Lattice-Theoretic Framework for Data-Flow Analysis

Last time
- Generalizing data-flow analysis

Today
- Introduce lattice-theoretic frameworks for data-flow analysis

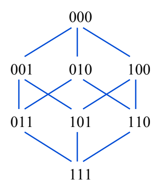
Context for Lattice-Theoretic Framework

Goals
- Provide a single formal model that describes all data-flow analyses
- Formalize the notions of “correct,” “conservative,” and “optimistic”
- Correctness proof for IDFA (iterative data-flow analysis)
- Place bounds on time complexity of data-flow analysis

Approach
- Define **domain** of program properties (flow values) computed by data-flow analysis, and organize the domain of elements as a **lattice**
- Define flow functions and a merge function over this domain using lattice operations
- Exploit lattice theory in achieving goals

Lattices

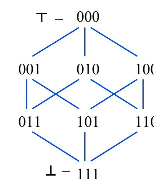
Define lattice L = (V, $\sqcap$)
- V is a set of elements of the lattice
- $\sqcap$ is a binary relation over the elements of V (**meet** or **greatest lower bound**)



Properties of $\sqcap$
- $x,y \in V \Rightarrow x \sqcap y \in V$ (closure)
- $x,y \in V \Rightarrow x \sqcap y = y \sqcap x$ (commutativity)
- $x,y,z \in V \Rightarrow (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$ (associativity)

Lattices (cont)

Under ($\sqsubseteq$)
- Imposes a partial order on V
- $x \sqsubseteq y \Leftrightarrow x \sqcap y = x$



Top ($\top$)
- A unique “greatest” element of V (if it exists)
- $\forall x \in V - \{\top\}, x \sqsubseteq \top$

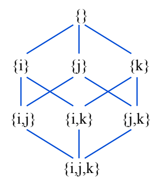
Bottom ($\perp$)
- A unique “least” element of V (if it exists)
- $\forall x \in V - \{\perp\}, \perp \sqsubseteq x$

Height of lattice L
- The longest path through the partial order from greatest to least element (top to bottom)

Data-Flow Analysis via Lattices

Relationship

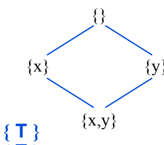
- Elements of the lattice (V) represent flow values (in[] and out[] sets)
  - *e.g.*, Sets of live variables for liveness
- $\top$ represents "best-case" information (initial flow value)
  - *e.g.*, Empty set
- $\perp$ represents "worst-case" information
  - *e.g.*, Universal set
- $\sqcap$ (meet) merges flow values
  - *e.g.*, Set union
- If $x \sqsubseteq y$, then x is a conservative approximation of y
  - *e.g.*, Superset



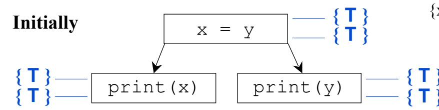
Data-Flow Analysis via Lattices (cont)

Remember what these flow values represent

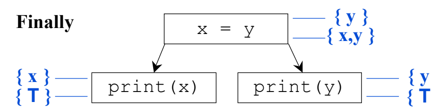
- At each program point a lattice element represents an in[] set or an out[] set



Initially



Finally



Data-Flow Analysis Frameworks

Data-flow analysis framework

- A set of **flow values** (V)
- A binary **meet operator** ($\sqcap$)
- A set of **flow functions** (F) (also known as **transfer functions**)

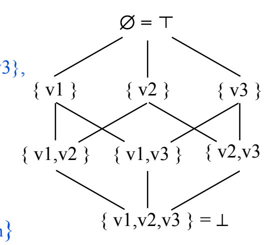
Flow Functions

- F = {f: V→V}
  f describes how each node in CFG affects the flow values
- Flow functions map program behavior onto lattices

Visualizing DFA Frameworks as Lattices

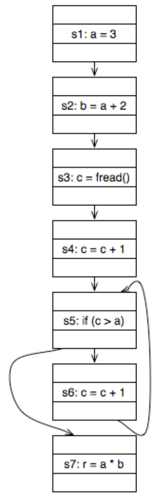
Example: Liveness analysis with 3 variables
U = {v1, v2, v3}

- V: **2^S =** {{v1,v2,v3}, {v1,v2},{v1,v3},{v2,v3}, {v1},{v2},{v3}, ∅}
- Meet ($\sqcap$): $\cup$
  - $\sqsubseteq$: $\supseteq$
  - Top($\top$): $\varnothing$
  - Bottom ($\perp$): **U**
- F: $\{f_n(X) = Gen_n \cup (X - Kill_n), \forall n\}$



Inferior solutions are lower on the lattice
More conservative solutions are lower on the lattice

Lattice Example



What is the data-flow set for liveness?

What is the meet operation for liveness?

What partial order does the meet operation induce?

What is the liveness lattice for this example?

