Calculus 1 — Final Examination

Practice Problems with Full Solutions

Instructions: Solve all problems. Full solutions with justifications are required. Proofs should be complete.

1 Sequences and Limits

Problem 1

Define (a_n) recursively by $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3 + 2a_n}$ for $n \geq 1$.

(a) Show that (a_n) is monotone increasing and bounded above by 3.

(b) Prove (a_n) converges and compute $\lim_{n\to\infty} a_n$.

(c) Establish the contraction $|a_{n+1} - 3| \leq \frac{2}{5}|a_n - 3|$ and deduce exponential convergence.

Problem 2

Consider the limit

$$L = \lim_{x\to 0} \frac{1 - \cos(x)}{x^2}.$$

(a) Compute L without l'Hôpital's rule.

(b) Use the identity $1 - \cos(x) = 2\sin^2(x/2)$ and the squeeze theorem.

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x^3-8}{x-2} & x \neq 2, \\ k & x = 2. \end{cases}$$

(a) Determine the value of k for which f is continuous at $x = 2$.

(b) Verify continuity using the ε-δ definition.

(c) Is f differentiable at $x = 2$ for this choice of k? If so, compute $f'(2)$.

Problem 4

Compute the following limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{9x^2 - 5x} + 3x}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

(c) $\displaystyle\lim_{x\to 0} x \left\lfloor \frac{1}{x} \right\rfloor$

2 Differentiation

Problem 5

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^3 \sin(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) Prove f is differentiable at $x = 0$ and compute $f'(0)$.

(b) Find $f'(x)$ for $x \neq 0$.

(c) Determine whether f' is continuous at $x = 0$.

Problem 6

Let $C \subset \mathbb{R}^2$ be the curve defined implicitly by $x^2y + y^3 = 4x$.

(a) Compute $\frac{dy}{dx}$ via implicit differentiation.

(b) Find the tangent line to C at $(1, 1)$.

(c) Express the differential dy in terms of dx at $(1, 1)$ and interpret geometrically.

Problem 7

Let $h(x) = e^{x^2} \cos(3x)$.

(a) Compute $h'(x)$ and $h''(x)$.

(b) Locate all critical points of h in $[0, \pi]$.

(c) Use the differential to approximate $h(0.1)$ given $h(0) = 1$.

Problem 8

Using the limit definition of the derivative:

(a) Prove that if $f(x) = x^3$, then $f'(x) = 3x^2$.

(b) Prove that if $g(x) = \frac{1}{x}$, then $g'(x) = -\frac{1}{x^2}$.

(c) Use these results to verify the quotient rule for $h(x) = \frac{x^3}{x} = x^2$.

3 Applications of Derivatives

Problem 9

(Fermat's Principle) Light travels from $A = (0, 2)$ in medium 1 (speed $v_1 = 3 \cdot 10^8$ m/s) to $B = (4, -1)$ in medium 2 (speed $v_2 = 2 \cdot 10^8$ m/s), refracting at $(x, 0)$ on the boundary.

(a) Express total travel time $T(x)$ as a function of x.

(b) Find the critical point of T and show it satisfies Snell's law.

Problem 10

A particle moves along $y = x^3 - 3x + 1$ with $\frac{dx}{dt} = 2$ cm/s.

(a) Find $\frac{dy}{dt}$ when $x = 1$.

(b) At which points is $\frac{dy}{dt} = 0$?

(c) Compute the rate of change of distance from the origin when $x = 2$.

Problem 11

A rectangular box with square base and open top has volume 256 cm^3.

(a) Express surface area S as a function of base side length x.

(b) Find dimensions minimizing S.

(c) Use the differential to estimate ΔS when x increases by 0.5 cm from the optimal value.

Problem 12

A 5-meter ladder leans against a wall; its base slides away at 0.8 m/s.

(a) Determine how fast the top descends when the base is 3 m from the wall.

(b) Find the rate of change of the angle θ between ladder and ground at this instant.

