STARKWARE STARK 101

Stark 101: Part 2

Polynomial Constraints

What Do We Want to Prove?

There is a number x such that:

$a_0 = 1$

$a_1 = x$

$a_{1022} = 2338775057$

For $\{a_n\}$ FibonacciSq: $a_{n+2} = a_{n+1}^2 + a_n^2$, for any n

We will use part I:

Trace - a

Generator of G - g

Trace Polynomial - $f(x)$

Constraints on $\{a_n\}$

We need:

$a_0 = 1$

$a_{1022} = 2338775057$

$a_{n+2} = a_{n+1}^2 + a_n^2$

Where are We Heading?

Constraints on $\{a_n\}$:

$a_0 = 1$

$a_{1022} = 2338775057$

$a_{n+2} = a_{n+1}^2 + a_n^2$

Reductions

Another statement

Implies

Where are We Heading?

Constraints on $\{a_n\}$:

$a_0 = 1$

$a_{1022} = 2338775057$

$a_{n+2} = a_{n+1}^2 + a_n^2$

Reductions

Exists a polynomial $f(x)$

such that:

3 rational functions

$p_0(x)$, $p_1(x)$, $p_2(x)$ are **polynomials**

Where are We Heading?

Trace polynomial

Constraints on {a_n}:

$a_0 = 1$

$a_{1022} = 2338775057$

$a_{n+2} = a_{n+1}^2 + a_n^2$

Reductions

Exists a polynomial $f(x)$

such that:

3 rational functions

$$\frac{a(x)}{b(x)}$$

$p_0(x)$, $p_1(x)$, $p_2(x)$ are **polynomials**

Step I - From $\{a_n\}$ to $f(x)$

Trace polynomial

3 constraints on $\{a_n\}$ ------► **3 constraints on $f(x)$**

$a_0 = 1$ ------► $f(x) = 1$, for $x = g^0$

$a_{1022} = 2338775057$ ------► $f(x) = 2338775057$, for $x = g^{1022}$

$a_{n+2} = a_{n+1}^{\ 2} + a_n^{\ 2}$ ------► $f(g^2x) = f(gx)^2 + f(x)^2$,

for $x = g^i$, $0 \le i \le 1020$

x	$f(x)$
g^0	a_0
g^1	a_1
g^2	a_2
...	...
g^{1022}	a_{1022}

Third $f(x)$ Constraint

$$a_{n+2} = a_{n+1}^{\;2} + a_n^{\;2} \dashrightarrow f(g^2x) = f(gx)^2 + f(x)^2,$$

for $x = g^i,\ 0 \le i \le 1020$

Example: for $x = g^5$:

$$\underbrace{f(g^2 \cdot g^5)}_{g^7} = \underbrace{f(g \cdot g^5)}_{g^6}{}^2 + \underbrace{f(g^5)}_{g^5}{}^2$$

x	$f(x)$
g^0	a_0
g^1	a_1
g^2	a_2
...	...
g^5	a_5
g^6	a_6
g^7	a_7
...	...
g^{1022}	a_{1022}

Step I - From $\{a_n\}$ to $f(x)$

3 constraints on $\{a_n\}$ - - - - - - ► **3 constraints on $f(x)$**

$a_0 = 1$ - - - - - - ► $f(x) = 1$, for $x = g^0$

$a_{1022} = 2338775057$ - - - - - - ► $f(x) = 2338775057$, for $x = g^{1022}$

$a_{n+2} = a_{n+1}{}^2 + a_n{}^2$ - - - - - - ► $f(g^2x) = f(gx)^2 + f(x)^2$,

for $x = g^i$, $0 \leq i \leq 1020$

Step I - From $\{a_n\}$ to $f(x)$

3 constraints on $\{a_n\}$	---▶	**3 constraints on $f(x)$**
$a_0 = 1$	---▶	$f(x) = 1$, for $x = g^0$
$a_{1022} = 2338775057$	---▶	$f(x) = 2338775057$, for $x = g^{1022}$
$a_{n+2} = a_{n+1}^2 + a_n^2$	---▶	$f(g^2x) = f(gx)^2 + f(x)^2$, for $x = g^i$, $0 \le i \le 1020$

$$\boxed{\text{If } f(x) \text{ satisfies constraints} \longrightarrow \text{Original statement is true}}$$

Step II - From Constraints to Roots

$\boldsymbol{f(x) - 1} = 0$, for $x = g^0$ $\dashrightarrow$ root: g^0

$\boldsymbol{f(x) - 2338775057} = 0$, for $x = g^{1022}$ $\dashrightarrow$ root: g^{1022}

$\boldsymbol{f(g^2x) - f(gx)^2 - f(x)^2} = 0$, for $x = g^i$, $0 \leq i \leq 1020$ $\dashrightarrow$ roots: $\{g^i \mid 0 \leq i \leq 1020\}$

Step II - From Constraints to Roots

$\boldsymbol{f(x) - 1} = 0$, for $x = g^0$ ⟶ root: g^0

$\boldsymbol{f(x) - 2338775057} = 0$, for $x = g^{1022}$ ⟶ root: g^{1022}

$\boldsymbol{f(g^2x) - f(gx)^2 - f(x)^2} = 0$, for $x = g^i$, $0 \le i \le 1020$ ⟶ roots: $\{g^i \mid 0 \le i \le 1020\}$

g^0 *is a root of* $\boldsymbol{f(x) - 1}$

g^{1022} *is a root of* $\boldsymbol{f(x) - 2338775057}$

$\{g^i \mid 0 \le i \le 1020\}$ *are roots of* $\boldsymbol{f(g^2x) - f(gx)^2 - f(x)^2}$