Recall Liveness Analysis

Data-flow equations for liveness

$\text{in}[n] = \textbf{use}[n] \cup (\text{out}[n] - \textbf{def}[n])$

$\text{out}[n] = \bigcup_{s \in \text{succ}[n]} \text{in}[s]$

Liveness equations in terms of Gen and Kill

$\text{in}[n] = \textbf{gen}[n] \cup (\text{out}[n] - \textbf{kill}[n])$

$\text{out}[n] = \bigcup_{s \in \text{succ}[n]} \text{in}[s]$

A use of a variable generates liveness
A def of a variable kills liveness

Gen: New information that's added at a node
Kill: Old information that's removed at a node

Can define (almost) any data-flow analysis in terms of Gen and Kill

More Examples

Reaching definitions
- V: **2^S (S = set of all defs)**
- $\sqcap$: $\cup$
  - $\sqsubseteq$: $\supseteq$
  - Top($\top$): $\varnothing$
  - Bottom ($\perp$): **U**
- F: . . .

Reaching Constants
- V: **$2^{v \times c}$, variables v and constants c**
- $\sqcap$: $\cap$
  - $\sqsubseteq$: $\subseteq$
  - Top($\top$): **U**
  - Bottom ($\perp$): $\varnothing$
- F: . . .

Direction of Flow

Backward data-flow analysis
- Information at a node is based on what happens later in the flow graph
  i.e., in[] is defined in terms of out[]

$\text{in}[n] = \text{gen}[n] \cup (\text{out}[n] - \text{kill}[n])$
$\text{out}[n] = \bigcup_{s \in \text{succ}[n]} \text{in}[s]$

n: in ↑ out — liveness

Forward data-flow analysis
- Information at a node is based on what happens earlier in the flow graph
  i.e., out[] is defined in terms of in[]

$\text{in}[n] = \bigcup_{p \in \text{pred}[n]} \text{out}[p]$
$\text{out}[n] = \text{gen}[n] \cup (\text{in}[n] - \text{kill}[n])$

n: in ↓ out — reaching definitions

Some problems need both forward and backward analysis
- *e.g.,* Partial redundancy elimination (uncommon)

Merging Flow Values

Live variables and reaching definitions

– Merge **flow values** via set union

Reaching Definitions

$$in[n] = \bigcup_{p \in pred[n]} out[s]$$

$$out[n] = gen[n] \cup (in[n] - kill[n])$$

Live Variables

$$out[n] = \bigcup_{s \in succ[n]} in[s]$$

$$in[n] = gen[n] \cup (out[n] - kill[n])$$

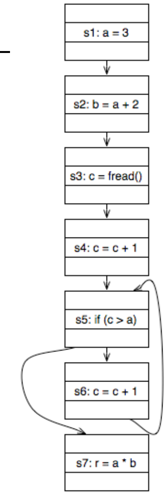
Why?

When might this be inappropriate?

Reaching Defs Example

What is the initial guess?

What is the meet operation?



Available Expressions (cont)

Data-Flow Equations

$$in[n] = \bigcap_{p \in pred[n]} out[p]$$

$$out[n] = gen[n] \cup (in[n] - kill[n])$$

Plug it in to our general DFA algorithm

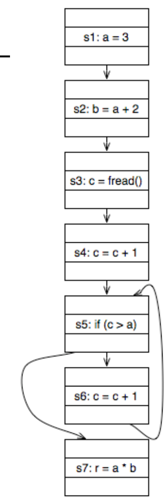
for each node n
  in[n] $= \upsilon$; out[n] $= \upsilon$
repeat
  for each n
    in′[n] = in[n]
    out′[n] = out[n]
    $in[n] = \bigcap_{p \in pred[n]} out[p]$
    $out[n] = gen[n] \cup (in[n] - kill[n])$
until in′[n]=in[n] and out′[n]=out[n] for all n

Available Expressions Example

What is the initial guess?

What is the meet operation?

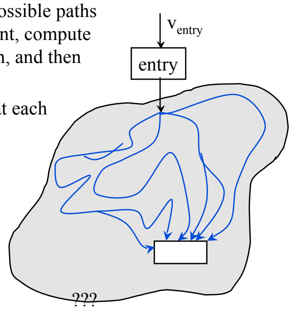
What does the lattice look like?



Solving Data-Flow Analyses

Goal

- For a forward problem, consider all possible paths from the entry to a given program point, compute the flow values at the end of each path, and then meet these values together
- Meet-over-all-paths (MOP) solution at each program point
- $\sqcap_{\text{all paths } n1, n2, \ldots, ni} (f_{ni}(\ldots f_{n2}(f_{n1}(v_{entry}))))$



Solving Data-Flow Analyses (cont)

Problems

- Loops result in an infinite number of paths
- Statements following merge must be analyzed for all preceding paths
  - Exponential blow-up

Solution

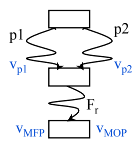
- Compute meets early (at merge points) rather than at the end
- Maximum fixed-point (MFP)

Questions

- Is this correct?
- Is this efficient?
- Is this accurate?