(c) When does the top descend fastest?

4 Mean Value Theorem and Foundations

Problem 13

(a) State the Mean Value Theorem with all hypotheses.

(b) For $f(x) = x^3 - 3x$ on $[-2, 2]$, find all c satisfying the MVT.

(c) Prove: if $|f'(x)| \leq M$ on $[a, b]$, then $|f(b) - f(a)| \leq M(b - a)$.

Problem 14

Let $p(x) = x^3 - 3x + 1$.

(a) Prove p has exactly three real roots.

(b) Use IVT to locate intervals containing each root.

(c) Apply Newton's method with $x_0 = 2$ to approximate the largest root (three iterations).

Problem 15

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at a.

(a) Prove f is continuous at a.

(b) Show $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a)$.

(c) Give an example showing continuity does not imply differentiability.

Problem 16

(a) Prove $\lim_{x\to 3}(2x + 1) = 7$ using ε-δ.

(b) Prove $\lim_{x\to 2} x^2 = 4$ using ε-δ.

(c) Show that differentiability at a means: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{f(a+h) - f(a)}{h} - f'(a)\right| < \varepsilon \quad \text{whenever } 0 < |h| < \delta.$$

5 Differentials as Linear Maps

Problem 17

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + xy$.

(a) Compute the differential $df_{(2,3)} : \mathbb{R}^2 \to \mathbb{R}$ as a linear map.

(b) Use this to approximate $f(2.1, 2.9)$.

(c) Compare with the exact value.

Problem 18

For $g(x) = x^3$, interpret $g'(a)$ as defining a linear map $L : \mathbb{R} \to \mathbb{R}$ by $L(h) = 3a^2h$.

(a) Prove $g(a + h) = g(a) + L(h) + o(h)$ where $\lim_{h\to 0} \frac{o(h)}{h} = 0$.

(b) Identify the error term $o(h)$ explicitly.

(c) Estimate 2.1^3 using this linear approximation.

Problem 19

Let $f(x) = x^2$ and $g(x) = \sin(x)$.

(a) Compute $d(g \circ f)_{\pi/2}$ directly.

(b) Compute $df_{\pi/2}$ and $dg_{f(\pi/2)}$ separately and verify $d(g \circ f) = dg \circ df$.

(c) Interpret this as composition of linear approximations.

Problem 20

For $f(x) = \sqrt{x}$ near $x = 4$, the differential is $df_4(h) = \frac{h}{4}$.

(a) Derive this from the definition.

(b) Use this to approximate $\sqrt{4.2}$, $\sqrt{3.8}$, and $\sqrt{5}$.

(c) Which approximation is most accurate? Explain using f''.

Solutions

Solution 1

Proof. **(a)** We use induction. *Base:* $a_1 = \sqrt{3} < 3$. *Step:* If $a_n \leq 3$, then

$$a_{n+1} = \sqrt{3 + 2a_n} \leq \sqrt{3 + 6} = 3.$$

For monotonicity, compute $a_2 = \sqrt{3 + 2\sqrt{3}} > \sqrt{3} = a_1$. Suppose $a_{n+1} > a_n$. Then

$$a_{n+2} = \sqrt{3 + 2a_{n+1}} > \sqrt{3 + 2a_n} = a_{n+1}.$$

Thus (a_n) is increasing and bounded above by 3.

(b) Since (a_n) is monotone and bounded, it converges to some $L \in \mathbb{R}$. Taking limits in the recurrence:

$$L = \sqrt{3 + 2L} \implies L^2 = 3 + 2L \implies L^2 - 2L - 3 = 0.$$

Factoring: $(L - 3)(L + 1) = 0$. Since $L > 0$, we have $L = 3$.