⟶ Original statement is true

Step III - From Roots to Rational Functions

Thm: z is a root of $p(x)$ ⇔ $(x - z)$ divides $p(x)$

Def: $(x - z)$ divides $p(x)$ if $p(x) / (x - z)$ is a polynomial

Polynomial

$$\frac{x^2 - 3x + 2}{x - 2} = \frac{(x-2)(x-1)}{x-2} = x - 1$$

2 is a root

Not polynomial

$$\frac{x^2 - 7x + 6}{x - 2} = \frac{(x-1)(x-6)}{x-2}$$

2 is NOT a root

Step III - From Roots to Rational Functions

Thm: z is a root of $p(x)$ ⇔ $(x - z)$ divides $p(x)$

Def: $(x - z)$ divides $p(x)$ if $p(x) / (x - z)$ is a polynomial

g^0 is a root of $f(x) - 1$ ---▶ $\dfrac{f(x) - 1}{x - g^0}$ is a polynomial

g^{1022} is a root of $f(x) - 2338775057$ ---▶ $\dfrac{f(x) - 2338775057}{x - g^{1022}}$ is a polynomial

Step III - From Roots to Rational Functions

$\{g^i \mid 0 \le i \le 1020\}$ are roots of $f(g^2x) - f(gx)^2 - f(x)^2$

$$\frac{f(g^2x) - f(gx)^2 - f(x)^2}{\prod_{i=0}^{1020}(x - g^i)}$$ is a polynomial

Not Efficient

Step III - From Roots to Rational Functions

$\{g^i \mid 0 \le i \le 1020\}$ are roots of $f(g^2x) - f(gx)^2 - f(x)^2$ - - - ►

$$\frac{f(g^2x) - f(gx)^2 - f(x)^2}{\prod_{i=0}^{1020}(x - g^i)}$$ is a polynomial

$$\prod_{i=0}^{1023}(x - g^i) = x^{1024} - 1$$

fix:

$$\frac{f(g^2x) - f(gx)^2 - f(x)^2}{(x^{1024} - 1)/[(x - g^{1021})(x - g^{1022})(x - g^{1023})]}$$

3 Rational Functions

$$p_0(x) = \frac{f(x) - 1}{x - g^0}$$

$$p_1(x) = \frac{f(x) - 2338775057}{x - g^{1022}}$$

$$p_2(x) = \frac{f(g^2x) - f(gx)^2 - f(x)^2}{(x^{1024} - 1)/[(x - g^{1021})(x - g^{1022})(x - g^{1023})]}$$

3 Rational Functions

$$p_0(x) = \frac{f(x) - 1}{x - g^0}$$

$$p_1(x) = \frac{f(x) - 2338775057}{x - g^{1022}}$$

$$p_2(x) = \frac{f(g^2x) - f(gx)^2 - f(x)^2}{(x^{1024} - 1)/\left[(x - g^{1021})(x - g^{1022})(x - g^{1023})\right]}$$

If $p_0(x)$, $p_1(x)$, $p_2(x)$ are polynomials $\longrightarrow$ Original statement is true!

Where are We Heading?

Constraints on $\{a_n\}$:

$a_0 = 1$

$a_{1022} = 2338775057$

$a_{n+2} = a_{n+1}^2 + a_n^2$

Reductions

$\dashrightarrow$

Exists a polynomial $f(x)$

such that:

3 rational functions

$p_0(x)$, $p_1(x)$, $p_2(x)$ are **polynomials**

Reduction Overview - First Constraint

Reductions

$a_0 = 1$ $\dashrightarrow$

Exists a polynomial

$f(x)$ such that:

$p_0(x)$ *is a* **polynomial**

Reduction Overview - First Constraint

$a_0 = 1$

Reductions

Step I

$f(x) = 1$

for $x = g^0$

Step II

g^0 is a root

of $f(x) - 1$

Step III

Exists a polynomial

$f(x)$ such that:

$$p_0(x) = \frac{f(x) - 1}{x - g^0}$$

is a **polynomial**

3 Rational Functions

$$p_0(x) = \frac{f(x) - 1}{x - g^0}$$

$$p_1(x) = \frac{f(x) - 2338775057}{x - g^{1022}}$$

$$p_2(x) = \frac{f(g^2x) - f(gx)^2 - f(x)^2}{(x^{1024} - 1)/\left[(x - g^{1021})(x - g^{1022})(x - g^{1023})\right]}$$

If $p_0(x)$, $p_1(x)$, $p_2(x)$ are polynomials $\longrightarrow$ Original statement is true!

Combining $p_i(x)$'s

Random linear combination:

Composition Polynomial

$$CP = \alpha_0 \cdot p_0(x) + \alpha_1 \cdot p_1(x) + \alpha_2 \cdot p_2(x)$$

With high probability:

CP is a polynomial $\Leftrightarrow$ all p_i's are polynomials

Commiting on CP with Merkle Tree

What's Next?

Part 3 - how to prove that *CP* is a polynomial?

But first - coding.....

1) $p_0(x), p_1(x), p_2(x)$

2) $CP = \alpha_0 \cdot p_0(x) + \alpha_1 \cdot p_1(x) + \alpha_2 \cdot p_2(x)$

3) Commit on *CP*

Thank you