Correctness

"Is v_{MFP} correct?" ≡ "Is $v_{MFP} \sqsubseteq v_{MOP}$?"



Look at Merges

$v_{MOP} = F_r(v_{p1}) \sqcap F_r(v_{p2})$
$v_{MFP} = F_r(v_{p1} \sqcap v_{p2})$
$v_{MFP} \sqsubseteq v_{MOP} \equiv F_r(v_{p1} \sqcap v_{p2}) \sqsubseteq F_r(v_{p1}) \sqcap F_r(v_{p2})$

Observation

$\forall x,y \in V$

$$\mathbf{f(x \sqcap y) \sqsubseteq f(x) \sqcap f(y) \quad \Leftrightarrow \quad x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)}$$

∴ v_{MFP} correct when F_r (really, the flow functions) are monotonic

Monotonicity

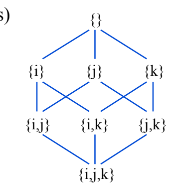
Monotonicity: $(\forall x,y \in V)[x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)]$

- If the flow function f is applied to two members of V, the result of applying f to the "lesser" of the two members will be under the result of applying f to the "greater" of the two
- Giving a flow function more conservative inputs leads to more conservative outputs (never more optimistic outputs)

Why else is monotonicity important?

For monotonic F over domain V

- The maximum number of times F can be applied to self w/o reaching a fixed point is height(V) – 1
- IDFA is guaranteed to terminate if the flow functions are monotonic and the lattice has finite height



Efficiency

Parameters
- n: Number of nodes in the CFG
- k: Height of lattice
- t: Time to execute one flow function

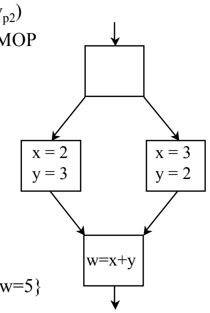
Complexity
- O(nkt)

Example
- Reaching definitions?

Accuracy

Distributivity
- $f(u \sqcap v) = f(u) \sqcap f(v)$
- $v_{MFP} \sqsubseteq v_{MOP} \equiv F_r(v_{p1} \sqcap v_{p2}) \sqsubseteq F_r(v_{p1}) \sqcap F_r(v_{p2})$
- If the flow functions are distributive, MFP = MOP

Examples
- Reaching definitions?
- Reaching constants?



$f(u \sqcap v) = f(\{x=2,y=3\} \sqcap \{x=3,y=2\})$

$= f(\varnothing) = \varnothing$

$f(u) \sqcap f(v) = f(\{x=2,y=3\}) \sqcap f(\{x=3,y=2\})$

$= [\{x=2,y=3,w=5\} \sqcap \{x=2,y=2,w=5\}] = \{w=5\}$

$\Rightarrow$ MFP $\neq$ MOP

Tuples of Lattices

Problem
- Simple analyses may require very complex lattices (*e.g.,* Reaching constants)

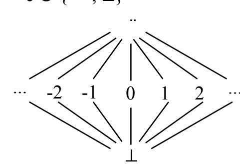
Solution
- Use a tuple of lattices, one per variable

$\mathbf{L = (V, \sqcap) \equiv (L_T = (V_T, \sqcap_T))^N}$
- $V = (V_T)^N$
- Meet ($\sqcap$): point-wise application of $\sqcap_T$
- $(\ldots, v_i, \ldots) \sqsubseteq (\ldots, u_i, \ldots) \equiv v_i \sqsubseteq u_i, \forall\, i$
- Top ($\top$): tuple of tops ($\top_T$)
- Bottom ($\perp$): tuple of bottoms ($\perp_T$)
- Height (L) = N * height(L_T)

Tuples of Lattices Example

Reaching constants (previously)
- $P = v \times c$, for variables v & constants c
- V: 2^P

Alternatively
- $V = c \cup \{\top, \perp\}$



The whole problem is a tuple of lattices, one for each variable

Examples of Lattice Domains

Two-point lattice ($\top$ and $\perp$)
- Examples?
- Implementation?

Set of incomparable values (and $\top$ and $\perp$)
- Examples?

Powerset lattice (2^S)
- $\top = \emptyset$ and $\perp = S$, or vice versa
- Isomorphic to tuple of two-point lattices

Concepts

Lattices
- Conservative approximation
- Optimistic (initial guess)
- Data-flow analysis frameworks
- Tuples of lattices

Data-flow analysis
- Fixed point
- Meet-over-all-paths (MOP)
- Maximum fixed point (MFP)
- Legal/safe/correct (monotonic)
- Efficient
- Accurate (distributive)

Next Time

Lecture
- Some transformations that you can implement for Project 4
  - Copy propagation
  - Constant propagation
  - Common sub-expression elimination