(c) Set $e_n := a_n - 3$. Then $e_n < 0$ for all n (since $a_n < 3$). We have

$$e_{n+1} = \sqrt{3 + 2a_n} - 3 = \sqrt{9 + 2e_n} - 3.$$

Rationalizing:

$$e_{n+1} = \frac{(9 + 2e_n) - 9}{\sqrt{9 + 2e_n} + 3} = \frac{2e_n}{\sqrt{9 + 2e_n} + 3}.$$

Since $e_n < 0$, we have $9 + 2e_n > 9 - 6 = 3$, so $\sqrt{9 + 2e_n} > \sqrt{3}$. Thus

$$|e_{n+1}| = \frac{2\,|e_n|}{\sqrt{9 + 2e_n} + 3} < \frac{2\,|e_n|}{\sqrt{3} + 3} < \frac{2\,|e_n|}{5}.$$

By iteration, $|e_n| \leq (2/5)^{n-1}\,|e_1| \to 0$ exponentially. □

Solution 2

Proof. **(a)** Use the identity $1 - \cos(x) = 2\sin^2(x/2)$. Set $u = x/2$, so $x = 2u$ and $x \to 0 \iff u \to 0$. Then

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{u \to 0} \frac{2\sin^2(u)}{4u^2} = \frac{1}{2} \lim_{u \to 0} \left(\frac{\sin(u)}{u}\right)^2 = \frac{1}{2}.$$

(b) For $x > 0$ small, the inequalities $\sin(\theta) \leq \theta$ and $\sin(\theta) \geq \theta - \theta^3/6$ give

$$\frac{2\sin^2(x/2)}{x^2} \leq \frac{2(x/2)^2}{x^2} = \frac{1}{2}$$

and

$$\frac{2\sin^2(x/2)}{x^2} \geq \frac{2((x/2) - (x/2)^3/6)^2}{x^2} \to \frac{1}{2}.$$

By the squeeze theorem, the limit is $1/2$. □

Solution 3

Proof. **(a)** For $x \neq 2$, factor:

$$f(x) = \frac{x^3 - 8}{x - 2} = \frac{(x-2)(x^2+2x+4)}{x-2} = x^2 + 2x + 4.$$

Thus $\lim_{x\to 2} f(x) = 4 + 4 + 4 = 12$. Continuity requires $k = 12$.

(b) Let $\varepsilon > 0$. For $x \neq 2$,

$$|f(x) - 12| = \left|x^2 + 2x - 8\right| = |(x-2)(x+4)|.$$

If $|x - 2| < 1$, then $1 < x < 3$, so $|x + 4| < 7$. Choose $\delta = \min(1, \varepsilon/7)$. Then for $0 < |x - 2| < \delta$:

$$|f(x) - 12| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon.$$

(c) With $k = 12$, we have $f(x) = x^2 + 2x + 4$ everywhere (after continuous extension). This is a polynomial, hence differentiable, with $f'(x) = 2x + 2$. At $x = 2$: $f'(2) = 6$. □

Solution 4

Proof. **(a)** For $x > 0$ large, factor out x from the radical:

$$\sqrt{9x^2 - 5x} = x\sqrt{9 - 5/x}.$$

Thus

$$\lim_{x\to\infty} \frac{x\sqrt{9 - 5/x} + 3x}{x} = \lim_{x\to\infty} \left(\sqrt{9 - 5/x} + 3\right) = 3 + 3 = 6.$$

(b) Rationalize the numerator:

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})} = \frac{2}{\sqrt{1+x} + \sqrt{1-x}}.$$

As $x \to 0$: $\dfrac{2}{1+1} = 1$.

(c) For $x > 0$, the floor function satisfies

$$\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}.$$

Multiplying by x:

$$1 - x < x \left\lfloor \frac{1}{x} \right\rfloor \leq 1.$$

By squeeze theorem, $\lim_{x\to 0^+} x\lfloor 1/x \rfloor = 1$. Similarly for $x < 0$, the limit is 1. □

Solution 5

Proof. **(a)** By definition,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^3 \sin(1/h)}{h} = \lim_{h\to 0} h^2 \sin(1/h).$$

Since $|\sin(1/h)| \le 1$, we have $|h^2 \sin(1/h)| \le h^2 \to 0$. Thus $f'(0) = 0$.

