Sequences — Problem Set 3

1. Let (a_n) be defined by
$$a_n = \frac{n^2+1}{n^2+n}.$$
   (a) Compute a_1, a_2, a_3.
   (b) Determine $\lim_{n\to\infty} a_n$.
   (c) Show using the ε-N definition that the limit is correct.
   (d) Find N such that $|a_n - L| < 0.01$ for all $n \geq N$.

2. Consider the sequence
$$b_n = \frac{\ln(n+1)}{n}.$$
   (a) Find two sequences $x_n \leq b_n \leq y_n$ that squeeze b_n.
   (b) Use the squeeze theorem to determine $\lim_{n\to\infty} b_n$.
   (c) Explain why your choice of x_n and y_n works.

3. Let (c_n) be defined by
$$c_n = \frac{(-1)^n}{n} + 2.$$
   (a) Determine whether (c_n) is monotone.
   (b) If not, split it into monotone *subsequences*, one even and one for odd n.
   (c) Use the Monotone Convergence Theorem to find the limit of each subsequence.
   (d) Determine $\lim_{n\to\infty} c_n$.

4. Let (x_n) and (y_n) satisfy $x_n + y_n \to 5$ and $x_n \to 2$.
   (a) Determine $\lim_{n\to\infty} y_n$.
   (b) Suppose $x_n \cdot y_n \to 6$ and $x_n \to 2$. Find $\lim_{n\to\infty} y_n$.
   (c) Explain which limit laws are used.

5. Let
$$a_n = \frac{n\sin(1/n)}{1+1/n}.$$
   (a) Show that (a_n) is bounded.

(b) Find sequences to squeeze a_n.

(c) Determine $\lim_{n\to\infty} a_n$.

Solutions

1. (a)
$$a_1 = 2, \quad a_2 = \frac{5}{3}, \quad a_3 = \frac{10}{6} = \frac{5}{3}.$$
   (b)
$$\lim_{n\to\infty} a_n = 1.$$
   (c) Given $\varepsilon > 0$, choose $N > 1/\varepsilon$. Then for all $n \geq N$,
$$|a_n - 1| = \left|\frac{n^2+1}{n^2+n} - 1\right| = \left|\frac{1-n}{n^2+n}\right| = \frac{1}{n} < \varepsilon.$$
   (d) For $|a_n - 1| < 0.01$, choose $N = 100$. Then $n \geq 100$ satisfies the inequality.

2. (a) $0 \leq \frac{\ln(n+1)}{n} \leq \frac{n}{n} = 1$ for $n \geq 1$.
   (b) By the squeeze theorem, $\lim_{n\to\infty} b_n = 0$.
   (c) Because $\ln(n+1) < n$ for all $n \geq 1$, the inequalities hold and both bounds converge to 0.

3. (a) Not monotone due to the alternating sign of $(-1)^n$.
   (b) Split into subsequences $c_{2n} = \frac{1}{2n} + 2$, $c_{2n+1} = -\frac{1}{2n+1} + 2$, which are monotone.
   (c) $c_{2n} \to 2$, $c_{2n+1} \to 2$, so both converge to 2.
   (d) $\lim_{n\to\infty} c_n = 2$.

4. (a) $y_n = 5 - x_n \to 5 - 2 = 3$.
   (b) $y_n = 6/x_n \to 6/2 = 3$.
   (c) Uses sum, difference, and quotient limit laws.

5. (a) Bounded since $0 < \sin(1/n) < 1$ and $1 + 1/n > 1$, so $0 < a_n < n/(1+1/n) < \infty$.
   (b) $0 \leq a_n \leq \frac{n\cdot 1}{1+1/n} = \frac{n}{1+1/n}$. Also, $\cos(1/n) < \frac{\sin(1/n)}{1/n} < 1$, so $a_n \approx n \cdot 1/n = 1$.
   (c) $\lim_{n\to\infty} a_n = 1$.