by $X_n := \log(\frac{S_n}{S_{n-1}})$, $n \geq 2$, the log return of the stock. As a model for the conditional distribution of X_{n+1} at time t_{n+1} given the stock prices up to time t_n, we assume a normal distribution $N(\hat{\mu}_{n+1}, \hat{\sigma}^2_{n+1})$ and take for $\hat{\mu}_{n+1}$ and $\hat{\sigma}^2_{n+1}$ the empirical mean and variance as estimators, i.e.

$$\hat{\mu}_{n+1} := \frac{1}{251} \sum_{k=n-250}^{n} X_k \quad \text{and} \quad \hat{\sigma}^2_{n+1} := \frac{1}{250} \sum_{k=n-250}^{n} (X_k - \hat{\mu}_{n+1})^2 \text{ for } n \geq 252\,.$$

(We ignore the days of the first trading year.)

(a) Write a *Python* function

```
VaR_log_normal(s,alpha),
```

that computes for given stock prices $s = (s_{n-251}, ..., s_n)$ of the past trading year the Value at Risk (VaR) at level α of the next trading day's loss L_{n+1} for this stock.

(b) Assume that the DAX time series data from the first exercise sheet follow this model. Compute for each day after the first 252 days the $\mathrm{VaR}_{90\%}$ and the $\mathrm{VaR}_{95\%}$ of the DAX time series and visualize the violations, i.e. the days when the actual loss lies above the computed VaR. How much violations do you expect theoretically, how much do you observe?

Hint: The VaR of the loss L_{n+1} at time t_{n+1} is given by

$$\mathrm{VaR}_\alpha(L_{n+1}) = S_n(1 - \exp(\mu_{n+1} + \sigma_{n+1} q_{1-\alpha})),$$

where $q_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the standard normal distribution.

Please label the diagrams and comment your solution.
Useful Python commands: `scipy.stats.norm`

T-Exercise 5

Prove Example 1.14: Let $L = s(1 - e^X)$ be the random loss of a portfolio, where $s > 0$ is a constant and $X \sim N(\mu, \sigma^2)$, i.e. X follows a Gaussian law with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Show that

$$\mathrm{ES}_\alpha(L) = s\left(1 - \frac{1}{1-\alpha} e^{\mu + \sigma^2/2} \Phi(-\Phi^{-1}(\alpha) - \sigma)\right)$$

holds for $\alpha \in (0, 1)$, where Φ denotes the cdf of a standard Gaussian distributed random variable.

C-Exercise 6

Assume that we have a sample $\nu = (\nu_1, ..., \nu_m) \in \{0,1\}^m$ of i.i.d. random variables with $P(\nu_1 = 1) = p \in [0,1]$. Hence, their sum $\sum_{k=1}^m v_k$ follows a $\mathrm{Bin}(m,p)$-distribution, i.e. a binomial distribution with m experiments and success probability p. Design a two sided statistical test at significance level $\beta \in (0,1)$ for the null hypothesis $H_0 \colon p = p_0$ and implement this test in a *Python* function called

`test_binomial(v,p0,beta)`

This function is supposed to return the value 1, if the null hypothesis is rejected, and 0 otherwise.

We want to apply this test on the results from C-Exercise 4(b): Apply your function on the violation vectors from the DAX time series using a significance level of $\beta = 0.05$.

Hint: You may construct an exact test based on the cumulative distribution function of $\mathrm{Bin}(m,p)$ *or use a normal approximation to derive a test with asymptotic level* β*.*

Please comment your solution.

Useful Python commands: `scipy.stats.binom, math.floor, math.ceil`

Please comment your solution. Please include your name(s) as comment in the beginning of the file.

Submit until: Wednesday, 09.11.2022, 12:00

Discussion in tutorial: Monday, 14.11.2022 and Tuesday, 15.11.2022