(b) For $x \neq 0$, use the product rule:

$$f'(x) = 3x^2 \sin(1/x) + x^3 \cos(1/x) \cdot \left(-\frac{1}{x^2}\right) = 3x^2 \sin(1/x) - x\cos(1/x).$$

(c) As $x \to 0$, both terms vanish:

$$\left|3x^2 \sin(1/x)\right| \le 3x^2 \to 0 \quad \text{and} \quad |x\cos(1/x)| \le |x| \to 0.$$

By squeeze theorem, $\lim_{x\to 0} f'(x) = 0 = f'(0)$, so f' is continuous at 0. □

Solution 6

Proof. **(a)** Differentiate implicitly with respect to x:

$$2xy + x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 4.$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(x^2 + 3y^2) = 4 - 2xy \implies \frac{dy}{dx} = \frac{4 - 2xy}{x^2 + 3y^2}.$$

(b) At $(1,1)$: $\frac{dy}{dx} = \frac{4-2}{1+3} = \frac{1}{2}$. The tangent line is

$$y - 1 = \frac{1}{2}(x - 1) \implies y = \frac{x+1}{2}.$$

(c) At $(1,1)$, the differential is $dy = \frac{1}{2}dx$. Geometrically: if x increases by dx, then y increases by approximately $\frac{1}{2}dx$. This is the linearization $\Delta y \approx \frac{1}{2}\Delta x$—the best linear approximation to the curve near $(1,1)$. □

Solution 7

Proof. **(a)** By the product rule:

$$h'(x) = 2xe^{x^2} \cos(3x) + e^{x^2}(-3\sin(3x)) = e^{x^2}(2x\cos(3x) - 3\sin(3x)).$$

For $h''(x)$, apply the product rule again:

$$\begin{aligned} h''(x) &= 2xe^{x^2}(2x\cos(3x) - 3\sin(3x)) + e^{x^2}(2\cos(3x) - 6x\sin(3x) - 9\cos(3x)) \\ &= e^{x^2}((4x^2 - 7)\cos(3x) - 12x\sin(3x)). \end{aligned}$$

(b) Critical points satisfy $h'(x) = 0$, i.e., $2x\cos(3x) = 3\sin(3x)$, or $\tan(3x) = \frac{2x}{3}$. This transcendental equation requires numerical solution in $[0, \pi]$.

(c) At $x = 0$: $h(0) = 1$ and $h'(0) = 0$. The differential gives $dh = 0 \cdot dx = 0$, so

$$h(0.1) \approx h(0) + dh = 1 + 0 = 1.$$

$\square$

Solution 8

Proof. **(a)** By definition:

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.$$

(b) For $x \neq 0$:

$$g'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x - (x+h)}{hx(x+h)} = \lim_{h\to 0} \frac{-h}{hx(x+h)} = -\frac{1}{x^2}.$$

(c) For $h(x) = x^2$, compute directly: $h'(x) = 2x$. Using the quotient rule on $f(x)/g(x) = x^3/x$:

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{3x^2 \cdot x - x^3 \cdot 1}{x^2} = \frac{2x^3}{x^2} = 2x.$$

$\square$

Solution 9

Proof. **(a)** The path lengths are $\sqrt{x^2+4}$ in medium 1 and $\sqrt{(4-x)^2+1}$ in medium 2. Thus

$$T(x) = \frac{\sqrt{x^2+4}}{v_1} + \frac{\sqrt{(4-x)^2+1}}{v_2} = \frac{\sqrt{x^2+4}}{3 \cdot 10^8} + \frac{\sqrt{(4-x)^2+1}}{2 \cdot 10^8}.$$

