Sequences of Real Numbers — Problem Set 1

Problem 1. Let (a_n) be the sequence defined by

$$a_n = \frac{3n-1}{n+2}, \qquad n \in \mathbb{N}.$$

(a) Compute a_1, a_2, and a_3.

(b) Write the first four terms of the sequence explicitly as an ordered list.

(c) Show that a_n may be rewritten in the form

$$a_n = 3 - \frac{7}{n+2}.$$

(d) Using this expression, explain why the change from a_n to a_{n+1} becomes smaller as n increases.

Problem 2. Consider the recursively defined sequence (b_n) given by

$$b_0 = 1, \qquad b_{n+1} = 2b_n + 1 \quad \text{for } n \geq 0.$$

(a) Compute b_1, b_2, and b_3.

(b) Guess a closed-form expression for b_n.

(c) Prove that your formula is correct using mathematical induction.

(d) Explain one advantage and one disadvantage of defining a sequence recursively rather than explicitly.

Problem 3. Define a sequence (c_n) by

$$c_n = (-1)^n \frac{n+1}{n+2}.$$

(a) Write out the first six terms of the sequence.

(b) Separate the sequence into its even-indexed terms and odd-indexed terms.

(c) Describe the pattern of signs in the sequence.

(d) Describe how the magnitudes of the terms change as n increases.

Problem 4. A quantity is measured once per unit time, producing values

$$x_1, x_2, x_3, \ldots$$

(a) Explain why this data is naturally modeled as a sequence rather than a function defined on $\mathbb{R}$.

(b) Give an explicit formula for a sequence that could model steady linear growth.

(c) Give an explicit formula for a sequence that could model oscillatory behavior.

(d) Give a recursive rule that could describe how x_{n+1} depends on x_n.

Solutions

Problem 1. (a) We compute directly from the definition

$$a_n = \frac{3n-1}{n+2}.$$

Thus

$$a_1 = \frac{3(1)-1}{1+2} = \frac{2}{3}, \qquad a_2 = \frac{6-1}{4} = \frac{5}{4}, \qquad a_3 = \frac{9-1}{5} = \frac{8}{5}.$$

(b) The first four terms are

$$\left\{ \frac{2}{3}, \frac{5}{4}, \frac{8}{5}, \frac{11}{6} \right\}.$$

(c) We rewrite a_n by separating the numerator:

$$\frac{3n-1}{n+2} = \frac{3(n+2)-7}{n+2} = 3 - \frac{7}{n+2}.$$

(d) From the expression

$$a_n = 3 - \frac{7}{n+2},$$

the only part that changes with n is the fraction $\frac{7}{n+2}$. As n increases, the denominator grows, so the fraction becomes smaller in magnitude. Consequently, the difference between successive terms a_{n+1} and a_n decreases as n increases.

Insight. Writing a formula as a constant plus a small correction term often makes long-term behavior easier to understand.

Problem 2. (a) Using the recursion,

$$b_1 = 2b_0 + 1 = 3, \qquad b_2 = 2b_1 + 1 = 7, \qquad b_3 = 2b_2 + 1 = 15.$$

(b) From the values

$$1,\ 3,\ 7,\ 15, \ldots$$

we observe that each term is one less than a power of 2. This suggests the formula

$$b_n = 2^{n+1} - 1.$$

(c) *Proof.* We prove the formula by induction.
Base case: For $n = 0$,

$$2^{0+1} - 1 = 1 = b_0,$$

so the formula holds.
Inductive step: Assume $b_n = 2^{n+1} - 1$ for some $n \geq 0$. Then

$$b_{n+1} = 2b_n + 1 = 2(2^{n+1} - 1) + 1 = 2^{n+2} - 1.$$

Thus the formula holds for $n + 1$, completing the proof. □

(d) An advantage of recursive definitions is that they describe how the sequence evolves step by step, often reflecting real processes. A disadvantage is that computing b_n typically requires knowing all previous terms, whereas an explicit formula allows direct computation.

Insight. Recursive rules describe dynamics; explicit formulas describe structure.

Problem 3. (a) The first six terms are

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}.$$

(b) For even indices $n = 2k$,

$$c_{2k} = \frac{2k+1}{2k+2}.$$

For odd indices $n = 2k+1$,

$$c_{2k+1} = -\frac{2k+2}{2k+3}.$$

(c) The factor $(-1)^n$ causes the sign to alternate: positive for even n, negative for odd n.

(d) The magnitudes satisfy

$$|c_n| = \frac{n+1}{n+2},$$

which increases toward 1 as n increases, though it never reaches 1.

Insight. Alternating signs and changing magnitudes often indicate competing effects within a formula.

Problem 4. (a) The data is indexed by discrete time steps, not by all real numbers. Since values are only recorded at integer times, the natural model is a sequence rather than a function on $\mathbb{R}$.

(b) A sequence modeling steady linear growth is

$$x_n = an + b,$$

where a represents the constant increase per time step.

(c) A sequence modeling oscillatory behavior is

$$x_n = A(-1)^n,$$

or more generally,

$$x_n = A\sin(\omega n),$$

for constants A and ω.

(d) A recursive rule could be

$$x_{n+1} = x_n + d,$$

for constant growth, or

$$x_{n+1} = -x_n,$$

for alternating behavior.

Insight. Explicit formulas describe global behavior, while recursive rules describe how each step depends on the previous one.