(b) Compute $T'(x) = 0$ (ignoring constant factors):

$$\frac{x}{v_1\sqrt{x^2+4}} - \frac{4-x}{v_2\sqrt{(4-x)^2+1}} = 0.$$

Define $\sin\theta_1 = \frac{x}{\sqrt{x^2+4}}$ and $\sin\theta_2 = \frac{4-x}{\sqrt{(4-x)^2+1}}$ (angles of incidence/refraction). Then

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \implies v_1\sin\theta_1 = v_2\sin\theta_2.$$

This is Snell's law. $\square$

Solution 10

Proof. **(a)** The curve gives $y = x^3 - 3x + 1$, so $\frac{dy}{dx} = 3x^2 - 3$. When $x = 1$:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3 - 3) \cdot 2 = 0 \text{ cm/s}.$$

(b) We need $\frac{dy}{dt} = (3x^2 - 3) \cdot 2 = 0$, so $x^2 = 1$, giving $x = \pm 1$. The points are $(1, -1)$ and $(-1, 3)$.

(c) Distance from origin: $D = \sqrt{x^2 + y^2}$. When $x = 2$: $y = 8 - 6 + 1 = 3$, so $D = \sqrt{13}$. Also $\frac{dy}{dx}\Big|_{x=2} = 9$.

$$\frac{dD}{dt} = \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

At $x = 2$: $\frac{dy}{dt} = 9 \cdot 2 = 18$. Thus

$$\frac{dD}{dt} = \frac{2 \cdot 2 + 3 \cdot 18}{\sqrt{13}} = \frac{58}{\sqrt{13}} \text{ cm/s}.$$

□

Solution 11

Proof. **(a)** Let the base be $x \cdot x$ and height h. Volume constraint: $x^2 h = 256 \implies h = \frac{256}{x^2}$. Surface area (no top):

$$S = x^2 + 4xh = x^2 + \frac{1024}{x}.$$

(b) Compute $S'(x) = 2x - \frac{1024}{x^2} = 0 \implies x^3 = 512 \implies x = 8$. Then $h = \frac{256}{64} = 4$.

Check: $S''(x) = 2 + \frac{2048}{x^3} > 0$, confirming a minimum.

(c) At the optimal $x = 8$, we have $S'(8) = 0$. The differential $dS = S'(8)\Delta x = 0$ to first order. For a better estimate, use

$$\Delta S \approx \frac{1}{2}S''(8)(\Delta x)^2 = \frac{1}{2}\left(2 + \frac{2048}{512}\right)(0.5)^2 = \frac{1}{2}(6)(0.25) = 0.75 \text{ cm}^2.$$

□

Solution 12

Proof. **(a)** Let x be the distance from wall, y the height. Constraint: $x^2 + y^2 = 25$. Differentiate:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 3$: $y = 4$ and $\frac{dx}{dt} = 0.8$, so

$$\frac{dy}{dt} = -\frac{3}{4}(0.8) = -0.6 \text{ m/s}.$$

(b) We have $\cos\theta = \frac{x}{5}$, so $-\sin\theta \cdot \frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$. When $x = 3$: $\sin\theta = \frac{4}{5}$, giving

$$\frac{d\theta}{dt} = -\frac{1}{5\sin\theta}\frac{dx}{dt} = -\frac{0.8}{5 \cdot 4/5} = -0.2 \text{ rad/s}.$$

(c) From $\left|\frac{dy}{dt}\right| = \frac{x}{y} \cdot 0.8$, the rate is maximized when $\frac{x}{y}$ is largest. As $x \to 5$ (and $y \to 0$), this ratio grows without bound—the top descends fastest just before impact. □

Solution 13

Proof. **(a)** *Mean Value Theorem:* If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) For $f(x) = x^3 - 3x$ on $[-2, 2]$: $f(-2) = -2$ and $f(2) = 2$. The MVT gives

$$f'(c) = \frac{2 - (-2)}{4} = 1.$$

Since $f'(x) = 3x^2 - 3$, we solve $3x^2 - 3 = 1 \implies x^2 = \frac{4}{3} \implies x = \pm\frac{2\sqrt{3}}{3}$.

Both values lie in $(-2, 2)$.

(c) By the MVT, there exists $c \in (a, b)$ with $f(b) - f(a) = f'(c)(b - a)$. Taking absolute values:

$$|f(b) - f(a)| = |f'(c)| \cdot (b - a) \le M(b - a).$$

□

Solution 14

Proof. **(a)** Let $p(x) = x^3 - 3x + 1$. Then $p'(x) = 3x^2 - 3 = 3(x^2 - 1)$, with critical points at $x = \pm 1$.

Evaluate: $p(-1) = -1 + 3 + 1 = 3 > 0$, $p(1) = 1 - 3 + 1 = -1 < 0$. Also $p(-2) = -1 < 0$ and $p(2) = 3 > 0$.

By IVT, p has roots in each of the intervals $(-2, -1)$, $(-1, 1)$, and $(1, 2)$. Since p has only two critical points and is cubic (hence monotone between criticals after accounting for behavior at $\pm\infty$), there are exactly three roots.

(b) The intervals are $(-2,-1)$, $(-1,1)$, and $(1,2)$ from above.

(c) Newton's method: $x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = x_n - \frac{x_n^3-3x_n+1}{3x_n^2-3}$.

$x_0 = 2$: $p(2) = 3$, $p'(2) = 9$, so $x_1 = 2 - \frac{3}{9} = \frac{5}{3} \approx 1.667$.

$x_1 = 5/3$: $p(5/3) \approx 0.63$, $p'(5/3) \approx 5.33$, so $x_2 \approx 1.549$.

$x_2 = 1.549$: $p(1.549) \approx 0.07$, $p'(1.549) \approx 4.20$, so $x_3 \approx 1.532$. □

Solution 15

Proof. **(a)** Since f is differentiable at a, we can write

$$f(a+h) = f(a) + f'(a)h + o(h) \quad \text{where } \lim_{h\to 0} \frac{o(h)}{h} = 0.$$

Thus

$$\lim_{h\to 0}(f(a+h) - f(a)) = \lim_{h\to 0}(f'(a)h + o(h)) = 0,$$

proving continuity.

(b) Write

$$\frac{f(a+h)-f(a-h)}{2h} = \frac{1}{2}\left(\frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h}\right).$$

As $h \to 0$, the first term approaches $f'(a)$ and the second approaches $\frac{f(a)-f(a-h)}{h} = \frac{f(a-(-h))-f(a)}{-h} \cdot \frac{-h}{h} \to f'(a)$. Hence the limit is $f'(a)$.

(c) Let $f(x) = |x|$. Then f is continuous at $x = 0$, but

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = 1 \quad \text{and} \quad \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = -1.$$

Since these one-sided derivatives differ, f is not differentiable at 0. □

Solution 16

Proof. **(a)** Let $\varepsilon > 0$. We need $|(2x+1)-7| = 2|x-3| < \varepsilon$. Choose $\delta = \frac{\varepsilon}{2}$. Then for $0 < |x-3| < \delta$:

$$|(2x+1)-7| = 2|x-3| < 2\delta = \varepsilon.$$

(b) Let $\varepsilon > 0$. We have $|x^2-4| = |x-2||x+2|$. If $|x-2| < 1$, then $1 < x < 3$, so $|x+2| < 5$. Choose $\delta = \min(1, \frac{\varepsilon}{5})$. Then for $0 < |x-2| < \delta$:

$$\left|x^2-4\right| < 5|x-2| < 5\delta \le \varepsilon.$$

(c) This is precisely the definition of $f'(a)$: the limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a)$$

means that for any $\varepsilon>0$, there exists $\delta>0$ such that the inequality holds whenever $0<|h|<\delta$. $\square$

Solution 17

Proof. **(a)** Compute partial derivatives: $\frac{\partial f}{\partial x}=2x+y$ and $\frac{\partial f}{\partial y}=x$. At $(2,3)$:

$$\left.\frac{\partial f}{\partial x}\right|_{(2,3)}=7 \quad\text{and}\quad \left.\frac{\partial f}{\partial y}\right|_{(2,3)}=2.$$

The differential is the linear map $df_{(2,3)}:\mathbb{R}^2\to\mathbb{R}$ given by

$$df_{(2,3)}(h_1,h_2)=7h_1+2h_2.$$

(b) We have $f(2,3)=4+6=10$. The linear approximation gives

$$f(2.1,2.9)\approx f(2,3)+df_{(2,3)}(0.1,-0.1)=10+7(0.1)+2(-0.1)=10.5.$$

(c) Exact value: $f(2.1,2.9)=(2.1)^2+(2.1)(2.9)=4.41+6.09=10.5$. Error: 0. $\square$

Solution 18

Proof. **(a)** Expand: $g(a+h)=(a+h)^3=a^3+3a^2h+3ah^2+h^3$. Set $L(h):=3a^2h$ and $o(h):=3ah^2+h^3$. Then

$$g(a+h)=g(a)+L(h)+o(h).$$

Moreover,

$$\lim_{h\to 0}\frac{o(h)}{h}=\lim_{h\to 0}(3ah+h^2)=0.$$

(b) The error term is $o(h)=3ah^2+h^3=h^2(3a+h)$.

(c) For $a=2$ and $h=0.1$:

$$2.1^3\approx g(2)+L(0.1)=8+3(4)(0.1)=8+1.2=9.2.$$

Exact: $2.1^3=9.261$. Error: 0.061. $\square$

Solution 19

Proof. **(a)** We have $(g \circ f)(x) = \sin(x^2)$. Differentiating:

$$(g \circ f)'(x) = \cos(x^2) \cdot 2x.$$

At $x = \pi/2$:

$$d(g \circ f)_{\pi/2}(h) = \pi \cos(\pi^2/4) \cdot h.$$

(b) Compute $df_{\pi/2}(h) = 2(\pi/2)h = \pi h$. Next, $f(\pi/2) = \pi^2/4$, so

$$dg_{\pi^2/4}(k) = \cos(\pi^2/4) \cdot k.$$

The composition is

$$(dg_{f(\pi/2)} \circ df_{\pi/2})(h) = dg_{\pi^2/4}(\pi h) = \pi \cos(\pi^2/4) \cdot h.$$

This matches part (a).

(c) The chain rule says the differential of a composition is the composition of differentials. Geometrically: approximate f near $\pi/2$ by its tangent line (slope π), then approximate g at the new point by its tangent line (slope $\cos(\pi^2/4)$). Composing these linear approximations gives the correct derivative. □

Solution 20

Proof. **(a)** By definition:

$$f'(4) = \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} = \frac{1}{4}.$$

Thus $df_4(h) = \frac{h}{4}$.

(b) Linear approximation: $\sqrt{4+h} \approx 2 + \frac{h}{4}$.

- $\sqrt{4.2} \approx 2 + \frac{0.2}{4} = 2.05$. (Exact: 2.049, error ≈ 0.001.)
- $\sqrt{3.8} \approx 2 + \frac{-0.2}{4} = 1.95$. (Exact: 1.949, error ≈ 0.001.)
- $\sqrt{5} \approx 2 + \frac{1}{4} = 2.25$. (Exact: 2.236, error ≈ 0.014.)

(c) The approximation is most accurate for $\sqrt{4.2}$ (smallest $|h|$), least accurate for $\sqrt{5}$ (largest $|h|$). The second derivative is

$$f''(x) = -\frac{1}{4x^{3/2}},$$

which measures the curvature of the graph. Larger $|h|$ produces a greater accumulated effect of curvature, so the linear approximation deviates more significantly from the actual function value. □